A study on the impact of strategic behaviour in financial networks

Hao Zhou

A thesis submitted for the degree of
Doctor of Philosophy

School of Computer Science and Electronic Engineering
University of Essex

November 24, 2023

Abstract

A financial system comprises a set of institutions, such as banks, that engage in financial transactions. The interconnections showing the *liabilities* among the banks are represented by a network and can be highly complex. Hence, the utility (well-being) of a bank depends on the entire network and not just on the well-being of its immediate borrowers/debtors. For example, the possible bankruptcy of a bank and the corresponding damage to its immediate lenders/creditors resulting from the bank's failure to repay, can be propagated through the financial network by causing the creditors' (and other banks', in sequence) inability to repay their debts, thus having a global effect.

In this thesis, we study the global impact induced by a sequence of individual strategic operations in the realm of financial networks. In particular, we investigated a range of financial operations including *cash injection*, *debt removal*, *debt transfer*, and *priority-proportional payments*[1], as adopted by banks. We study each operation in both centralized and decentralized manner. In the centralized context where the financial authority has the power to control the behavior of each bank as desired, we aim to determine if there exists an algorithm that efficiently computes the optimal combination of these operations to achieve desired systemic objectives. In cases where no efficient algorithm exists, we provide hardness results on the computation aspects and turn our attention to the development of approximate algorithms. In the game-theoretic (decentralized) setting, we consider each bank as a utility-maximizing agent that can be strategic about the corresponding operations. Regarding the games in financial networks, we study the existence and quality of equilibria, as well as the computational complexity of equilibrium-related problems. Overall, our findings contribute to a good understanding of the impact of strategic behavior on financial networks.

[1] A payment scheme where the liabilities can be divided to various priorities, while the liabilities with the same priority will be paid proportionally.

Acknowledgment

To begin, I would like to express my sincere gratitude to my supervisors, Dr Maria Kyropoulou, and Dr Panagiotis Kanellopoulos, for being there whenever I needed your help in research and in other non-research related situations. Without your guidance and help, I cannot imagine how I could complete this 4-year doctoral program. In the past four years, I have been inspired a lot by your strong passion and scientific attitude. And also, I have benefited greatly from how you treat work, manage time, and keep a work-life balance. In addition, I am forever grateful for the countless hours that we spent together reading papers, discussing research questions, and solving problems. Indeed, you are the gift that I will never be able to thank you enough for.

I extremely appreciate my entire family. You have always supported and trusted me so that I have enough courage to start and complete this journey. Your meticulous care and encouragement are something that I will never forget. I am extremely grateful for my wife. I appreciate you a lot for all the support and sacrifices you have made for me. In fact, you were playing an excellent role model in my mind, and your perseverance and bravery have always inspired me to move forward.

I also extend my gratitude to the members of termly panel meetings: Jon Chamberlain, Jinya Su, Michael Kampouridis and Zulfiqar Ali for their useful comments on my research, presentation, and the structure of this thesis.

I am thankful for meeting these friends during previous years: Bin Li, Zheng Gong, Fuli Wang, Tao Chen, Yunfei Long, and Abbas Salami. Additionally, special thanks go to all the great people in the Centre for Computational Finance and Economic Agents (CCFEA) for creating such a nice atmosphere. Finally, I would like to thank the School of Computer Science and Electronic Engineering at the University of Essex for the financial support that made my PhD studies possible and gave me the privilege to work with all these great people.

Preface

My main area of research is algorithmic game theory in financial networks. In particular, for a financial network game corresponding to a specific type of strategic operation, the existence, computation, and quality of equilibria are the main research questions I focus on. My contributions on these topics can be best summarized by the following papers[2].

1. **“ Financial Network Games ”**

   Panagiotis Kanellopoulos, Maria Kyropoulou, and Hao Zhou. *In Proceedings of the 2nd ACM International Conference on Finance in AI (ICAIF 2021)*

2. **“ Forgiving Debts in Financial Network Games (Extended Abstract) ”**

   Panagiotis Kanellopoulos, Maria Kyropoulou, and Hao Zhou. *In Proceeding of the 21st International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2022)*

3. **“ Forgiving Debts in Financial Network Games ”**

   Panagiotis Kanellopoulos, Maria Kyropoulou, and Hao Zhou. *In Proceedings of the 31st International Joint Conference on Artificial Intelligence (IJCAI 2022)*

4. **“ Debt Transfers in Financial Networks: Complexity and Equilibria ”**

   Panagiotis Kanellopoulos, Maria Kyropoulou, and Hao Zhou. *In Proceeding of the 22nd International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2023)*

[2] The order of authors is alphabetical.

Contents

Chapter 1

Introduction

In the contemporary financial landscape, financial institutions are extensively interconnected, giving rise to complex networks. While these interconnections can potentially provide a stabilizing effect by spreading losses across multiple institutions, they can also facilitate spill-over effects and amplify systemic risk, thereby jeopardizing the entire financial system. The failure of Lehman Brothers during the 2008 financial crisis is a compelling illustration of such risk, as emphasized by Acharya et al. [4]. With the benefit of hindsight, it has become clear that Lehman's failure contained substantial systemic risk and triggered a chain reaction that nearly led to the collapse of the financial system.

In the aftermath of the 2008 financial crisis, there has been an increased interest in studying the systemic risks in financial networks. Numerous studies on this topic, such as Morris [100], Morris and Shin [101], Allen and Babus [5], and Babus [18], have focused on issues related to systemic risk, risk contagion, and financial network stability (see, e.g., in [30, 3, 64]). A financial network can be represented as a graph where the nodes are financial institutions ("banks" for short), and the edges are financial contracts, e.g., *liability relations* (see, e.g., [48, 116, 121, 118]). One of the key challenges in analyzing financial networks is understanding the strategic behavior of banks, which can have a significant impact on the overall stability and resilience of the entire system. Banks may engage in strategic behavior to manage their own liquidity, protect their interests in the derivatives market, or manipulate market prices, among other things. In the existing literature, some efforts have been made to demonstrate strategic interactions within financial networks, especially towards understanding the game-theoretic issues. For instance, a bank's default can trigger a chain reaction of defaults among other banks it has financial contracts with, leading to a contagion effect. Battiston et al. [22], and Elliott et al. [49] have found that banks can strategically choose their counterparties to minimize their contagion risk; Gai and Kapadia [59] pointed out a bank may intentionally choose to default on its financial obligations if it expects other banks to do the same, or if it believes it can benefit from the default in some way (e.g., by renegotiating contracts on more favorable terms), and strategic default can lead to a domino effect, causing other banks to default as well; besides, banks may take on more risk in order to increase their profits, but this can also increase the systemic risk of the financial network. Acemoglu et al. [3] have found that banks may strategically take on more risk in response to changes in the network structure or to the behavior of other banks.

