$C^{1,\alpha}$-REGULARITY OF QUASILINEAR EQUATIONS ON THE HEISENBERG GROUP

SHIRSHO MUKHERJEE

Abstract. In this article, we reproduce results of classical regularity theory of quasilinear elliptic equations in the divergence form, in the setting of Heisenberg Group. The considered cases encompass a very wide class of equations with isotropic growth conditions that are generalizations of the p-Laplacian type equation and also include equations with polynomial or exponential type growth. Some more general conditions have also been explored.

1. Introduction

Regularity theory for weak solutions of second order quasilinear elliptic equations in the Euclidean spaces, has been well-developed over a long period of time since the pioneering work of De Giorgi [13] and has involved significant contributions of many authors. For more details on this topic, we refer to [50, 14, 51, 29, 27, 54, 20, 37], etc. and references therein. A comprehensive study of the subject can be found in the nowadays classical books by Gilbarg-Trudinger [31], Ladyzhenskaya-Ural'tseva [36] and Morrey [43].

The goal of this paper is to obtain regularity results in the setting of Heisenberg Group $\mathbb{H}^n$, that are previously known in the Euclidean setting. We consider the equation

$$\mathrm{Q}u = \mathrm{div}_H\, A(x,u,\mathfrak{X}u) + B(x,u,\mathfrak{X}u) = 0 \tag{1.1}$$

in a domain $\Omega \subset \mathbb{H}^n$ for any $n \geq 1$, where $\mathfrak{X}u = (X_1 u, \ldots, X_{2n}u)$ is the horizontal gradient of a function $u : \Omega \to \mathbb{R}$ and div_H is the horizontal divergence of a vector field (see Section 2 for details). Here $A : \Omega \times \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and $B : \Omega \times \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$ are given locally integrable functions. We also assume that A is differentiable and the $(2n \times 2n)$ matrix $D_p\, A(x,z,p) = (\partial A_i(x,z,p)/\partial p_j)_{ij}$ is non-negative definite for every $x \in \Omega, z \in \mathbb{R}$ and $p = (p_1, \ldots, p_{2n}) \in \mathbb{R}^{2n}$. Thus, the results of this setting can also be applied to minimizers of a variational integral

$$I(u) = \int_\Omega f(x,u,\mathfrak{X}u)\, dx$$

for a smooth scalar function $f : \Omega \times \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$; the Euler-Lagrange equation corresponding to the functional I, would be an equation of the form (1.1). The equations in settings similar to ours, are often referred as sub-elliptic equations.

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In addition to A and B, we consider a C^1-function $g : [0,\infty) \to [0,\infty)$ also as given data, which satisfies $g(0) = 0$ and there exists constants $g_0 \geq \delta \geq 0$ such that the following holds,

$$\delta \leq \frac{tg'(t)}{g(t)} \leq g_0 \quad \text{for all } t > 0. \tag{1.2}$$

The function g shall be used in the hypothesis of growth and ellipticity conditions satisfied by A and B, as given below. The condition (1.2) appears in the work of Lieberman [39], in the Euclidean setting. In the case of Heisenberg Groups, a special class of quasilinear equations with growth conditions involving (1.2), has been recently studied in [44]. We remark that the special case $g(t) = t^{p-1}$ for $1 < p < \infty$, would correspond to equations with p-laplacian type growth. For a more detailed discussion on the relevance of the condition (1.2) and more examples of such function g, we refer to [39, 41, 1], etc.

The study of regularity theory for sub-elliptic equations goes back to the fundamental work of Hörmander [33] which was followed by the work of Folland-Stein [23], Folland [22] and Rothschild-Stein [48] that developed a comprehensive literature for sub-elliptic linear equations. However, the corresponding picture for quasi-linear equations remained open for some time and involved the work of many authors. The Poincaré inequality on vector fields was first proved by Jerison [34] leading to the Sobolev embedding theorem which has been later shown in different levels of generality later by Garofalo-Nhieu [26], Franchi-Lu-Wheeden [25], Hajlasz-Koskela [32], etc. Some notable works thereafter include [5, 9, 11, 21]. The higher regularity for the gradient turned out to be quite non-trivial due to the non-commuting nature of the vector fields. Towards this goal, notable works include Domokos-Manfredi [17, 18] and [42, 40, 16], etc. Finally, the regularity of the gradient were established for the Heisenberg group in the works of Ricciotti [47], Zhong [55] and Mukherjee-Zhong [45], the techniques of these have been adopted in this paper. Further notable works in more general vector fields can be found in Capogna et al. [7], Domokos-Manfredi [19] and Citti et al. [12].

The structure conditions for the equation (1.1) used in this paper, have been introduced in [39], which are generalizations of the so called natural conditions for elliptic equations in divergence form; these have been extensively studied by Ladyzhenskaya-Ural'tseva in [36] for equations in the Euclidean setting. The first structure condition is as follows.

Given some non-negative constants a_1, a_2, a_3, b_0, b_1 and χ, we assume that A and B satisfies

$$\begin{aligned}
&\big\langle A(x,z,p), p\big\rangle \geq |p| g(|p|) - a_1\, g\Big(\frac{|z|}{R}\Big)\frac{|z|}{R} - g(\chi)\chi;\\
&|A(x,z,p)| \leq a_2\, g(|p|) + a_3\, g\Big(\frac{|z|}{R}\Big) + g(\chi);\\
&|B(x,z,p)| \leq \frac{1}{R}\Big[b_0\, g(|p|) + b_1\, g\Big(\frac{|z|}{R}\Big) + g(\chi)\Big],
\end{aligned} \tag{1.3}$$

where $(x,z,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^{2n}$ and $0 < R < \frac{1}{2}\operatorname{diam}(\Omega)$. Similar growth conditions have been considered previously in [31],[36] and [52] for the special case $g(t) = t^{\alpha-1}$ for $\alpha > 1$.

For weak solutions of equation (1.1) with the above structure conditions, the appropriate domain is the Horizontal Orlicz-Sobolev space $HW^{1,G}(\Omega)$ (see Section 2 for the definition), where $G(t) = \int_0^t g(s)ds$. The following is the first result of this paper.

Theorem 1.1. *Let $u \in HW^{1,G}(\Omega) \cap L^\infty(\Omega)$ be a weak solution of the equation* (1.1)*, with $G(t) = \int_0^t g(s)ds$ and $|u| \leq M$ in Ω. Suppose the structure condition* (1.3) *holds for some*

$\chi \geq 0$, $0 < R \leq R_0$ and a function g satisfying (1.2) with $\delta > 0$, then there exists $c > 0$ and $\alpha \in (0,1)$ dependent on $n, \delta, g_0, a_1, a_2, a_3, b_0 M, b_1$ such that $u \in C^{0,\alpha}_{\rm loc}(\Omega)$ and

$$\operatorname{osc}_{B_r} u \leq c \left(\frac{r}{R}\right)^{\alpha} \Big(\operatorname{osc}_{B_R} u + \chi R\Big), \tag{1.4}$$

whenver $B_{R_0} \subset\subset \Omega$ and B_r, B_R are concentric to B_{R_0} with $0 < r < R \leq R_0$.

The above theorem follows as a consequence of Harnack inequalities, Theorem 3.4 and Theorem 3.5 in Section 3. Similar Harnack inequalities in the sub-elliptic setting, has also been shown in [9] for the special case of polynomial type growth. The proof of these are standard imitations of the corresponding classical results due to Serrin [49], see also [52, 39].

Theorem 1.1 is necessary for our second result, the $C^{1,\alpha}$-regularity of weak solutions. This is new and relies on some recent development in [44]. The structure conditions considered for this, are as follows. Given the constants $L, L' \geq 1$ and $\alpha \in (0,1]$, we assume

$$\begin{aligned} \frac{g(|p|)}{|p|}\,|\xi|^2 \leq \big\langle D_p\, A(x,z,p)\,\xi,\xi\big\rangle &\leq L\,\frac{g(|p|)}{|p|}\,|\xi|^2; \\ |A(x,z,p)-A(y,w,p)| &\leq L'\big(1+g(|p|)\big)\Big(|x-y|^{\alpha}+|z-w|^{\alpha}\Big); \\ |B(x,z,p)| &\leq L'\big(1+g(|p|)\big)|p|, \end{aligned} \tag{1.5}$$

for every $x, y \in \Omega$, $z, w \in [-M_0, M_0]$ and $p, \xi \in \mathbb{R}^{2n}$, where $M_0 > 0$ is another given constant. The following theorem is the second result of this paper.

Theorem 1.2. *Let $u \in HW^{1,G}(\Omega) \cap L^\infty(\Omega)$ be a weak solution of the equation (1.1), with $G(t) = \int_0^t g(s)ds$ and $\|u\| \leq M_0$ in Ω. Suppose the structure condition (1.5) holds for some $L, L' \geq 1, \alpha \in (0,1]$ and a function g satisfying (1.2) with $\delta > 0$, then there exists a constant $\beta = \beta(n, \delta, g_0, \alpha, L) \in (0,1)$ such that $u \in C^{1,\beta}_{\rm loc}(\Omega)$ and for any open $\Omega' \subset\subset \Omega$, we have*

$$|\mathfrak{X}u|_{C^{0,\beta}(\Omega',\mathbb{R}^{2n})} \leq C\big(n, \delta, g_0, \alpha, L, L', M_0, g(1), \operatorname{dist}(\Omega', \partial\Omega)\big). \tag{1.6}$$

Pertaining to the growth conditions involving (1.2), local Lipshcitz continuity for the class of equations of the form $\operatorname{div}_H \mathcal{A}(\mathfrak{X}u) = 0$, has been shown in [44]. As a follow up, here we show the $C^{1,\alpha}$-regularity for this case as well, with a robust gradient estimate unlike (1.6).

Theorem 1.3. *Let $u \in HW^{1,G}(\Omega)$ be a weak solution of the equation $\operatorname{div}_H \mathcal{A}(\mathfrak{X}u) = 0$, where $\mathcal{A} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and the following structure condition holds,*

$$\begin{gathered} \frac{g(|p|)}{|p|}\,|\xi|^2 \leq \big\langle D\mathcal{A}(p)\,\xi,\xi\big\rangle \leq L\,\frac{g(|p|)}{|p|}\,|\xi|^2; \\ |\mathcal{A}(p)| \leq L\,g(|p|). \end{gathered} \tag{1.7}$$

for every $p, \xi \in \mathbb{R}^{2n}$, $L \geq 1$ is a given constant and g satisfies (1.2) with $\delta > 0$. Then $\mathfrak{X}u$ is locally Hölder continuous and there exists $\sigma = \sigma(n, g_0, L) \in (0,1)$ and $c = c(n, \delta, g_0, L) > 0$ such that for any $B_{r_0} \subset \Omega$ and $0 < r < r_0/2$, we have

$$\max_{1\leq l\leq 2n} \fint_{B_r} G(|X_l u - \{X_l u\}_{B_r}|)\,dx \;\leq\; c\Big(\frac{r}{r_0}\Big)^{\sigma} \fint_{B_{r_0}} G(|\mathfrak{X}u|)\,dx. \tag{1.8}$$

The proof of the above theorem, follows similarly along the line of that in [45]. It involves Caccioppoli type estimates of the horizontal and vertical vector fields along with the use of an integrability estimate of [55] and a double truncation of [51] and [38].

We remark that the spaces $C^{0,\alpha}$ and $C^{1,\alpha}$ considered in this paper, are in the sense of Folland-Stein [24]. In other words, the spaces are defined with respect to the homogeneous metric of the Heisenberg Group, see Section 2 for details. No assertions are made concerning the regularity of the vertical derivative.

This paper is organised as follows. In Section 2, we provide a brief review on Heisenberg Group and Orlicz spaces. Then in Section 3, first we prove a global maximum principle exploring some generalised growth conditions along the lines of [39]; then we prove the Harnack inequalities, thereby leading to the proof of Theorem 1.1. The whole of Section 4 is devoted to the proof of Theorem 1.3. Finally in Section 5, the proof of Theorem 1.2 is provided and some possible extensions of the structure conditions are discussed.

2. Preliminaries and Previous results

In this section, we fix the notations used and provide a brief introduction of the Heisenberg Group $\mathbb{H}^n$. Also, we provide some essential facts on Orlicz spaces and the Horizontal Sobolev spaces and sub-elliptic equations, which are required for the purpose of this paper.

2.1. **Heisenberg Group.**

Here we provide the definition and properties of Heisenberg group that would be useful in this paper. For more details, we refer the reader to [2],[10], etc.

Definition 2.1. For $n \geq 1$, the *Heisenberg Group* denoted by $\mathbb{H}^n$, is identified to the Euclidean space $\mathbb{R}^{2n+1}$ with the group operation

$$x \cdot y := \Big(x_1 + y_1,\ \dots,\ x_{2n} + y_{2n},\ t + s + \frac{1}{2}\sum_{i=1}^{n}(x_i y_{n+i} - x_{n+i} y_i)\Big) \tag{2.1}$$

for every $x = (x_1, \dots, x_{2n}, t)$, $y = (y_1, \dots, y_{2n}, s) \in \mathbb{H}^n$.

Thus, $\mathbb{H}^n$ with the group operation (2.1) forms a non-Abelian Lie group, whose left invariant vector fields corresponding to the canonical basis of the Lie algebra, are

$$X_i = \partial_{x_i} - \frac{x_{n+i}}{2}\partial_t, \quad X_{n+i} = \partial_{x_{n+i}} + \frac{x_i}{2}\partial_t,$$

for every $1 \leq i \leq n$ and the only non zero commutator $T = \partial_t$. We have

$$[X_i\,, X_{n+i}] = T \quad \text{and} \quad [X_i\,, X_j] = 0 \ \ \forall\ j \neq n+i. \tag{2.2}$$

We call $X_1, \dots, X_{2n}$ as *horizontal vector fields* and T as the *vertical vector field*. For a scalar function $f : \mathbb{H}^n \to \mathbb{R}$, we denote $\mathfrak{X} f = (X_1 f, \dots, X_{2n} f)$ and $\mathfrak{X}\mathfrak{X} f = (X_i(X_j f))_{i,j}$ as the *Horizontal gradient* and *Horizontal Hessian*, respectively. From (2.2), we have the following trivial but nevertheless, an important inequality $|Tf| \leq 2|\mathfrak{X}\mathfrak{X} f|$. For a vector valued function $F = (f_1, \dots, f_{2n}) : \mathbb{H}^n \to \mathbb{R}^{2n}$, the *Horizontal divergence* is defined as

$$\mathrm{div}_H(F) = \sum_{i=1}^{2n} X_i f_i.$$

The Euclidean gradient of a function $g : \mathbb{R}^k \to \mathbb{R}$, shall be denoted by $\nabla g = (D_1 g, \dots, D_k g)$ and the Hessian matrix by $D^2 g$.

The *Carnot-Carathèodory metric* (CC-metric) is defined as the length of the shortest horizontal curves, connecting two points. This is equivalent to the *Korànyi metric*, denoted as $d_{\mathbb{H}^n}(x,y) = \|y^{-1}\cdot x\|_{\mathbb{H}^n}$, where the Korànyi norm for $x=(x_1,\ldots,x_{2n},t)\in\mathbb{H}^n$ is

$$\|x\|_{\mathbb{H}^n} := \Big(\ \sum_{i=1}^{2n} x_i^2 + |t|\ \Big)^{\frac{1}{2}}. \tag{2.3}$$

Throughout this article we use CC-metric balls denoted by $B_r(x) = \{y\in\mathbb{H}^n : d(x,y)<r\}$ for $r>0$ and $x\in\mathbb{H}^n$. However, by virtue of the equivalence of the metrics, all assertions for CC-balls can be restated to Korànyi balls.

The Haar measure of $\mathbb{H}^n$ is just the Lebesgue measure of $\mathbb{R}^{2n+1}$. For a measurable set $E\subset\mathbb{H}^n$, we denote the Lebesgue measure as $|E|$. For an integrable function f, we denote

$$\{f\}_E = ⨍_E f\,dx = \frac{1}{|E|}\int_E f\,dx.$$

The Hausdorff dimension with respect to the metric d is also the homogeneous dimension of the group $\mathbb{H}^n$, which shall be denoted as $Q=2n+2$, throughout this paper. Thus, for any CC-metric ball B_r, we have that $|B_r| = c(n)r^Q$.

For $1\le p<\infty$, the *Horizontal Sobolev space* $HW^{1,p}(\Omega)$ consists of functions $u\in L^p(\Omega)$ such that the distributional horizontal gradient $\mathfrak{X}u$ is in $L^p(\Omega,\mathbb{R}^{2n})$. $HW^{1,p}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{HW^{1,p}(\Omega)} = \ \|u\|_{L^p(\Omega)} + \|\mathfrak{X}u\|_{L^p(\Omega,\mathbb{R}^{2n})}. \tag{2.4}$$

We define $HW^{1,p}_{\mathrm{loc}}(\Omega)$ as its local variant and $HW^{1,p}_0(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $HW^{1,p}(\Omega)$ with respect to the norm in (2.4). The Sobolev Embedding theorem has the following version in the setting of Heisenberg group (see [9],[10]).

Theorem 2.2 (Sobolev Embedding)**.** *Let $B_r\subset\mathbb{H}^n$ and $1<q<Q$. For all $u\in HW^{1,q}_0(B_r)$, there exists constant $c=c(n,q)>0$ such that*

$$\Big(⨍_{B_r} |u|^{\frac{Qq}{Q-q}}\,dx\Big)^{\frac{Q-q}{Qq}} \le\ c\,r\Big(⨍_{B_r} |\mathfrak{X}u|^q\,dx\Big)^{\frac{1}{q}}. \tag{2.5}$$

Hölder spaces with respect to homogeneous metrics have appeared in Folland-Stein [24] and therefore, are sometimes called are known as Folland-Stein classes and denoted by Γ^α or $\Gamma^{0,\alpha}$ in some literature. However, here we maintain the classical notation and define

$$C^{0,\alpha}(\Omega) = \{u\in L^\infty(\Omega) : |u(x)-u(y)| \le c\,d(x,y)^\alpha\ \forall\ x,y\in\Omega\} \tag{2.6}$$

for $0<\alpha\le 1$, which are Banach spaces with the norm

$$\|u\|_{C^{0,\alpha}(\Omega)} = \|u\|_{L^\infty(\Omega)} + \sup_{x,y\in\Omega}\frac{|u(x)-u(y)|}{d(x,y)^\alpha}. \tag{2.7}$$

These have standard extensions to classes $C^{k,\alpha}(\Omega)$ for $k\in\mathbb{N}$, which consists of functions having horizontal derivatives up to order k in $C^{0,\alpha}(\Omega)$. The local counterparts are denoted as $C^{k,\alpha}_{\mathrm{loc}}(\Omega)$. Now, the definition of Morrey and Campanato spaces in sub-elliptic setting differs in different texts. Here, we adopt the definition similar to the classical one.

