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Fear of exclusion: the dynamics of club formation

Priyanka Joshi[1]

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Abstract
This paper explores a dynamic model of sequential club formation in which identical individuals join or leave clubs over time and their preferences depend solely on the number of members in the club. There exists a unique optimal-sized club which maximises per-period payoff of each individual. To study the implications of the dynamic setting, we use a benchmark game of a finite number of periods which mimics the static framework. Implied by the dynamic nature of the problem, we find a new source of inefficiency that is caused by so called *fear of exclusion* phenomenon where individuals fear being excluded from a relatively superior sustainable club, which is not necessarily optimal. An unusual behaviour may be observed in which individuals strictly prefer to form sub-optimal sized clubs. A specific class of equilibria is analysed to examine such behaviour.

Keywords Non-cooperative games · Club goods · Fear of exclusion · Stable club structure · Infinite horizon · Optimal club

1 Introduction

Clubs form for the purpose of sharing the benefits and costs of a club good. When individuals decide to become members of a particular club, they generally care about about how many other members are there and how those other members behave. In this paper, we consider situations in which individuals only care about the size of the club they intend to form or join. One can think of gyms as an example. If gym etiquettes are commonly understood and all members follow the norms, then what

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Priyanka Joshi
priyanka.joshi@essex.ac.uk

[1] Department of Economics, University of Essex, Colchester, UK

really defines members' preferences is the number of members using the gym facilities. It is desirable to have some members in the gym because it provides incentives and motivation to exercise, but too many would lead to congestion and long waiting times. Therefore, it can be argued that an individual would prefer to join a gym which is neither too crowded nor too empty: we call them optimal-sized clubs. The question then arises whether individuals always join such an optimal sized club? To analyse such situations, we consider dynamic environment in which individuals take decisions over long periods and found that inefficiencies (potential members joining sub-optimal sized clubs) may arise. The aim of this paper is to uncover the underlying mechanism behind the source of such inefficiencies. We investigate the process where individuals decide to form clubs and decisions involve changing the size of the club; more specifically, the paper explores the dynamic decision making process of club formation and aims to find a new source of inefficiency, which we show in the paper is generated by an endogenous "fear of exclusion" (FOE) phenomenon. Fear of exclusion is characterized by a situation in which an individual feels anxious about being left out of the what is considered as a relatively better outcome, and this exact fear might generate inefficiency endogenously.

The study of club goods has spawned a huge literature in economics going back to at least Tiebout (1956) and Buchanan (1965). Club theory, as developed by Buchanan (1965), views club goods as public goods that are excludable and partially rivalrous: there is excludability in the sense that the club goods are restricted only to the members of the club; and there is rivalry because of the crowding effect. If the cost of provision is shared equally among members then increasing the size of the club reduces the cost. This, in turn, increases the utility of each member but only until reaching a point where congestion may set in. Therefore, individuals' preferences will incorporate a trade-off between cost reduction and crowding as size increases. Several extensions and refinements of Buchanan's theory of club goods have been modelled and analysed, but the analysis of membership size in a dynamic context to explore varied inefficiencies is largely missing in the club literature.[1]

The early literature on club theory mainly aimed at examining the welfare aspect of club formation (optimal provision of the club good and optimal membership size) in a static setting. However, it is evident that players can join or leave clubs repeatedly. The future implications of such decisions may be particularly important if the agents in the economy are far-sighted and patient enough. Konishi et al. (1997) studied a coalition formation game with free mobility of players where the population partitions itself into clubs, but the game ends as soon as the stable club structure is reached. Stiglitz (1977) analysed club formation with a median voting rule by assuming that the current changes would not lead to future changes. Klevorick and Kramer (1973) also considered a median voter rule in a one period game with single-peaked preferences over the decision variable. Layard (1990) studied a bargaining model of wages for a democratic trade union where the median voter's choice of wage is bargained with the firm. Under the assumption of zero discounting, the

[1] These refinements include different forms of the congestion functions, hedonic games (the composition of members), different types of mechanism for exclusion, optimality of clubs, financing of clubs, etc.

equilibrium level of wages and employment was analysed. An important distinction between these papers and ours is that the clubs' decisions (not just membership) are agreed at the formation stage through negotiations.

To capture dynamic considerations, we define a non-cooperative game of club formation in which each player's strategy is either to propose to form a new club or respond (say yes or no) to the proposal offered to her over time. The order in which players propose is given by an exogenous protocol. Each player only cares about the size of her club, rather than the identity of its members.[2] Utility increases initially with size, and then decreases because of a crowding effect. There is, therefore, a unique optimal sized club, which maximises its members' utility. Since all individuals are identical and have the same preferences over the size of the club, they would all like to be members of an optimal sized club. The objective of this exercise is to obtain equilibria where fear of exclusion arises endogenously. A class of FOE equilibria are the ones in which agents refrain from forming optimal sized clubs. To this end, we concentrate on a very specific class of stationary Markov perfect equilibria (SMPEs). In the infinite-horizon game, we consider those equilibria in which clubs of only two sizes prevail in the long-run: a good (bigger size) club whose members receive a higher utility; and a bad (smaller size) club whose members receive a low utility. We establish the existence of SMPEs of interest of such a game by construction.

One can argue that it is well-known from Folk theorems that repeated interactions with rational and patient enough players can allow for many SPE outcomes that satisfy the properties of Individual Rationality and Feasibility. We would like to bring attention to two points here. First, that standard Folk theorems do not directly apply to Markov Prefect equilibria because of the changing and evolving states, on which players can base their decision. The Folk-Theorem-like results, even for Markovian equilibria, typically rely on mixed-strategy equilibria, randomization devices, and intricate constructions. The equilibria we have, on top of being Markovian, are as refined and intuitive as it gets: in particular, they are pure and absorbing. In addition, while the Folk-Theorem literature typically focuses on the set of equilibrium payoff vectors, our focus is on equilibrium behaviour, and in particular the emergence of the fear of exclusion as an equilibrium phenomenon. Hörner et al. (2011) studied stochastic game and proved a Folk theorem under some conditions. The game is stochastic in the sense that each state depends on the action and previous state and is probabilistic, whereas in our paper, only the order of moves of players is probabilistic. Also, it is assumed in their paper that limiting set of equilibrium payoff is independent of the initial state and in our set up, we do not assume this, but it rather emerges as a property of the class of FOE equilibria. Their model is more complex and hence stricter assumptions are imposed to prove the desired results. In comparison, our model is much simpler, but still provide intuitive insights without imposing many restrictive assumptions.

[2] The size of the club can appear directly in the players' utility function through the cost sharing of the club good or indirectly through decisions taken by a club of a given size.

Second point concerns the results: Hörner et al. (2011) proved a Folk theorem for the stochastic model, whereas our results differ from theirs and hence from Folk theorems. We find that there is a unique equilibrium payoff in finite repetition game and characterize multiple equilibria in infinite repetition. If the payoff of the finite game coincides with that of the minmax payoff of the stage game or if we use the same version of the Folk theorem as Friedman (1971), then multiplicity of equilibria in the infinite game reveals that all but one equilibrium payoff are either same or worse than the minmax payoff or Nash equilibrium payoffs of one stage game. In fact, we can support any equilibrium in the infinite game under some mild conditions, whereas Folk theorem supports individually rational and feasible payoffs of the stage game. In terms of Pareto optimality, the outcome of finite game is Pareto optimal, but most of the outcome in infinite game is Pareto inefficient. This stands in contrast to Folk Theorems.

A club structure is just a partition of individuals, i.e. grouping of individuals into non empty clubs in such a way that every individual is included in exactly one club. We follow Acemoglu et al. (2012) to find (dynamically) stable club structures, i.e. club structure which does not change once it has been formed. Under the assumption that players are forward looking, we characterize and prove the existence of a very specific class of stable club structures. The properties we consider when providing such a characterization are: (a) existence of stable club structures, and (b) formation of optimal sized clubs in stable club structures. We explore the different kinds of behaviour individuals might exhibit, depending on whether they are in finite or infinite horizon game.

Our main conclusions are derived from comparing the results of finite and infinite games given in Propositions 1, 2, and 3. If the players are patient enough, we show that club structures with no optimal sized clubs may be stable in an infinite horizon game. The assumption of a high discount factor is essential and enables us to get the desired results; but it is also empirically relevant. A high discount factor is natural when club formation and dissolution take little time.

Why may we observe stable club structures which have no optimal sized clubs in the infinite horizon game? Despite the dynamic nature of the game, we can point out two straightforward reasons that dictate the equilibrium behaviour which leads to such club structures: (1) one or more individuals have pessimistic views that optimal sized clubs are not sustainable, and (2) fear of exclusion. Players might believe, in equilibrium, that optimal sized clubs would not survive for long. They might then fear exclusion from a club of sub-optimal size which provides a larger return than the other sustainable club. In other words, they fear exclusion from their best available option *among clubs that are sustainable in the future*. Fear of exclusion manifests itself in players' behaviour in such a way that it compels disloyalty to optimal sized clubs and other non-stable clubs. By contrast, these arguments do not apply in finite period game. Even though membership commitment is not feasible in our set-up, players' ability to see the end outcome guarantees non-betrayal.

The rest of the paper is organised as follows. Section 2 reviews the relevant literature. We present our model in Sect. 3, and provide results for finite and infinite period games in Sects. 4 and 5, respectively. In Sect. 6, we discuss the results and intuitions, and provide a brief overview of a potential extension of the model which could potentially explain how extreme groups form and are sustained and weather FOE, among factors, can explain the emergence of such groups. Section 7 concludes.

2 Related literature

Buchanan (1965) was one of the first studies that looked at the welfare analysis of the in-between case of pure public and private goods. Buchanan developed a general theory of clubs to address how the size of the club influences the provision of public goods. For predecessors of club theory such as Tiebout (1956), Wiseman (1957) and Olson (1965) among others, the provision was exogenous and therefore, it was not clear how the provision of shared good and membership interacted. In contrast, in Buchanan clubs, membership size is an endogenous choice which depends on the provision decision. Several models and extensions have built on this concept of Buchanan clubs: earlier papers include Pauly (1970), Wooders (1978) and Shubik and Wooders (1982, 1983); more recent papers include Page and Wooders (2007), Banerjee et al. (2001) and Bogomolnaia and Jackson (2002).