In the strategic contexts mentioned above, Algorithmic Game Theory (AGT) [117, 105] has emerged as a powerful tool for modeling strategic interactions between financial institutions and analyzing their behavior in financial networks. First of all, it provides a formal framework to study the strategic behavior of banks in financial systems. Through game-theoretic models of financial networks, we can analyze the incentives and predict outcomes of the strategic interaction between individual banks or groups of banks. Furthermore, as financial networks in reality can be complex and difficult to analyze, AGT provides computational tools to analyze the computational complexity of relevant aspects, like determining the existence and quality of equilibria. In addition, AGT provides tools to design algorithms that can optimize the behavior of banks in financial networks. By designing algorithms that incentivize banks to behave in certain ways, we can achieve desirable outcomes in financial systems. For all the above reasons, AGT is a suitable tool for analyzing the strategic behavior in financial networks.

The research presented in this thesis builds on the seminal work of Eisenberg and Noe [48] and explores the impact of various strategic behaviors on financial networks, including cash injection, debt removal, debt transfer, and priority-proportional payments, from a game-theoretic perspective. This research aims to contribute to the intersection between financial networks and AGT by addressing primary research questions related to the strategic behavior of banks in financial systems. In the following, we will provide a high-level introduction to financial networks, algorithmic game theory, and the intersection between them, as well as the primary research questions considered in this thesis.

1.1 Background

1.1.1 Financial networks

A financial system comprises a set of institutions, such as banks, that engage in financial transactions. The interconnections showing the liabilities (financial obligations or debts) among the banks can be represented by a network, where the nodes correspond to banks and the edges correspond to liability relations. Each bank initially has a fixed amount of *external assets* (not affected by the network) which are measured in the same currency as the liabilities. A bank's total assets comprise its external assets and its incoming payments, which can be used for (outgoing) payments to its lenders. If a bank's assets are not enough to cover its liabilities, that bank will be *in default* (or, *insolvent*), and the value of its assets will be decreased (e.g., by liquidation); the extent of this decrease is captured by *default costs* and essentially implies that the corresponding bank will have only a part of its total assets available for making payments.

On the liquidation day (also known as clearing), each bank in the system has to pay its debts in accordance with the following three principles of bankruptcy law (see, e.g., [48]): i) *absolute priority*, i.e., banks must first pay their liabilities in full in order to have positive equity, ii) *limited liability*, i.e., banks cannot pay more than their total assets, and iii) *proportionality*, i.e., in case of default, payments to lenders are made in proportion to the respective liability. Payments that satisfy the above properties are called *clearing payments* and (perhaps surprisingly) these payments are not uniquely defined for a given financial system. However, maximal clearing payments, i.e., ones that point-wise maximize all corresponding payments, are known to exist and can be efficiently computed [116]. Note that computing clearing payments is a necessary step before performing a game-theoretic analysis since it is associated with banks' utility.

1.1.2 Algorithmic game theory (AGT)

Game theory is the study of mathematical models of strategic interactions among rational agents [103]. The first general mathematical formulation of game theory, a concept known to economists, social scientists, and biologists, was introduced by John von Neuman and Oskar Morgenstern [99]. Initially, game theory occupied a niche within economics during the 1960s and 1970s, but there was growing anticipation and excitement around it, particularly in the 1980s and early 1990s. As computers became more prevalent, problems were tailored to be solvable by computers, and algorithms were developed to compute solutions. Computer scientists then began the task of determining which algorithms were better in terms of running time and complexity. The emergence of the Internet brought together game theory and computer science, as both fields sought to understand and optimize internet applications. This intersection of disciplines gave rise to Algorithmic Game Theory (AGT), which explores the interplay between algorithms, computation, and game theory in complex systems such as the Internet.

Overall, in the domain of Algorithmic Game Theory, there exists an intersection between the disciplines of Economics and Theoretical Computer Science. Broadly speaking, economists concentrate on forecasting the equilibrium behavior of agents, while computer scientists are primarily concerned with elucidating the mechanisms by which equilibria are reached. Specifically, economists explore the presence and inefficiency of equilibria, whereas computer scientists investigate the computational complexity of determining the existence of equilibria, computing established equilibria, and designing algorithms possessing desired properties. The integration of these inquiries represents a pivotal area of investigation in the AGT community.

1.1.3 When AGT meets financial networks

When AGT meets financial networks, the main research topics of this thesis come out naturally. The intersection between AGT and financial network analysis has emerged as a promising area of research in recent years. It involves applying game theory concepts and techniques to analyze financial networks and understand how their structure affects the stability of the financial system. This combination also has the potential to provide insights into the behavior of financial markets and the effectiveness of policy interventions aimed at stabilizing them. Researchers have developed various game-theoretic models and algorithms for analyzing financial networks, such as models of network formation [77, 21], strategic default [7, 13, 38], and contagion [110, 91, 101]. Additionally, these models have been used to study various aspects of financial networks, such as the role of intermediaries, the effects of network structure on stability, and the impact of regulatory policies.

Overall, the intersection of AGT and financial networks represents a promising area of research with significant implications for both theory and practice. By developing better models and algorithms for analyzing financial systems, we can gain a deeper understanding of the behavior and dynamics of these systems, and design better policies to manage them.

1.2 Literature Review

In the following, we review the literature that covers several aspects shared among Chapters 3, 4 and 5, while more concrete discussion will be presented in the corresponding chapter as required.

1.2.1 Financial network models

Financial networks are intricate systems characterized by the interconnectedness of numerous institutions in various ways. The primary means of linkage among these entities are financial contracts, through which they engage in lending and borrowing activities aimed at mitigating idiosyncratic liquidity fluctuations and meeting deposit obligations. In addition, they collaborate in exploring investment prospects and engage in asset repackaging and resale activities, resulting in chain-like operations. These interdependent networks serve as a primary subject of extensive research in the literature.