For any domain $\Omega \subset \mathbb{H}^n$ and $\lambda > 0$, we define the *Morrey space* as

$$\mathcal{M}^{1,\lambda}(\Omega) = \left\{ u \in L^1_{\mathrm{loc}}(\Omega) : \int_{B_r} |u|\, dx < c\, r^\lambda \ \forall\ B_r \subset \Omega, r > 0 \right\} \tag{2.8}$$

and the *Campanato space* as

$$\mathcal{L}^{1,\lambda}(\Omega) = \left\{ u \in L^1_{\mathrm{loc}}(\Omega) : \int_{B_r} \left| u - \{u\}_{B_r} \right| dx < c\, r^\lambda \ \forall\ B_r \subset \Omega, r > 0 \right\}, \tag{2.9}$$

where in both definitions B_r represents balls with metric d. These spaces are Banach spaces and have properties similar to the classical spaces in the Euclidean setting. We shall use the fact that for every $0 < \alpha < 1$ and $Q = 2n + 2$, we have

$$\mathcal{L}^{1,Q+\alpha}(\Omega) \subset C^{0,\alpha}(\Omega), \tag{2.10}$$

where the inclusion is to be understood as taking continuous representatives. For details on classical Morrey and Campanato spaces, we refer to [35] and for the sub-elliptic setting we refer to [10].

2.2. **Orlicz-Sobolev Spaces.**
In this subsection, we recall some basic facts on Orlicz-Sobolev functions, which shall be necessary later. Further details can be found in textbooks e.g. [35],[46].

Definition 2.3 (Young function)**.** If $\psi : [0,\infty) \to [0,\infty)$ is an non-decreasing, left continuous function with $\psi(0) = 0$ and $\psi(s) > 0$ for all $s > 0$, then any function $\Psi : [0,\infty) \to [0,\infty]$ of the form

$$\Psi(t) = \int_0^t \psi(s)\, ds \tag{2.11}$$

is called a *Young function.* A continuous Young function $\Psi : [0,\infty) \to [0,\infty)$ satisfying $\Psi(t) = 0$ iff $t = 0$, $\lim_{t\to\infty} \Psi(t)/t = \infty$ and $\lim_{t\to 0} \Psi(t)/t = 0$, is called *N-function.*

There are several different definitions available in various references. However, within a slightly restricted range of functions (as in our case), all of them are equivalent. We refer to the book of Rao-Ren [46], for a more general discussion.

Definition 2.4 (Conjugate)**.** The *generalised inverse* of a montone function ψ is defined as $\psi^{-1}(t) := \inf\{s \geq 0 \mid \psi(s) > t\}$. Given any Young function $\Psi(t) = \int_0^t \psi(s) ds$, its *conjugate* function $\Psi^* : [0,\infty) \to [0,\infty]$ is defined as

$$\Psi^*(s) := \int_0^s \psi^{-1}(t)\, dt \tag{2.12}$$

and (Ψ, Ψ^*) is called a *complementary pair*, which is *normalised* if $\Psi(1) + \Psi^*(1) = 1$.

A Young function Ψ is convex, increasing, left continuous and satisfies $\Psi(0) = 0$ and $\lim_{t\to\infty} \Psi(t) = \infty$. The generalised inverse of Ψ is right continuous, increasing and coincides with the usual inverse when Ψ is continuous and strictly increasing. In general, the inequality

$$\Psi(\Psi^{-1}(t)) \leq t \leq \Psi^{-1}(\Psi(t)) \tag{2.13}$$

is satisfied for all $t \geq 0$ and equality holds when $\Psi(t)$ and $\Psi^{-1}(t) \in (0,\infty)$. It is also evident that that the conjugate function Ψ^* is also a Young function, $\Psi^{**} = \Psi$ and for any constant $c > 0$, we have $(c\,\Psi)^*(t) = c\,\Psi^*(t/c)$.

Here are two standard examples of complementary pair of Young functions.

(1) $\Psi(t) = t^p/p$ and $\Psi^*(t) = t^{p^*}/p^*$ when $1 < p, p^* < \infty$ and $1/p + 1/p^* = 1$.

(2) $\Psi(t) = (1+t)\log(1+t) - t$ and $\Psi^*(t) = e^t - t - 1$.

The following Young's inequality is well known. We refer to [46] for a proof.

Theorem 2.5 (Young's Inequality)**.** *Given a Young function $\Psi(t) = \int_0^t \psi(s)ds$, we have*

$$st \le \Psi(s) + \Psi^*(t) \tag{2.14}$$

for all $s, t > 0$ and equality holds if and only if $t = \psi(s)$ or $s = \psi^{-1}(t)$.

A Young function Ψ is called *doubling* if there exists a constant $C_2 > 0$ such that for all $t \ge 0$, we have $\Psi(2t) \le C_2\,\Psi(t)$. By virtue of (1.2), the structure function g is doubling with the doubling constant $C_2 = 2^{g_0}$ and hence, we restrict to Orlicz spaces of doubling functions.

Definition 2.6. Let $\Omega \subset \mathbb{R}^m$ be Borel and ν be a σ-finite measure on Ω. For a doubling Young function Ψ, the *Orlicz space* $L^\Psi(\Omega, \nu)$ is defined as the vector space generated by the set $\{u : \Omega \to \mathbb{R} \mid u \text{ measurable}, \int_\Omega \Psi(|u|)\, d\nu < \infty\}$. The space is equipped with the following *Luxemburg norm*

$$\|u\|_{L^\Psi(\Omega,\nu)} := \inf\Big\{k > 0 : \int_\Omega \Psi\Big(\frac{|u|}{k}\Big)\, d\nu \le 1\Big\} \tag{2.15}$$

If ν is the Lebesgue measure, the space is denoted by $L^\Psi(\Omega)$ and any $u \in L^\Psi(\Omega)$ is called a Ψ-integrable function.

The function $u \mapsto \|u\|_{L^\Psi(\Omega,\nu)}$ is lower semi continuous and $L^\Psi(\Omega, \nu)$ is a Banach space with the norm in (2.15). The following theorem is a generalised version of Hölder's inequality, which follows easily from the Young's inequality (2.14), see [46] or [53].

Theorem 2.7 (Hölder's Inequality)**.** *For every $u \in L^\Psi(\Omega, \nu)$ and $v \in L^{\Psi^*}(\Omega, \nu)$, we have*

$$\int_\Omega |uv|\, d\nu \le 2\, \|u\|_{L^\Psi(\Omega,\nu)} \|v\|_{L^{\Psi^*}(\Omega,\nu)} \tag{2.16}$$

Remark 2.8. The factor 2 on the right hand side of the above, can be dropped if (Ψ, Ψ^*) is normalised and one is replaced by $\Psi(1)$ in the definition (2.15) of Luxemburg norm.

The *Orlicz-Sobolev space* $W^{1,\Psi}(\Omega)$ can be defined similarly by L^Ψ norms of the function and its gradient, see [46], that resembles $W^{1,p}(\Omega)$. But here for $\Omega \subset \mathbb{H}^n$, we require the notion of *Horizontal Orlicz-Sobolev spaces*, analoguous to the horizontal Sobolev spaces defined in the previous subsection.

Definition 2.9. We define the space $HW^{1,\Psi}(\Omega) = \{u \in L^\Psi(\Omega) \mid \mathfrak{X}u \in L^\Psi(\Omega, \mathbb{R}^{2n})\}$ for an open set $\Omega \subset \mathbb{H}^n$ and a doubling Young function Ψ, along with the norm

$$\|u\|_{HW^{1,\Psi}(\Omega)} := \|u\|_{L^\Psi(\Omega)} + \|\mathfrak{X}u\|_{L^\Psi(\Omega,\mathbb{R}^{2n})}.$$

The spaces $HW^{1,\Psi}_{\text{loc}}(\Omega)$, $HW^{1,\Psi}_0(\Omega)$ are defined, similarly as earlier.

We remark that, all these notions can be defined for a general metric space, equipped with a doubling measure. We refer to [53] for the details.

The following theorem, so called (Ψ, Ψ)-Poincaré inequality, has been proved (see Proposition 6.23 in [53]) in the setting of a general metric space with a doubling measure and metric upper gradient. We provide the statement in the setting of Heisenberg Group.

Theorem 2.10. *Given any doubling N-function Ψ with doubling constant $c_2 > 0$, every $u \in HW^{1,\Psi}(\Omega)$ satisfies the following inequality for every $B_r \subset \Omega$ and some $c = c(n, c_2) > 0$,*

$$\fint_{B_r} \Psi\left(\frac{|u - \{u\}_{B_r}|}{r}\right) dx \leq c \fint_{B_r} \Psi(|\mathfrak{X}u|)\, dx. \tag{2.17}$$

In case of $\Psi(t) = t^p$, the inequality is referred as (p, p)-Poincaré inequality. The following corrollary follows easily from (2.17) and the $(1, 1)$-Poincaré inequality on $\mathbb{H}^n$.

Corollary 2.11. *Given a convex doubling N-function Ψ with doubling constant $c_2 > 0$, there exists $c = c(n, c_2)$ such that for every $B_r \subset \Omega$ and $u \in HW^{1,\Psi}(\Omega) \cap HW_0^{1,1}(\Omega)$, we have*

$$\fint_{B_r} \Psi\left(\frac{|u|}{r}\right) dx \leq c \fint_{B_r} \Psi(|\mathfrak{X}u|)\, dx. \tag{2.18}$$

Given a domain $\Omega \subset \mathbb{H}^n$, using (2.18) and arguments with chaining method (see [32]), it is also possible to show that for u, Ψ and $c = c(n, c_2) > 0$ as in Corrollary 2.11, we have

$$\fint_{\Omega} \Psi\left(\frac{|u|}{\operatorname{diam}(\Omega)}\right) dx \leq c \fint_{\Omega} \Psi(|\mathfrak{X}u|)\, dx. \tag{2.19}$$

Now we enlist some important properties of the function g that satisfies (1.2).

Lemma 2.12. *Let $g \in C^1([0, \infty))$ be a function that satisfies (1.2) for some constant $g_0 > 0$ and $g(0) = 0$. If $G(t) = \int_0^t g(s)ds$, then the following holds.*

$$(1)\quad G \in C^2([0, \infty)) \text{ is convex}; \tag{2.20}$$

$$(2)\quad tg(t)/(1 + g_0) \leq G(t) \leq tg(t) \quad \forall \quad t \geq 0; \tag{2.21}$$

$$(3)\quad g(s) \leq g(t) \leq (t/s)^{g_0} g(s) \quad \forall \quad 0 \leq s < t; \tag{2.22}$$

$$(4)\quad G(t)/t \text{ is an increasing function} \quad \forall \quad t > 0; \tag{2.23}$$

$$(5)\quad tg(s) \leq tg(t) + sg(s) \quad \forall \quad t, s \geq 0. \tag{2.24}$$

The proof is trivial (see Lemma 1.1 of [39]), so we omit it. Notice that (2.22) implies that g is increasing and doubling, with $g(2t) \leq 2^{g_0} g(t)$ and

$$\min\{1, \alpha^{g_0}\} g(t) \leq g(\alpha t) \leq \max\{1, \alpha^{g_0}\} g(t) \quad \text{for all } \alpha, t \geq 0. \tag{2.25}$$

Since G is convex, an easy application of Jensen's inequality yields

$$\fint_{\Omega} G(|w - \{w\}_\Omega|)\, dx \;\leq\; c(g_0) \min_{k\in\mathbb{R}} \fint_{\Omega} G(|w - k|)\, dx \quad \forall\, w \in L^G(\Omega) \tag{2.26}$$

All the above properties hold even if $\delta = 0$ in (1.2) and they are purposefully kept that way. However, the properties corresponding to $\delta > 0$, shall be required in some situations. For this case, (2.21) and (2.22) becomes

$$tg(t)/(1 + g_0) \leq G(t) \leq tg(t)/(1 + \delta) \quad \forall \quad t \geq 0; \tag{2.27}$$

$$(t/s)^{\delta} g(s) \leq g(t) \leq (t/s)^{g_0} g(s) \quad \forall \quad 0 \leq s < t, \tag{2.28}$$

and hence $t \mapsto g(t)/t^{g_0}$ is decreasing and $t \mapsto g(t)/t^{\delta}$ is increasing.

2.3. **Previous Results on sub-elliptic equations.**
Here we provide some results that are known and previously obtained, which would be essential for our purpose. For more details, we refer to [45, 44] and references therein.

In a domain $\Omega \subset \mathbb{H}^n$, let us consider

$$\operatorname{div}_H(\mathcal{A}(\mathfrak{X}u)) = 0 \quad \text{in } \Omega, \tag{2.29}$$

where $\mathcal{A} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a given C^1 function. We denote $\mathcal{A}(p) = (\mathcal{A}_1(p), \ldots, \mathcal{A}_{2n}(p))$ for all $p \in \mathbb{R}^{2n}$ and $D\mathcal{A}(p)$ as the $2n \times 2n$ Jacobian matrix $(\partial \mathcal{A}_i(p)/\partial p_j)_{ij}$. Here onwards, throughout this paper, we fix the notations

$$\mathrm{F}(t) := g(t)/t \quad \text{and} \quad G(t) := \int_0^t g(s)\, ds, \tag{2.30}$$

where $g : [0, \infty) \to [0, \infty)$ is a given C^1 function satisfying (1.2) and $g(0) = 0$ and we rewrite the conditions (1.7) as follows;

$$\begin{gathered} \mathrm{F}(|p|)|\xi|^2 \leq \big\langle D\mathcal{A}(p)\, \xi, \xi \big\rangle \leq L\, \mathrm{F}(|p|)|\xi|^2; \\ |\mathcal{A}(p)| \leq L\, |z| \mathrm{F}(|p|), \end{gathered} \tag{2.31}$$

for every $p, \xi \in \mathbb{R}^{2n}$ and $L \geq 1$. The following monotonicity and ellipticity inequalities follow easily from (2.31), which are essential to show the existence of a weak solution $u \in HW^{1,G}(\Omega)$ of the equation (2.29);

$$(1)\ \big\langle \mathcal{A}(p) - \mathcal{A}(q), p - q \big\rangle \geq c(g_0) \begin{cases} |p-q|^2\, \mathrm{F}(|p|) & \text{if } |p-q| \leq 2|p| \\ |p-q|^2\, \mathrm{F}(|p-q|) & \text{if } |p-q| > 2|p| \end{cases} \tag{2.32}$$

$$(2)\ \big\langle \mathcal{A}(p), p \big\rangle \geq c(g_0)\, |p|^2 \mathrm{F}(|p|) \geq c(g_0)\, G(|p|) \tag{2.33}$$

for all $p, q \in \mathbb{R}^{2n}$ and some constant $c(g_0) > 0$.

The local boundedness of $\mathfrak{X}u$ for a weak solution u of (2.29), has been established (see Theorem 1.1 of [44]) in the case when $D\mathcal{A}$ is symmetric. However, we remark that it also works generally if the Riemmanian approximation method is used instead of Hilbert-Haar theory in [44]. The Riemmanian approximation involves taking the approximate gradients

$$\mathfrak{X}^\varepsilon u = (X_1^\varepsilon u, \ldots, X_{2n+1}^\varepsilon u) = (X_1 u, \ldots, X_{2n} u, \varepsilon T u)$$

and $\operatorname{div}_H^\varepsilon$ is similarly defined. Hence, $\mathfrak{X}^\varepsilon u \to (\mathfrak{X}u, 0)$ and $|\mathfrak{X}^\varepsilon u|_\varepsilon^2 = |\mathfrak{X}u|^2 + \varepsilon^2 |Tu|^2 \to |\mathfrak{X}u|^2$ as $\varepsilon \to 0^+$. This is done so that the equation $\operatorname{div}_H^\varepsilon(\mathcal{A}_\varepsilon(\mathfrak{X}^\varepsilon u)) = 0$ is uniformly elliptic with $\mathcal{A}_\varepsilon \to \mathcal{A}$, and hence, has all necessary regularities to carry out integral estimates. The gradient bound being obtained independent of ε, the limit can be passed to obtain the gradient bound for the sub-elliptic equation. We refer to [8, 6, 7, 12], etc. for more details.

The local boundedness of the gradient (Theorem 1.1 of [44]) is restated in the following.

Theorem 2.13. *Let $u \in HW^{1,G}(\Omega)$ be a weak solution of equation* (2.29) *satisfying structure condition* (2.31) *and g satisfies* (1.2) *with $\delta > 0$. Then $\mathfrak{X}u \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$; moreover for any $B_r \subset \Omega$, we have*

$$\sup_{B_{\sigma r}} G(|\mathfrak{X}u|) \leq \frac{c}{(1-\sigma)^Q} \fint_{B_r} G(|\mathfrak{X}u|)\, dx \tag{2.34}$$

for any $0 < \sigma < 1$, where $c = c(n, g_0, \delta, L) > 0$ is a constant.

3. Hölder continuity of weak solutions

In this section, we show that weak solutions of quasilinear equations in the Heisenberg Group satisfy the Harnack inequalities, which leads to the Hölder continuity, thereby proving Theorem 1.1. The techniques are standard, based on appropriate modifications of similar results in the Euclidean setting, by Trudinger [52] and Lieberman [39].

On a domain $\Omega \subset \mathbb{H}^n$, we consider the prototype quasilinear operator in divergence form

$$\mathrm{Q}u = \mathrm{div}_H\, A(x,u,\mathfrak{X}u) + B(x,u,\mathfrak{X}u) \tag{3.1}$$

throughout this paper, where $A : \Omega \times \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and $B : \Omega \times \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$ are given functions. Appropriate additional hypothesis on structure conditions satisfied by A and B, shall be assumed in the following subsections, accordingly as required.

We remark that the conditions chosen for A, always ensure some sort of ellipticity for the operator (3.1) and the existence of weak solutions $u \in HW^{1,G}(\Omega)$ for $\mathrm{Q}u = 0$ is always assured. Any pathological situation, where this does not hold, is avoided.

3.1. **Global Maximum principle.**

Given weak solution $u \in HW^{1,G}(\Omega)$ for $\mathrm{Q}u = 0$, here we show global L^∞ estimates of u under appropriate boundary conditions. The method and techniques are adaptations of similar classical results in [39] for quasilinear equations in the Euclidean setting.

Here, we assume that u satisfies the boundary condition $u - u_0 \in HW_0^{1,G}(\Omega)$ for some $u_0 \in L^\infty(\bar{\Omega})$. In addition, we assume that there exists $b_0 > 0$ and $M \geq \|u_0\|_{L^\infty}$ such that

$$\big\langle A(x,z,p), p\big\rangle \geq |p| g(|p|) - f_1(|z|); \tag{3.2}$$

$$zB(x,z,p) \leq b_0 \big\langle A(x,z,p), p\big\rangle + f_2(|z|), \tag{3.3}$$

holds for all $x \in \Omega,\ |z| \geq M$ and $p \in \mathbb{R}^{2n}$, where f_1, f_2 and g are non-negative increasing functions. Also, we require $\big\langle A(x,u,\mathfrak{X}u), \mathfrak{X}u\big\rangle \in L^1(\Omega)$ and $u \in L^\infty(\Omega)$. The first condition (3.2), can be viewed as a weak ellipticity condition.

Additional conditions on f_1 and f_2, yields apriori integral estimates as in the following lemma. Similar results in Euclidean setting, can be found in [31] and [36].

Lemma 3.1. *Let $u \in HW^{1,G}(\Omega)$ be a weak solution of $\mathrm{Q}u = 0$ in Ω along with the conditions* (3.2) *and* (3.3) *and $u - u_0 \in HW_0^{1,G}(\Omega)$. If the functions f_1, f_2 and g satisfy*

$$(1)\ tg(t) \leq a_1 G(t); \tag{3.4}$$

$$(2)\ tg(t) f_1(Rt) + G(t) f_2(Rt) \leq a_1 G(t)^2, \tag{3.5}$$

for some $a_1 \geq 1, R > 0$ and every $t > M/R$, then there exists $c(n) > 0$ such that for $Q = 2n+2$ and $c = c(n)[(1+a_1)(1+2b_0)]^Q$, we have

$$\sup_\Omega G(|u|/R) \leq \max\Big\{ \frac{c}{R^Q} \int_\Omega G(|u|/R)\, dx\,,\ (1+a_1)G(M/R) \Big\}. \tag{3.6}$$

Proof. The proof is similar to that of Lemma 2.1 in [39] (see also Lemma 10.8 in [31]) and follows from standard Moser's iteration. We provide a brief outline.