This paper is closely related to the models which allow for congestion effects. Milchtaich (1996) studied a class of non-cooperative games in which the utility of a player derived from using a specific strategy depends only on the total numbers of players who are employing that same specific strategy, and the utility decreases with that numbers in a way which is defined for that particular player. Similarly, Holzman and Law-Yone (1997) looked at a more specific case of congestion games, which reflects the negative effects of the congestion. Konishi et al. (1997) analysed a non-cooperative game where individuals have the same congestion function. They proved existence for the general case and showed that existence may fail without the assumption of a common congestion function. Hollard (2000) studied a similar model with an anonymous congestion function which allowed for externalities on non-members. While these studies prove existence and find stable Nash equilibrium strategy choices, the static nature of these models cannot explain how an equilibrium is reached, if at all. Our first contribution is therefore to the limited literature on the dynamics of club formation. To do this, we follow Acemoglu et al. (2012) by allowing for repeated interactions, and characterize dynamically stable states.

This paper also relates to the literature on non-cooperative coalition formation. Chatterjee et al. (1993) analysed a game of coalitional bargaining with n players where players can transfer utility and investigated the efficiency (no-delay and formation of grand coalition) properties of SSPEs. They show that inefficiently small coalitions may arise in equilibrium and/or agreement may be delayed. However, the sequence in which the players move significantly affects the efficiency of the equilibrium. Seidmann and Winter (1998) studied endogenous bargaining games with reversible and irreversible actions and show that if players are allowed to renegotiate then the process results in the grand coalition, but this might not happen otherwise. The kind of commitment device (players leaving the game after forming the club in Chatterjee et al. (1993)) and (reversible and irreversible action in Seidmann and Winter (1998)) drives their results to some extent. We refrain from imposing such commitment, which emerges endogenously in our framework.

Ray and Vohra (1999) studied a coalition formation game where players have utilities which depends on the whole coalition structure. They formulated the analysis after defining a partition function, and assumed that players can write binding agreements. They provided sufficient conditions for no-delay equilibria, and used these finding in the context of Cournot oligopoly. Hyndman and Ray (2007) studied a coalition formation game in which agreements are binding until all affected parties agree to renegotiate it. Allowing for history-dependent strategies, they showed that for characteristic function games, efficiency is achieved on every equilibrium path. In contrast to these studies, members of a club in our model cannot write binding contracts; in other words, they cannot formally provide commitment to members within or outside the club they belong to.

Anesi and Seidmann (2015) studied a dynamic committee voting model in which committee members decide on the division of a pie. They employed Markov Perfect Equilibrium concept to demonstrate the existence and uniqueness of equilibrium payoffs. Using two different voting rules, they analysed whether the entire pie is divided among a minimal winning majority and whether the equilibria are Pareto efficient. The logic underlying the equilibrium construction in our paper is reminiscent of Anesi and Seidmann (2015), but the voting-game structure of their extensive form is different from ours. In particular, their construction relies on the fact that as soon as a winning coalition reaches an agreement, all players are "stuck" with that absorbing state. This is similar to our construction, where players are stuck in *dynamically stable state* (optimal or suboptimal) once it has been reached on the equilibrium path. However, unlike their voting game structure, in our club-formation context, once a club reaches an agreement (even on the equilibrium path), other players can still form clubs, and the subgame that starts after the formation of that club is strategically different from the subgame that started in the previous period. Moreover, due to the different context, their primary arguments on efficiencies are based on comparing the outcomes of unanimity games with those of non-unanimity games, while our main focus is on identifying the long-term inefficiencies that arise in the dynamics of club formation games.

Arnold and Wooders (2015) studied a dynamic club formation game where the mobility of players is modelled explicitly, and players are myopic. In their model, there is no restriction on how clubs are formed and dissolved, in the sense that players simultaneously choose locations/clubs, and individuals who choose the same location or club form a club. The result is that existing members of a club cannot restrict further entry. Therefore, an equilibrium fails because of the indivisibility problem: when the population size is not an integer multiple of the optimal size club, the remaining individuals who are not members of optimal sized club prevent the process from settling down. To solve this, they restrict attention to approximate Nash club equilibrium. We depart from this and emphasize on forward looking players and follow a different approach to restrict entry by introducing unanimity rules within a club. This formulation allows us to investigate equilibria in which players rationally avoid forming optimal sized clubs.

Our model is a simple version of the dynamic club good game analysed by, among others, Roberts (2015) and Barbera et al. (2001). The former article considers voting by majority rule, and individuals are ordered according to single-crossing preferences, meaning individuals have different preferences over the size of the club depending on the order. The later article proved existence of a pure strategy perfect equilibrium for finite horizon games in which any member of the club can vote to include a non-member/s unilaterally and the utility of the players depends on the stream of the members included in the club. Even though these models incorporate more complex elements of club formation process and provide many useful insights, a model like ours is much needed to analyse the emergence of clubs in a relatively simple setting of homogeneous preferences over the size of the club without the factors at play considered in the above mentioned papers.

3 The model

In this section, we introduce the model of club formation. There are n identical individuals denoted by $N \equiv \{1, \dots, n\}$ with $n \geq 3$. The players are indexed by i. A *club* S is a non-empty subset of players. We write $s \in N$ as the size of club S or the number of members of S. A *club structure*, π is partition of N. Let Π be the set of all club structures. For any non-empty subset of K of N, the set of partitions of K is denoted by Π_K, with typical element π_K.

The focus of this paper is on the size of the club; so, we assume that the each individual's preferences do not depend on the identity of the members of her club, but only on the size of the club. An individual who belongs to club S gains a per-period payoff of $v(s)$.[3] For each $i \in N$ and each $\pi \in \Pi$, define $v_i(\pi)$ as $v_i(\pi) = v(s)$ where $i \in S \in \pi$ and $s = |S|$. We assume that v is strictly concave and single-peaked. This implies that there exists an optimal club size for each individual.

Let P be the set of permutations of N and $\mathcal{N} \equiv 2^N \backslash \{\emptyset\}$. A *rule of order* $\rho \in \Delta(P)$ is an exogenous protocol, where ρ determines the order of moves of proposers at the beginning of each period t in the sequential game of club formation. Suppose that Nature chooses the sequence of proposers $(\imath_1, \dots, \imath_n)$, then each agent is chosen as a proposer exactly once, but her order in the sequence is randomly chosen by Nature in each period.

In each period $t = 1, 2..., T$ (where T may be infinite) the sequential game of club formation proceeds as follows. The first proposer, $\imath_1$, starts the game by proposing to form a club $S \ni \imath_1$ or passes. If $\imath_1$ proposes S to which she belongs then all $i \in S$ simultaneously decide whether to accept or reject $\imath_1$'s proposal. If all members accept, the members of S form a coalition and each member then incurs a positive cost of forming a club, denoted $\varepsilon > 0$, in that period and the period ends. The simultaneous move is considered for the ease of writing responders' strategies.[4] It has no

[3] Buchanan (1965) assumed that there are n identical individuals, with utility represented by $U(x, s, G)$, where x is the private good consumed, s is the number of club members, and G is the provision of the club good. Utility increases in x and G, and decreases in s (congestion from overcrowding).

[4] The significance of sequential versus simultaneous moves lies in when a responder chooses to accept or reject a proposal. A responder is pivotal when her rejection changes the outcome, while she is non-pivotal when she would accept regardless. In sequential moves, the responder accepts if all prior players have done so. In simultaneous moves, her choice depends on others' strategies: if others reject, she is indifferent, but if all accept, she does too.

implications on the results as we restrict our attention to those equilibria in which no player uses weakly dominated strategy as established by Bernheim (1984) and further supported by Milgrom and Roberts (1991).[5]

Note that the cost of forming a new club could, in principle, depend on the size of the club but, here, we only introduce this cost as a tie breaking device. Therefore, we assume this cost to be given and fixed. If any $i \in S$ rejects the proposal or the proposer passes then the next proposer, ι_2, makes a proposal $S' \ni \iota_2$ or passes. The stage game then proceeds as before, until a new club has been formed or the last proposer has proposed.[6] The next period then starts and the game proceeds with the first proposer in the order selected by Nature. Finally, all agents seek to maximise their average discounted per-period payoff, and share a common discount factor $\delta \in [0, 1)$.

Thus, at the end of each period t, we obtain a coalition structure π^t, which contains: (i) a new club (possible empty) that has been formed in the current period; (ii) clubs in π^{t-1} that have not been affected by the moves in the current period (none of their members have successfully proposed or agreed to form another club); and (iii) the broken clubs (one or more whose members has left the club to join the new club). We assume that members of the broken club remain in their old club after some of the members have left.

Period t starts with the club structure obtained from the last period, $t-1$. In any period t, the club structure from the last period, π^{t-1}, can be altered at most once. Thus, when a player i gets to move, the club structure from the last period is intact and she has the opportunity to change it, either as a proposer or as a respondent. Hence, the current coalition structure for an active player i in period t is π^{t-1}. The above set up implies that in each period, only one club is formed, if at all. In any period t, if all the proposers pass or if all the proposals are unsuccessful then no club is formed in that period. The assumption that maximum of one club can be formed in each period is for expositional purposes and our results do not depend on this.[7]

A history at any stage of the game is a complete list of all proposals, acceptances and rejections that took place in all the previous periods and stages. A player is called *active* at history h^t at date t if it is her turn to move after history h^t. We assume that all players possess perfect recall and that the game is of perfect information. At any stage, each player observes and recalls everything that has previously transpired in the game, which we call a complete history.