Financial networks and their related properties have been analyzed in various works that follow the standard (non-strategic) model developed by Eisenberg and Noe [48]. They introduce a financial network model allowing debt-only contracts among banks with non-negative external assets that make proportional payments. Following the model of Eisenberg and Noe [48], a series of papers extend and enrich the model by adding default costs [116, 75, 128, 61, 133, 32] and cross-ownership [49, 130, 2, 60]. Compared to the setting with only non-negative external assets, Demange [43] proposes a model that allows for banks to be indebted to entities outside the network, and captures this by allowing negative external assets. Additionally, linkages formed between banks and insurance companies can also act as potential channels of financial contagion. These linkages are formed and resolved in a way that is different from normal bank loans, e.g., Credit Default Swaps (CDS) [89, 121].

Therefore, Schuldenzucker et al. [121] and Papp and Wattenhofer [106] enhance the model by allowing for Credit Default Swap (CDS) contracts, while Mundt and Minca [83] present a more established model that incorporates reinsurance contracts. Furthermore, Allouch and Jalloul [7], Kuznetsov and Veraart [87], and Banerjee et al. [19] account for the evolving nature of financial networks, which static models struggle to capture. Specifically, these authors consider a liability structure with discrete multiple maturities. Sonin and Sonin [124] further explore the dynamics of financial networks through a continuous-time model.

1.2.2 Simulating financial networks

Network simulation is a tool used to study the behavior of financial networks and assess their stability in different scenarios. One of the earliest approaches to generating financial networks was proposed by Erdős and Rényi [52] in 1960. Their random graph model assumes that each pair of nodes has a fixed probability of being connected by an edge. Later, Barabási and Albert [20] introduce the preferential attachment model, which assumes that new nodes are more likely to attach to highly connected nodes, leading to the emergence of scale-free networks. More recently, a number of studies have focused on generating financial networks based on empirical data [22, 26], as well as applying machine learning techniques to recreate financial networks [85, 111].

With respect to distribution assumptions for generating synthetic financial networks, previous studies have assumed that the amount of liabilities and the edge degrees in such networks follow a power-law distribution, while the recovery rate and the bank assets follow a bimodal distribution (as in [82]) or the normal distribution (as in [136]). Leventides et al. [92] simulate contagion dynamics in fully random networks based on the uniform distribution, while in [97] a similar approach is followed in terms of liabilities and external assets. Finally, Chen et al. [35] develop a dynamic model to study systemic risk.

1.2.3 Clearing payments

Given a complex financial network, one of the natural questions that come to mind is how much actual payment each bank has to make on liquidation, which is called *clearing problem*. One interpretation of the clearing problem is that in a financial crisis, a clearing authority (e.g., a central bank) observes the whole network of contracts, seeks to solve the clearing problem, and prescribes to each bank how much it has to pay to every other bank [121]. *Clearing payments* are the payments that comply with the following principles during the clearing. *i) absolute priority* requires that all creditors must be paid off before a bank's stakeholders can split assets; *ii) limited liability principle* implies that a bank that does not have enough assets to pay its liabilities in full has to spend all its remaining assets to pay its creditors; while *iii) proportionality* requires the debtor to pay its creditors in proportion to the liabilities.

In debt-only networks, Eisenberg and Noe [48] prove that there always exist maximum and minimum clearing payments and identify sufficient conditions so that uniqueness is guaranteed; they also present an efficient iterative algorithm that computes them. Furthermore, Rogers and Veraart [116] show the existence of maximal clearing payments in the networks with default costs and provide an algorithm that computes them in polynomial time. However, once Credit Default Swaps (CDS) are considered in the model, the clearing problem becomes much more complicated. In particular, Schuldenzucker et al. [121] consider CDS contracts and show that, in general, there can be zero or multiple clearing payments, additionally in [120] they show that finding clearing payment in networks with CDS is hard from computational perspective. In the same spirit, Ioannidis et al. [72] examine the complexity of the clearing problem in financial networks with derivatives and priorities among creditors, while in [73] they study the clearing problem from the point of view of irrationality and strength of approximation. Papp and Wattenhofer [108] study which banks are in default, and how much of their liabilities these can pay.

In addition, most recently, Stachurski [125] shows that the condition of *regular* network is not necessary to guarantee the unique clearing payments under the model by Eisenberg and Noe [48], and shows that there exists one unique clearing payments if each node in the network is *cash accessible*, which is a weaker condition than *regular*. Note that the definition of *cash accessible* exactly coincides with that of *proper* which we define in Chapter 5.

1.2.4 Financial operations

In the aftermath of the 2008 financial crisis, central banks have resorted to strategic bailouts [95, 123, 47] as a means to mitigate potential collapses of businesses and organizations.

A strategic bailout involves an injection of funds to a failing entity that would otherwise face imminent failure. Notably, the US government implemented one of the largest bailouts in history during the global financial crisis in 2008, while Ireland bailed out the Anglo-Irish Bank Corporation to the tune of 29.3 billion euros in 2010 [46]. On the other hand, the bail-in mechanism i.e., debt removals, which entails debt cancellation or forgiveness, provides relief to a financial institution on the brink of failure by requiring the cancellation of debts owed to creditors and depositors. Cyprus and European Union resolutions provide two examples of bail-in mechanisms in action [112].

Cash injection is also known as a bail-out. Lambrecht and Tse [88] develop a continuous-time model to investigate the effects of bailouts versus liquidation or bail-ins, which involve debt-to-equity conversion or debt write-downs, on lending and risk-taking behavior. Altinoglu and Stiglitz [8] analyze how systemically important institutions can emerge as a consequence of banks' anticipation of public bailouts. In contrast, Erol [53] demonstrates that the expectation of bailouts leads to higher interconnectedness and a core-periphery network structure. Additionally, when the financial regulator has available funds to bail out each bank of the network, [75] characterizes the minimum bailout budget needed to ensure systemic solvency and prove that computing it is an NP-hard problem. When the financial authority has a limited bailout budget, [43] proposes the threat index as a means to determine which banks should receive cash during a default episode and suggests a greedy algorithm for this process. [47] focuses on how central banks should decide which insolvent banks to bail out and formulate corresponding optimization problems. [45] introduces an efficient greedy-based clearing algorithm for an extension of the Eisenberg-Noe model, while also studying bailout policies when banks in default have no assets to distribute. We note that the problem of injecting cash (as subsidies) in financial networks has been studied (in a different context) in microfinance markets [74].