Note that, we can assume $|u| \geq M$ without loss of generality, as otherwise we are done; we provide the proof for $u \geq M$, the proof for $u \leq -M$ is similar. The test function $\varphi = h(u)$ is used for the equation $\mathrm{Q}u = 0$, where letting $G = G(|u|/R)$ and $\tau = G(M/R)$, we choose

$$h(u) = uG^\beta \big| \big(1 - \tau/G\big)^+ \big|^{Q\beta+1},$$

for $\beta \geq 2b_0$ and $Q = 2n+2$. Thus $\varphi/u \geq 0$ and $\varphi = 0$ on $\partial\Omega$, since $M \geq \|u_0\|_{L^\infty}$. Hence, applying φ as a test function and using (3.3), we get

$$\begin{aligned} \int_\Omega \big\langle A(x,u,\mathfrak{X}u), \mathfrak{X}\varphi \big\rangle \, dx &= \int_\Omega B(x,u,\mathfrak{X}u)\varphi \, dx \\ &\leq \int_\Omega \Big[b_0 \big\langle A(x,u,\mathfrak{X}u), \mathfrak{X}u \big\rangle + f_2(|u|) \Big] \frac{\varphi}{u} \, dx. \end{aligned} \tag{3.7}$$

Note that $\mathfrak{X}\varphi = h'(u)\mathfrak{X}u$ and we have

$$h'(u) = \frac{\varphi}{u} + \left[\beta\Big(1 - \frac{\tau}{G}\Big) + (Q\beta+1)\frac{\tau}{G} \right] G^{\beta-1} \big|(1-\tau/G)^+\big|^{Q\beta} g\Big(\frac{|u|}{R}\Big)\frac{u}{R},$$

which implies $h'(u) \geq (\beta+1)\varphi/u$ and $h'(u) \leq a_1(Q+2)(\beta+1)|(1-\tau/G)^+|^{Q\beta}G^\beta$ from (3.4). For every $\beta \geq 2b_0$, we obtain that

$$\begin{aligned} \frac{1}{2}\int_\Omega h'(u) g(|\mathfrak{X}u|)|\mathfrak{X}u| \, dx &\leq \int_\Omega \Big(h'(u) - b_0\varphi/u\Big) \Big[\big\langle A(x,u,\mathfrak{X}u), \mathfrak{X}u \big\rangle + f_1(|u|) \Big] \, dx \\ &\leq \int_\Omega \Big[f_2(|u|)\varphi/u + \Big(h'(u) \; - b_0\varphi/u\Big) f_1(|u|) \Big] \, dx, \end{aligned} \tag{3.8}$$

where we have used $h'(u) \geq 2b_0\,\varphi/u$ and (3.2) for the first inequality and (3.7) for the second inequality of the above. From (3.8) and (3.5), we obtain

$$\frac{1}{2}\int_\Omega h'(u) g(|\mathfrak{X}u|)|\mathfrak{X}u| \, dx \leq a_1(\beta+1)(2n+4) \int_\Omega \big|(1-\tau/G)^+\big|^{Q\beta} G^{\beta+1} \, dx. \tag{3.9}$$

Now, leting $w = \psi(G) = \frac{1}{2}G^{\beta+1}|(1-\tau/G)^+|^{Q\beta+1}$, note that $|\psi'(G)| \leq h'(u)g(|u|/R)|\mathfrak{X}u|/R$. Then, we use (2.24) of Lemma 2.12 with $t = |\mathfrak{X}u|$ and $s = |u|/R$, to obtain

$$\begin{aligned} \int_\Omega |\mathfrak{X}w| \, dx \leq \int_\Omega h'(u) g\Big(\frac{|u|}{R}\Big)\frac{|\mathfrak{X}u|}{R} \, dx &\leq \int_\Omega h'(u) \left[g\Big(\frac{|u|}{R}\Big)\frac{|u|}{R^2} + g(|\mathfrak{X}u|)\frac{|\mathfrak{X}u|}{R} \right] dx \\ &\leq \frac{c(n)}{R} a_1(\beta+1) \int_\Omega \big|(1-\tau/G)^+\big|^{Q\beta} G^{\beta+1} \, dx \end{aligned} \tag{3.10}$$

for some $c(n) > 0$, where for the last inequality of the above, we have used (3.9) and (3.4). Recalling Sobolev's inequality (2.5) with $q = 1$, we have

$$\Big(\int_\Omega w^\kappa \, dx \Big)^{1/\kappa} \leq c(n) \int_\Omega |\mathfrak{X}w| \, dx$$

for $\kappa = Q/(Q-1) = (2n+2)/(2n+1)$. Combining this with (3.10), we obtain

$$\Big(\int_\Omega \big|(1-\tau/G)^+\big|^{\kappa(Q\beta+1)} G^{\kappa(\beta+1)} \, dx \Big)^{1/\kappa} \leq \frac{c(n)}{R} a_1(\beta+1) \int_\Omega \big|(1-\tau/G)^+\big|^{Q\beta} G^{\beta+1} \, dx \tag{3.11}$$

which can be reduced to $\|v\|_{L^{\kappa\gamma}(\Omega,\mu)} \leq (\gamma/\gamma_0)^{1/\gamma}\|v\|_{L^\gamma(\Omega,\mu)}$, where $v = G|(1-\tau/G)^+|^Q$, $\gamma = \beta+1, \gamma_0 = 2b_0+1$ and the measure μ satisfying $d\mu = (\frac{c(n)}{R}a_1\gamma_0)^Q(1-\tau/G)^{-Q}dx$. Iterating with $\gamma_m = \kappa^m\gamma_0$ for $m = 0,1,2,\ldots$ and taking $m \to \infty$, we finally obtain

$$\sup_\Omega G|(1-\tau/G)^+|^Q \leq c(n)\Big(\frac{a_1(2b_0+1)}{R}\Big)^Q \int_\Omega G \, dx$$

for some $c(n) > 0$. It is easy to see that this yields (3.6), since $\sup_\Omega G > (1+a_1)\tau$ implies $\sup_\Omega G|(1-\tau/G)^+|^Q \geq \left(\frac{a_1}{1+a_1}\right)^Q \sup_\Omega G$. Thus, the proof is finished. □

Now, we are ready to prove the global maximum principle. For the Euclidean setting, similar theorems have been proved before, see e.g. Theorem 10.10 in [31].

Theorem 3.2. *Let $u \in HW^{1,G}(\Omega)$ be a weak solution of $\mathrm{Q}u = 0$ in Ω with $\sup_{\partial\Omega}|u| < \infty$. We assume that there exists non-negative increasing functions f_1, f_2 and g such that the conditions (3.2) and (3.3) hold for $R = \operatorname{diam}(\Omega)$ and $0 < b_0 < 1$; furthermore we assume $\Psi(t) = tg(t)$ is convex and g satisfies (3.4) for some $a_1 \geq 1$. Then there exists $c_0 = c_0(n, a_1)$ sufficiently small such that, if f_1 and f_2 satisfy*

$$f_1(|z|) + \frac{f_2(|z|)}{1-b_0} \leq c_0 \Psi\left(\frac{|z|}{R}\right) \tag{3.12}$$

for all $|z| \geq \sup_{\partial\Omega}|u|$, then for some $c(n, b_0, a_1) > 0$, we have

$$\sup_\Omega G(|u|/R) \leq c(n, b_0, a_1) \sup_{\partial\Omega} G(|u|/R) \tag{3.13}$$

Proof. First notice that, since $\Psi(t) = tg(t)$ and g is increasing, we have $G(t) \leq \Psi(t)$ and from (3.4), we have $\Psi(t) \leq a_1 G(t)$. These together imply that G is convex and doubling and so is Ψ, with 2^{a_1} as their doubling constant.

Let us denote $M = \sup_{\partial\Omega}|u|$ and $\Omega^+ = \{u > M\}$. We choose $\varphi = (u-M)^+$ as a test function for $\mathrm{Q}u = 0$ and use (3.3) to get

$$\begin{aligned}
\int_{\Omega^+} \langle A(x,u,\mathfrak{X}u), \mathfrak{X}u\rangle\, dx &= \int_{\Omega^+} (u-M) B(x,u,\mathfrak{X}u)\, dx \\
&\leq \int_{\Omega^+} \big(1 - M/u\big)\Big[b_0\big\langle A(x,u,\mathfrak{X}u), \mathfrak{X}u\big\rangle + f_2(|u|)\Big]\, dx \\
&\leq \int_{\Omega^+} b_0\big\langle A(x,u,\mathfrak{X}u), \mathfrak{X}u\big\rangle\, dx + \int_{\Omega^+} f_2(|u|)\, dx,
\end{aligned} \tag{3.14}$$

and then we use (3.14) together with (3.2) and (3.12) to obtain

$$\int_{\Omega^+} \Psi(|\mathfrak{X}u|)\, dx \leq \int_{\Omega^+} \left[f_1(|u|) + \frac{f_2(|u|)}{1-b_0}\right] dx \leq c_0 \int_{\Omega^+} \Psi\left(\frac{|u|}{R}\right) dx. \tag{3.15}$$

Now, from the Poincaré inequality (2.19), we have

$$\int_\Omega \Psi\left(\frac{\varphi}{R}\right) dx \leq c(n, a_1) \int_\Omega \Psi(|\mathfrak{X}\varphi|)\, dx = c(n, a_1) \int_{\Omega^+} \Psi(|\mathfrak{X}u|)\, dx. \tag{3.16}$$

We have $\Psi(2\varphi/R) \leq 2^{a_1}\Psi(\varphi/R)$ from the doubling condition and letting $\Omega^* = \{u > 2M\}$, notice that $\Psi(u/R) \leq \Psi(2\varphi/R)$ on Ω^*. Using these together with (3.16) and (3.15), we get

$$\int_{\Omega^*} \Psi\left(\frac{|u|}{R}\right) dx \leq \tau_0 \int_{\Omega^+} \Psi\left(\frac{|u|}{R}\right) dx = \tau_0 \left[\int_{\Omega^*} \Psi\left(\frac{|u|}{R}\right) dx + \int_{\Omega^+ \setminus \Omega^*} \Psi\left(\frac{|u|}{R}\right) dx\right] \tag{3.17}$$

where $\tau_0 = 2^{a_1} c(n, a_1) c_0 < 1$ for small enough c_0. Hence, from (3.17), we arrive at

$$(1-\tau_0) \int_{\Omega^*} \Psi\left(\frac{|u|}{R}\right) dx \leq \tau_0 \int_{\Omega^+ \setminus \Omega^*} \Psi\left(\frac{|u|}{R}\right) dx,$$

which, after adding $(1-\tau_0)\int_{\Omega^+\setminus\Omega^*}\Psi(|u|/R)dx$ on both sides, imply

$$(1-\tau_0)\int_{\Omega^+}\Psi\left(\frac{|u|}{R}\right)dx \le \int_{\Omega^+\setminus\Omega^*}\Psi\left(\frac{|u|}{R}\right)dx \le |\Omega^+|\Psi(2M/R). \tag{3.18}$$

From a similar argument with $\Omega^- = \{u < -M\}$, we can obtain

$$(1-\tau_0)\int_{\Omega^-}\Psi\left(\frac{|u|}{R}\right)dx \le |\Omega^-|\Psi(2M/R). \tag{3.19}$$

Now for $\Omega^0 = \{|u| \le M\}$, we directly have

$$(1-\tau_0)\int_{\Omega^0}\Psi\left(\frac{|u|}{R}\right)dx \le |\Omega^0|\Psi(2M/R) \tag{3.20}$$

since Ψ is increasing. Thus, adding (3.18),(3.19) and (3.20), we obtain

$$(1-\tau_0)\int_{\Omega}\Psi\left(\frac{|u|}{R}\right)dx \le |\Omega|\Psi(2M/R). \tag{3.21}$$

Now, if $c_0 < 1/a_1$, notice that multiplying $\Psi(|z|/R)$ on both sides of (3.12) and using inequality $G(t) \le \Psi(t) \le a_1 G(t)$, we can obtain

$$\Psi(|z|/R)f_1(|z|) + G(|z|/R)\frac{f_2(|z|)}{1-b_0} \le a_1 G(|z|/R)^2$$

which is similar to (3.5). Hence, we can combine (3.6) of Lemma 3.1 with (3.21) and conclude $\sup_\Omega G(|u|/R) \le c(n, b_0, a_1)G(M/R)$, which completes the proof. □

Remark 3.3. With minor modifications of the above arguments, the global bound can also be shown corresponding to u^+ for weak supersolutions u i.e. for $\mathrm{Q}u \ge 0$.

3.2. **Harnack Inequality.**

Here we show that weak solutions of $\mathrm{Q}u = 0$, satisfy Harnack inequality. The proofs are standard modifications of those in [52] and [39] for the Euclidean setting. We also refer to [9] for the Harnack inequalities on special cases, in the sub-elliptic setting.

In this subsection, we consider

$$\big\langle A(x,z,p),p\big\rangle \ge |p|g(|p|) - a_1\, g\left(\frac{|z|}{R}\right)\frac{|z|}{R} - g(\chi)\chi \tag{3.22}$$

$$|A(x,z,p)| \le a_2\, g(|p|) + a_3\, g\left(\frac{|z|}{R}\right) + g(\chi) \tag{3.23}$$

for given non-negative constants a_1, a_2, a_3, and $\chi, R > 0$.

Theorem 3.4. *In $B_R \subset \Omega$, let $u \in HW^{1,G}(B_R)\cap L^\infty(B_R)$ be a weak supersolution, $\mathrm{Q}u \ge 0$ with $|u| \le M$ in B_R and with the structure conditions* (3.22),(3.23) *and*

$$\mathrm{sign}(z)B(x,z,p) \le \frac{1}{R}\left[b_0\, g(|p|) + b_1\, g\left(\frac{|z|}{R}\right) + g(\chi)\right] \tag{3.24}$$

for given non-negative constants a_1, a_2, a_3, b_0, b_1 and $g \in C^1([0,\infty))$ that satisfies (1.2) *with $\delta \ge 0$. Then for any $q > 0$ and $0 < \sigma < 1$, there exists $c = c(n, g_0, a_1, a_2, a_3, b_0M, b_1, q) > 0$*

such that, letting $Q = 2n+2$, we have

$$\sup_{B_{\sigma R}} u^+ \leq \frac{c}{(1-\sigma)^{(1+g_0)Q/q}} \Big[\Big(\fint_{B_R} |u^+|^q \, dx \Big)^{\frac{1}{q}} + \chi R. \Big] \tag{3.25}$$

Proof. The proof is based on Moser's iteration, similar to that of Theorem 1.2 in [39]. We provide an outline. First notice that, using $\bar z = z + \chi R$, the structure conditions (3.22),(3.23) and (3.24) can be reduced to

$$\big\langle A(x,z,p), p \big\rangle \geq |p| g(|p|) - (1+a_1) g\big(|\bar z|/R\big)|\bar z|/R; \tag{3.26}$$

$$|A(x,z,p)| \leq a_2\, g(|p|) + (1+a_3) g\big(|\bar z|/R\big); \tag{3.27}$$

$$\bar z B(x,z,p) \leq b_0 |p| g(|p|) + (1+b_0+b_1) g\big(|\bar z|/R\big)|\bar z|/R. \tag{3.28}$$

To obtain (3.28), we multiply $\bar z$ on (3.24) and use (2.24) of Lemma 2.12 with $t = |\bar z|/R$ and $s = |p|$.

Hence, we use $\bar u = u^+ + \chi R$ for the proof. Given any $\sigma \in (0,1)$, we choose a standard cutoff function $\eta \in C_0^\infty(B_R)$ such that $0 \leq \eta \leq 1$, $\eta = 1$ in $B_{\sigma R}$ and $|\mathfrak{X}\eta| \leq 2/(1-\sigma)R$. Then, for some $\gamma \in \mathbb{R}$ and $\beta \geq 1 + |\gamma|$ which are chosen later, we use

$$\varphi = \eta^\gamma \bar u \, G(\eta \bar u/R)^{\beta-1} e^{b_0 \bar u}$$

as a test function for $\mathrm{Q}u \geq 0$, to get

$$\begin{aligned}
&(1+b_0) \int_{B_R} \eta^\gamma G(\eta\bar u/R)^{\beta-1} e^{b_0\bar u} \big\langle A(x,u,\mathfrak{X}u), \mathfrak{X}\bar u \big\rangle \, dx \\
&\quad + \frac{\beta-1}{R} \int_{B_R} \eta^\gamma \bar u \, G(\eta\bar u/R)^{\beta-2} g(\eta\bar u/R) e^{b_0\bar u} \big\langle A(x,u,\mathfrak{X}u), \mathfrak{X}\bar u \big\rangle \, dx \\
&\quad\leq -\frac{\beta-1}{R} \int_{B_R} \eta^\gamma |\bar u|^2 G(\eta\bar u/R)^{\beta-2} g(\eta\bar u/R) e^{b_0\bar u} \big\langle A(x,u,\mathfrak{X}u), \mathfrak{X}\eta \big\rangle \, dx \\
&\qquad - \gamma \int_{B_R} \eta^{\gamma-1} \bar u \, G(\eta\bar u/R)^{\beta-1} e^{b_0\bar u} \big\langle A(x,u,\mathfrak{X}u), \mathfrak{X}\eta \big\rangle \, dx \\
&\qquad + \int_{B_R} \eta^\gamma G(\eta\bar u/R)^{\beta-1} e^{b_0\bar u} \bar u \, B(x,u,\mathfrak{X}u) \, dx.
\end{aligned} \tag{3.29}$$

Now we use the structure condition (3.26) for the left hand side and (3.27),(3.28) for the right hand side of the above inequality. Then, we use (2.21) and (2.22) of Lemma 2.12 and also the fact that $e^{b_0\chi R} \leq e^{b_0\bar u} \leq e^{b_0(M+\chi R)}$, since $|u| \leq M$ in B_R. We obtain

$$\begin{aligned}
&\beta \int_{B_R} \eta^\gamma G(\eta\bar u/R)^{\beta-1} g(|\mathfrak{X}\bar u|)|\mathfrak{X}\bar u| \, dx \\
&\quad\leq \frac{a_2 \beta\, e^{b_0 M}}{(1-\sigma)} \int_{B_R} \eta^{\gamma-1} G(\eta\bar u/R)^{\beta-1} \frac{\bar u}{R} g(|\mathfrak{X}\bar u|) \, dx \\
&\quad\quad + \beta(1+g_0) C_1 e^{b_0 M} \int_{B_R} \eta^{\gamma-1} G(\eta\bar u/R)^{\beta-1} g\Big(\frac{\bar u}{R}\Big) \frac{\bar u}{R} \, dx \\
&\quad = I_1 + I_2
\end{aligned} \tag{3.30}$$

where $C_1 = (1+a_1)(1+b_0) + (1+b_0+b_1) + (1+a_3)/(1-\sigma)$. Here onwards, we use $c = c(n, g_0, a_1, a_2, a_3, b_0 M, b_1) > 0$ as a large enough constant, throughout the rest of the proof. Now we estimate both I_1 and I_2 as follows.