[5] By imposing stage-undominated strategies, we eliminate situations where non-pivotal players vote against their dynamic preferences, ensuring more consistent voting behavior. This also excludes uninteresting equilibria in which all responders reject new proposals, preventing the formation of any new club with three or more members. As a result, the focus shifts to more meaningful outcomes, where responders' decisions align with their evolving preferences, promoting relevant and stable group formation.

[6] The period ends when either a club is formed or the last proposer has proposed.

[7] This is just for the ease of clarity to show how a club forms. We could, in fact, allow for multiple clubs to form in each period and the results will still hold. The reasoning behind this is the fact that what matters is how many members are there in each club, not who is in the club. If an individual can successfully form a desired club in period $t > 1$, then she could also form that club in period 1 if it is possible and she gets the opportunity. This would result in same equilibrium club structure with the only difference that who belongs to those clubs might differ depending on the protocol.

3.1 Equilibrium and strategies

Note that games with infinitely repeated interactions and patient players may have many equilibrium outcomes (cf. the folk theorem). It is therefore easy to find an equilibrium in which clubs of inefficient sizes form. Following the lead of Acemoglu et al. (2012), we therefore use the Markovian solution concept. Specifically, a player's strategy can, generally, depend on the complete history, i.e. everything that has transpired in all stages of all previous periods; but we focus on Markovian strategies (cf. Maskin and Tirole (2001)). We will therefore shed unneeded generality and only provide a formal definition for such strategies.

The definition of Markov strategies begins with the set of payoff-relevant states, which in this case are of two types: proposer states, and respondent states. A *proposer state* is a pair of a club structure $\pi \in \Pi$ and a list of remaining proposers $(\imath_\ell, \dots, \imath_n) \in N^\ell$, for some $\ell \in \{1, \dots, n\}$. (We identify $\imath_\ell$ as the identity of the proposer, whose turn it is to make a proposal). A *responder state* is a pair of a proposer state $(\pi, ((\imath_\ell, \dots, \imath_n))$ and a proposal to form a club $S \in 2^N$.

For a player who is a proposer and active at history h^t, the only payoff-relevant variables of the history are the current coalition structure π, and the remaining sequence of proposers; for a respondent, the payoff-relevant variables are the current coalition structure, the remaining sequence of proposers, and the proposal just made to her, S. (We interpret $S = \emptyset$ as passing). The sets of proposer and responder states are denoted by K^p and K^r respectively, with generic element κ.

Let K_i^p be the set of proposer states in which it is player i's turn to make a proposal i.e., $K_i^p \equiv \left\{ \left(\pi, (\imath_\ell, \dots, \imath_n)\right) \in K^p \,:\, \imath_\ell = i \right\}$; and let K_i^r be the set of responder states in which i has to respond to a proposal i.e., $K_i^r \equiv \left\{ \left(\pi, (\imath_\ell, \dots, \imath_n), S\right) \in K^r \,:\, S \ni i \right\}$. A *(stationary) Markov strategy* for player $i \in N$ is a pair $\sigma_i = (\alpha_i, \beta_i)$ of a proposer strategy and a responder strategy, where $\alpha_i : K_i^p \to 2^N$, and $\beta_i : K_i^r \to \{\text{ yes , no }\}$. Note that strategy profile σ_i is stationary in the sense that it only depends on history via payoff-relevant states.

A *stationary Markov perfect equilibrium* (SMPE) is a subgame perfect equilibrium in which all players use stationary Markov strategies. We follow the standard approach of concentrating throughout on equilibria in stage-undominated responder strategies, i.e., those in which, at any response stage, no player uses a weakly dominated strategy. In particular, we are interested in finding pure strategy SMPEs. Henceforth, any reference to 'equilibria' is to the pure strategy SMPEs in stage-undominated responder strategies.

Optimality or efficiency. In our analysis of sequential club formation, we define optimality as follows: optimal club structures are the ones which have all possible optimal-sized clubs; on the other extreme, sub-optimal club structures are the ones in which none of the clubs are of optimal size. In the literature of club goods or public goods, it is not clear whether efficiency or Pareto-optimality should be viewed from the point of social welfare or the welfare of the representative member of a club [see Helpman and Hillman (1977), Ng (1973, 1978)]. As a first step in this paper, we refrain from the complete welfare analysis of equilibrium club structure and follow the approach of Buchanan clubs to find optimal club structure, so we use the word 'optimal' even though we are not using it in the sense of welfare analysis.[8] In some instances, we will comment on whether an equilibrium club structure is Pareto-optimal.

4 Benchmark: short-run interactions

In this section we study the game with finite number of periods, i.e. $1 \leq T < \infty$. The main purpose of this benchmark is to characterize equilibria of the finite game, so as to compare equilibrium outcomes in the finite and infinite games. We start by studying the special case of $T = 1$ period.

We know that v is strictly concave and let $s^* = min\{n, \arg\max_{s \in \mathbb{N}} v(s)\}$ be the optimal club size. Throughout the paper we assume that $v(s) > v(s') \Rightarrow v(s) - \varepsilon > v(s')$. This assumption, plus the concavity of v imply that

$$v(s^*) - \varepsilon > v(s) \quad \forall s \neq s^* \tag{1}$$

where $\varepsilon > 0$.

Proposition 1 *Let k^* be implicitly defined by $k^* s^* \leq n \leq (k^* + 1)s^*$. If $T = 1$ then in any subgame perfect equilibrium, the following holds: (i) no club of size-s^* breaks up; and (ii) if at the start of the period there are fewer than k^* clubs of size-s^* then an additional club of size-s^* forms.*

The proof of proposition 1 is in the Appendix. We use the standard backward induction argument to prove the claims made. The way it proceeds is that the last proposer will either form an optimal sized club if she is already not a member of one and there are enough individuals left out of the optimal sized club. If the last propose is already a member of optimal sized club, then it is best for her to remain in her club: she already has highest possible payoff and this is the only payoff she cares about. Same reasoning is applied to respondents. If a respondent is a member of an optimal sized club, then she remains in that club. Given this reasoning, no optimal sized club will dissolve after the last proposer has proposed. Moving back to the best response of the second last proposer and the respondents, same reasoning can be applied to their responses because they can anticipate that no optimal sized club will dissolve until the end of the game. Thus, no optimal size club break. Given this, individuals will form an optimal sized if they have the opportunity because they can anticipate that club will remain in place.

The Proposition 1 shows that, in a single period game, an optimal sized club forms if it is possible to form one, i.e. if s^* or more than s^* individuals do not belong to a club of size-s^* at the start of the period. It also shows that members of optimal

[8] In deriving the Pareto-optimality conditions, Buchanan equilibrium clubs maximise the benefits of a representative club member rather than maximising the total net benefit of the whole population.

sized club never break their clubs (pass if they are proposer or reject any proposal if they are respondents). This implies that if k^* optimal sized clubs already exist at the start of the game then k^* clubs would remain intact until the end of the game. Note that size-$(n - s^*k^*)$ is the next best club after size-s^* club since v is strictly concave. Then, Proposition 1 also implies that if k^* optimal sized clubs exist at the start of the game then the individuals who are not in optimal sized club form a club of size-$(n - s^*k^*)$ if they already do not belong to size-$(n - s^*k^*)$ club; otherwise every proposer passes. Individuals know that they will get payoff only once and therefore they strive to get the highest payoff, i.e to form or join an optimal sized club as soon as they get an opportunity. If they are already in optimal sized club then they can anticipate that none of the members of their club would break their club because optimal sized club provide them the highest payoff since forming new clubs incurs a cost. We now consider the more general case of $T > 1$ finite.

Proposition 2 *If T is finite then the following holds in each period*: (*i*) *no club of size-s^* breaks up*; *and* (*ii*) *if at the start of the period there are fewer than k^* clubs of size-s^* then an additional club of size-s^* forms*.

The proof of proposition 2 is in the appendix. Proposition 2 implies that in every period an optimal sized club would form if it is possible to form one and that no existing optimal sized club would break until the end of the game. Individuals can anticipate that optimal sized clubs remain intact until the end of very last period if they form or join an optimal sized club. Then, from there onwards they can get the highest payoff in every period. Thus, if there are enough periods to form all optimal sized clubs then there will be k^* clubs of size-s^* in the stable club structure.

As mentioned before, the cost of forming a new club is assumed to be positive as a tie breaking rule. The essence of our main results remains valid even if $\epsilon = 0$. To demonstrate Proposition 2 if $\epsilon = 0$, we will focus exclusively on SMPEs to eliminate any trigger strategies.[9] Given that we concentrate on Markov strategies, assume $\epsilon = 0$. In the last period, there exists at least one club of size s^*. If no optimal sized club exists in the last period, then s^* individual will successfully form one. If all or some optimal sized clubs exist, then if a new club forms it must be of size s^*. The players in the second last period anticipate that they will either be in s^* club or some other club $s' \neq s^*$. Then, in the second last period, if a club is successfully formed, it must be of size s^* because $v(s^*) + \delta[\alpha v(s^*) + (1 - \alpha)vs'] > v(s'') + \delta[\alpha v(s^*) + (1 - \alpha)vs']$ where α is the probability of ending up in optimal sized club. Note that a player's strategy and her being in optimal club cannot guarantee that she will remain in optimal club in the next period. Some members of her current club might break the club and form a new optimal sized club. However, she knows for sure that she will either be in s^* or $s' \neq s^*$.

[9] Even without Markov strategies, the core results would still hold. The difference is that in some equilibria, players might use trigger strategies to support forming sub-optimal club except in the last period, when it will eventually break down. Therefore, sub-optimal club structure is not sustainable in the finite period game.

Corollary 1 *Let m^* (possibly 0) be the number of size-s^* clubs at the start of period 1. Then, at the end of period $t = (k^* - m^*)$ there exist k^* clubs of size-s^*.*

This is a direct consequence of Proposition 2. We know from Proposition 2 that a club of size-s^* never breaks. Then, after all the optimal-sized clubs are formed, all the proposers who are members of a size-s^* club would pass when given the opportunity to propose, and all the respondents who are members of a size-s^* club reject any offer to form a new club. Thus, all clubs at the end of period $k^* - m^*$ are optimal sized, and there are some left over individuals who cannot form an optimal-sized club because there are not enough individuals left to form one. Also, for any initial club structure, the protocol and the proposer and responder optimal strategies determine the equilibrium, but not the outcome, i.e., the club structure. In other words, the stable club structure remains the same in every equilibrium.