Debt removal, also known as bail-in, has been widely regarded as the principal measure for resolving failing systemically important banks [28, 36]. It aims to address the moral hazard issue by making stakeholders responsible for bearing the losses while minimizing the adverse effects of a bank's failure on the economy and the financial system [28]. Klimek et al. [84] employ an agent-based network model to assess the economic and financial implications of bail-in, while Huser et al. [71] evaluate the systemic consequences of bail-in in the

European Union, utilizing a calibrated multi-layered network model of bank debt and equity cross-holdings. Bernard et al. [23] investigate the impact of strategic negotiations with regulators on banks' willingness to participate voluntarily in a bail-in, while [119] considers banks' incentive to approve the removal of a set of liabilities forming a directed cycle in financial networks.

In addition to cash injection and debt removal, other financial operations and instruments, such as liability netting [57, 11], pairwise netting [68], portfolio compression [129, 119], and debt swap [42, 66, 107], have received significant attention in academic research. Amini et al. [11] demonstrate that partial netting of interbank liabilities can increase bank shortfall and reduce clearing asset price and aggregate bank surplus, as compared to full multilateral netting, and prove that partial multilateral netting can be worse than no netting at all. Schuldenzucker et al. [119] analyze the impact of compression on social welfare and the banks' willingness to accept a compression proposal. They also examine the necessary and sufficient conditions to ensure that compression results in a Pareto improvement for all banks. Papp and Wattenhofer [107] investigate the properties of debt swapping operations in financial networks. They show that in a static financial system, there can be no positive swap, which is strictly beneficial for banks involved in swapping, while a positive swap may exist under worst-case shock models. They also explore the computational complexity of finding a positive swap when it is possible. Furthermore, Froese et al. [56] also analyze the computational complexity of debt swapping but in networks with ranking-based clearing. Apart from these operations aforementioned, a common financial operation known as *debt transfer* [81], also referred to as claim assignment [67] or debt assignment [127], has gained attention. Debt transfer allows a bank to transfer its right to claim a debt to one of its lenders if doing so would alleviate its own debt to that lender. To the best of our knowledge, it has been extensively studied in the law literature (e.g., [1, 127]), but not in computer science literature.

Redefining the priorities on payments is also a critical financial operation, see, e.g., in [106, 72, 12]. Although the principle of proportionality is common in actual bankruptcy law, Moulin [102], Chatterjee and Eyigungor [34], and Flores-Szwagrzak [55] argue priority to be another important principle. Also, as discussed in [62, 40, 31, 113], the creditors of different standings may have different priorities. Gaffeo et al. [58] investigate the network where interbank debt obligations are with a higher priority compared to the repayments claimed by external depositors, and find that the corresponding clearing algorithm could potentially underestimate the systemic liquidity shortage. Papp and Watenhofer [106] consider adjusting the priorities on payments as a strategic behavior, and prove that banks can increase their utilities by rearranging the payment priorities.

1.2.5 Financial optimization objectives

In the realm of financial research, a significant body of work investigates methods for assessing and optimizing the well-being of a financial system. In empirical research, the number of banks that have defaulted is commonly used as a natural measure of risk assessment (e.g., [9, 10, 51, 29]). Furthermore, the importance of systemic *liquidity* – the amount of cash flowing within a financial system – in maintaining financial stability has been extensively emphasized (e.g., [6, 70, 37, 90, 90]). Similarly, a variety of theoretical studies also consider these factors to be critical criteria for evaluating the financial health of a system.

When the financial regulator has available funds to bail out each bank of the network, Jackson et al. [75] characterize the minimum bailout cost needed to ensure systemic solvency, i.e., none of the banks in the system are in default. They prove that finding the minimum bailout budget which can guarantee systemic solvency is an NP-hard problem. In the case where the financial authority has a limited bailout budget, Demange [43] proposes the *threat index* as a means to determine which banks should receive cash during a default episode and suggests a greedy algorithm for this process, in order to maximize the systemic liquidity, i.e., the sum of clearing payments, as much as possible. The recent work of Egressy and Wattenhofer [47] is also very relevant to our setting. They focus on how central banks should decide which insolvent banks to bail out and they formulate corresponding optimization problems, e.g., maximizing the market value, maximizing the number of banks saved, as well as minimizing the welfare loss. They prove that most of these optimization problems are NP-hard, and in some cases even hard to approximate. Finally, Dong et al. [45] introduce an efficient greedy-based clearing algorithm for an extension of the Eisenberg-Noe model, and then further study the bailout policies, e.g., computing the minimum bailout fund to save one single insolvent bank, in a setting where banks in default have no assets to distribute.

1.2.6 Game theory in networks

The concept of game theory has been extensively explored in various types of networks, such as social networks [109, 114], energy networks[126, 76], economic networks [104, 41], and blockchain networks [94, 94]. Game-theoretic approaches have been used to explain a variety of phenomena in these networks. For example, Bindel et al. [27] use a game-theoretic lens to explain the behavior of individuals in opinion formation networks. Even-Dar et al. [54] study the network formation game in bipartite exchange economic networks. Recently, the computational finance community has also explored financial networks from a game-theoretic standpoint. Csóka and Herings [39] examine a liability game in which insolvent banks strategically distribute their assets among creditors, while Babaioff et al. [17] study optimal collaterals in multi-enterprise investment networks. Finally, Avarikioti et al. [15] analyze payment channel networks in blockchain through the lens of network creation games. These works demonstrate the utility of game theory in modeling and understanding various types of networks.