For I_1, we use (2.24) with $t = \frac{2}{(1-\sigma)} a_2 e^{b_0 M} \bar{u}/\eta R$ and $s = |\mathfrak{X}\bar{u}|$, to obtain

$$
\begin{aligned}
I_1 &\leq \frac{\beta}{2} \int_{B_R} \eta^\gamma G(\eta\bar{u}/R)^{\beta-1} g(|\mathfrak{X}\bar{u}|)|\mathfrak{X}\bar{u}|\, dx \\
&\quad + \frac{c\beta}{(1-\sigma)} \int_{B_R} \eta^\gamma G(\eta\bar{u}/R)^{\beta-1} \frac{\bar{u}}{\eta R}\, g\Big(\frac{\bar{u}}{(1-\sigma)\eta R}\Big)\, dx \\
&\leq \frac{\beta}{2} \int_{B_R} \eta^\gamma G(\eta\bar{u}/R)^{\beta-1} g(|\mathfrak{X}\bar{u}|)|\mathfrak{X}\bar{u}|\, dx \\
&\quad + \frac{c\beta}{(1-\sigma)^{1+g_0}} \int_{B_R} \eta^{\gamma-(2+2g_0)} G(\eta\bar{u}/R)^\beta\, dx,
\end{aligned} \tag{3.31}
$$

where we have used $g(\bar{u}/\eta R) \leq \eta^{-2g_0} g(\eta\bar{u}/R)$ for the latter inequality of the above.

For I_2, we trivially have

$$
I_2 \leq \frac{c\beta}{(1-\sigma)} \int_{B_R} \eta^{\gamma-1} G(\eta\bar{u}/R)^\beta\, dx. \tag{3.32}
$$

Letting $\theta = 2+2g_0$ and combining (3.30) with (3.31) and (3.32), we obtain

$$
\frac{\beta}{2} \int_{B_R} \eta^\gamma G(\eta\bar{u}/R)^{\beta-1} g(|\mathfrak{X}\bar{u}|)|\mathfrak{X}\bar{u}|\, dx \leq \frac{c\beta}{(1-\sigma)^{\theta/2}} \int_{B_R} \eta^{\gamma-\theta} G(\eta\bar{u}/R)^\beta\, dx. \tag{3.33}
$$

Now, we use Sobolev inequality

$$
\Big(\int_{B_R} |w|^\kappa\, dx\Big)^{\frac{1}{\kappa}} \leq c(n) \int_{B_R} |\mathfrak{X}w|\, dx
$$

for $\kappa = Q/(Q-1) = (2n+2)/(2n+1)$ and $w = \eta^\gamma G(\eta\bar{u}/R)^\beta$ with the choice of $\gamma = -(Q-1)\theta$, so that $\kappa\gamma = -Q\theta = \gamma - \theta$. Combining with (3.33), we obtain

$$
\Big(\int_{B_R} \eta^{-Q\theta} G(\eta\bar{u}/R)^{\kappa\beta}\, dx\Big)^{\frac{1}{\kappa}} \leq \frac{c\beta}{(1-\sigma)^{\theta/2}} \int_{B_R} \eta^{-Q\theta} G(\eta\bar{u}/R)^\beta\, dx.
$$

Iterating the above with $\beta_0 = q \geq Q\theta$ and $\beta_m = \kappa^m \beta_0$ and letting $m \to \infty$, we get

$$
\sup_{B_R} G(\eta\bar{u}/R) \leq \frac{c(q)}{(1-\sigma)^{Q\theta/2q}} \Big(\fint_{B_R} \eta^{-Q\theta} G(\eta\bar{u}/R)^q\, dx\Big)^{\frac{1}{q}}. \tag{3.34}
$$

Hence, using (2.21), we get

$$
\sup_{B_{\sigma R}} \bar{u} \leq \frac{c(q)}{(1-\sigma)^{Q\theta/2q}} \Big(\fint_{B_R} |\bar{u}|^q\, dx\Big)^{\frac{1}{q}}
$$

for all $q \geq Q\theta$ and $c(q) = c(n, g_0, a_1, a_2, a_3, b_0 M, b_1, q) > 0$. Then from the interpolation argument in [15], we get the above for all $q > 0$. This concludes the proof. ☐

Theorem 3.5. *In $B_R \subset \Omega$, let $u \in HW^{1,G}(B_R) \cap L^\infty(B_R)$ be a weak subsolution,* $\mathrm{Q}u \leq 0$ *with $0 \leq u \leq M$ in B_R and with the structure conditions* (3.22),(3.23) *and*

$$
\operatorname{sign}(z) B(x, z, p) \geq -\frac{1}{R}\Big[b_0\, g(|p|) + b_1\, g\Big(\frac{|z|}{R}\Big) + g(\chi)\Big] \tag{3.35}
$$

for given non-negative constants a_1, a_2, a_3, b_0, b_1 and $g \in C^1([0,\infty))$ that satisfies (1.2) *with $\delta > 0$. Then there exists positive constants q_0 and c depending on $n, \delta, g_0, a_1, a_2, a_3, b_0 M, b_1$ such that, letting $Q = 2n+2$, we have*

$$\left(\fint_{B_{R/2}} u^{q_0}\,dx\right)^{\frac{1}{q_0}} \le c\left(\inf_{B_{R/4}} u + \chi R\right) \tag{3.36}$$

Proof. Taking $\bar u = u + \chi R$ and $\eta \in C_0^\infty(B_{R/2})$ similarly as in the proof of Theorem 3.4, we can use the test function $\varphi = \eta^\gamma \bar u G(\bar u/\eta R) e^{-b_0 \bar u}$ on $\mathrm{Q}u \le 0$ and obtain

$$\left(\fint_{B_{R/2}} \bar u^{-q}\,dx\right)^{-\frac{1}{q}} \le c(q) \inf_{B_{R/4}} \bar u \tag{3.37}$$

for any $q > 0$. Now for any $0 < r \le R$, we choose $\eta \in C_0^\infty(B_r)$ such that $0_{\mathbb{H}} \le \eta \le 1, \eta = 1$ in $B_{r/2}$ and $|\mathfrak{X}\eta| \le 2/r$. Then we choose test function $\varphi = \eta^{g_0} \bar u G(\bar u/r)^{-1}$ in $\mathrm{Q}u \le 0$. Here we use the fact that g satisfies (1.2) with $\delta > 0$, so that from (2.27) and (2.28), we have

$$G(\bar u/r)^{-1} - G(\bar u/r)^{-2} g(\bar u/r)\bar u/r \le -G(\bar u/r)^{-1}\delta/(1+\delta).$$

Thus, using test function φ and structure conditions (3.26),(3.27) and (3.35), we obtain

$$\int_{B_r} \eta^{g_0}\, \frac{g(|\mathfrak{X}\bar u|)|\mathfrak{X}\bar u|}{G(\bar u/r)}\,dx \le c\int_{B_r}\left[(a_1+a_3+b_0+b_1)\,\frac{g(\bar u/r)\bar u/r}{G(\bar u/r)}\right]dx \le c r^Q$$

where we suppress the dependence of a_i, b_j, g_0, δ and denote constant as c. Now, recalling (2.24), we use $t \le tg(t)/g(s) + s$, with $t = |\mathfrak{X}u|$ and $s = \bar u/r$, to obtain

$$\begin{aligned} \int_{B_{r/2}} \frac{|\mathfrak{X}\bar u|}{\bar u}\,dx &\le \int_{B_{r/2}} \Big[\frac{g(|\mathfrak{X}\bar u|)|\mathfrak{X}\bar u|}{\bar u g(\bar u/r)} + \frac{1}{r}\Big]\,dx \\ &\le \frac{c}{r}\int_{B_r}\Big[\eta^{g_0}\,\frac{g(|\mathfrak{X}\bar u|)|\mathfrak{X}\bar u|}{G(\bar u/r)} + 1\Big]\,dx \le c\, r^{Q-1} \end{aligned} \tag{3.38}$$

Taking $w = \log(\bar u)$, we use Poincaré inequality and (3.38) to get

$$\fint_{B_{r/2}} |w - \{w\}_{B_{r/2}}|\,dx \le c\, r \fint_{B_{r/2}} |\mathfrak{X}w|\,dx = \frac{c}{r^{Q-1}}\int_{B_{r/2}} \frac{|\mathfrak{X}\bar u|}{\bar u}\,dx \le c,$$

which shows that $w \in \mathrm{BMO}(B_{r/2})$. John-Nirenberg type inequalities in the setting of metric spaces with doubling measures, is known; we refer to [3]. This is applicable in our setting and the above inequality imples exponential inetegrability for $w = \log(\bar u)$. Thus there exists $q_0 > 0$ and $c_0 > 0$ such that

$$\left(\fint_{B_{r/2}} \bar u^{-q_0}\,dx\right)\left(\fint_{B_{r/2}} \bar u^{q_0}\,dx\right) \le \left(\fint_{B_{r/2}} e^{q_0|w - \{w\}_{B_{r/2}}|}\,dx\right)^2 \le c_0^2. \tag{3.39}$$

for any $r \le R$. Thus, (3.37) with $q = q_0$ and (3.39), concludes the proof. □

From Theorem 3.4 and Theorem 3.5, the following corrollary is immediate.

Corollary 3.6. *In $B_R \subset \Omega$, let $u \in HW^{1,G}(B_R) \cap L^\infty(B_R)$ be a weak solution of $\mathrm{Q}u = 0$ with $0 \le u \le M$ in B_R and with the structure conditions* (3.22),(3.23) *and*

$$|B(x,z,p)| \le \frac{1}{R}\left[b_0\, g(|p|) + b_1\, g\left(\frac{|z|}{R}\right) + g(\chi)\right] \tag{3.40}$$

for given non-negative constants a_1, a_2, a_3, b_0, b_1 *and* $g \in C^1([0,\infty))$ *that satisfies* (1.2) *with* $\delta > 0$. *Then there exists* $c = c(n, \delta, g_0, a_1, a_2, a_3, b_0M, b_1) > 0$ *such that we have*

$$\sup_{B_{R/4}} u \leq c\Big(\inf_{B_{R/4}} u + \chi R\Big) \tag{3.41}$$

Thus, bounded weak solutions satisfy the Harnack inequality (3.41), which implies the Hölder continuity of weak solutions. By standard arguments, it is possible to show that there exists $\alpha = \alpha(n, \delta, g_0, a_1, a_2, a_3, b_0M, b_1) \in (0,1)$ and $c = c(n, \delta, g_0, a_1, a_2, a_3, b_0M, b_1) > 0$ such that, we have

$$\operatorname{osc}_{B_r} u \leq c\left(\frac{r}{R}\right)^{\alpha}\Big(\operatorname{osc}_{B_R} u + \chi R\Big). \tag{3.42}$$

for every $0 < r < R$ and $B_R \subset \Omega$. This is enough to prove Theorem 1.1.

Remark 3.7. The growth and ellipticity conditions (3.22),(3.23) and (3.40) are special cases of the more general conditions in (3.2) and (3.3). When g satisfies (1.2), it is easy to see that (3.4) holds with $a_1 = 1 + g_0$ and (3.2), (3.3) and (3.5) holds if $f_1(|z|), f_2(|z|) \sim g(|z|/R)|z|/R + g(\chi)\chi$. Therefore, it is not restrictive to assume $|u| \leq M$ since we have Theorem 3.2 for the above cases. Furthermore, (3.40) can be relaxed to

$$|zB(x,z,p)| \leq b_0|p|g(|p|) + b_1 g\left(\frac{|z|}{R}\right)\frac{|z|}{R} + g(\chi)\chi \tag{3.43}$$

so that, in this case (3.28) can be obtained immediately.

4. Hölder continuity of Horizontal gradient

In this section, we consider the quasilinear equation (2.29), i.e. $\operatorname{div} \mathcal{A}(\mathfrak{X}u) = 0$. Estimates for this equation shall be necessary in Section 5. However, all results in this section are obtained independently, without any reference to the rest of this paper, apart from the usage of the structure function $\mathrm{F}(t) = g(t)/t$.

Without loss of generality, we need the to temporarily remove possible singularities of the function F. Here onwards, the following shall be assumed until the end of this section.

$$(A): \text{ There exists } 0 < m_1, m_2 < \infty, \text{ such that } \lim_{t\to 0} \mathrm{F}(t) = m_1 \text{ and } \lim_{t\to\infty} \mathrm{F}(t) = m_2. \tag{4.1}$$

This combined with the local boundedness of $\mathfrak{X}u$ from Theorem 2.13, makes the equation (2.29) to be uniformly elliptic and enables us to conlcude

$$\mathfrak{X}u \in HW^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n}) \cap C^{0,\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n}), \quad Tu \in HW^{1,2}_{\mathrm{loc}}(\Omega) \cap C^{0,\alpha}_{\mathrm{loc}}(\Omega) \tag{4.2}$$

from Theorem 1.1 and Theorem 3.1 of Capogna [5]. However, every estimates in this section, are independent of the constants m_1 and m_2 and (4.1) shall be ultimately removed.

The regularity (4.2) is necessary to (weakly) differentiate the equation (2.29) and obtain the equations satisfied by $X_l u$ and Tu, as shown in the following two lemmas. The proofs are simple and omitted here, we refer to [44] and [55] for details.

Lemma 4.1. *If* $u \in HW^{1,G}(\Omega)$ *is a weak solution of* (2.29), *then* Tu *is a weak solution of*

$$\sum_{i,j=1}^{2n} X_i(D_j\mathcal{A}_i(\mathfrak{X}u)X_j(Tu)) = 0. \tag{4.3}$$

Lemma 4.2. *If $u \in HW^{1,G}(\Omega)$ is a weak solution of (2.29), then for any $l \in \{1, \ldots, n\}$, we have that $X_l u$ is weak solution of*

$$\sum_{i,j=1}^{2n} X_i(D_j\mathcal{A}_i(\mathfrak{X}u)X_jX_lu) + \sum_{i=1}^{2n} X_i(D_i\mathcal{A}_{n+l}(\mathfrak{X}u)Tu) + T(\mathcal{A}_{n+l}(\mathfrak{X}u)) = 0 \tag{4.4}$$

and similarly, $X_{n+l}u$ is weak solution of

$$\sum_{i,j=1}^{2n} X_i(D_j\mathcal{A}_i(\mathfrak{X}u)X_jX_{n+l}u) - \sum_{i=1}^{2n} X_i(D_i\mathcal{A}_l(\mathfrak{X}u)Tu) - T(\mathcal{A}_l(\mathfrak{X}u)) = 0. \tag{4.5}$$

We enlist some Caccioppoli type inequalitites, that are very similar to those in [55] and [45]. They will be essential for the estimates in the next subsection.

The following lemma is similar to Lemma 3.3 in [55], the proof is trivial and omitted here.

Lemma 4.3. *For any $\beta \geq 0$ and all $\eta \in C_0^\infty(\Omega)$, we have, for some $c = c(n, g_0, L) > 0$, that*

$$\int_\Omega \eta^2 \,\mathrm{F}\,(|\mathfrak{X}u|)\,|Tu|^\beta |\mathfrak{X}(Tu)|^2\,dx \leq \frac{c}{(\beta+1)^2}\int_\Omega |\mathfrak{X}\eta|^2 \mathrm{F}\,(|\mathfrak{X}u|)\,|Tu|^{\beta+2}\,dx.$$

The following lemma is similar to Corollary 3.2 of [55] and Lemma 2.5 of [45]. This is crucial for the proof of the Hölder continuity of the horizontal gradient. The proof of the lemma is similar to that in [55] and involves few other Caccioppoli type estimates.

Lemma 4.4. *For any $q \geq 4$ and all non-negative $\eta \in C_0^\infty(\Omega)$, we have that*

$$\int_\Omega \eta^q \,\mathrm{F}\,(|\mathfrak{X}u|)\,|Tu|^q\,dx \leq c(q)K^{q/2}\int_{\mathrm{supp}(\eta)} \mathrm{F}\,(|\mathfrak{X}u|)\,|\mathfrak{X}u|^q\,dx, \tag{4.6}$$

where $K = \|\mathfrak{X}\eta\|_{L^\infty}^2 + \|\eta T\eta\|_{L^\infty}$ and $c(q) = c(n, g_0, L, q) > 0$.

The following corollary follows immediately from Lemma 4.3 and Lemma 4.4.

Corollary 4.5. *For any $q \geq 4$ and all non-negative $\eta \in C_0^\infty(\Omega)$, we have*

$$\int_\Omega \eta^{q+2} \,\mathrm{F}\,(|\mathfrak{X}u|)\,|Tu|^{q-2}|\mathfrak{X}(Tu)|^2\,dx \leq c(q)K^{\frac{q+2}{2}}\int_{spt(\eta)} \mathrm{F}\,(|\mathfrak{X}u|)\,|\mathfrak{X}u|^q\,dx,$$

where $K = \|\mathfrak{X}\eta\|_{L^\infty}^2 + \|\eta T\eta\|_{L^\infty}$ and $c(q) = c(n, g_0, L, q) > 0$.

4.1. **The truncation argument.**
In this subsection, we follow the technique of [45] and prove Caccioppoli type inequalities invovling a double truncation of horizontal derivatives. In the setting of Euclidean spaces, similar ideas have been implemented previously by Tolksdorff [51] and Lieberman [38].

Here onwards, throughout this section, we shall denote $u \in HW^{1,G}(\Omega)$ as a weak solution of (2.29) and equipped with local Lipschitz continuity from Theorem 2.13, we denote

$$\mu_i(r) = \sup_{B_r} |X_i u|, \quad \mu(r) = \max_{1\leq i\leq 2n} \mu_i(r). \tag{4.7}$$

for a fixed ball $B_r \subset \Omega$.

We fix any $l \in \{1, 2, .., 2n\}$ and consider the following double truncation

$$v := \min\big(\mu(r)/8\,,\, \max\,(\mu(r)/4 - X_l u, 0)\big). \tag{4.8}$$

It is important to note that, from the regularity (4.2), we have

$$\mathfrak{X}v \in L^2_{\mathrm{loc}}(\Omega;\mathbb{R}^{2n}), \quad Tv \in L^2_{\mathrm{loc}}(\Omega) \tag{4.9}$$

and moreover, letting

$$E = \{x\in\Omega : \mu(r)/8 < X_l u < \mu(r)/4\}, \tag{4.10}$$

we have that

$$\mathfrak{X}v = \begin{cases} -\mathfrak{X}X_l u & \text{a.e. in } E; \\ 0 & \text{a.e. in } \Omega\setminus E, \end{cases} \quad \text{and} \quad Tv = \begin{cases} -TX_l u & \text{a.e. in } E; \\ 0 & \text{a.e. in } \Omega\setminus E. \end{cases} \tag{4.11}$$

The properties of this truncation shall be exploited for proving all the following Caccioppoli type estimates. In particular, notice that

$$\mu(r)/8 \le |\mathfrak{X}u| \le (2n)^{1/2}\mu(r) \quad \text{in } E\cap B_r; \tag{4.12}$$

since $\mathrm{F}(t) = g(t)/t$, (4.12) combined with (2.25) implies

$$\frac{1}{8^{g_0}(2n)^{1/2}}\mathrm{F}(\mu(r)) \le \mathrm{F}(|\mathfrak{X}u|) \le 8(2n)^{g_0/2}\mathrm{F}(\mu(r)) \quad \text{in } E\cap B_r, \tag{4.13}$$

which shall be used several times during the estimates that follow in this subsection. The main lemma required to prove Theorem 1.3, is the following.