The analysis of the finite-horizon game implies that the outcome (number of clubs and club sizes) of all possible equilibria is unique. Depending on the protocol and the initial club structure, some individuals will end up in an optimal sized club and those who did not get the opportunity to become member of an optimal sized club would end up in a sub-optimal sized club. Agents anticipate that at the end of the game, there will only be two types of clubs: one club of size-$(n - m^*s^*)$ and rest of the clubs would be of size-s^*. The finite horizon allows members of the optimal sized club to implicitly "commit" not to break their club if agents are aware that the process of club formation ends at some point. Note that outcome in the finite game is also Pareto optimal.

The club structure at the end of period $k^* - m^*$ would have k^* clubs of size s^*. In the next period, players who are not in a size-s^* club would form a club among themselves, i.e. a club of size-$(n - k^*s^* < s^*)$, if they already do not belong to a club of size-$(n - k^*s^*)$: a club of size-$(n - k^*s^*)$ is the next best club after a size-s^* club. Then, the club structure at the end of period $k^* - m^* + 1$ would have one club of size-$(n - k^*s^*)$ and k^* clubs of size-s^*. This club structure would never change because we know from Proposition 2 that size-s^* clubs never break, and size-$(n - k^*s^*)$ is the next best club after size-s^* club. All the possible stable club structures of the finite period game have the maximum possible number of clubs of optimal size.

5 Long-run interactions: $T = \infty$

In this section, we analyse how clubs of inefficient size may form in the infinite horizon game. We observed in Section 4 that, in the case of short-term interactions, all the possible optimal-sized clubs form after a finite number of periods and never dissolve. Once all the optimal-sized clubs have been formed, the rest of the agents form the next best club and remain in that club forever. As a result, the equilibrium stable club structure does not change once all the optimal-sized clubs and next best club have been formed. Thus, optimality (stable optimal club structure) is achieved in the case of finite periods. The objective of this section is to show that this may not hold if there is no deadline. In particular, fear of exclusion may appear endogenously in an equilibrium. If agents are far-sighted then fear of exclusion from a relatively better club can make agents behave differently as compared to the case of finite periods. A change in behaviour, in turn, might change the club structure obtained, since membership in a club is itself influenced by the behaviour of agents.

5.1 (Sub-optimal) dynamic stable club structure

Now we characterise the equilibria of this game. To this end we will use the approach of Acemoglu et al. (2012) to define *dynamically stable club structures*.

Definition 1 We say that a club structure $\pi \in \Pi$ is *dynamically stable* if there exists a threshold $\bar{\delta} \in (0, 1)$ such that for all $\delta \in (\bar{\delta}, 1)$, there exists a SMPE in which the following holds: (i) π is formed after a finite number of periods with positive probability on the equilibrium path; and (ii) whenever π is formed (on or off the path), it remains in place in all future periods.

In other words, π is dynamically stable if it does not change once it has been formed in some period. Our objective in this section is to show that there exists a dynamically stable club structure π which only consists of clubs of sub-optimal size, i.e. where none of the clubs are of size-s^*. We now state the main result of this section.

Now, we state Proposition 3 and provide a proof and intuition for it. We will use the following notation to prove the Proposition. For any integer $\bar{s} < n$, let $m(\bar{s})$ be the maximum number of clubs of size-$\bar{s}$ that can possibly be formed, i.e. $m(\bar{s}) \in \mathbb{N}$ is implicitly defined by $m(\bar{s})\bar{s} \leq n \leq (m(\bar{s}) + 1)\bar{s}$.

Proposition 3 *For any $\bar{s} \in \mathbb{N}$ that satisfies $m(\bar{s})\bar{s} < n$ and $v(n - m(\bar{s})\bar{s}) < v(\bar{s})$, there is a threshold $\bar{\delta} < 1$ such that, if $\delta > \bar{\delta}$ then there exists an equilibrium in which the following holds in each period*: (*i*) *no club of size-$\bar{s}$ ever breaks up*; *and* (*ii*) *if at the start of any period there are fewer than $m(\bar{s})$ clubs of size-$\bar{s}$ then an additional club of size-$\bar{s}$ forms*.

For notational ease, let $m(\bar{s}) \equiv \bar{m}$. In Proposition 3 we show that the dynamic stable structures that exhibit FOE can be obtained with stationary strategies. In these stable club structures, members of size-$(n - \bar{m}\bar{s})$ are excluded from size-$\bar{s}$ clubs. We proved through Propositions 1 and 2 that every possible stable club structure of the finite horizon game has as many optimal sized clubs as possible. Proposition 3 shows that the result of the finite horizon game might not hold in the infinite horizon game where it is possible to generate equilibria in which none of the clubs in the stable club structure are of optimal size. We provide the intuition for this result in Sect. 6 and discuss why this result cannot be replicated in the finite horizon game.

Proof The proof of Proposition 3 is constructive and proceeds in four steps. In Step 1, we construct a function W_i for each player i, and establish some properties which will be useful in the next steps. Step 2 defines a stationary Markov strategy σ. Step 3 defines the continuation values; and finally, in Step 4 and 5, we show that σ is an

SMPE, thus completing the proof of the Proposition. Note that $\bar{s} \leq s^*$ is consistent with the premise and some $\bar{s} > s^*$ may also satisfy the premise.

Step 1: Preliminaries. We begin with the construction of n real functions $W_1, \ldots, W_n$. The domain of each W_i is the union of the set of proposer states K^p and the set of coalition structures Π: $\mathbf{K} \equiv K^p \cup \Pi$.

To define W_i, $i \in N$, we need to establish some additional notation. First define the order $\lhd_i$ on $N \setminus \{i\}$ as follows: if $i = 1$ or $i = n$, then $\lhd_i =<$, where $<$ is equivalent to strictly less than; otherwise

$$i + 1 \lhd_i \ldots \lhd_i n \lhd_i 1 \lhd_i 2 \lhd_i i - 1.$$

Proposer i proposes to other agents to form a club as defined by the order $\lhd_i$. Now for each club structure $\pi \in \Pi$, let $\bar{A}(\pi)$ be the set of agents who are members of club of size-$\bar{s}$, that is

$$\bar{A}(\pi) \equiv \{i \in N : i \in S \text{ for some } S \in \pi \text{ such that } s = \bar{s}\}.$$

For each $i \in N$ and $\pi \in \Pi$, we define $S_i(\pi) \in 2^N$ as the club comprising the first $\bar{s} - 1$ agents (according to order $\lhd_i$) in $N \setminus \bar{A}(\pi)$ if $n - |\bar{A}(\pi)| \geq \bar{s}$, and as $N \setminus \bar{A}(\pi)$ otherwise.

We are now in a position to define $W_i(\mathbf{k})$ for every $\mathbf{k} \in \mathbf{K}$. To this end, consider the following path of the game that begins with proposer state $\mathbf{k} = (\pi, (\imath_\ell, \ldots, \imath_n))$, i.e., player $\imath_\ell$ is about to move in a proposal stage at which the current club structure is π and the list of remaining proposers is $(\imath_\ell, \ldots, \imath_n)$:

1. In any proposal stage at which she is called upon to propose, player i behaves as follows:

   (a) If i belongs to $\bar{A}(\pi)$, or if the current club structure comprises $\bar{m}$ size-$\bar{s}$ clubs and one size-$(n - \bar{m}\bar{s})$ club, then she passes.
   (b) Otherwise she proposes to form club $S_i(\pi)$.

2. All proposals made (on this path) are successful.

It is readily checked that on this path, the game reaches a dynamically stable club structure after a finite number of rounds, which comprises $\bar{m}$ size-$\bar{s}$ clubs and one club of size-$(n - \bar{m}\bar{s})$. Let $W_i(\mathbf{k})$ be the expected payoff to player i resulting from this path (where expectations are taken over the distributions of proposer orders in each period).

To complete the definition of W_i, we must define the values it takes when Nature has not yet selected the sequence of proposers and $\mathbf{k}$ contains some coalition structure $\pi \in \Pi$. In this case, we set $W_i(\pi) \equiv \mathbb{E}_{\boldsymbol{\rho}}\left[W_i\left(\pi, (\imath_1, \ldots, \imath_n)\right)\right]$, where the expectation is taken over the set of proposer lists $(\imath_1, \ldots, \imath_n)$ which Nature may select at the start of any period using probability distribution ρ.

Lemma 1 *For each* $i \in N$, *let* $W_i : \mathbf{K} \to \mathbb{R}$ *be defined as above. Then there exists* $\bar{\delta} \in (0,1)$ *such that the following holds for all* $\delta > \bar{\delta}$:

(i) $\bar{W} \equiv \max \left\{ W_i(\pi) \, : \, i \notin \bar{A}(\pi) \right\} < v(\bar{s}) - \varepsilon$;
(ii) $(1-\delta)v(s^*) + \delta W_i(\pi) < v(\bar{s}) - \varepsilon$, *for all* $i \in N$ *and* $\pi \in \Pi$ *such that* $i \notin \bar{A}(\pi)$;
(iii) $W_i(\mathbf{k}) \le v(\bar{s})$, *for all* $\mathbf{k} \in \mathbf{K}$ *and* $i \in N$.

The proof of Lemma 1 is in the appendix. Through this Lemma we establish some properties of W_i that we will use in the rest of the proof. It essentially specifies what values W_i can take if it follows a path defined in (a) and (b). These values are provide for different conditions and depending on the discount factor being significantly high. These are inequalities between the maximum values of W_i on the path and the deviation values.