A large body of recent work considers game-theoretic aspects of financial networks. Papp and Wattenhofer [106] consider the incentives of banks to remove incoming edges, redefine the seniorities of liabilities, as well as to donate external assets, while in [107] they consider the impact of debt swapping in mitigating risk. Kanellopoulos et al. [80] study a game where banks can remove incoming edges and also allow for a bailout from a central authority. Bertschinger et al. [25] and Kanellopoulos et al. [79] study strategic behavior under payment schemes other than the proportional one. In very recent work, Hoefer and Wilhelmi [69] consider clearing games with different seniorities. Bertschinger et al. [24] study the existence and structure of equilibria in a game modeling fire sales, as well as the convergence of best-response dynamics. Additionally, Schuldenzucker et al. [119] study the impact of portfolio compression in financial networks and derive sufficient conditions leading to a Pareto improvement for all banks.

1.3 Roadmap and Summary of Results

Inspired by these related works, e.g., [25, 7], we define various game-theoretic models and initiate the study of associated games that arise from different types of strategic operations by banks, i.e., *cash injection*, *debt removal*, *debt transfer*, and *priority-proportional payments* respectively, in financial networks.

With respect to each operation, the problems considered from a centralized perspective can be summarized as follows.

a) *Is there an efficient algorithm available to achieve desirable systemic objectives?*

b) *If not, are there any approximate algorithms with a favorable approximation ratio, and what is the quantitative measure of their effectiveness?*

On the other hand, we study the game-theoretic setting where each bank can strategically take corresponding operations to maximize its well-being (i.e., utility). For each corresponding game in financial networks, we then focus on:

a) *Do pure Nash equilibria always exist?*

b) *How does the quality of equilibria compare to social optima?*

c) *How to detect and compute equilibria that arise in these games?*

In the following, we will outline the key findings and provide a roadmap for each chapter.

Chapter 2: Preliminaries. This chapter presents the necessary notions and definitions, as well as provides examples to showcase the financial networks.

Chapter 3: Debt Removal in Debt-Only Networks. In this chapter, we study cash injection and debt removal operations from both centralized and decentralized perspectives. We start with additional preliminaries in Section 3.1. In Section 3.2 we investigate the computational complexity of finding optimal cash injection policies, and optimal debt removals, respectively. Then, in Section 3.3, we turn attention to the edge-removal games, focusing on the existence, quality, and computational aspects of equilibria for such games.

The results of this chapter can be summarised as follows. First of all, in a centralized manner, we study the computation of optimal cash injection policies and debt removal policies respectively to optimize some desired social objectives, e.g., maximizing the sum of payments, and minimizing the number of defaulted banks. Although we show that the optimal cash injection policy can be computed in polynomial time using linear programming when default costs do not exist, the LP-based algorithm lacks the desirable property of monotonicity, so, we then investigate a greedy algorithm with that property, and quantify its approximation ratio. With respect to computing optimal debt removals, we also provide a series of hardness results corresponding to various social objectives. Furthermore, from a game-theoretic standpoint, we consider each bank in the financial network as an intelligent agent that can strategically remove incoming edges to maximize its utility, i.e., total assets, which naturally forms a so-called *edge-removal game*. We showcase a network lacking pure Nash equilibria and indicate that it is possible that the worst equilibria may improve the original network significantly, but it is also possible that the best equilibria can deviate considerably from social optima. We also examine the computational complexity of detecting equilibria and finding pure Nash equilibria when they exist.

Chapter 4: Debt Transfer in Debt-Only Networks. In this chapter, we study debt transfer operations in financial networks, where banks can transfer their right to claim debts to their lenders. We begin with some additional preliminary definitions in Section 4.1. In Section 4.2 we consider the computational complexity of selecting a collection of debt transfers that optimizes certain objectives, e.g., maximizing total payments or equity. In Section 4.3 we introduce debt transfer games that emerge when banks can strategically transfer their debt claims. Our game-theoretic analysis considers two different definitions of utility motivated by the financial literature, namely total assets, or equity, respectively. We analyze each variant with respect to the existence, computational complexity and quality of the Nash equilibria that arise. Specifically, we show a network without pure Nash equilibria when banks wish to maximize their total assets and default costs are applied, while there always exists a Nash equilibrium in games where players wish to maximize their equity and no default costs apply. In terms of quality, we prove that Nash equilibria can have arbitrarily worse social welfare than the optimal state, but they may also have arbitrarily better social welfare than the initial network. We also investigate the computational aspects of equilibria-related problems in debt transfer games. In Section 4.4 we complement our theoretical results with an empirical analysis on synthetic networks. In particular, we examine the performance of simple heuristics for finding a collection of debt transfers according to various performance measures, and we also study the dynamics of game-playing and the quality of associated equilibria. Overall, our analysis provides evidence supporting the use of debt transfers for improving the financial well-being of a system.

Chapter 5: Priority-Proportional Payments in Networks with CDS. In this chapter, we quantify the extent to which strategic behavior of the banks affects the welfare of society, by analyzing the *priority-proportional payment games* that are defined by a particular utility function, and possibly allow the presence of several financial instruments, e.g., Credit Default Swap (CDS). We identify the Nash equilibria of such financial network games and argue about their (in)efficiency in terms of the total welfare of the system. In particular, we consider financial network games under *priority-proportional strategies*, defined for different utility functions, such as total assets or equities, and which potentially allow CDS contracts, default costs, or negative external assets. We derive structural results that have to do with the existence, computation, and the properties of clearing payments for fixed payment decisions in a non-strategic setting, and/or the existence and quality of equilibrium strategies. In particular, in Section 5.2 we prove the existence of maximal clearing payments under priority-proportional strategies, even in the presence of default costs, and provide an algorithm that computes them efficiently. We are then able to prove the existence of equilibria when the utility is defined as the equity, but show that equilibria are not guaranteed to exist when the utility is captured by the total assets. We then turn our attention to the efficiency of equilibria and provide an almost complete picture of the Price of Anarchy [86] and the Price of Stability [122, 14]. Our results for total assets appear in Section 5.3.1, while the case of equities is treated in Section 5.3.2. Finally, in Section 5.4 we provide hardness results not only on equilibria-related problems but also on the optimization of societal objectives through central planning by a financial authority.

Chapter 6: Conclusions and Future Work. In this chapter, we conclude and offer potential directions for future research.

Chapter 2

Preliminaries

In this chapter, we present the concepts and notations shared throughout this thesis, while additional definitions are deferred to the corresponding chapters for ease of exposition. The critical notions and their graphical representation are presented in an example (Figure 2.1), at the end of this chapter.