Lemma 4.6. *Let v be the truncation (4.8) and $\eta\in C_0^\infty(B_r)$ be a non-negative cut-off function such that $0\le\eta\le 1$ in B_r, $\eta = 1$ in $B_{r/2}$ and that $|\mathfrak{X}\eta|\le 4/r$, $|\mathfrak{X}\mathfrak{X}\eta|\le 16n/r^2$. Then we have the following Caccioppoli type inequality*

$$\int_{B_r}\eta^{\beta+4}v^{\beta+2}|\mathfrak{X}v|^2\,dx \le c(\beta+2)^2\frac{|B_r|^{1-1/\gamma}}{r^2}\mu(r)^4\Big(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\,dx\Big)^{1/\gamma} \tag{4.14}$$

for all $\beta\ge 0$ and $\gamma>1$, where $c = c(n,g_0,L,\gamma)>0$ is a constant.

In the setting of equations with p-laplace type growth, the above lemma has been shown previously in [45](see Lemma 1.1). The proof is going to be similar. Hence, we would require two auxillary lemmas, similarly as in [45].

We also remark that the inequality (4.14) also holds corresponding to the truncation

$$v' = \min\big(\mu(r)/8, \max(\mu(r)/4 + X_l u, 0)\big),$$

and the proof can be carried out in the same way as that of Lemma 4.6.

The following lemma is the analogue of Lemma 3.1 of [45]. The proof is similar and lengthy, and involves only minor modifications.

Lemma 4.7. *For any $\beta\ge 0$ and all non-negative $\eta\in C_0^\infty(\Omega)$, we have that*

$$\begin{aligned}\int_\Omega &\eta^{\beta+2}v^{\beta+2}\mathrm{F}(|\mathfrak{X}u|)\,|\mathfrak{X}u|^2|\mathfrak{X}\mathfrak{X}u|^2\,dx \\ &\le c(\beta+2)^2\int_\Omega\eta^\beta\big(|\mathfrak{X}\eta|^2+\eta|T\eta|\big)v^{\beta+2}\mathrm{F}(|\mathfrak{X}u|)\,|\mathfrak{X}u|^4\,dx \\ &\quad + c(\beta+2)^2\int_\Omega\eta^{\beta+2}v^\beta\mathrm{F}(|\mathfrak{X}u|)\,|\mathfrak{X}u|^4|\mathfrak{X}v|^2\,dx \\ &\quad + c\int_\Omega\eta^{\beta+2}v^{\beta+2}\mathrm{F}(|\mathfrak{X}u|)\,|\mathfrak{X}u|^2|Tu|^2\,dx,\end{aligned} \tag{4.15}$$

where v is as in (4.8) *and* $c = c(n, g_0, L) > 0$.

Throughout the rest of this subsection, we fix a ball $B_r \subset \Omega$ and a cut-off function $\eta \in C_0^\infty(B_r)$ that satisfies

$$0 \le \eta \le 1 \quad \text{in } B_r, \quad \eta = 1 \quad \text{in } B_{r/2} \tag{4.16}$$

and

$$|\mathfrak{X}\eta| \le 4/r, \quad |\mathfrak{X}\mathfrak{X}\eta| \le 16n/r^2, \quad |T\eta| \le 32n/r^2 \quad \text{in } B_r. \tag{4.17}$$

The following technical lemma, that is required for the proof of Lemma 4.6, is a weighted Caccioppoli inequality for Tu involving v similar to that in Lemma 3.2 of [45]. We provide the proof here for sake of completeness.

Lemma 4.8. *Let $B_r \subset \Omega$ be a ball and $\eta \in C_0^\infty(B_r)$ be a cut-off function satisfying* (4.16) *and* (4.17). *Let $\tau \in (1/2, 1)$ and $\gamma \in (1, 2)$ be two fixed numbers. Then, for any $\beta \ge 0$, we have the following estimate,*

$$\int_\Omega \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^4 |\mathfrak{X}(Tu)|^2\, dx \le c(\beta+2)^{2\tau} \frac{|B_r|^{1-\tau}}{r^{2(2-\tau)}} \mathrm{F}(\mu(r))\mu(r)^6\, J^\tau, \tag{4.18}$$

where $c = c(n, g_0, L, \tau, \gamma) > 0$ *and*

$$J = \int_{B_r} \eta^{\beta+4} v^{\beta+2} |\mathfrak{X}v|^2\, dx \;+\; \mu(r)^4 \frac{|B_r|^{1-\frac{1}{\gamma}}}{r^2} \Big(\int_{B_r} \eta^{\gamma\beta} v^{\gamma\beta}\, dx \Big)^{\frac{1}{\gamma}}. \tag{4.19}$$

Proof. We denote the left hand side of (4.18) by M,

$$M = \int_\Omega \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^4 |\mathfrak{X}(Tu)|^2\, dx, \tag{4.20}$$

where $1/2 < \tau < 1$. Now we use $\varphi = \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)} |\mathfrak{X}u|^4\, Tu$ as a test function for the equation (4.3). We obtain that

$$\begin{aligned}
&\int_\Omega \sum_{i,j=1}^{2n} \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)} |\mathfrak{X}u|^4 D_j \mathcal{A}_i(\mathfrak{X}u) X_j Tu\, X_i Tu\, dx \\
&= -\left(\tau(\beta+2)+4\right) \int_\Omega \sum_{i,j=1}^{2n} \eta^{\tau(\beta+2)+3}\, v^{\tau(\beta+4)} |\mathfrak{X}u|^4 Tu D_j \mathcal{A}_i(\mathfrak{X}u) X_j Tu\, X_i\eta\, dx \\
&\quad - \tau(\beta+4) \int_\Omega \sum_{i,j=1}^{2n} \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)-1} |\mathfrak{X}u|^4 Tu D_j \mathcal{A}_i(\mathfrak{X}u) X_j Tu\, X_i v\, dx \\
&\quad - 4 \int_\Omega \sum_{i,j,k=1}^{2n} \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)} |\mathfrak{X}u|^2 X_k u Tu D_j \mathcal{A}_i(\mathfrak{X}u) X_j Tu\, X_i X_k u\, dx \\
&= K_1 + K_2 + K_3,
\end{aligned} \tag{4.21}$$

where the integrals in the right hand side of (4.21) are denoted by K_1, K_2, K_3 in order. To prove the lemma, we estimate both sides of (4.21) as follows.

For the left hand side, we have by the structure condition (2.31) that

$$\text{left of (4.21)} \ge \int_\Omega \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^4 |\mathfrak{X}(Tu)|^2\, dx = M, \tag{4.22}$$

and for the right hand side of (4.21), we estimate each item $K_i, i = 1, 2, 3$, one by one.

To this end, we denote

$$\tilde{K} = \int_\Omega \eta^{(2\tau-1)(\beta+2)+6}\, v^{(2\tau-1)(\beta+4)} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^4 |Tu|^2 |\mathfrak{X}(Tu)|^2 \, dx. \tag{4.23}$$

First, we estimate K_1 by the structure condition (2.31) and Hölder's inequality, to get

$$\begin{aligned} |K_1| \leq & c(\beta+2) \int_\Omega \eta^{\tau(\beta+2)+3}\, v^{\tau(\beta+4)} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^4 |Tu| |\mathfrak{X}(Tu)| |\mathfrak{X}\eta| \, dx \\ \leq & c(\beta+2) \tilde{K}^{\frac{1}{2}} \Big(\int_\Omega \eta^{\beta+2} v^{\beta+4} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^4 |\mathfrak{X}\eta|^2 \, dx \Big)^{\frac{1}{2}}, \end{aligned} \tag{4.24}$$

where $c = c(n, g_0, L, \tau) > 0$.

Second, we estimate K_2 also by the structure condition (2.31) and Hölder's inequality,

$$\begin{aligned} |K_2| \leq & c(\beta+2) \int_\Omega \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)-1} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^4 |Tu| |\mathfrak{X}(Tu)| |\mathfrak{X}v| dx \\ \leq & c(\beta+2) \tilde{K}^{\frac{1}{2}} \Big(\int_\Omega \eta^{\beta+4} v^{\beta+2} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^4 |\mathfrak{X}v|^2 \, dx \Big)^{\frac{1}{2}}. \end{aligned} \tag{4.25}$$

Finally, we estimate K_3. In the following, the first inequality follows from the structure condition (2.31), the second from Hölder's inequality and the third from Lemma 4.7. We have

$$\begin{aligned} |K_3| & \leq c \int_\Omega \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^3 |Tu| |\mathfrak{X}(Tu)| |\mathfrak{X}\mathfrak{X}u| \, dx \\ & \leq c \tilde{K}^{\frac{1}{2}} \Big(\int_\Omega \eta^{\beta+4} v^{\beta+4} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^2 |\mathfrak{X}\mathfrak{X}u|^2 \, dx \Big)^{\frac{1}{2}} \\ & \leq c\, \tilde{K}^{\frac{1}{2}}\, I^{\frac{1}{2}}, \end{aligned} \tag{4.26}$$

where I is the right hand side of (4.15) in Lemma 4.7

$$\begin{aligned} I = & c(\beta+2)^2 \int_\Omega \eta^{\beta+2} v^{\beta+4} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^4 \big(|\mathfrak{X}\eta|^2 + \eta |T\eta| \big) \, dx \\ & + c(\beta+2)^2 \int_\Omega \eta^{\beta+4} v^{\beta+2} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^4 |\mathfrak{X}v|^2 \, dx \\ & + c \int_\Omega \eta^{\beta+4} v^{\beta+4} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^2 |Tu|^2 \, dx. \end{aligned} \tag{4.27}$$

where $c = c(n, g_0, L) > 0$. Notice that the integrals on the right hand side of (4.24) and (4.25) are both controlled from above by I. Hence, we can combine (4.24), (4.25) and (4.26) to obtain

$$|K_1| + |K_2| + |K_3| \leq \; c\tilde{K}^{\frac{1}{2}} I^{\frac{1}{2}},$$

from which, together with the estimate (4.22) for the left hand side of (4.21), it follows that

$$M \leq c\tilde{K}^{\frac{1}{2}} I^{\frac{1}{2}}, \tag{4.28}$$

where $c=c(n,g_0,L,\tau)>0$. Now, we estimate $\tilde{K}$ by Hölder's inequality as follows.

$$
\begin{aligned}
\tilde{K} \le & \Big(\int_\Omega \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|\mathfrak{X}(Tu)|^2\,dx\Big)^{\frac{2\tau-1}{\tau}}\\
& \times\Big(\int_\Omega \eta^{\frac{2\tau}{1-\tau}+4}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|Tu|^{\frac{2\tau}{1-\tau}}|\mathfrak{X}(Tu)|^2\,dx\Big)^{\frac{1-\tau}{\tau}}\\
= & M^{\frac{2\tau-1}{\tau}}H^{\frac{1-\tau}{\tau}},
\end{aligned}
\tag{4.29}
$$

where M is as in (4.20) and we denote by H the second integral on the right hand side of (4.29)

$$
H=\int_\Omega \eta^{\frac{2\tau}{1-\tau}+4}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|Tu|^{\frac{2\tau}{1-\tau}}|\mathfrak{X}(Tu)|^2\,dx. \tag{4.30}
$$

Combining (4.29) and (4.28), we get

$$
M\le cH^{1-\tau}I^\tau, \tag{4.31}
$$

for some $c=c(n,g_0,L,\tau)>0$. To estimate M, we estimate H and I from above. We estimate H by Corollary 4.5 with $q=2/(1-\tau)$ and monotonicity of g, to obtain

$$
\begin{aligned}
H &\le c\mu(r)^4\int_\Omega \eta^{q+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|Tu|^{q-2}|\mathfrak{X}(Tu)|^2\,dx\\
&\le \frac{c}{r^{q+2}}\mu(r)^4\int_{B_r}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^q\,dx\le \frac{c}{r^{q+2}}|B_r|\mathrm{F}(\mu(r))\mu(r)^{q+4},
\end{aligned}
\tag{4.32}
$$

where $c=c(n,g_0,L,\tau)>0$.

Now, we fix $1<\gamma<2$ and estimate each term of I in (4.27) as follows. For the first term of I, we have by Hölder's inequality and monotonicity of g that

$$
\begin{aligned}
&\int_\Omega \eta^{\beta+2}v^{\beta+4}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4\big(|\mathfrak{X}\eta|^2+\eta|T\eta|\big)\,dx\\
&\qquad\le \frac{c}{r^2}\mathrm{F}(\mu(r))\mu(r)^8|B_r|^{1-\frac{1}{\gamma}}\Big(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\,dx\Big)^{\frac{1}{\gamma}}.
\end{aligned}
\tag{4.33}
$$

For the second term of I, we similarly have

$$
\int_\Omega \eta^{\beta+4}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|\mathfrak{X}v|^2\,dx\le c\mathrm{F}(\mu(r))\mu(r)^4\int_{B_r}\eta^{\beta+4}v^{\beta+2}|\mathfrak{X}v|^2\,dx. \tag{4.34}
$$

For the third term of I, we have that

$$
\begin{aligned}
&\int_\Omega \eta^{\beta+4}v^{\beta+4}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^2|Tu|^2\,dx\\
&\qquad\le\Big(\int_\Omega \eta^{\frac{2\gamma}{\gamma-1}}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^2|Tu|^{\frac{2\gamma}{\gamma-1}}\,dx\Big)^{1-\frac{1}{\gamma}}\\
&\qquad\quad\times\Big(\int_\Omega \eta^{\gamma(\beta+2)}v^{\gamma(\beta+4)}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^2\,dx\Big)^{\frac{1}{\gamma}}\\
&\qquad\le\frac{c}{r^2}\mathrm{F}(\mu(r))\mu(r)^8|B_r|^{1-\frac{1}{\gamma}}\Big(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\,dx\Big)^{\frac{1}{\gamma}}
\end{aligned}
\tag{4.35}
$$

where $c = c(n, g_0, L, \gamma) > 0$. Here in the above inequalities, the first one follows from Hölder's inequality and the second from Lemma 4.4 and monotonicity of g. Combining the estimates for three items of I above (4.33), (4.34) and (4.35), we get the following estimate for I,

$$I \le c(\beta+2)^2 \mathrm{F}(\mu(r))\mu(r)^4 J, \tag{4.36}$$

where J is defined as in (4.19)

$$J = \int_{B_r} \eta^{\beta+4} v^{\beta+2} |\mathfrak{X} v|^2\, dx \;+\; \mu(r)^4 \frac{|B_r|^{1-\frac{1}{\gamma}}}{r^2} \Big(\int_{B_r} \eta^{\gamma\beta} v^{\gamma\beta}\, dx \Big)^{\frac{1}{\gamma}}.$$

Now from the estimates (4.32) for G and (4.36) for I, we obtain the desired estimate for M by (4.31). Combing (4.32), (4.36) and (4.31), we end up with

$$M \le c(\beta+2)^{2\tau} \frac{|B_r|^{1-\tau}}{r^{2(2-\tau)}} \mathrm{F}(\mu(r))\mu(r)^6 J^\tau, \tag{4.37}$$

where $c = c(n, g_0, L, \tau, \gamma) > 0$. This completes the proof. □

Now we provide the proof of Lemma 4.6, for the sake of completeness.

Proof of Lemma 4.6. First, notice that we may assume $\gamma < 3/2$, since otherwise we can apply Hölder's inequality to the integral in the right hand side of the claimed inequality (4.14). Also, we recall from (4.8), that for some $l \in \{1, \dots, 2n\}$,

$$v = \min\big(\mu(r)/8\,,\, \max(\mu(r)/4 - X_l u, 0)\big).$$

We prove the lemma assuming $l \in \{1, \dots, n\}$; the case for $l \in \{n+1, \dots, 2n\}$ can be proven similarly. Henceforth, we fix $1 < \gamma < 3/2$ and $l \in \{1, \dots, n\}$ throughout the rest of the proof. Let $\beta \ge 0$ and $\eta \in C_0^\infty(B_r)$ be a cut-off function satisfying (4.16) and (4.17). Using test function $\varphi = \eta^{\beta+4} v^{\beta+3}$ for the equation (4.4), we obtain

$$\begin{aligned}
-(\beta+3) &\int_\Omega \sum_{i,j=1}^{2n} \eta^{\beta+4} v^{\beta+2} D_j \mathcal{A}_i(\mathfrak{X} u) X_j X_l u X_i v\, dx \\
&= (\beta+4) \int_\Omega \sum_{i,j=1}^{2n} \eta^{\beta+3} v^{\beta+3} D_j \mathcal{A}_i(\mathfrak{X} u) X_j X_l u X_i \eta\, dx \\
&\quad + (\beta+4) \int_\Omega \sum_{i=1}^{2n} \eta^{\beta+3} v^{\beta+3} D_i \mathcal{A}_{n+l}(\mathfrak{X} u) T u X_i \eta\, dx \\
&\quad + (\beta+3) \int_\Omega \sum_{i=1}^{2n} \eta^{\beta+4} v^{\beta+2} D_i \mathcal{A}_{n+l}(\mathfrak{X} u) X_i v\, T u\, dx \\
&\quad - \int_\Omega \eta^{\beta+4} v^{\beta+3}\, T\big(\mathcal{A}_{n+l}(\mathfrak{X} u)\big)\, dx.
\end{aligned} \tag{4.38}$$

Now notice that from (2.2), we have

$$\sum_{i,j=1}^{2n} D_j\mathcal{A}_i(\mathfrak{X}u)X_jX_luX_i\eta + \sum_{i=1}^{2n} D_i\mathcal{A}_{n+l}(\mathfrak{X}u)TuX_i\eta$$
$$= \sum_{i,j=1}^{2n} D_j\mathcal{A}_i(\mathfrak{X}u)X_lX_juX_i\eta = \sum_{i=1}^{2n} X_l\big(\mathcal{A}_i(\mathfrak{X}u)\big)X_i\eta.$$

Thus, we can combine the first two integrals in the right hand side of (4.38) by the above equality. Then (4.38) becomes

$$\begin{aligned}
&-(\beta+3)\int_\Omega \sum_{i,j=1}^{2n} \eta^{\beta+4}v^{\beta+2}D_j\mathcal{A}_i(\mathfrak{X}u)X_jX_luX_iv\,dx \\
&= (\beta+4)\int_\Omega \sum_{i=1}^{2n} \eta^{\beta+3}v^{\beta+3}X_l\big(\mathcal{A}_i(\mathfrak{X}u)\big)X_i\eta\,dx \\
&\quad + (\beta+3)\int_\Omega \sum_{i=1}^{2n} \eta^{\beta+4}v^{\beta+2}D_i\mathcal{A}_{n+l}(\mathfrak{X}u)X_ivTu\,dx \\
&\quad - \int_\Omega \eta^{\beta+4}v^{\beta+3}T\big(\mathcal{A}_{n+l}(\mathfrak{X}u)\big)\,dx \\
&= I_1+I_2+I_3,
\end{aligned} \tag{4.39}$$

where we denote the terms in the right hand side of (4.39) by I_1, I_2, I_3, respectively.