Step 2: Construction of the strategy profile $\sigma = (\sigma_1, \dots, \sigma_n)$. For each player $i \in N$, strategy σ_i prescribes her the following behaviour. At any proposer state $\left(\pi, (\imath_\ell, \dots, \imath_n)\right) \in K^p$:

(a) If i belongs to $\bar{A}(\pi)$, or if the current club structure comprises $\bar{m}$ size-$\bar{s}$ clubs and one size-$(n - \bar{m}\bar{s})$ club, then she passes.
(b) Otherwise she proposes club $S_i(\pi)$.

Note that the club structure at the start of the period can be altered at most once per period. For every club structure $\pi \in \Pi$ and any proposal (offered club) $S \in 2^N \backslash \{\emptyset\}$, let π_S denote the coalition structure obtained at the end of the period if S forms. At any responder state $\left(\pi, (\imath_\ell, \dots, \imath_n), S\right) \in K^r$:

(c) If i belongs to $\bar{A}(\pi)$, then she rejects proposal S.
(d) If i does not belong to $\bar{A}(\pi)$, and either $|S| = \bar{s}$ or $\left[|\bar{A}(\pi)| = \bar{m}\bar{s} \;\&\; S = N \backslash \bar{A}(\pi)\right]$, then she accepts proposal S.
(e) Otherwise, she accepts the proposal if and only if:

$$(1-\delta)v\left(|S|\right) + \delta W_i(\pi_S) > \begin{cases} (1-\delta)v(s_0) + \delta W_i(\pi) & \text{if } \ell = 1, \\ W_i\left(\pi, (\imath_2, \dots, \imath_\ell)\right) & \text{if } \ell \ge 2, \end{cases}$$

where s_0 is the size of the club i belongs to in π.

Note that σ defined above depends on K^r or K^p which only contain the payoff-relevant part of the history; so σ is Markovian.

Step 3: Continuation values. We now define the continuation values from play according to the strategy profile define above. Let $V_i^\sigma(\kappa)$ be the continuation values of player i from play that begins in state κ (according to σ).[10]

From conditions (a) and (c) in the definition of σ, agents in $\bar{A}(\pi)$ always pass when they are proposers and reject when they are responders. As a consequence, no existing club of size-$\bar{s}$ will ever break up. Note that (a) and (b) in Step 2 correspond to (1) in the path defined in Step 1. Similarly, (c) and (d) in Step 2 correspond to (2) in the path defined in Step 1. We know from Step 1 that, for any $\mathbf{k} \in \mathbf{K}$, the payoff of player i is $W_i(\mathbf{k})$ if she plays according to (a)–(d). Hence, conditions (a)–(d) imply that $V_i^\sigma(\kappa) = W_i(\mathbf{k})$ for $i \in N$, where $\mathbf{K} \equiv K^p \cup \Pi$ and κ is either a proposer state or respondent state.

Step 4: Verification that players do not accept stage-dominated proposals. Consider an arbitrary responder state, in which a responder has been proposed to form club S. Then, $V_i^\sigma(\kappa) = W_i(\mathbf{k})$, together with (e) in Step 2 ensures that each responder accepts the proposal S only if $V_i^\sigma(S \mid \kappa) > V_i^\sigma(s_0 \mid \kappa)$, and only rejects proposal S if $V_i^\sigma(S \mid \kappa) \leq V_i^\sigma(s_0 \mid \kappa)$.

Step 5: Verification that σ is an SPE. Let $\bar{\delta}$ be defined as in Lemma 1 and, from now on, assume that $\delta \geq \bar{\delta}$. We know from Step 2 that σ is a stationary Markov strategy profile. To complete the proof, therefore, it remains to establish that σ is a subgame perfect equilibrium. By the One-shot Deviation Principle, it suffices to check that there is no state at which an agent has a profitable deviation from the prescribed strategy profile.

One-shot deviations at proposer states. We begin with proposer states. Take an arbitrary state $\mathbf{k} = (\pi, (j_1, \ldots, j_l))$ and let i be the agent whose turn it is to propose.

(a) We show that there is no profitable deviation from (a), as defined in Step 2.

a.1 Consider first the case when $i \in \bar{A}(\pi)$. Strategy σ prescribes her to pass. If she does so then, from (c), her current club will never be dissolved. Hence, her total discounted payoff from playing according to σ is $v(\bar{s})$. By deviating from σ, either (i) she makes an unsuccessful proposal, in which case her payoff is $v(\bar{s})$, or (ii) she forms another club of size-$\bar{s}$, so her payoff is $(1-\delta)(v(\bar{s}) - \varepsilon) + \delta v(\bar{s})$, or (iii) she successfully proposes S with $s \neq \bar{s}$. In cases (i) and (ii), she does not have a profitable deviation. In case (iii), she dissolves her size-$\bar{s}$ to form a club of a different size. The payoff she obtains by doing so is bounded above by

$$(1-\delta)(v(s^*) - \varepsilon) + \delta \bar{W} \leq v(\bar{s})$$

where the inequality follows from our assumption that $\delta \geq \bar{\delta}$ and Lemma 1(ii). Hence, proposer i cannot profitably deviate from σ.

[10] Note that the sets of proposer and responder states are denoted by K^p and K^r respectively, with generic element κ.

a.2 We now consider a case when $i \notin \bar{A}(\pi)$ and π is such that there are $\bar{m}$ clubs of size-$\bar{s}$ and one size-$(n-\bar{m}\bar{s})$ club. Strategy σ prescribes i to pass. If she passes then her payoff is $v(n-\bar{m}\bar{s})$. If she deviates, she can only make a successful proposal to agents who are not in $\bar{A}(\pi)$, since all agents in $\bar{A}(\pi)$ reject all future proposals. So, by deviating, i can only be in clubs of size $s' \le n-\bar{m}\bar{s}$. Since $v(s') \le v(n-\bar{m}\bar{s})$, there is no profitable deviation.

(b) We show that there is no profitable deviation from (b) defined in Step 2. We consider two cases: when $|\bar{A}(\pi)| < \bar{m}\bar{s}$ and when $|\bar{A}(\pi)| = \bar{m}\bar{s}$.

b.1 If $|\bar{A}(\pi)| < \bar{m}\bar{s}$, then $S_i(\pi)$ is the first $\bar{s}-1$ successors of i in $N\backslash\bar{A}(\pi)$. If she offers $S_i(\pi)$ then, from (d), she successfully forms a club of size-$\bar{s}$; so her payoff is $v(\bar{s})-\varepsilon$. By deviating from σ, either (i) she successfully forms another club S with $s \neq \bar{s}$, or (ii) she makes an unsuccessful proposal or passes. In case (i), her payoff is $v(\bar{s})-\varepsilon$. In case (ii), her payoff is

$$\begin{array}{ll}(1-\delta)v(s_0)+\delta W_i(\pi) & \text{if } \ell = 1,\\ W_i\big(\pi,(\iota_2,\dots,\iota_n)\big) & \text{if } \ell \ge 2,\end{array}$$

where s_0 is the size of her current club. If $\ell = 1$, then from Lemma 1(ii) her payoff is bounded above by $(1-\delta)v(s^*)+\delta\bar{W} \le v(\bar{s})-\varepsilon$. If $\ell \ge 2$, then from Lemma 1(i) her payoff is bounded above by $\bar{W} \le v(\bar{s})-\varepsilon$. Hence, i has no profitable deviation.

b.2 Let $|\bar{A}(\pi)| = \bar{m}\bar{s}$ such that $i \in S$ and $s < n-\bar{m}\bar{s}$. Then, if she plays according to σ, she successfully forms $S_i(\pi)$ such that $i \in S_i(\pi)$ and $s_i(\pi) = n-\bar{m}\bar{s}$. Her payoff then is $v(n-\bar{m}\bar{s})-\varepsilon$. If she deviates, she can only make a successful proposal to agents who are not in $\bar{A}(\pi)$, since all agents of $\bar{A}(\pi)$ reject all the future proposals. So, by deviating, i can only be in clubs of size $s' \le n-\bar{m}\bar{s}$. Since $v(s')-\varepsilon \le v(n-\bar{m}\bar{s})-\varepsilon$, she has no profitable deviation.

One-shot deviations at responder states. We now turn to responder states.

(c) We show that there is no profitable deviation from (c) defined in Step 2. Let $i \in \bar{A}(\pi)$. Strategy σ prescribes her to reject any offer. If she rejects, then we know from the equilibrium construction that her current club will never dissolve. Thus, her payoff is $v(\bar{s})$. If she rejects, she gets $V_i^\sigma(\boldsymbol{\kappa})$. From Step 3 we know that $W_i(\mathbf{k}) = V_i^\sigma(\boldsymbol{\kappa})$. We also know from Lemma 1(iii) that $v_i(\bar{s}) \ge W_i(\mathbf{k}) = V_i^\sigma(\mathbf{k})$ for all $\mathbf{k} \in \mathbf{K}$. Hence, i cannot profitably deviate.

(d) We show that there is no profitable deviation from (d) defined in Step 2.

(d.1) Suppose that $i \notin \bar{A}(\pi)$ and that i is offered $S \ni i$ with $s = \bar{s}$. Strategy σ prescribes her to accept the offer. If she does so then her payoff is $v(\bar{s})-\varepsilon$. By deviating to reject the offer, her payoff is

$$\begin{array}{ll}(1-\delta)v(s_0)+\delta W_i(\pi) & \text{if } \ell = 1,\\ W_i\big(\pi,(\iota_2,\dots,\iota_n)\big) & \text{if } \ell \ge 2,\end{array}$$

If $\ell = 1$ then her payoff is bounded above by $(1-\delta)v(s^*)+\delta\bar{W} \le v(\bar{s})-\varepsilon$. If $\ell \ge 2$, then from Lemma 1(i), her payoff is bounded above by $\bar{W} < v(\bar{s})-\varepsilon$. Hence, i cannot profitably deviate.

d.2 Suppose that $i \notin \bar{A}(\pi)$, $|\bar{A}(\pi)| = \bar{m}\bar{s}$ and that i is offered $S \ni i$ such that $s = n - \bar{m}\bar{s}$. Strategy σ prescribes her to accept the offer. Then, from the same argument as in (b.2), she has no profitable deviation.