2.1 Debt-Only Networks

A *debt-only financial network* $N = (V, E)$ consists of a set $V = \{v_1, \ldots, v_n\}$ of n banks, where each bank v_i initially has some non-negative *external assets* e_i corresponding to income received from entities outside the financial system. Banks have payment obligations, i.e., *liabilities*, among themselves. In particular, a *debt contract* creates a liability l_{ij} of bank v_i (the borrower) to bank v_j (the lender); we assume that $l_{ij} \geq 0$ and $l_{ii} = 0$. Note that $l_{ij} > 0$ and $l_{ji} > 0$ may both hold simultaneously. Also, let $L_i = \sum_j l_{ij}$ be the total liabilities of bank v_i and set $\mathbf{l} = (L_1, \ldots, L_n)$. Banks with sufficient funds to pay their obligations in full are called *solvent* banks, while ones that cannot are *in default* or, also, *insolvent*. The *recovery rate*, r_i, of a bank of v_i that is in default, is defined as the fraction of its total liabilities that it can fulfill, while the relative liability matrix $\Pi \in \mathbb{R}^{n \times n}$ is defined by

$$\pi_{ij} = \begin{cases} l_{ij}/L_i, & \text{if } L_i > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Let p_{ij} denote the actual payment[1] from v_i to v_j; we assume that $p_{ii} = 0$. These payments

[1]Note that the actual payment need not equal the liability, i.e., the payment obligation.

define a payment matrix $\mathbf{P} = (p_{ij})$ with $i, j \in [n]$, where by $[n]$ we denote the set of integers $\{1, \ldots, n\}$. We denote by $p_i = \sum_{j\in[n]} p_{ij}$ the total outgoing payments of bank v_i, while $\mathbf{p} = (p_1, \ldots, p_n)$ is the payment vector; this should not be confused with the breakdown of individual payments of bank v_i that is denoted by $\mathbf{p}_i = (p_{i1}, \ldots, p_{in})$. A bank in default may need to liquidate its external assets or make payments to entities outside the financial system (e.g., to pay wages). This is modeled using *default costs* defined by values $\alpha, \beta \in [0, 1]$. In particular, a bank in default can only use an α fraction of its external assets and a β fraction of its incoming payments (the case without default costs is captured by $\alpha = \beta = 1$). The regulatory principles of absolute priority and limited liability, as stated in the introduction, imply that a solvent bank must repay all its obligations to all its lenders, while a bank in default must repay as much of its debt as possible, taking default costs into account. Summarizing, it must hold that $\mathbf{P} = \Phi(\mathbf{P})$, where

$$\Phi(\mathbf{x})_{ij} = \begin{cases} l_{ij}, & \text{if } L_i \le e_i + \sum_{j=1}^n x_{ji} \\ \left(\alpha e_i + \beta \sum_{j=1}^n x_{ji}\right) \cdot \pi_{ij}, & \text{otherwise.} \end{cases} \tag{2.1}$$

Payments $\mathbf{P}$ that satisfy these constraints are called *clearing payments*[2]. *Proportional payments* have been frequently studied in the financial literature (e.g., in [43, 48, 116]). It is worth noting that the clearing payments matrix $\mathbf{P}$, which satisfies the equations as expressed in Equation 2.1, demonstrates a natural adherence to the principle of proportionality. Specifically, each $p_{ij} = l_{ij}$ when v_i is solvent, while $p_{ij} = \left(\alpha e_i + \beta \sum_{j\in[n]} p_{ji}\right) \pi_{ij}$ when v_i is in default.

Given clearing payments $\mathbf{P}$, the *total assets* $a_i(\mathbf{P})$ of bank v_i are defined as the sum of external assets plus incoming payments, i.e.,

$$a_i(\mathbf{P}) = e_i + \sum_{j\in[n]} p_{ji},$$

while the *equity* $E_i(\mathbf{P})$ is

$$E_i(\mathbf{P}) = \max\{0, a_i(\mathbf{P}) - L_i\}.$$

Maximal clearing payments, i.e., ones that point-wise maximize all corresponding payments (and hence total assets), are known to exist [48, 116] and can be computed in polynomial

[2]Clearing payments are not necessarily unique.

time [116]. Note that we only focus on maximal clearing payments in this thesis unless we specifically mention otherwise.

We measure the total *liquidity* of the system (also referred to as systemic liquidity) $\mathcal{F}(\mathbf{P})$ as the sum of payments traversing through the network, i.e.,

$$\mathcal{F}(\mathbf{P}) = \sum_{i\in[n]} \sum_{j\in[n]} p_{ji}.$$

2.2 Games in Financial Networks

Given a financial network N, games arise naturally when we view the banks as strategic agents. let $S_i(\cdot)$ be the set of all possible strategies for bank v_i. We denote the strategy of bank v_i by $s_i(\cdot)$, while the strategy profile of all players except v_1 by s_{-i}. The strategy profile of all players is denoted by $\mathbf{s} = (s_1(\cdot), \dots, s_n(\cdot))$ or $\mathbf{s} = (s_i(\cdot), s_{-i}(\cdot))$ interchangeably. Note that the concrete type of strategy depends on the actual game under consideration, and further details will be provided in the corresponding chapters. For instance, in the edge-removal games studied in Chapter 4, each participating bank can employ a distinct strategy, whereby they selectively eliminate a designated subset of incoming edges within the financial network.

Given clearing payments $\mathbf{P}$ which is consistent with strategy profile $\mathbf{s}$, we define bank v_i's *utility* $u_i(\mathbf{P})$ using either of the previously mentioned two notions, i.e., *total assets* $a_i(\mathbf{P})$ and *equity* $E_i(\mathbf{P})$, while the *social welfare* $SW(\mathbf{P})$ is the sum of the banks' utilities; the particular utility notion (total assets or equity) will be clear from the context. Also, unless stated otherwise $\mathbf{P}$ will refer to the maximal clear payments.