We will estimate both sides of (4.39) as follows. For the left hand side, denoting E as in (4.10) and using structure condition (2.31), we have

$$\begin{aligned}
\text{left of (4.39)} &\ge (\beta+3)\int_E \eta^{\beta+4}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}v|^2\,dx \\
&\ge c_0(\beta+2)\mathrm{F}(\mu(r))\int_{B_r} \eta^{\beta+4}v^{\beta+2}|\mathfrak{X}v|^2\,dx,
\end{aligned} \tag{4.40}$$

for a constant $c_0 = c_0(n, g_0, L) > 0$. Here we have used (4.11) and (4.13).

For the right hand side of (4.39), we claim that each item I_1, I_2, I_3 satisfies

$$\begin{aligned}
|I_m| \le\ & \frac{c_0}{6}(\beta+2)\mathrm{F}(\mu(r))\int_{B_r} \eta^{\beta+4}v^{\beta+2}|\mathfrak{X}v|^2\,dx \\
&+ c(\beta+2)^3\frac{|B_r|^{1-1/\gamma}}{r^2}\mathrm{F}(\mu(r))\mu(r)^4\Big(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\,dx\Big)^{1/\gamma},
\end{aligned} \tag{4.41}$$

where $m = 1, 2, 3$, $1 < \gamma < 3/2$ and c is a constant depending only on n, g_0, L and γ. Then the lemma follows from the estimate (4.40) for the left hand side of (4.39) and the above claim (4.41) for each item in the right. Thus, we are only left with proving the claim (4.41).

In the rest of the proof, we estimate I_1, I_2, I_3 one by one. First for I_1, using integration by parts, we have that

$$I_1 = -(\beta+4)\int_\Omega \sum_{i=1}^{2n} \mathcal{A}_i(\mathfrak{X}u)X_l\big(\eta^{\beta+3}v^{\beta+3}X_i\eta\big)\,dx,$$

from which it follows by the structure condition (2.31), that

$$
\begin{aligned}
|I_1| &\le c(\beta+2)^2 \int_\Omega \eta^{\beta+2} v^{\beta+3} \mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|\big(|\mathfrak{X}\eta|^2+\eta|\mathfrak{X}\mathfrak{X}\eta|\big)\,dx\\
&\quad + c(\beta+2)^2 \int_\Omega \eta^{\beta+3} v^{\beta+2} \mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u||\mathfrak{X}v||\mathfrak{X}\eta|\,dx\\
&\le \frac{c}{r^2}(\beta+2)^2 \mathrm{F}(\mu(r))\mu(r)^4 \int_{B_r} \eta^\beta v^\beta\,dx\\
&\quad + \frac{c}{r}(\beta+2)^2 \mathrm{F}(\mu(r))\mu(r)^2 \int_{B_r} \eta^{\beta+2} v^{\beta+1}|\mathfrak{X}v|\,dx,
\end{aligned}
\tag{4.42}
$$

where $c=c(n,g_0,L)>0$. For the latter inequality of (4.42), we have used the fact that $g(t)=t\mathrm{F}(t)$ is monotonically increasing. Now we apply Young's inequality to the last term of (4.42) to end up with

$$
\begin{aligned}
|I_1| &\le \frac{c_0}{6}(\beta+2)\mathrm{F}(\mu(r)) \int_{B_r} \eta^{\beta+4} v^{\beta+2}|\mathfrak{X}v|^2\,dx\\
&\quad + \frac{c}{r^2}(\beta+2)^3 \mathrm{F}(\mu(r))\mu(r)^4 \int_{B_r} \eta^\beta v^\beta\,dx,
\end{aligned}
\tag{4.43}
$$

where $c=c(n,g_0,L)>0$ and c_0 is the same constant as in (4.40). The claimed estimate (4.41) for I_1, follows from the above estimate (4.43) and Hölder's inequality.

To estimate I_2, we have by the structure condition (2.31) that

$$
|I_2| \le c(\beta+2)\int_\Omega \eta^{\beta+4} v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}v||Tu|\,dx,
$$

from which it follows by Hölder's inequality that

$$
\begin{aligned}
|I_2| &\le c(\beta+2)\Big(\int_E \eta^{\beta+4} v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}v|^2\,dx\Big)^{\frac12}\\
&\qquad\times\Big(\int_E \eta^{\gamma(\beta+2)} v^{\gamma(\beta+2)}\mathrm{F}\left(|\mathfrak{X}u|\right)\,dx\Big)^{\frac{1}{2\gamma}}\\
&\qquad\times\Big(\int_\Omega \eta^q\,\mathrm{F}\left(|\mathfrak{X}u|\right)|Tu|^q\,dx\Big)^{\frac1q},
\end{aligned}
\tag{4.44}
$$

where $q=2\gamma/(\gamma-1)$. The fact that the integrals are on the set E, is crucial since we can use (4.13) and the following estimates can not be carried out unless the function F is increasing. We have the following estimates for the first two integrals of the above, using (4.13).

$$
\int_E \eta^{\beta+4} v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}v|^2\,dx \le c\mathrm{F}(\mu(r))\int_{B_r}\eta^{\beta+4} v^{\beta+2}|\mathfrak{X}v|^2\,dx,
\tag{4.45}
$$

and

$$
\int_E \eta^{\gamma(\beta+2)} v^{\gamma(\beta+2)}\mathrm{F}\left(|\mathfrak{X}u|\right)\,dx \le c\mathrm{F}(\mu(r))\mu(r)^{2\gamma}\int_{B_r}\eta^{\gamma\beta} v^{\gamma\beta}\,dx,
\tag{4.46}
$$

where $c = c(n, g_0, L) > 0$. We estimate the last integral in the right hand side of (4.44) by (4.6) of Lemma 4.4 and monotonicity of g, to obtain

$$\int_\Omega \eta^q \,\mathrm{F}\left(|\mathfrak{X}u|\right) |Tu|^q\, dx \le \frac{c}{r^q} \int_{B_r} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^q\, dx \le \frac{c|B_r|}{r^q} \mathrm{F}(\mu(r))\mu(r)^q, \tag{4.47}$$

where $c = c(n, g_0, L, \gamma) > 0$. Now combining the above three estimates (4.45), (4.46) and (4.47) for the three integrals in (4.44) respectively, we end up with the following estimate for I_2

$$|I_2| \le c(\beta+2)\frac{|B_r|^{\frac{\gamma-1}{2\gamma}}}{r}\mathrm{F}(\mu(r))\mu(r)^2\Big(\int_{B_r}\eta^{\beta+4}v^{\beta+2}|\mathfrak{X}v|^2\, dx\Big)^{\frac12}\Big(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\, dx\Big)^{\frac{1}{2\gamma}},$$

from which, together with Young's inequality, the claim (4.41) for I_2 follows.

Finally, we prove the claim (4.41) for I_3. Recall that

$$I_3 = -\int_\Omega \eta^{\beta+4}v^{\beta+3}T\big(\mathcal{A}_{n+l}(\mathfrak{X}u)\big)\, dx.$$

By virtue of the regularity (4.9) for v, integration by parts yields

$$\begin{aligned} I_3 &= \int_\Omega \mathcal{A}_{n+l}(\mathfrak{X}u)T\big(\eta^{\beta+4}v^{\beta+3}\big)\, dx \\ &= (\beta+4)\int_\Omega \eta^{\beta+3}v^{\beta+3}\mathcal{A}_{n+l}(\mathfrak{X}u)T\eta\, dx \\ &\quad + (\beta+3)\int_\Omega \eta^{\beta+4}v^{\beta+2}\mathcal{A}_{n+l}(\mathfrak{X}u)Tv\, dx = I_3^1 + I_3^2, \end{aligned} \tag{4.48}$$

where we denote the last two integrals in the above equality by I_3^1 and I_3^2, respectively. The estimate for I_3^1 easily follows from the structure condition (2.31) and monotonicity of g, as

$$\begin{aligned} |I_3^1| &\le c(\beta+2)\int_\Omega \eta^{\beta+3}v^{\beta+3}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u||T\eta|\, dx \\ &\le \frac{c}{r^2}(\beta+2)\mathrm{F}(\mu(r))\mu(r)^4\int_{B_r}\eta^\beta v^\beta\, dx. \end{aligned} \tag{4.49}$$

Thus by Hölder's inequality, I_3^1 satisfies estimate (4.41). To estimate I_3^2, note that by (4.11) and the structure condition (2.31) we have

$$|I_3^2| \le c(\beta+2)\int_E \eta^{\beta+4}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u||\mathfrak{X}(Tu)|\, dx, \tag{4.50}$$

where the set E is as in (4.10). For $1 < \gamma < 3/2$, we continue to estimate I_3^2 by Hölder's inequality as follows,

$$\begin{aligned} |I_3^2| \le c(\beta+2)&\Big(\int_E \eta^{(2-\gamma)(\beta+2)+4}v^{(2-\gamma)(\beta+4)}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^2|\mathfrak{X}(Tu)|^2\, dx\Big)^{\frac12} \\ &\times\Big(\int_E \eta^{\gamma(\beta+2)}v^{\gamma\beta+4(\gamma-1)}\mathrm{F}\left(|\mathfrak{X}u|\right)\, dx\Big)^{\frac12}. \end{aligned}$$

Since, we have (4.13) on the set E, hence

$$|I_3^2| \le c(\beta+2)\mathrm{F}(\mu(r))^{\frac12}\mu(r)^{2(\gamma-1)-1}M^{\frac12}\Big(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\, dx\Big)^{\frac12}, \tag{4.51}$$

where

$$M=\int_\Omega \eta^{(2-\gamma)(\beta+2)+4}\, v^{(2-\gamma)(\beta+4)}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|\mathfrak{X}(Tu)|^2\,dx. \tag{4.52}$$

Now we can apply Lemma 4.8 to estimate M from above. Note that Lemma 4.8 with $\tau=2-\gamma$, gives us that

$$M\le c(\beta+2)^{2(2-\gamma)}\frac{|B_r|^{\gamma-1}}{r^{2\gamma}}\mathrm{F}(\mu(r))\mu(r)^6\,J^{2-\gamma} \tag{4.53}$$

where $c=c(n,g_0,L,\gamma)>0$ and J is defined as in (4.19)

$$J=\int_{B_r}\eta^{\beta+4}v^{\beta+2}|\mathfrak{X}v|^2\,dx+\mu(r)^4\frac{|B_r|^{1-\frac{1}{\gamma}}}{r^2}\Big(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\,dx\Big)^{\frac{1}{\gamma}}. \tag{4.54}$$

Now, it follows from (4.53) and (4.51) that

$$|I_3^2|\le c(\beta+2)^{3-\gamma}\mathrm{F}(\mu(r))\mu(r)^{2\gamma}\,\frac{|B_r|^{\frac{\gamma-1}{2}}}{r^\gamma}\,J^{\frac{2-\gamma}{2}}\Big(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\,dx\Big)^{\frac{1}{2}}.$$

By Young's inequality, we end up with

$$\begin{aligned}|I_3^2|\le{}&\frac{c_0}{12}(\beta+2)\mathrm{F}(\mu(r))J\\&+c(\beta+2)^{\frac{4}{\gamma}-1}\mathrm{F}(\mu(r))\mu(r)^4\frac{|B_r|^{1-\frac{1}{\gamma}}}{r^2}\Big(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\,dx\Big)^{\frac{1}{\gamma}},\end{aligned}$$

where $c_0>0$ is the same constant as in (4.41). Note that, with J as in (4.54), I_3^2 satisfies an estimate similar to (4.41). Now the desired claim (4.41) for I_3 follows, since both I_3^1 and I_3^2 satisfy similar estimates. This concludes the proof of the claim (4.41), and hence the proof of the lemma. □

The following corollary follows from Lemma 4.6 by Moser's iteration. We refer to [45] for the proof.

Corollary 4.9. *There exists a constant $\theta=\theta(n,g_0,L)>0$ such that the following statements hold. If we have*

$$|\{x\in B_r: X_l u<\mu(r)/4\}|\le\theta|B_r| \tag{4.55}$$

for an index $l\in\{1,\dots,2n\}$ and for a ball $B_r\subset\Omega$, then

$$\inf_{B_{r/2}}X_l u\ge 3\mu(r)/16;$$

Analogously, if we have

$$|\{x\in B_r: X_l u>-\mu(r)/4\}|\le\theta|B_r|, \tag{4.56}$$

for an index $l\in\{1,\dots,2n\}$ and for a ball $B_r\subset\Omega$, then

$$\sup_{B_{r/2}}X_l u\le\ -3\mu(r)/16.$$

4.2. **Proof of Theorem 1.3.**
At the end of this subsection, we provide the proof of Theorem 1.3. As before, we denote $u \in HW^{1,G}(\Omega)$ as a weak solution of equation (2.29) We fix a ball $B_{r_0} \subset \Omega$. For all balls $B_r, 0 < r < r_0$, concentric to B_{r_0}, we denote for $l = 1, 2, ..., 2n$,

$$\mu_l(r) = \sup_{B_r} |X_l u|, \quad \mu(r) = \max_{1 \le l \le 2n} \mu_l(r),$$

and

$$\omega_l(r) = \operatorname{osc}_{B_r} X_l u, \quad \omega(r) = \max_{1 \le l \le 2n} \omega_l(r).$$

We clearly have $\omega(r) \le 2\mu(r)$. For any function w, we define

$$A^+_{k,\rho}(w) = \{x \in B_\rho : (w(x) - k)^+ = \max(w(x) - k, 0) > 0\};$$

and $A^-_{k,\rho}(w)$ is similarly defined.

The following lemma is similar to Lemma 4.1 of [45] and Lemma 4.3 of [55].

Lemma 4.10. *Let $B_{r_0} \subset \Omega$ be a ball and $0 < r < r_0/2$. Suppose that there is $\tau > 0$ such that*

$$|\mathfrak{X}u| \ge \tau\mu(r) \quad \text{in } A^+_{k,r}(X_l u) \tag{4.57}$$

for an index $l \in \{1, 2, ..., 2n\}$ and for a constant $k \in \mathbb{R}$. Then for any $q \ge 4$ and any $0 < r'' < r' \le r$, we have

$$\begin{aligned} &\int_{B_{r''}} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}(X_l u - k)^+|^2 \, dx \\ &\qquad \le \frac{c}{(r' - r'')^2} \int_{B_{r'}} \mathrm{F}\left(|\mathfrak{X}u|\right) |(X_l u - k)^+|^2 \, dx \, + \, cK|A^+_{k,r'}(X_l u)|^{1-\frac{2}{q}} \end{aligned} \tag{4.58}$$

where $K = r_0^{-2}|B_{r_0}|^{2/q}\mu(r_0)^2\mathrm{F}(\mu(r_0))$ and $c = c(n, p, L, q, \tau) > 0$.

Remark 4.11. Similarly, we can obtain an inequality, corresponding to (4.58), with $(X_l u-k)^+$ replaced by $(X_l u - k)^-$ and $A^+_{k,r}(X_l u)$ replaced by $A^-_{k,r}(X_l u)$.

Lemma 4.12. *There exists a constant $s = s(n, g_0, L) \ge 0$ such that for every $0 < r \le r_0/16$, we have the following,*

$$\omega(r) \le (1 - 2^{-s})\omega(8r) + 2^s\mu(r_0)\left(\frac{r}{r_0}\right)^\alpha, \tag{4.59}$$

where $\alpha = 1/2$ when $0 < g_0 < 1$ and $\alpha = 1/(1 + g_0)$ when $g_0 \ge 1$.

Proof. To prove the lemma, we fix a ball B_r concentric to B_{r_0}, such that $0 < r < r_0/16$.

Letting $\alpha = 1/2$ when $0 < g_0 < 1$ and $\alpha = 1/(1 + g_0)$ when $g_0 \ge 1$, we may assume that

$$\omega(r) \ge \mu(r_0)\left(\frac{r}{r_0}\right)^\alpha, \tag{4.60}$$

since, otherwise, (4.59) is true with $s = 0$. In the following, we assume that (4.60) is true and we divide the proof into two cases.

Case 1. For at least one index $l \in \{1, \dots, 2n\}$, we have either

$$|\{x \in B_{4r} : X_l u < \mu(4r)/4\}| \le \theta|B_{4r}| \tag{4.61}$$

or

$$|\{x \in B_{4r} : X_l u > -\mu(4r)/4\}| \le \theta |B_{4r}|, \tag{4.62}$$

where $\theta = \theta(n, g_0, L) > 0$ is the constant in Corollary 4.9. Assume that (4.61) is true; the case (4.62) can be treated in the same way. We apply Corollary 4.9 to obtain that

$$|X_l u| \ge 3\mu(4r)/16 \quad \text{in} \quad B_{2r}.$$

Thus we have

$$|\mathfrak{X} u| \ge 3\mu(2r)/16 \quad \text{in} \quad B_{2r}. \tag{4.63}$$

Due to (4.63), we can apply Lemma 4.10 with $q = 2Q$ to obtain

$$\begin{aligned} \int_{B_{r''}} |\mathfrak{X}(X_i u - k)^+|^2 \, dx \ &\le \frac{c}{(r' - r'')^2} \int_{B_{r'}} |(X_i u - k)^+|^2 \, dx \\ &\quad + cK\mathrm{F}(\mu(2r))^{-1} |A^+_{k,r'}(X_i u)|^{1-\frac{1}{Q}} \end{aligned} \tag{4.64}$$

where $K = r_0^{-2} |B_{r_0}|^{1/Q} \mu(r_0)^2 \mathrm{F}(\mu(r_0))$. The above inequality holds for all $0 < r'' < r' \le 2r$, $i \in \{1, \ldots, 2n\}$ and all $k \in \mathbb{R}$, which means that for each i, $X_i u$ belongs to the De Giorgi class $DG^+(B_{2r})$, see [55] for details. The corresponding version of Lemma 4.10 for $(X_i u - k)^-$, see Remark 4.11, shows that $X_i u$ also belong to $DG^-(B_{2r})$ and hence $X_i u$ belongs to $DG(B_{2r})$. Now we can apply Theorem 4.1 of [55] to conclude that there is $s_0 = s_0(n, p, L) > 0$ such that for each $i \in \{1, 2, ..., 2n\}$

$$\operatorname{osc}_{B_r} X_i u \le (1 - 2^{-s_0}) \operatorname{osc}_{B_{2r}} X_i u + cK^{\frac{1}{2}} \mathrm{F}(\mu(2r))^{-\frac{1}{2}} r^{\frac{1}{2}}. \tag{4.65}$$

Now, from doubling property of g, see (2.22) of Lemma 2.12, we have $g(\mu(r_0)) \le \big(\frac{\mu(r_0)}{\mu(2r)}\big)^{g_0} g(\mu(2r))$ whenever $2r \le r_0$ and hence

$$\mathrm{F}(\mu(r_0))/\mathrm{F}(\mu(2r)) \le \big(\mu(r_0)/\mu(2r)\big)^{g_0 - 1}.$$

Thus, notice that when $0 < g_0 < 1$, we have

$$\mathrm{F}(\mu(2r))^{-1} \le \mathrm{F}(\mu(r_0))^{-1}$$

and when $g_0 \ge 1$, our assumption (4.60) with $\alpha = 1/(1 + g_0)$ gives

$$\begin{aligned} \mathrm{F}(\mu(2r))^{-1} \le \left(\frac{\mu(r_0)}{\mu(2r)} \right)^{g_0 - 1} \mathrm{F}(\mu(r_0))^{-1} &\le 2^{g_0 - 1} \mathrm{F}(\mu(r_0))^{-1} \left(\frac{\mu(r_0)}{\omega(r)} \right)^{g_0 - 1} \\ &\le 2^{g_0 - 1} \mathrm{F}(\mu(r_0))^{-1} \left(\frac{r}{r_0} \right)^{\frac{1 - g_0}{1 + g_0}} \end{aligned}$$

where in the second inequality we used that $\mu(2r) \ge \omega(2r)/2 \ge \omega(r)/2$. In both cases, we find that (4.65) becomes

$$\operatorname{osc}_{B_r} X_i u \le (1 - 2^{-s_0}) \operatorname{osc}_{B_{2r}} X_i u + c\mu(r_0) \left(\frac{r}{r_0} \right)^{\alpha}, \tag{4.66}$$

where $c = c(n, g_0, L) > 0$, $\alpha = 1/2$ when $0 < g_0 < 1$ and $\alpha = 1/(1 + g_0)$ when $g_0 \ge 1$. This shows that the lemma holds in this case.