(e) Strategy σ prescribes player i to accept the offer S iff the expression on the LHS is strictly greater than the expression on the RHS of the inequality. The expression on the LHS is the payoff player i gets if she accepts offer S, and the expressions on the RHS are the payoffs she gets if she rejects offer S for $\ell = 1$ and $\ell \geq 2$. Thus, i cannot profitably deviate from accepting offer S if the expression on the LHS is strictly greater than the expression on the RHS. □

Corollary 2 *Let $\bar{s} \in \mathbb{N}$ that satisfies $m(\bar{s})\bar{s} < n$ and $v(n - m(\bar{s})\bar{s}) < v(\bar{s})$, then any club structure that comprises $\bar{m}$ clubs of size-$\bar{s}$ and one club of size-$(n - \bar{m}\bar{s})$ is dynamically stable.*

This is a direct consequence of Proposition 3. Note that there is a multiplicity of equilibria and hence there are also other equilibria which have properties different from the one defined above. If $\bar{s} < s^*$ then the premise in Proposition 3 is satisfied without any condition. If $\bar{s} > s^*$, then we need to impose the condition that $v(n - \bar{m}\bar{s}) < v(\bar{s})$ for the condition in the premise to be satisfied.

The construction of inefficient equilibria is straightforward and intuitive, and it relies more on the framework of the analysis than on the mathematics. The features of the model that are important in the characterization of the inefficient SMPEs are: that the individuals are homogeneous in their preferences in that they all care only about the size of the club they belong to; that utility function v is strictly concave and single peaked; and a high enough discount factor and a mild condition on the existence of different sized clubs in the stable state. In particular, that each player can either be in a bad club or a good club eventually.

In the club formation context, it is important that members can form new club in the next or future periods even if an optimal/good club has been just formed and that the unanimity is required to form a new club. In the club formation process, it turns out to be important that the members cannot commit not to break their current clubs or agreement. Even if they deviate, the need for unanimous agreement would prevent breaking up their current good club. The construction then relies on player specific punishment: any player who successfully form an optimal sized club is punished with a positive probability of ending up in a bad sized club. Of course, the protocol will dictate who will in good club and in bad club, but it does not influence the number of optimal/good club in the stable state. While a complete characterization would be desirable, our purpose here is to provide some support for a class of dynamic repeated interactions and to suggest that such an analysis will reveal, at time strikingly, different behaviour and hence equilibrium outcomes.

The following example illustrates Definition 1 and demonstrates the mechanism behind our equilibrium construction. This example provides some intuition for the general result in Proposition 3.

Example 1 Let $N = \{1, 2, ..., 8\}$ and $v(s) = 10s - s^2$ for all $i \in N$. Hence, the optimal club is of size 5. Take for example a club structure $\pi = \{S_1, S_2, S_3\}$ such that $s_1 = s_2 = 3$ and $s_3 = 2$. Let $\bar{A}$ be the set of players who are in a club of size 3 and let $X = [1, 2, 3, 4, 5, 6, 7, 8]$ be a ternary relation. If $\delta \geq 8/9$ then the following strategy profile forms a pure strategy SMPE in which π is a *dynamically stable club structure*:

- If proposer i belongs to a club of size 3 or π has been formed then she always passes, and otherwise proposes to the next two agents in $X \setminus \bar{A}$ to form a club of size 3 if $8 - |\bar{A}| \geq 3$, otherwise proposes to next agent in $X \setminus \bar{A}$ to form a club of size 2.
- Let s_0 be the size of the current club of respondent i and s be the size of the club she has been offered. If $s_0 = 3$ then i rejects any offer. If $s_0 \neq 3$, and either $s = 3$ or $|\bar{A}| = 6$ and $s = 2$ then she accepts the offer. Otherwise, she accepts the offer iff $(1 - \delta)v(s) + 205\delta/10 > (1 - \delta)v(s_0) + 205\delta/10$.

The intuition is as follows. Even though the result would hold for any club structure, assume that the club structure at the start of the game is such that there exists one club of size 3 and one club of size 5. It is readily checked that the strategy profile above leads to dynamically stable club structure π at the end of period 2: one club of size 3 forms in the first period and one club of size 2 forms in the second period, and never changed thereafter.

Note that every player can end up in one of the two clubs in the long run and remains in that club forever: a “good club” (club of size-3) in which she receives 21 in every period, and a “bad club” (club of size-2) in which she receives 16 in every period. A club structure at the start of any period is either a stable club structure π or would ultimately lead to a stable club structure π. In the former case, player i’s expected discounted payoff is 21 if i is in a good club, and 16 otherwise. In the latter case, player i receives $(1 - \delta)v(s)$ in the current period and total discounted payoff from next period onwards is $9/10 \times 21 + 1/10 \times 16 = 205/10$ (i belongs to a good club with probability 9/10: recall that Nature decides the order of moves). Her expected payoff is therefore $(1 - \delta)v(s) + (\delta 205/10)$, which is less than 21 for all values of s (recall that $\delta \geq 8/9$). The agents in good clubs would never want to leave their current club because of the fear of ending up in a bad club at some later stages of the game. (There is a positive probability that a member of a good club club ends up in a bad club if she decides to break her current (good) club now to form an optimal sized club).

Thus, every agent i wants to minimise the chance of ending up in a bad club. In respondent stages, this includes rejecting any proposal if respondent i already belongs to a good club: even if the proposal is to form an optimal sized club, the respondent knows that optimal sized clubs are not sustainable and fears ending up in a bad club. It also includes accepting any proposal $S \ni i$ such that $s = 3$ when i is not already in a good club. Any attempt by the members of a bad club to form a club which includes members of a good club would be unsuccessful; and any proposal to form a good club when enough agents are not in a good club is successful. In the proposal stage, it is therefore optimal for player i to pass if she already is in a good club and/or the current club structure is π, and otherwise to propose a good club if enough agents are outside a good club, and to propose a club with the rest of the agents otherwise.

This example illustrates why the result in the infinite horizon game is different from that in the finite period game. In particular, it explains how and why there might be situations where none of the clubs which form are of optimal size in equilibrium: any deviation to propose to form a optimal club would be either unsuccessful or is not profitable, as the club structure would revert back to (sub-optimal) dynamically stable club structure π. The role of the discount factor is crucial to get this result. If the discount factor is small then the agents do not regard future payoff highly enough. Therefore, they would form an optimal sized club as soon as they have an opportunity because only the current period payoff really would matter; and forming optimal sized club would earn them the highest payoff. In conclusion, the above mentioned strategy profile fails to survive in equilibrium if agents' discount factors are not high enough.

6 Implications

The analysis of dynamic club formation in infinite periods contrasts starkly with that of finite periods. In the finite period game, the optimal sized clubs form as long as there are enough agents left to form one; and that once an optimal club forms, it never dissolves. Players use backward induction to anticipate that once an optimal sized club forms, it does not dissolve until the end of the game. However, if the players are unsure about the sustainability of optimal sized clubs then they cannot always guarantee not to dissolve an optimal sized club.

The uncertainty mentioned above about the future leads to a natural lack of *commitment*: if players, for some reason, are pessimistic about the stability of optimal sized clubs then they would not commit to stay in the optimal sized club, as it might hurt them in future (if they end up in the bad club). Note that commitment here does not mean that players can write binding contracts; this is mutually understood among the members of a club based on their beliefs and time horizon. Therefore, for the results in Sect. 5 to hold, it is important that players do not have the means to formally commit.

In Proposition 3, we prove the existence of and characterize a (dynamically) stable club structure in which none of the clubs need be of optimal size. This characterization relies on the observation that sufficiently forward looking agents do not support a change which might ultimately lead to a situation in which they are worse off. Consider an equilibrium which prescribes clubs of size $\bar{s} < s^*$. Agents who can make changes in the current period to get a higher return by forming an optimal sized club cannot guarantee that they will remain in that club. This is because they may believe (based on what other agents do) that the optimal sized club is not sustainable in the future. Note that the agents who are currently in a bad club have high incentives to change the clubs: they know that if they are in a bad club and the stable state in reached, they will get stuck in the bad club forever. Those who immediately gain by forming an optimal sized club now cannot refrain from taking decisions later that would hurt some of their fellow agents who made it possible to form an optimal sized club.

Based on players' pessimistic views and their response to such views, a steady state will be reached at some point in time in which only clubs of two different sizes exist. Players strictly prefer to be included in the larger club, and therefore take advantage of an opportunity to join such a club, as they fear being excluded from such a club in the future. The presence of an inferior club in the stable state is a requisite condition for individuals to exhibit fear of exclusion. If this is not the case, players do not fear being excluded from sustainable (same size) clubs in the stable state: knowing that everyone would eventually end up in same size club, they could profitably deviate in the current period from forming sub-optimal sized to optimal sized clubs. Thus, in the absence of inferior clubs in the stable state, the strategy profile which prescribes players to form a sub-optimal sized club cannot survive in equilibrium.

Fear of exclusion (from a better sustainable club) is the main reason why players cannot assure that they will remain in the optimal sized club. This leads to two intuitive results. First, the stability of a club structure turns on whether there are enough players excluded from better clubs to jointly form such a club. It does not depend on whether players would prefer to be members of a club that is not in the structure. For instance, in Example 1, members of A and B can form a new club of size 5; but this club structure is not stable when the players fear exclusion. Second, a dynamically stable club structure can be inefficient, i.e. there might be another club structure whose payoff dominates the payoffs in the dynamically stable club structure. Again in Example 1, a club structure with one club of size 5 and one club of size 3 Pareto dominates the club structure with two clubs of size 3 and one club of size 2.