Definition 2.1 (Pure Nash Equilibrium)**.** Consider a strategy profile $\mathbf{s_{eq}}$, and the corresponding clearing payments $\mathbf{P_{eq}}$. $\mathbf{s_{eq}}$ is a pure Nash equilibrium if for each bank v_i with $i \in [n]$ and any alternative strategy $s_i(\cdot) \in S_i(\cdot)$ of v_i, it holds that $u_i(\mathbf{P_{eq}}) \geq u_i(\mathbf{P})$, where $\mathbf{P}$ denotes the clearing payments under the profile $\mathbf{s} = (s_i(\cdot), s_{-i}^{eq}(\cdot))$, where player v_i chooses strategy $s_i(\cdot)$ while all other players keep using their equilibrium strategy.

Definition 2.2 (Best Response)**.** Consider strategy profile $\mathbf{s}^* = (s_i^*(\cdot), s_{-i}(\cdot))$, and the corresponding clearing payments $\mathbf{P}^*$. $s_i^*(\cdot)$ is the best response for bank v_i to some strategy profile $s_{-i}(\cdot)$ if and only if $u_i(\mathbf{P}^*) \geq u_i(\mathbf{P})$ for all $s_i(\cdot) \in S_i(\cdot)$, where $\mathbf{P}$ corresponds to the clearing payments under the profile $\mathrm{s} = (s_i(\cdot), s_{-i}(\cdot))$.

Definition 2.3 (Dominant Strategy)**.** Consider strategy profile $\mathrm{s}' = (s_i'(\cdot), s_{-i}(\cdot))$ and the corresponding clearing payments $\mathbf{P}'$. A strategy $s_i'(\cdot)$ is a dominant strategy for bank v_i if and only if $u_i(\mathbf{P}') > u_i(\mathbf{P})$ for any $s_i(\cdot) \in S_i(\cdot)$ where $s_i(\cdot) \neq s_i'(\cdot)$, where $\mathbf{P}$ corresponds to the clearing payments under the profile $\mathrm{s} = (s_i(\cdot), s_{-i}(\cdot))$.

The optimal social welfare is denoted by OPT. To measure the quality of equilibria, the Price of Anarchy (PoA) of a game is defined as the worst-case ratio of the optimal social welfare over the social welfare achieved at any equilibrium over all possible networks. In contrast, the Price of Stability (PoS) measures how far the highest social welfare that can be achieved at equilibrium is from the optimal social welfare over all possible instances.

$$\text{PoA} = \max_{N} \max_{\mathbf{P} \in \mathbf{P_{eq}}} \frac{OPT}{SW(\mathbf{P})} \qquad \text{PoS} = \max_{N} \min_{\mathbf{P} \in \mathbf{P_{eq}}} \frac{OPT}{SW(\mathbf{P})}$$

The Price of Anarchy/Stability notions provide indications regarding the extent to which the individual objectives of the banks and the objective of the regulator are (not) aligned. Here, we also introduce two novel notions, namely, the Effect of Anarchy (EoA) and the Effect of Stability (EoS), which measure the discrepancy between the social welfare SW_N of the original network and that of the worst (best, respectively) Nash equilibrium over all possible instances. These are defined as follows mathematically.

$$\text{EoA} = \max_{N} \max_{\mathbf{P} \in \mathbf{P_{eq}}} \frac{SW_N}{SW(\mathbf{P})} \qquad \text{EoS} = \min_{N} \min_{\mathbf{P} \in \mathbf{P_{eq}}} \frac{SW_N}{SW(\mathbf{P})}$$

2.3 Example and Visual Representation

We represent a financial network by a graph as follows. Nodes correspond to banks and black solid edges correspond to debt-liabilities; a directed edge from node v_i to node v_j with label l_{ij} implies that bank v_i owes the bank v_j an amount of money equal to l_{ij}. Nodes are also labeled, their label appears in a rectangle and denotes their external assets; we omit these labels for banks with external assets equal to 0.

Figure 2.1 depicts a financial network with four banks having external assets $e_1 = e_2 = e_4 = 0$ and $e_3 = 2$. There exist four debt contracts, i.e., bank v_1 owes v_2 two units, while v_2 owes v_1 and v_4 three units and one unit, respectively; and v_3 owes v_2 three units.

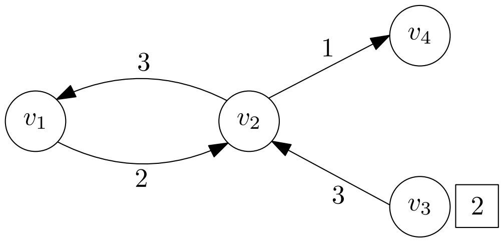


Figure 2.1: An example of a financial network.

- Let us consider a case without default costs, i.e., $\alpha = \beta = 1$. Then, the clearing payment vector would be $\mathbf{p} = (2, 4, 2, 0)$ with $\mathbf{p}_2 = (3, 0, 0, 1)$, while the detailed clearing payments are $p_{12} = 2$, $p_{21} = 3$, $p_{24} = 1$ and $p_{32} = 2$ with the total assets of the banks are $a_1(\mathbf{P}) = 3$, $a_2(\mathbf{P}) = 4$, $a_3(\mathbf{P}) = 2$, $a_4(\mathbf{P}) = 1$ and equities $E_1(\mathbf{P}) = 1$, $E_2(\mathbf{P}) = 0$, $E_3(\mathbf{P}) = 0$ and $E_4(\mathbf{P}) = 1$. The recovery rates are $r_1(\mathbf{P}) = r_2(\mathbf{P}) = r_4(\mathbf{P}) = 1$ and $r_3(\mathbf{P}) = \frac{2}{3}$ with all banks being solvent, except v_3.

- However, if $\alpha = \beta = \frac{1}{2}$, the clearing payment vector would be $\mathbf{p}' = (\frac{3}{13}, \frac{8}{13}, 1, 0)$ with $\mathbf{p}'_2 = (\frac{6}{13}, 0, 0, \frac{2}{13})$, while the detailed clearing payments are $p'_{12} = \frac{3}{13}$, $p'_{21} = \frac{6}{13}$, $p'_{24} = \frac{2}{13}$ and $p'_{32} = 1$ with the total assets of the banks being $a_1(\mathbf{P}') = \frac{6}{13}$, $a_2(\mathbf{P}') = 1 + \frac{3}{13} = \frac{16}{13}$, $a_3(\mathbf{P}') = 2$, $a_4(\mathbf{P}') = \frac{2}{13}$ and equities $E_1(\mathbf{P}') = E_2(\mathbf{P}) = E_3(\mathbf{P}') = 0$ and $E_4(\mathbf{P}') = \frac{2}{13}$. Furthermore, the recovery rates would become $r_1(\mathbf{P}') = \frac{3}{26}$, $r_2(\mathbf{P}') = \frac{2}{13}$, $r_3(\mathbf{P}') = \frac{1}{3}$ and $r_4(\mathbf{P}') = 1$ with all banks in default apart from v_4.