Case 2. If Case 1 does not happen, then for every $i \in \{1, \ldots, 2n\}$, we have

$$|\{x \in B_{4r} : X_i u < \mu(4r)/4\}| > \theta |B_{4r}|, \tag{4.67}$$

and

$$|\{x \in B_{4r} : X_i u > -\mu(4r)/4\}| > \theta |B_{4r}|, \tag{4.68}$$

where $\theta = \theta(n, g_0, L) > 0$ is the constant in Corollary 4.9.

Note that on the set $\{x \in B_{8r} : X_i u > \mu(8r)/4\}$, we trivially have

$$|\mathfrak{X} u| \geq \mu(8r)/4 \quad \text{in } A^+_{k,8r}(X_i u) \tag{4.69}$$

for all $k \geq \mu(8r)/4$. Thus, we can apply Lemma 4.10 with $q = 2Q$ to conclude that

$$\begin{aligned} \int_{B_{r''}} |\mathfrak{X}(X_i u - k)^+|^2 \, dx \ &\leq \frac{c}{(r' - r'')^2} \int_{B_{r'}} |(X_i u - k)^+|^2 \, dx \\ &\quad + cK \, \mathrm{F}(\mu(8r))^{-1} |A^+_{k,r'}(X_i u)|^{1-\frac{1}{Q}} \end{aligned} \tag{4.70}$$

where $K = r_0^{-2} |B_{r_0}|^{1/Q} \mu(r_0)^2 \mathrm{F}(\mu(r_0))$, whenever $k \geq k_0 = \mu(8r)/4$ and $0 < r'' < r' \leq 8r$. The above inequality is true all $i \in \{1, 2, ..., 2n\}$. We note that (4.67) trivially implies

$$|\{x \in B_{4r} : X_i u < \mu(8r)/4\}| > \theta |B_{4r}|.$$

Now we can apply Lemma 4.2 of [55] to conclude that there exists $s_1 = s_1(n, p, L) > 0$ such that the following holds,

$$\sup_{B_{2r}} X_i u \leq \sup_{B_{8r}} X_i u - 2^{-s_1} \big(\sup_{B_{8r}} X_i u - \mu(8r)/4 \big) + cK^{\frac{1}{2}} \mathrm{F}(\mu(8r))^{-1/2} r^{\frac{1}{2}}. \tag{4.71}$$

From (4.68), we can derive similarly, see Remark 4.11, that

$$\inf_{B_{2r}} X_i u \geq \inf_{B_{8r}} X_i u + 2^{-s_1} \big(- \inf_{B_{8r}} X_i u - \mu(8r)/4 \big) - cK^{\frac{1}{2}} \mathrm{F}(\mu(8r))^{-1/2} r^{\frac{1}{2}}. \tag{4.72}$$

The above two inequalities (4.71) and (4.72) yield

$$\operatorname{osc}_{B_{2r}} X_i u \leq \ (1 - 2^{-s_1}) \operatorname{osc}_{B_{8r}} X_i u + 2^{-s_1 - 1} \mu(8r) + cK^{\frac{1}{2}} \mathrm{F}(\mu(8r))^{-1/2} r^{\frac{1}{2}},$$

and hence

$$\omega(2r) \leq \big(1 - 2^{-s_1}\big) \omega(8r) + 2^{-s_1 - 1} \mu(8r) + cK^{\frac{1}{2}} \mathrm{F}(\mu(8r))^{-1/2} r^{\frac{1}{2}}. \tag{4.73}$$

By using doubling condition of g and the inequality $\mu(8r) \geq \omega(8r)/2 \geq \omega(r)/2$ along with the assumption (4.60), we proceed by the same argument as in the preceeding case, to conclude

$$\omega(2r) \leq \big(1 - 2^{-s_1}\big) \omega(8r) + 2^{-s_1 - 1} \mu(8r) + c\mu(r_0) \left(\frac{r}{r_0} \right)^{\alpha}$$

for $\alpha = 1/2$ when $0 < g_0 < 1$ and $\alpha = 1/(1 + g_0)$ when $g_0 \geq 1$.

Now we notice that (4.67) implies that $\inf_{B_{4r}} X_i u \leq \mu(4r)/4$ and (4.68) implies that $\sup_{B_{4r}} X_i u \geq -\mu(4r)/4$ for every $i \in \{1, \ldots, 2n\}$. Hence

$$\omega(8r) \geq \mu(8r) - \mu(4r)/4 \geq 3\mu(8r)/4.$$

Then from the above two inequalities we arrive at

$$\omega(2r) \leq \big(1 - 2^{-s_1 - 2}\big) \omega(8r) + c\mu(r_0) \left(\frac{r}{r_0} \right)^{\alpha},$$

where $c = c(n, g_0, L) > 0$, $\alpha = 1/2$ when $0 < g_0 < 1$ and $\alpha = 1/(1 + g_0)$ when $g_0 \geq 1$. This shows that also in this case the lemma is true. Thus, the proof of the lemma follows from choice of $s = \max(0, s_0, s_1 + 2, \log_2 c)$. □

Proof of Theorem 1.3.
We first consider the apriori assumption (4.1) so that, equipped with this assumption, we have the above lemma, Lemma 4.12. Now, by an iteration on (4.59), it is easy to see that

$$\omega(r) \leq c\Big(\frac{r}{r_0}\Big)^{\sigma}\big[\omega(r_0/2)+\mu(r_0/2)\big] \tag{4.74}$$

for some $\sigma=\sigma(n,g_0,L)\in(0,1)$, $r\leq r_0/2$ and $c=c(n,g_0,L)>0$. Using (4.74), observe that

$$\begin{aligned}\fint_{B_r} G(|X_l u-\{X_l u\}_{B_r}|)\,dx \leq c\,G(\omega_l(r)) &\leq c\,G\bigg(\Big(\frac{r}{r_0}\Big)^{\sigma}\big[\omega(r_0/2)+\mu(r_0/2)\big]\bigg)\\ &\leq c\Big(\frac{r}{r_0}\Big)^{\sigma}\sup_{B_{r_0/2}} G(|\mathfrak{X}u|)\end{aligned} \tag{4.75}$$

where we have used (2.26) for the first inequality and (2.21) for the last inequality of the above. Hence from (2.34), we end up with

$$\fint_{B_r} G(|X_l u-\{X_l u\}_{B_r}|)\,dx \leq c\Big(\frac{r}{r_0}\Big)^{\sigma}\fint_{B_{r_0}} G(|\mathfrak{X}u|)\,dx \tag{4.76}$$

which gives us the required estimate.

Now, to complete the proof, first we need to show that the estimate (4.76) is uniform, without the assumption (4.1). This involves a standard approximation argument, using the following regularization, as constructed [44];

$$\mathrm{F}_\varepsilon(t)=\mathrm{F}\Big(\min\{\,t+\varepsilon\,,\,1/\varepsilon\,\}\Big)\quad\text{and}\quad \mathcal{A}_\varepsilon(p)=\eta_\varepsilon(|p|)\mathrm{F}_\varepsilon(|p|)\,p+\Big(1-\eta_\varepsilon(|p|)\Big)\mathcal{A}(p) \tag{4.77}$$

where $0<\varepsilon<1$, $\eta_\varepsilon\in C^{0,1}([0,\infty))$ as in [44] and $\mathrm{F}(t)=g(t)/t$ for g satisfying (1.2) with $\delta>0$. Then, given $u\in HW^{1,G}(B_r)$ we consider u_ε that solves $\mathrm{div}_H(\mathcal{A}_\varepsilon(\mathfrak{X}u_\varepsilon))=0$ and $u_\varepsilon-u\in HW_0^{1,G}(B_r)$. We have $\mathcal{A}_\varepsilon\to\mathcal{A}$ and $\mathrm{F}_\varepsilon\to\mathrm{F}$ uniformly on compact subsets and F_ε satisfies the assumption (4.1) with $m_1=\mathrm{F}(\varepsilon)$ and $m_2=\mathrm{F}(1/\varepsilon)$. Since the estimate (4.76) are independent of m_1 and m_2, hence the limit $\varepsilon\to 0$ can be taken to obtain the uniform estimate, where the constant depends on n,δ,g_0,L.

Now, we show that the uniform estimate (4.76) implies that $X_l u$ is Hölder continuous for every $l\in\{1,\dots,2n\}$. Using (2.21) and Jensen's inequality on (4.76), notice that

$$\begin{aligned}&\Big(\fint_{B_r}|X_l u-\{X_l u\}_{B_r}|\,dx\Big)g\Big(\fint_{B_r}|X_l u-\{X_l u\}_{B_r}|\,dx\Big)\\ &\qquad\leq(1+g_0)G\Big(\fint_{B_r}|X_l u-\{X_l u\}_{B_r}|\,dx\Big)\leq c\Big(\frac{r}{r_0}\Big)^{\sigma}\fint_{B_{r_0}} G(|\mathfrak{X}u|)\,dx\end{aligned} \tag{4.78}$$

for some $c=c(n,\delta,g_0,L)>0$. Now, observe that if $\fint_{B_r}|X_l u-\{X_l u\}_{B_r}|\,dx\geq 1$ then,

$$\Big(\fint_{B_r}|X_l u-\{X_l u\}_{B_r}|\,dx\Big)g\Big(\fint_{B_r}|X_l u-\{X_l u\}_{B_r}|\,dx\Big)\geq g(1)\fint_{B_r}|X_l u-\{X_l u\}_{B_r}|\,dx;$$

otherwise if $\fint_{B_r}|X_l u-\{X_l u\}_{B_r}|\,dx\leq 1$, then from doubling condition

$$\Big(\fint_{B_r}|X_l u-\{X_l u\}_{B_r}|\,dx\Big)g\Big(\fint_{B_r}|X_l u-\{X_l u\}_{B_r}|\,dx\Big)\geq g(1)\Big(\fint_{B_r}|X_l u-\{X_l u\}_{B_r}|\,dx\Big)^{1+g_0}.$$

Notice that, both cases of the above when combined with (4.78), yield

$$\fint_{B_r} |X_l u - \{X_l u\}_{B_r}|\, dx \le C\Big(n, \delta, g_0, L, g(1), \|u\|_{HW^{1,G}(\Omega)}\Big)\Big(\frac{r}{r_0}\Big)^{\frac{\sigma}{1+g_0}} \tag{4.79}$$

which implies that $X_l u \in \mathcal{L}^{1,Q+\sigma'}(B_r)$ and hence, recalling (2.10), $X_l u \in C^{0,\sigma'}(B_r)$ with $\sigma' = \sigma/(1+g_0)$ for some $\sigma = \sigma(n, g_0, L) \in (0,1)$. This completes the proof. □

Remark 4.13. Let $B_R \subset B_{R_0} \subset\subset \Omega$ be concentric balls for $0 < R < R_0$. As illustrated in the above proof, if $w \in HW^{1,G}(\Omega)$ with $\|u\|_{HW^{1,G}(\Omega)} \le M$, satisfies the inequality

$$\int_{B_R} G(|\mathfrak{X} w - \{\mathfrak{X} w\}_{B_R}|)\, dx \le C(R/R_0)^{\lambda}$$

for some positive constants $C = C(n, \delta, g_0, R_0, M) > 0$ and $\lambda \in (0, Q+1)$ with $Q = 2n+2$, then we have $\mathfrak{X} w \in \mathcal{L}^{1,\lambda'}(B_R, \mathbb{R}^{2n})$; where if $\lambda \in (0, Q)$ then $\lambda' = \lambda$ and if $\lambda \in (Q, Q+1)$ then $\lambda' = Q + (\lambda - Q)/(1+g_0)$. This shall be used in the next section.

5. $C^{1,\alpha}$-regularity of weak solutions

In this section, we prove Theorem 1.2. In a fixed subdomain Ω' compactly contained in Ω, we show that the weak solutions are locally $C^{1,\beta}$ in Ω'. The proof is standard, based on the results of the preceeding section and a Campanato type perturbation technique. Similar arguments in the Euclidean setting, can be found in [14, 27, 39], etc.

5.1. **The perturbation argument.**

Given $\Omega' \subset\subset \Omega$, we fix $x_0 \in \Omega'$ and a ball $B_R = B_R(x_0) \subset \Omega'$ for $R \le R_0 = \frac{1}{2}\operatorname{dist}(\Omega', \partial\Omega)$ and consider $u \in HW^{1,G}(B_R) \cap L^\infty(B_R)$ as weak solution of $\mathrm{Q}u = 0$ in B_R, where Q is defined as in (3.1). We recall the structure conditions for Theorem 1.2, as follows;

$$\frac{g(|p|)}{|p|}\,|\xi|^2 \le \big\langle D_p\, A(x,z,p)\,\xi, \xi\big\rangle \le L\,\frac{g(|p|)}{|p|}\,|\xi|^2; \tag{5.1}$$

$$|A(x,z,p) - A(y,w,p)| \le L'\big(1 + g(|p|)\big)\Big(|x-y|^\alpha + |z-w|^\alpha\Big); \tag{5.2}$$

$$|B(x,z,p)| \le L'\big(1 + g(|p|)\big)|p| \tag{5.3}$$

for all $(x,z,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^{2n}$ and the matrix $D_p\, A(x,z,p)$ is symmetric. In addition, we recall the hypothesis of Theorem 1.2 that, there exists $M_0 > 0$ such that $\|u\| \le M_0$ in Ω'.

From structure condition (5.1), it is not difficult to check that $A(x,z,p)$ satisfies conditions reminiscent of (3.22) and (3.23); the condition on variable z for (3.22) and (3.23) are absolved in the constants L and L', since the solution u is bounded. However, the condition (5.3) on B is more relaxed than (3.40) and (3.43), which is necessary for $C^{1,\beta}$-regularity.

Thus, this allows us to apply Theorem 1.1 and conclude u is Hölder continuous with

$$\operatorname{osc}_{B_R} u \le \theta(R) = \gamma R^\tau \tag{5.4}$$

for some constant $\gamma = \gamma(M_0, \operatorname{dist}(\Omega', \partial\Omega)) > 0$ and $\tau \in (0,1)$ can be chosen to be as small as required. Here onwards, we suppress the dependence of the data $n, \delta, g_0, \alpha, L, L', \operatorname{dist}(\Omega', \partial\Omega)$; all positive constants depending on these shall be denoted as c and the constants dependent further on $g(1), M_0$ in addition, shall be denoted as C throughout this subsection.

The proof of Theorem 1.2 involves a standard freezing technique as followed previously in the Euclidean setting in [39]. However, the integral oscillation estimate of the previous section is weaker than that of the Euclidean setting (see [39, (5.3b) of p. 339], which is, in fact, false in the setting of $\mathbb{H}^n$ due to the non-zero terms from the commutators). Therefore, [39] can not be followed entirely. For the present case, the proof is carried out with an extra step in the argument, using Lemma 5.2 and Proposition 5.4 below.

Let us denote $\mathcal{A} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ as

$$\mathcal{A}(p) = A(x_0, u(x_0), p), \tag{5.5}$$

so that from (5.1), $\mathcal{A}$ satisfies the structure condition (2.31) and hence also the monotonicity and ellipticity conditions (2.32) and (2.33) (with possible dependence on g_0 and δ). Hence, for the problem

$$\begin{cases} \operatorname{div}_H(\mathcal{A}(\mathfrak{X}\tilde{u})) = 0 \ \text{ in } B_R; \\ \tilde{u} - u \in HW_0^{1,G}(B_R). \end{cases} \tag{5.6}$$

we can use the monotonicity inequalities and uniform estimates from Section 4.

Lemma 5.1. *If $u \in HW^{1,G}(B_R) \cap C(\bar{B}_R)$ is given, then there exists a unique weak solution $\tilde{u} \in HW^{1,G}(B_R) \cap C(\bar{B}_R)$ for the problem (5.6), which satisfies the following:*

$$(i) \ \sup_{B_R} |u - \tilde{u}| \le \operatorname{osc}_{B_R} u \ ; \tag{5.7}$$

$$(ii) \ \int_{B_R} G(|\mathfrak{X}\tilde{u}|)\, dx \le c \int_{B_R} G(|\mathfrak{X}u|)\, dx. \tag{5.8}$$

Proof. Existence and uniqueness is standard from monotonicity of $\mathcal{A}$, we refer to [44] for more details. Also, (5.7) follows easily from Comparison principle and the fact that

$$\inf_{\partial B_R} u \le \tilde{u} \le \sup_{\partial B_R} u \quad \text{in } B_R,$$

which is easy to show by considering $\varphi = (\tilde{u} - \sup_{\partial B_R} u)^+$ (and similarly the other case) as a test function for (5.6), see Lemma 5.1 in [14].

The proof of (5.8) is also standard. Using test function $\varphi = \tilde{u} - u$ on (5.6), we get

$$\int_{B_R} \big\langle \mathcal{A}(\mathfrak{X}\tilde{u}), \mathfrak{X}\tilde{u} \big\rangle\, dx = \int_{B_R} \big\langle \mathcal{A}(\mathfrak{X}\tilde{u}), \mathfrak{X}u \big\rangle\, dx. \tag{5.9}$$

Now we choose $k = k(\delta, g_0, L) > 0$ such that combining ellipticity (2.33) and boundedness of $\mathcal{A}$, we have $\big\langle \mathcal{A}(p), p \big\rangle \ge (2/k)|p||\mathcal{A}(p)|$. Hence, we obtain

$$\begin{aligned} \int_{B_R} \big\langle \mathcal{A}(\mathfrak{X}\tilde{u}), \mathfrak{X}u \big\rangle\, dx &\le \frac{1}{k} \int_{|\mathfrak{X}\tilde{u}| \ge k|\mathfrak{X}u|} |\mathcal{A}(\mathfrak{X}\tilde{u})||\mathfrak{X}\tilde{u}|\, dx \, + \int_{|\mathfrak{X}\tilde{u}| < k|\mathfrak{X}u|} |\mathcal{A}(\mathfrak{X}\tilde{u})||\mathfrak{X}u|\, dx \\ &\le \frac{1}{2} \int_{B_R} \big\langle \mathcal{A}(\mathfrak{X}\tilde{u}), \mathfrak{X}\tilde{u} \big\rangle\, dx \, + \, k^{g_0} c \int_{B_R} g(|\mathfrak{X}u|)\, |\mathfrak{X}u|\, dx. \end{aligned}$$

which combined with (5.9) and the ellipticity (2.33), concludes the proof. □

We require the following comparison lemma.