An important point to note is that the FOE is generated endogenously, which is conceptually quite interesting, but it can also provide insights about the formation and sustainability of extreme groups and whether FOE, in addition to other phenomena, can explain the rise in religious fundamentalism in modern society. One limitation of our model is that individuals have homogeneous preferences on one dimension. More complicated formulation might consider heterogeneous preferences on more than one characteristic of club goods. Thus, we do not regard the present study as an explanation for how extreme groups come about and how they are sustained, but rather as an exemplar and first approach towards a promising enquiry. This could complement studies like Fan et al. (2021), which conducted simulations to analyse the causes of fundamentalism. The following very primitive stylized expose attempts to showcase an application if the current model is refined to include heterogeneous preferences over multiple characteristics of the club goods.

Consider a society in which individuals join groups based on their religious beliefs. To incorporate heterogeneous preferences, one could think of three different types of clubs or groups: fundamentalists, atheists and moderates. Assume that fundamentalists are minority. The dynamics here would present how the group representatives could make their group bigger to an optimal one to attain sustainability and influence over time. It would be interesting to see under what conditions which groups become sustainable and what is the underlying mechanism behind such observation and what role FOE plays in that. A comparative analysis could then be conducted to see the prevalence of fundamentalism across short and long run as the parameters of club goods' characteristics change. This model highlights the interplay of two different influences on the emergence of different groups: direct interaction between insiders and outsiders in the process of group formation, and peer group effects within groups (club goods). The increased complication because of heterogeneous preferences over multiple dimensions would probably require simplifying the interactions among agents, but it might lead to new insights.

7 Conclusion

This paper has examined a process of club formation in a dynamic setting when individuals are far-sighted. An important feature of the dynamic club formation process is that given the rules that govern the formation and dissolution of clubs, the current decision makers have an opportunity to take a decision to their advantage that affects their and others' choices in the future. This implies that dynamic club formation must recognize that current decision-makers make choices knowing that their decision will have an impact on their future choices, and, therefore, their current decision might depend on what they know or believe about future outcomes.

We developed a framework to study this problem of dynamic club formation when individuals are only concerned about the size of the club they belong to. The model is very simple with forward looking players who decide whether to form a new club or not over time. We use the same approach as Acemoglu et al. (2012) in finding stable club structures. We have investigated and provided the characterization for stable club structures that have all possible optimal sized clubs and stable club structures that have no optimal sized clubs (sub-optimal club structures). In the infinite horizon game, because of the multiplicity of equilibria we focus only on a very specific class of equilibria to find stable club structures with interest in the properties that all clubs are of suboptimal size in a stable club structure.

We first analyse the benchmark in Sect. 4 for finite horizon game with which to compare the results in Sect. 5. The most notable difference we observe is that that sub-optimal stable club structures can only exist in infinite horizon games. Pessimistic beliefs about the optimal club and fear of exclusion from better club exhibit a lack of commitment on the part of players. However, this is never observed in the finite horizon, where equilibrium outcomes are Pareto efficient though we cannot always comment on the Pareto efficiency of the outcomes in infinite horizon. Future studies could delve into this aspect further.

The theory in this paper has several interesting extensions. In our study, individuals whose sole aim is to join a club only care about the size of the club. It would be interesting to look at the hedonic setting in which the individuals have different tastes, and preferences not only over the club size but also who they share the club with. Another interesting extension would be to investigate other possible equilibria (in the infinite horizon case) such as clubs of three different sizes, all different from the optimal size and to examine the behaviour which supports these kinds of club structures. We could also do robustness checks on the equilibria of the infinite period game by proving that there are no joint deviations.

Appendix

Proof of Proposition 1 Let π be the initial club structure and $\overline{A}$ be the set of agents who are members of a size-s^* club at the the start of the game; that is

$$\overline{A} \equiv \{i \in N : \exists S \subseteq N \text{ such that } i \in S \in \pi \,\&\, s = s^*\}.$$

Part (i) To prove part i, it suffices to show that whenever a proposer offers to form a club which includes member/s of $\overline{A}$, the proposal is rejected. To see this, consider the last proposer, ι_n. Suppose the last proposer offers to form a club of size-s which includes member/s of $\overline{A}$. The utility of the agents in $\overline{A}$ is $v(s^*)$.

If a respondent in $\overline{A}$ accepts the offer, she gets a payoff of $v(s) - \varepsilon$, whereas, if she rejects then she receives a payoff of $v(s^*)$. Thus, each respondent $i \in \overline{A}$ rejects the offer from ι_n to form a club of size-s because $v(s^*) > v(s) - \varepsilon \quad \forall s$.[11] As a result, the last proposer is unsuccessful in breaking size-s^* club/s.

Consider now the second last proposer, ι_{n-1} and suppose that she offers to form a club which includes member/s of $\overline{A}$. Anticipating that the club of size-s^* will remain intact if she rejects proposal from ι_{n-1}, a respondent $i \in \overline{A}$ rejects the offer and the proposal is unsuccessful for the same reason as above. Proceeding along the same lines, we can conclude that a proposer ι_i is unsuccessful in breaking size-s^* club if all the proposers in the order from ι_i onwards till the last proposer, ι_n, are unsuccessful in their attempt to break club/s of size-s^*. We proved that ι_n is unsuccessful in breaking size-s^* club and therefore by inductive reasoning, no proposer is successful in forming a club which includes members of $\overline{A}$, proving part (i).

Part (ii) Let m be the number of clubs of size-s^* at the start of the game. Note that only one club is formed in every period and therefore we just need to prove that whenever a club is formed, it must be of size-s^* and that a club forms if $m < k^*$. If $m < k^*$ then the number of remaining players, $s^*(k^* - m)$, are a multiple of s^* who are outside $\overline{A}$. Now, consider the last proposer ι_n.

• If $\iota_n \in \overline{A}$ then no new club forms. From part (i) we know that the proposer either passes or proposes an unsuccessful club.

• If $\iota_n \notin \overline{A}$ then she successfully forms a size-s^* club. Given condition (2.1), all the respondents in $N \setminus \overline{A}$ would accept a proposal to form the size-s^* club. If ι_n successfully offers the members of $N \setminus \overline{A}$ to form a size-s^* club then she gets a payoff of $v(s^*) - \varepsilon$.[12] If she passes or offers an unsuccessful proposal then she gets a payoff of $v(s) < v(s^*) - \varepsilon$, and, therefore, she forms a size-s^* club in every equilibrium.

[11] Condition 1 implies $v(s^*) > v(s^*) - \varepsilon \geq v(s) - \varepsilon \quad \forall s$.

[12] If ι_n offers a proposal to a member/s in $\overline{A}$ then she is unsuccessful. Thus, the only way she can form a new club is by offering the proposal to the agents in $N \setminus \overline{A}$. Also, she would offer to form a size-s^* club because it gives her the highest payoff.

Consider now, the penultimate proposer.

• If $\iota_{n-1} \in \overline{A}$ then no new club forms. From part (i) we know that such a proposer would either pass or offer an unsuccessful proposal.

• If $\iota_{n-1} \notin \overline{A}$ then a club of size-s^* forms, either now or at some later stage. If ι_{n-1} anticipates that she would not otherwise be a part a size-s^* club which forms at some later stage then she successfully proposes to the members of $N \setminus \overline{A}$ to form a club of size-s^*. She is indifferent between forming a size-s^* club now and passing if she anticipates that she would be included in the club of size-s^* at some later stage. In either case, a size-s^* club forms.

Proceeding recursively, any proposer who does not belong to the optimal sized club would:

1. Successfully propose to form a new club of size-s^* if she anticipates that she would not otherwise be part of club of size-s^* which forms at some later stage or
2. Passes or successfully proposes to form a new club of size-s^* if she anticipates joining a size-s^* club at some later period if she passes. In either case, a club of size-s^* forms.

As $m < k^*$, some individual, and therefore some proposer must not be in $\overline{A}$: for every protocol ρ. Consequently, a club of size-s^* must form, proving part (ii). □

Proof of Proposition 2 We start by proving **Part (i)**.

In the last period T, no club of size-s^* breaks up. The last period is equivalent to the one period game. Hence, the same argument applies to the last period as to the one period game.

In period $t = T - 1$, no club of size-s^* breaks up. We begin with the last proposer. We show that last proposer, ι_n, is unsuccessful in breaking any size-s^* club, either by forming a club which includes member/s who are in the optimal sized club or by successfully forming a new club when she is already a member of an optimal-sized club. Suppose that she proposes to form a new club of size-s^* to a member of an optimal-sized club. This respondent anticipates that her club will remain in place in the next (last) period. If she accepts the offer then she gets a total discounted payoff of no more than $v(s) - \varepsilon + \delta v(s)$ and she gets a payoff of $v(s^*) + \delta v(s^*)$ if she rejects. The respondent then rejects the offer because $v(s^*) + \delta v(s^*) > v(s) - \varepsilon + \delta v(s)$. Thus, the last proposer in the last period is unsuccessful in forming a new club which includes any member of an optimal-sized club. Finally, suppose that $\iota_n \in \overline{A}$. In equilibrium, she cannot break her current club: anticipating that her club remains intact in the next period, ι_n gets a payoff of $v(s^*) + \delta v(s^*)$ if she passes or offers an unsuccessful proposal. This is the maximum payoff she can get. If she successfully forms a new club, she incurs a cost of ε. Thus, the last proposer in period $T - 1$ cannot break any size-s^* club.

Any respondent in a size-s^* club, would reject any offer to form a new club if she anticipates that her club will never be dissolved in all the later stages and periods, i.e. if all the proposers in later stages and periods are unsuccessful in breaking any size-s^* club. In other words, a proposer in period t is unsuccessful in breaking any size-s^* club in that period if all subsequent proposers in that and in later periods fail to break up any size-s^* club. We know that no proposer in the last period is successful in breaking a size-s^* club and that the last proposer in the second last period is unsuccessful. Hence by inductive reasoning, no proposer in any period is successful in breaking any optimal-sized club, proving part (i).

We now prove **Part (ii)**.

Consider a period t in which there are m^t clubs of size-s^* and that $m^t < k^*$ and $\overline{A}$ is a set of agents who belong to a size-s^* club in that period. Since $m^t < k^*$, $s^*(k^* - m^t) \geq s^*$ individuals are not in $\overline{A}$.

Now, consider a proposer ι_k in period t.

