Anthony C Davison and Igor Rodionov's contribution to the Discussion of 'Estimating means of bounded random variables by betting' by Waudby-Smith and Ramdas

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We congratulate the authors on a wide-ranging contribution to a statistical hardy perennial. The restriction that the support be known and bounded appears highly restrictive, yet Proposition 3 suggests much wider potential for applications, one of which we briefly explore below.

The Pickands–Balkema–de Haan theorem (Balkema & de Haan, 1974; Pickands, 1975) establishes that the distribution of a random variable X, conditioned on its exceeding a high threshold u can in rather wide generality be approximated by the generalised Pareto distribution (GPD). This result, a cornerstone of the statistical analysis of rare events, is typically applied by choosing a threshold u empirically from a random sample of n observations, fitting the GPD to the k observations that exceed u, and using this fit to estimate quantiles or other measures of risk. Numerous procedures have been suggested to choose u, or equivalently k, often based on threshold-stability properties of the GPD; informal graphical approaches were proposed by Davison and Smith (1990), and in many settings more formal procedures are complemented by these or other graphs. The Hill plot (Hill, 1975) is used when X lies in the Fréchet max-domain of attraction with tail index $1/\gamma$ and is related to the Rényi representation for exponential order statistics. One potential use of the ideas in Section 4.4 of the paper is to aid in the choice of k, using the sample order statistics $X_{(1)} \le \cdots \le X_{(n)}$ and the simple martingale

$$M_k = (X_{(n-k)}/X_{(n-k+1)})^k, \quad k = 1, \ldots, n;$$

both (M_k) and its expectation $1/(1+\gamma)$ when the scaled differences $k(\log X_{(n-k+1)} - \log X_{(n-k)})$ are independent exponential variables lie in the unit interval. This leads to a conservative overall approach to choosing the number of upper order statistics k used in the Hill estimator.

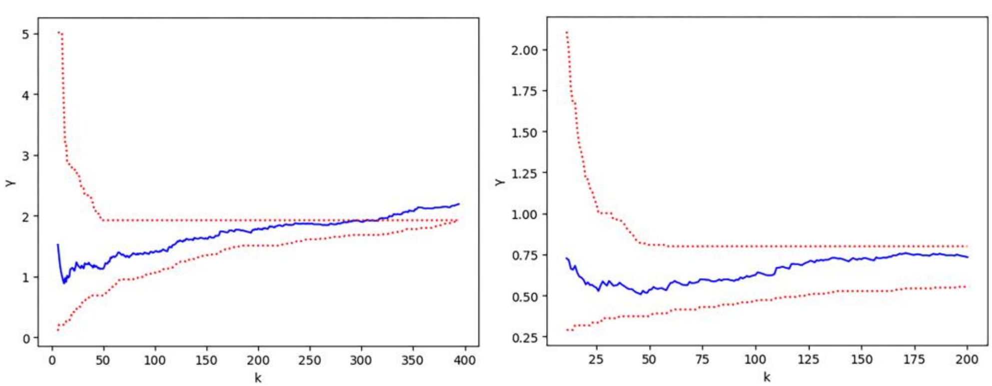

Figure 1. Hill plots showing the estimate of γ (solid) and martingale 95% confidence limits (dotted) based on a log-normal sample (left) and the Danish insurance data (right).

Figure 1 shows confidence intervals for γ for a log-normal sample and for the Danish fire insurance data (Embrechts et al., 1997). The first uses $\lambda_k^+ = (\gamma + 1)/4$ and $\lambda_k^- = (\gamma + 1)/(4\gamma)$, and the second uses the λs suggested in expressions (25) and (26) of the paper. The theory does not apply to the first, for which $\gamma = 0$, but extremes of finite log-normal samples are typically well-approximated by a distribution with a heavier tail; the martingale shows the expected lack of stability. The second is more stable, suggesting that $\gamma \approx 0.75$; the corresponding confidence interval is similar to that in the top left-hand panel of Figure 6.4.3 of Embrechts et al. (1997).

It should be clear that we found the paper very stimulating.

Conflict of interests: None declared.

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Steven R Howard's contribution to the Discussion of 'Estimating means of bounded random variables by betting' by Waudby-Smith and Ramdas

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I congratulate the authors on this creative and well-written contribution. It's always a delight to see new insights into such a fundamental problem as inference for the mean of bounded observations, and certainly inspires excitement about the future of game-theoretic statistical inference. Besides the impressive technical contributions, I appreciate how the authors draw previously unappreciated connections between work in information theory, finance, and online learning, hopefully increasing future collaboration between researchers in these fields.

In their conclusion, the authors allude to the possibility of estimating functionals other than the mean, using the quantile as an example. Are there fruitful applications of betting methods to estimation of general, possibly vector-valued functionals θ defined by possibly vector-valued estimating equations of the form $\mathbb{E}\left[\psi(X_t, \theta) \mid X_1, \ldots, X_{t-1}\right] = 0$ a.s. for all t, assuming such a value of θ exists (Angelopoulos et al., 2023)? This would be interesting even for univariate functionals