Lemma 5.2. *There exists $\sigma = \sigma(n, g_0, L) \in (0,1)$ and $c = c(n, g_0, \delta, L) > 0$ such that, for every $0 < \varrho < R/2$, the following estimate holds:*

$$\fint_{B_\varrho} G(|\mathfrak{X}u - \{\mathfrak{X}u\}_{B_\varrho}|)\, dx \ \le \ c\Big(\frac{\varrho}{R}\Big)^{\sigma} \fint_{B_R} G(|\mathfrak{X}u|)\, dx + c\Big(\frac{R}{\varrho}\Big)^{Q} \fint_{B_R} G(|\mathfrak{X}u - \mathfrak{X}\tilde{u}|)\, dx.$$

Proof. From (2.26) and triangle inequality, we have

$$\fint_{B_\varrho} G(|\mathfrak{X}u - \{\mathfrak{X}u\}_{B_\varrho}|)\,dx \le c\fint_{B_\varrho} G(|\mathfrak{X}\tilde u - \{\mathfrak{X}\tilde u\}_{B_\varrho}|)\,dx + c\fint_{B_\varrho} G(|\mathfrak{X}u - \mathfrak{X}\tilde u|)\,dx. \tag{5.10}$$

Now, we shall estimate both terms of the right hand side of (5.10) seperately.

The Theorem 1.3 being proved in the previous section, we estimate the first term of (5.10) using the estimate (1.8) for $\mathfrak{X}\tilde u$ as

$$\begin{aligned}\fint_{B_\varrho} G(|\mathfrak{X}\tilde u - \{\mathfrak{X}\tilde u\}_{B_\varrho}|)\,dx &\le c\Big(\frac{\varrho}{R}\Big)^{\sigma}\fint_{B_R} G(|\mathfrak{X}\tilde u|)\,dx\\ &\le c\Big(\frac{\varrho}{R}\Big)^{\sigma}\fint_{B_R} G(|\mathfrak{X}u|)\,dx + c\Big(\frac{\varrho}{R}\Big)^{\sigma}\fint_{B_R} G(|\mathfrak{X}\tilde u - \mathfrak{X}u|)\,dx\end{aligned}$$

The second term of (5.10) is estimated simply as

$$\fint_{B_\varrho} G(|\mathfrak{X}u - \mathfrak{X}\tilde u|)\,dx \le c\Big(\frac{R}{\varrho}\Big)^{Q}\fint_{B_R} G(|\mathfrak{X}u - \mathfrak{X}\tilde u|)\,dx,$$

and combining estimates of both terms of (5.10), the proof is finished. □

To proceed with the proof of Theorem 1.2, we shall need the following technical lemma which is a variant of a lemma of Campanato [4]. This is a fundamental lemma which has been extensively used in the literature. We refer to [30] or [28, Lemma 2.1] for a proof.

Lemma 5.3. *Let $\phi : (0,\infty) \to [0,\infty)$ be a non-decreasing function and $A, B > 1, \alpha > 0$ be fixed constants. Suppose that for any $\rho < r \le R_0$ and $\epsilon > 0$, we have*

$$\phi(\rho) \le A\left[\left(\frac{\rho}{r}\right)^{\alpha} + \kappa\right]\phi(r) + Br^{\alpha-\epsilon};$$

then there exists a constant $\kappa_0 = \kappa_0(\alpha, A, B) > 0$ such that if $\kappa < \kappa_0$, we have

$$\phi(\rho) \le c\left(\frac{\rho}{r}\right)^{\alpha-\epsilon}\left[\phi(r) + Br^{\alpha-\epsilon}\right]$$

for all $\rho < r \le R_0$, where $c = c(\alpha, \epsilon, A) > 0$ is a constant.

The proof of Theorem 1.2 requires the following intermediary step, which can be regarded as an almost-Lipschitz estimate. This is a consequence of the uniform Lipschitz estimate of $\tilde u$ from [44] and the above perturbation lemma.

Proposition 5.4. *Let $u \in HW^{1,G}(\Omega)$ be a weak solution of $\mathrm{Q}u = 0$, then for any $0 < \varepsilon < 1$ and $0 < r \le R \le R_0/2$, we have $\mathfrak{X}u \in \mathcal{L}^{1,Q-\varepsilon}_{\mathrm{loc}}(\Omega)$ and*

$$\int_{B_r} G(|\mathfrak{X}u|)\,dx \le c\left(\frac{r}{R}\right)^{Q-\varepsilon}\left[\int_{B_R} G(|\mathfrak{X}u|)\,dx + CR^{Q-\varepsilon}\right]. \tag{5.11}$$

Proof. For $B_R \subset \Omega' \subset\subset \Omega$, we have $\|u\| \le M_0$ in $\bar B_R$ and up to a representative we can regard that $u \in HW^{1,G}(B_R) \cap C(\bar B_R)$. Let us denote

$$I = \int_{B_R} \big\langle \mathcal{A}(\mathfrak{X}u), (\mathfrak{X}u - \mathfrak{X}\tilde u)\big\rangle\,dx, \tag{5.12}$$

where $\mathcal{A}$ is as in (5.5) and $\tilde u \in HW^{1,G}(B_R) \cap C(\bar B_R)$ is the weak solution of (5.6). Since $u = \tilde u$ in ∂B_R, the function $u - \tilde u$ can be used to test the equations satisfied by u and $\tilde u$, which shall be used to estimate I to obtain both lower and upper bounds.

First, using $u-\tilde{u}$ as test function for $\mathrm{Q}u=0$, we obtain

$$
\begin{aligned}
I &= \int_{B_R} \big\langle A(x_0,u(x_0),\mathfrak{X}u) - A(x,u,\mathfrak{X}u), (\mathfrak{X}u-\mathfrak{X}\tilde{u})\big\rangle\, dx \\
&\qquad + \int_{B_R} B(x,u,\mathfrak{X}u)(u-\tilde{u})\, dx \\
&\le c\big(R^\alpha+\theta(R)^\alpha\big)\int_{B_R} g(1+|\mathfrak{X}u|)|\mathfrak{X}u-\mathfrak{X}\tilde{u}|\, dx \\
&\qquad + c\,\theta(R)\int_{B_R} g(1+|\mathfrak{X}u|)|\mathfrak{X}u|\, dx
\end{aligned} \tag{5.13}
$$

with $\theta(R)$ as in (5.4), where we have used structure condition (5.2) and (5.3) for the first term and (5.7) for the second term of the right hand side of (5.13). Now we use (2.24) of Lemma 2.12 and (5.8) of Lemma 5.1 to estimate the first term of the above and obtain that

$$
I \le c\,\theta(R)^\alpha \int_{B_R} G(1+|\mathfrak{X}u|)\, dx. \tag{5.14}
$$

Secondly, to obtain the upper bound for I, we shall use the monotonicity inequality (2.32). Let us denote $S_1=\{x\in B_R : |\mathfrak{X}u-\mathfrak{X}\tilde{u}|\le 2|\mathfrak{X}u|\}$ and $S_2=\{x\in B_R : |\mathfrak{X}u-\mathfrak{X}\tilde{u}|> 2|\mathfrak{X}u|\}$. Taking $u-\tilde{u}$ as test function for (5.6) and using (2.32), we obtain

$$
\begin{aligned}
I &= \int_{B_R} \big\langle \mathcal{A}(\mathfrak{X}u)-\mathcal{A}(\mathfrak{X}\tilde{u}), (\mathfrak{X}u-\mathfrak{X}\tilde{u})\big\rangle\, dx \\
&\ge c\int_{S_1} \mathrm{F}(|\mathfrak{X}u|)|\mathfrak{X}u-\mathfrak{X}\tilde{u}|^2\, dx + c\int_{S_2}\mathrm{F}(|\mathfrak{X}u-\mathfrak{X}\tilde{u}|)|\mathfrak{X}u-\mathfrak{X}\tilde{u}|^2\, dx
\end{aligned} \tag{5.15}
$$

Recalling $G(t)\le t^2\mathrm{F}(t)$ from (2.21), we have from (5.14) and (5.15), that

$$
\int_{S_2} G(|\mathfrak{X}u-\mathfrak{X}\tilde{u}|)\, dx \le c\,\theta(R)^\alpha\int_{B_R} G(1+|\mathfrak{X}u|)\, dx. \tag{5.16}
$$

Now since $|\mathfrak{X}u-\mathfrak{X}\tilde{u}|\le 2|\mathfrak{X}u|$ in S_1 by definition, we obtain the following from (2.21), monotonicity of g and Hölder's inequality;

$$
\begin{aligned}
\int_{S_1} G(|\mathfrak{X}u-\mathfrak{X}\tilde{u}|)\, dx &\le c\Big(\int_{S_1}\mathrm{F}(|\mathfrak{X}u|)|\mathfrak{X}u-\mathfrak{X}\tilde{u}|^2\, dx\Big)^{\frac{1}{2}}\Big(\int_{S_1} G(|\mathfrak{X}u|)\, dx\Big)^{\frac{1}{2}} \\
&\le c\,\theta(R)^{\alpha/2}\int_{B_R} G(1+|\mathfrak{X}u|)\, dx
\end{aligned} \tag{5.17}
$$

where the latter inequality of the above follows from (5.14) and (5.15). Now, we add (5.16) and (5.17) to obtain the estimate of the integral over whole of B_R,

$$
\int_{B_R} G(|\mathfrak{X}u-\mathfrak{X}\tilde{u}|)\, dx \le c\,\theta(R)^{\alpha/2}\int_{B_R} G(1+|\mathfrak{X}u|)\, dx. \tag{5.18}
$$

Recalling (2.34) and (5.8), note that for any $0<r\le R/2$, we have

$$
\int_{B_r} G(|\mathfrak{X}\tilde{u}|)\, dx \le r^Q \sup_{B_{R/2}} G(|\mathfrak{X}\tilde{u}|) \le c\Big(\frac{r}{R}\Big)^Q\int_{B_R} G(|\mathfrak{X}\tilde{u}|)\, dx \le c\Big(\frac{r}{R}\Big)^Q\int_{B_R} G(|\mathfrak{X}u|)\, dx.
$$

where $Q = 2n + 2$. Combining the above with (5.18), we obtain

$$\int_{B_r} G(|\mathfrak{X}u|)\,dx \;\le\; c\left(\frac{r}{R}\right)^{Q}\int_{B_R} G(|\mathfrak{X}u|)\,dx + c\,\theta(R)^{\alpha/2}\int_{B_R} G(1+|\mathfrak{X}u|)\,dx. \tag{5.19}$$

Now, we follow the bootstrap technique of Giaquinta-Giusti [27].

For $0 < \rho \le R_0$, let us denote $\Phi(\rho) = \int_{B_\rho} G(|\mathfrak{X}u|)\,dx$, so that we rewrite (5.19) as

$$\Phi(r) \le c\left(\frac{r}{R}\right)^{Q}\Phi(R) + cR^{\vartheta}\int_{B_R} G(1+|\mathfrak{X}u|)\,dx \tag{5.20}$$

where $\vartheta = \tau\alpha/2$ with $\tau \in (0,1)$ as in (5.4). We proceed by induction, with the hyposthesis

$$\int_{B_R} G(1+|\mathfrak{X}u|)\,dx \le CR^{(k-1)\vartheta} \quad \text{for some } k \in \mathbb{N},\ k\vartheta < Q. \tag{5.21}$$

The hypothesis clearly holds for $k = 1$. Assuming the hypothesis (5.21) holds for some $k \in \mathbb{N}$, first notice that by virtue of (2.26), we have

$$\int_{B_R} G(|\mathfrak{X}u - \{\mathfrak{X}u\}_{B_R}|)\,dx \le CR^{(k-1)\vartheta}$$

which further implies that $\mathfrak{X}u \in \mathcal{L}^{1,(k-1)\vartheta}(\Omega', \mathbb{R}^{2n})$, see Remark 4.13. Now using (5.21) in (5.20), we apply Lemma 5.3 to obtain that

$$\Phi(R) \le c\left(\frac{R}{R_0}\right)^{k\vartheta}\big[\Phi(R_0) + C\big],$$

which, from definition of Φ, implies the hypothesis (5.21) for $k+1$ and $\mathfrak{X}u \in \mathcal{L}^{1,k\vartheta}(\Omega', \mathbb{R}^{2n})$ from Remark 4.13. We choose can choose ϑ small enough and carry on a finite induction for $k = 0, 1, \ldots m$ where m is chosen such that $m\vartheta < Q \le (m+1)\vartheta < Q+1$. Thus, after the last induction step, we conclude that

$$\Phi(r) \le c\left(\frac{r}{R}\right)^{m\vartheta}\big[\Phi(R) + CR^{m\vartheta}\big]$$

and $\mathfrak{X}u \in \mathcal{L}^{1,m\vartheta}(\Omega', \mathbb{R}^{2n})$. Given any $0 < \varepsilon < 1$, we can choose ϑ small enough such that $Q - \varepsilon \le m\vartheta$ and the proof is finished. □

Furthermore, using the estimate (5.18) above together with Lemma 5.2, we can get

$$\begin{aligned}\int_{B_\varrho} G(|\mathfrak{X}u - \{\mathfrak{X}u\}_{B_\varrho}|)\,dx &\le c\Big(\frac{\varrho}{R}\Big)^{Q+\sigma}\int_{B_R} G(|\mathfrak{X}u|)\,dx + c\int_{B_R} G(|\mathfrak{X}u - \mathfrak{X}\tilde{u}|)\,dx\\ &\le c\Big(\frac{\varrho}{R}\Big)^{Q+\sigma}\int_{B_R} G(|\mathfrak{X}u|)\,dx + cR^{\theta}\int_{B_R} G(1+|\mathfrak{X}u|)\,dx.\end{aligned} \tag{5.22}$$

where $\theta > 0$ is chosen similarly as in the above proof, which can be made small enough. This shall be required to prove Theorem 1.2 along with the estimate of Proposition 5.4.

Proof of Theorem 1.2.

Let us take $0 < \varrho \le r < R_0/2$ and we rewrite (5.22) as

$$\int_{B_\varrho} G(|\mathfrak{X}u - \{\mathfrak{X}u\}_{B_\varrho}|)\,dx \le c\Big(\frac{\varrho}{r}\Big)^{Q+\sigma}\int_{B_r} G(|\mathfrak{X}u|)\,dx + c\,r^{\theta}\int_{B_r} G(1+|\mathfrak{X}u|)\,dx. \tag{5.23}$$

From (5.11) of Proposition 5.4 we have,

$$\int_{B_r} G(|\mathfrak{X}u|)\,dx \le c\left(\frac{r}{R_0}\right)^{Q-\varepsilon}\left[\int_{B_{R_0}} G(|\mathfrak{X}u|)\,dx + CR_0^{Q-\varepsilon}\right]. \tag{5.24}$$

Using (5.24) on (5.23) to obtain the following estimate,

$$\begin{aligned}\int_{B_\varrho} G(|\mathfrak{X}u-\{\mathfrak{X}u\}_{B_\varrho}|)\,dx &\le c\Big(\frac{\varrho^{Q+\sigma}}{r^{\sigma+\varepsilon}R_0^{Q-\varepsilon}}\Big)\Big[\int_{B_{R_0}} G(|\mathfrak{X}u|)\,dx + CR_0^{Q-\varepsilon}\Big] + Cr^{Q+\theta-\varepsilon}\\ &\le C\Big(\frac{\varrho^{Q+\sigma}}{r^{\sigma+\varepsilon}} + r^{Q+\theta_0}\Big)\end{aligned} \tag{5.25}$$

where $\theta_0 = \theta-\varepsilon > 0$ is chosen with a choice of a small enough $\varepsilon > 0$. For some $0<\kappa<1$ we rewrite the above with the choice $r = \varrho^\kappa$ to have

$$\int_{B_\varrho} G(|\mathfrak{X}u-\{\mathfrak{X}u\}_{B_\varrho}|)\,dx \le C\big(\varrho^{Q+(1-\kappa)\sigma-\kappa\varepsilon} + \varrho^{\kappa(Q+\theta_0)}\big) \le C\varrho^{Q+\gamma}, \tag{5.26}$$

where the latter inequality follows when $Q+\gamma \le \min\{Q+(1-\kappa)\sigma-\kappa\varepsilon\,,\ \kappa(Q+\theta_0)\}$; indeed we can make sure that this is true with the choice of $\kappa=\kappa(\gamma)$ such that

$$\frac{Q+\gamma}{Q+\theta_0} \le \kappa \le \frac{\sigma-\gamma}{\sigma+\varepsilon},$$

for any $0<\gamma<(\sigma\theta_0-Q\varepsilon)/(Q+\sigma+\theta_0+\varepsilon)$, where $0<\varepsilon<\sigma\theta_0/Q$. Furthermore, having γ,ε small enough, $\kappa=\kappa(\gamma)$ can be chosen close enough to 1 and therefore, we can make sure $\varrho^\kappa < R_0/2$, whenever $0<\varrho<R_0/2$. Thus, from (5.26) we have obtained

$$\fint_{B_\varrho} G(|\mathfrak{X}u-\{\mathfrak{X}u\}_{B_\varrho}|)\,dx \le C\varrho^\gamma,$$

for any $0<\varrho<R_0/2$, which imlpies $\mathfrak{X}u \in C^{0,\beta}(\Omega',\mathbb{R}^{2n})$ with $\beta=\gamma/(1+g_0)$ from Remark 4.13 and the proof is finished. □

5.2. **Concluding Remarks.**
Here we discuss some possible extensions of the structure conditions that can be included and results similar to the above can be obtained with minor modifications of the arguments.

(1) Any dependence of x in structure conditions for $A(x,z,p)$ and $B(x,z,p)$ has been suppressed so far, for sake of simplicity. However, we remark that for some given non-negative measurable functions a_1,a_2,a_4,a_5,b_1,b_2, the structure condition

$$\begin{aligned}&\big\langle A(x,z,p),p\big\rangle \ge |p|g(|p|) - a_1(x)\,g\bigg(\frac{|z|}{R}\bigg)\frac{|z|}{R} - a_2(x);\\ &|A(x,z,p)| \le a_3\,g(|p|) + a_4(x)\,g\bigg(\frac{|z|}{R}\bigg) + a_5(x);\\ &|B(x,z,p)| \le \frac{1}{R}\bigg[b_0\,g(|p|) + b_1(x)\,g\bigg(\frac{|z|}{R}\bigg) + b_2(x)\bigg],\end{aligned}$$

can also be considered for obtaining the Harnack inequalities. In this case, we would require $a_1,a_2,a_4,a_5,b_1,b_2 \in L^q_{\rm loc}(\Omega)$ for some $q>Q$. Similar arguments can be carried out with a choice of $\chi>0$, such that $\|a_5\|_{L^q(B_R)} + \|b_2\|_{L^q(B_R)} \le g(\chi)$ and

$\|a_2\|_{L^q(B_R)} \leq g(\chi)\chi$. We refer to [39] and [9] for more details of such cases.

(2) The function $g(t)/t$ in the growth conditions can be replaced by $f(t)$, where f is a continuous doubling positive function on $(0,\infty)$ and $t \mapsto f(t)t^{1-\delta}$ is non-decreasing. A C^1-function $\tilde{g}$ can be found satisfying (1.2) and $\tilde{g}(t) \sim tf(t)$(see [39, Lemma 1.6]), which is sufficient to carry out all of the above arguments.

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(S. Mukherjee) Department of Mathematics, University of Bologna, Piazza di Porta San Donato 5, 40126 - Bologna (BO), Italy.

Email address: shirsho.mukherjee2@unibo.it