• If $\iota_k \in \overline{A}$, she passes or offers an unsuccessful proposal. To see this note that:

1. If she passes or offers an unsuccessful proposal, she gets $v(s^*) + \sum_{\tau=t+1}^{T} \delta^{\tau-t} v(s^*)$
2. If she successfully forms a new club, say of size-$\bar{s}$, she gets a payoff of $v(\bar{s}) - \varepsilon + \sum_{\tau=t+1}^{T} \delta^{\tau-t} v(s)$

We know that $v(s^*) \geq v(\bar{s})$ and that $\sum_{\tau=t+1}^{T} \delta^{\tau-t} v(s^*) \geq \sum_{\tau=t+1}^{T} \delta^{\tau-t} v(s)$. Therefore, $\iota_k \in \overline{A}$ passes or offers an unsuccessful proposal because $v(s^*) + \sum_{\tau=t+1}^{T} \delta^{\tau-t} v(s^*) > v(\bar{s}) - \varepsilon + \sum_{\tau=t+1}^{T} \delta^{\tau-t} v(s)$.[13]

Consequently, any new club must be offered by a proposer who does not belong to $\overline{A}$.

• If $\iota_k \notin \overline{A}$ then a new club of size-s^* forms, either now or at some later stage of the current period.

Suppose that $k = n$ and that the proposer belongs to a club of size $s_k \neq s^*$. The proposer gets a payoff of $v(s_k) + \sum_{\tau=t+1}^{T} \delta^{\tau-t} v(s)$ if she passes or offers an unsuccessful proposal. The proposer can successfully offer to the agents in $N \setminus \overline{A}$.[14] Since s^* is the unique maximum and $s^*(k^* - m^t) \geq s^*$, the proposer offers to individuals in $N \setminus \overline{A}$ to form a club of size s^*. Condition (2.1) implies that all respondents accept. The proposer gets a payoff of $v(s^*) - \varepsilon + \sum_{\tau=t+1}^{T} \delta^{\tau-t} v(s^*)$ if she offers a club of size-s^* to individuals in $N \setminus \overline{A}$. Thus, the proposer forms a new club of size-s^* because $v(s^*) - \varepsilon + \sum_{\tau=t+1}^{T} \delta^{\tau-t} v(s^*) > v(s_k) + \sum_{\tau=t+1}^{T} \delta^{\tau-t} v(s)$.

Suppose $k \neq n$ and that the proposer belongs to a club of size-$s_k \neq s^*$. She successfully forms a club of size-s^* if she anticipates that she would not otherwise be part of size-s^* club at some later stage. She is indifferent between forming a club now and passing if she anticipates joining a size-s^* club at some later stage. In either case, a club of size-s^* forms in the period t.

[13] Since there is discounting, the agents do not wait to become part of size-s^* club till the next period and they take the opportunity to be part of a size-s^* club either by forming one or by agreeing to the proposer who wants to form one.

[14] From part (i) we know that no club of size-s^* breaks and therefore individuals who belong to $\overline{A}$ reject any offer.

In sum, a club of size-s^* forms in period t if $\imath_k \notin \overline{A}$. As $m^t < k^*$, some individual, and therefore some proposer must not be in $\overline{A}$: for every protocol ρ. Consequently, a club of size-s^* must form, proving part (ii). $\square$

Proof of Lemma 1 *(i)* To establish the first part of the Lemma, it suffices to show that $W_i(\pi) < v(\bar{s}) - \varepsilon$, for all $i \in N$ and $\pi \in \Pi$ such that $i \notin \bar{A}(\pi)$. As there is only a finite number of pairs (i, π) such that $i \notin \bar{A}(\pi)$, this indeed guarantees that $\bar{W} < v(\bar{s}) - \varepsilon$.

Let $i \in N$ and $\pi \in \Pi$ be such that $i \notin \bar{A}(\pi)$, and suppose first that $\left|\bar{A}(\pi)\right| = \bar{m}\bar{s}$. By definition of W_i, this implies that

$$W_i(\pi) = \begin{cases} v(n - \bar{m}\bar{s}) & \text{if } N \setminus \bar{A}(\pi) \in \pi, \\ v(n - \bar{m}\bar{s}) - \varepsilon & \text{otherwise;} \end{cases}$$

so that $W_i(\pi) < v(\bar{s}) - \varepsilon$ because $v(n - \bar{m}\bar{s}) < v(\bar{s})$, where ε is the cost of forming a new club[15]. Indeed, if $N \setminus \bar{A}(\pi) \notin \pi$, then all proposers in $\bar{A}(\pi)$ pass, and the next proposer outside $\bar{A}(\pi)$ successfully forms the coalition $N \setminus \bar{A}(\pi)$, which contains i.

Now suppose that $\left|\bar{A}(\pi)\right| < \bar{m}\bar{s}$. It follows that the next proposer who is not in $\bar{A}(\pi)$, say j, forms the size-$\bar{s}$ club $S_j(\pi)$. If i is a member of $S_j(\pi)$ then she receives $v(\bar{s}) - \varepsilon < v(\bar{s})$. If i is not a member of $S_j(\pi)$ then it follows from the definition of the path in Step 1 that she will either end up in a size-$\bar{s}$ club or in a size-$(n - \bar{m}\bar{s})$ club after a (random) finite number, say τ, of periods. As her stage payoff is bounded above by $v(s^*)$ and $\tau \leq \bar{m} + 1$, i's expected payoff (conditional on $i \notin S_j(\pi)$) is itself bounded above by

$$(1 - \delta) \sum_{s=1}^{\bar{m}} \delta^{s-1} v(s^*) + \delta^{\bar{m}}(1 - \delta) \sum_{s=1}^{\infty} \delta^{s-1} \left(\mathbb{E}\left[v(\tilde{s})\right] - \varepsilon\right) = (1 - \delta^{\bar{m}}) v(s^*) + \delta^{\bar{m}} \left(\mathbb{E}\left[v(\tilde{s})\right] - \varepsilon\right),$$

where $\tilde{s} \in \{\bar{s}, n - \bar{m}\bar{s}\}$ is the random variable describing the size of i's club from period $\bar{m} + 1$ onward, and the expectation is computed from the distribution of proposer orders. As $v(n - \bar{m}\bar{s}) < v(\bar{s})$ and the probability of i ending up in the $(n - \bar{m}\bar{s})$-sized club is positive, $\mathbb{E}\left[v(\tilde{s})\right] < v(\bar{s})$. Moreover, as $\lim_{\delta \to 1} (1 - \delta^{\bar{m}}) v(s^*) + \delta^{\bar{m}} \left(\mathbb{E}\left[v(\tilde{s})\right] - \varepsilon\right) = \mathbb{E}\left[v(\tilde{s})\right] - \varepsilon$, there exists $\bar{\delta}_1 \in (0, 1)$ such that $(1 - \delta^{\bar{m}}) v(s^*) + \delta^{\bar{m}} \left(\mathbb{E}\left[v(\tilde{s})\right] - \varepsilon\right) < v(\bar{s}) - \varepsilon$, for all $\delta \in (\bar{\delta}_1, 1)$. This in turn implies that $W_i(\pi) < v(\bar{s}) - \varepsilon$. Henceforth, we assume that $\delta > \bar{\delta}_1$.

(ii) Let $i \in N$ and $\pi \in \Pi$ be such that $i \notin \bar{A}(\pi)$. By definition of $\bar{W}$, we have

$$\lim_{\delta \to 1} \left[(1 - \delta) v(s^*) + \delta W_i(\pi)\right] \leq \lim_{\delta \to 1} \left[(1 - \delta) v(s^*) + \delta \bar{W}\right] = \bar{W} < v(\bar{s}) - \varepsilon,$$

where the last inequality follows from part *(i)* and $\delta > \delta_1$. Hence, there exists $\bar{\delta}_2(i, \pi) \in (0, 1)$ such that $(1 - \delta) v(s^*) + \delta W_i(\pi) < v(\bar{s}) - \varepsilon$ whenever $\delta > \bar{\delta}_2(i, \pi)$. We obtain the result for all $i \in N$ and $\pi \in \Pi$ be such that $i \notin \bar{A}(\pi)$ by imposing $\delta > \bar{\delta}_2 \equiv \max \left\{\bar{\delta}_2(i, \pi) \,:\, i \notin \bar{A}(\pi)\right\}$. Henceforth, we assume that $\delta > \bar{\delta}_2$.

[15] Remember the assumption that $v(s) > v(s') \Rightarrow v(s) - \varepsilon > v(s')$

(iii) Let $i \in N$ and $\mathbf{k} \in \mathbf{K}$ and let π_S denote the club structure obtained at the end of the period if S forms. Suppose first that $i \in \bar{A}(\pi)$. Then, by definition $W_i(\mathbf{k}) = v(\bar{s})$. Suppose now that $i \notin \bar{A}(\pi)$ and $|\bar{A}(\pi)| < \bar{m}\bar{s}$. It follows that the next proposer (not in $\bar{A}(\pi)$), say j, forms the size-$\bar{s}$ club $S_j(\pi)$. If i is a member of $S_j(\pi)$, then she receives $v(\bar{s}) - \varepsilon < v(\bar{s})$. Otherwise, her payoff is $(1-\delta)v(s_0) + \delta W_i(\pi_{S_j(\pi)}) < v(\bar{s})$, where s_0 is the size of her current club and the inequality follows from from part *(ii)*.

Suppose now that $i \notin \bar{A}(\pi)$ and $\left|\bar{A}(\pi)\right| = \bar{m}\bar{s}$. By definition of W_i, this implies that

$$W_i(\pi) = \begin{cases} v(n - \bar{m}\bar{s}) & \text{if } N \setminus \bar{A}(\pi) \in \pi, \\ v(n - \bar{m}\bar{s}) - \varepsilon & \text{otherwise;} \end{cases}$$

Setting $\bar{\delta} \equiv \max\{\bar{\delta}_1, \bar{\delta}_2\}$, we obtain the Lemma. □

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