Chapter 3

Debt Removals in Debt-Only Networks

Recall that a financial system is represented by a network, where nodes correspond to banks, and directed labeled edges correspond to debt contracts between banks. In a debt-only network with default costs, the total *liquidity* of a financial system is measured by the sum of payments made at clearing, and is a natural metric for the well-being of the system [6, 70, 37, 90, 90]. Financial authorities, e.g., governments or other regulators, wish to keep the systemic liquidity as high as possible and they might interfere, if their involvement is necessary and would considerably benefit the system. For example, in the not so far past, the Greek government (among others) took loans in order to bailout banks that were in danger of defaulting, to avert collapse. In this work, we study the possibility of a financial regulating authority performing cash injections (i.e., bailouts) to selected bank(s) and/or forgiving debts selectively, with the aim of maximizing the total liquidity of the system (total money flow). Similarly to cash injections, it is a fact that debt removal can have a positive effect on systemic liquidity. Indeed, the existence of default costs can lead to the counter-intuitive phenomenon whereby removing a debt/edge from the financial network might result in increased money flow, e.g., if the corresponding borrower avoids default costs because of the removal.

Even more surprising than the increase of liquidity by the removal of debts, is the fact that the removal of an edge from borrower b to lender l might result in l receiving more incoming payments, e.g., if b avoids default costs and there is an alternative path in the network where money can flow from b to l. This motivates the definition of an edge-removal game on financial networks, where banks act as strategic agents who wish to maximize their total assets and might intentionally give up a part of their due incoming payments towards this goal. As implied earlier, removing an incoming debt could rescue the borrower from financial default, thereby avoiding the activation of default cost, and potentially increasing the lender's utility (total assets). This strategic consideration is meaningful both in the context where a financial authority performs cash injections or not. We consider the existence, quality, and computation of equilibria that arise in such games.

3.1 The Model and Definitions

Based on the preliminaries presented in Chapter 2, we complement with necessary concepts and notations utilized in this chapter and provide an example (Figure 3.1.3) to show how to compute these notions in detail.

3.1.1 Debt removal & Cash injection

Outside a given debt-only network, we assume that there exists a financial authority (a *regulator*) who aims to maximize systemic liquidity. In particular, the regulator can decide to remove certain debts (also called *debt removal*) from the network or inject cash into some banks (also called *cash injection*). In the latter case, we assume the regulator has a total *budget* M available in order to perform *cash injections* to individual banks. We sometimes refer to the total increased liquidity, $\Delta\mathcal{F}$, (as opposed to total liquidity $\mathcal{F}$) which measures the difference in the systemic liquidity before and after the cash injections[1].

A *cash injection policy* is a sequence of pairs of banks and associated transfers $((i_1, t_1), (i_2, t_2), \ldots (i_L, t_L)) \in (V \times \mathbb{R})^L$, such that the regulator gives capital t_1 to bank i_1, t_2 to bank i_2, etc. These actions naturally define two corresponding optimization problems on the total (increased) liquidity, i.e., *optimal cash injection* and *optimal debt removal*.

[1]This is necessary as in some cases, like the proof of the approximability of the greedy algorithm in Theorem 3.2, we cannot argue about the total liquidity but we can argue about the total increased liquidity.

3.1.2 Threat index

We will find useful the notion of the *threat index*[2], μ_i, of bank v_i, which captures how many units of total increased liquidity will be realized if the financial authority injects one unit of cash into bank v_i's external assets [43]; a unit of cash represents a small enough amount of money so that the set of banks in default would not change after the cash injection. We remark that for the maximum total increased liquidity it holds $\Delta\mathcal{F} \leq M \cdot \mu_{\mathtt{max}}$, where $\mu_{\mathtt{max}}$ is the maximum threat index. Naturally, the threat index of solvent banks is 0, while the threat index of banks in default will be at least 1. Formally, the threat index is defined as

$$\mu_i = \begin{cases} 1 + \sum_{j \in D} \pi_{ij}\mu_j, & \text{if } a_i(\mathbf{P}) < L_i \\ 0, & \text{otherwise,} \end{cases}$$

where $D = \{j | a_j(\mathbf{P}) < L_j\}$ is the set of banks who are in default[3].

3.1.3 An example

Figure 3.1 provides an example of a financial network, inspired by an example in [43]. The clearing payments are as follows: $p_{21} = 4.4$, $p_{32} = 3.2$, and $p_{43} = p_{45} = 1$, implying that banks v_2, v_3 and v_4 are in default. We assume that there are no default costs, i.e., $\alpha = \beta = 1$. The threat indexes are computed as follows: $\mu_1 = \mu_5 = 0$, $\mu_2 = 1 + \mu_1$, $\mu_3 = 1 + \mu_2$, and $\mu_4 = 1 + \frac{1}{2}\mu_3 + \frac{1}{2}\mu_5$, implying that $\mu_3 = \mu_4 = 2$, $\mu_2 = 1$, while $\mu_1 = \mu_5 = 0$.

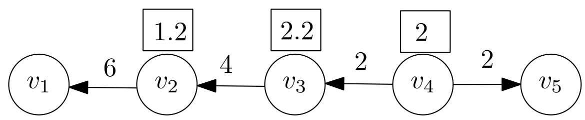


Figure 3.1: A simple financial network. Nodes correspond to agents, edges are labeled with the respective liabilities, while external assets are in a rectangle above the relevant agents.

[2] The term threat index aims to capture the "threat" posed to the network by a decrease in a bank's cash flow or even the bank's default; this index can be thought of as counting all the defaulting creditors that would be affected by a potential default of the said bank.

[3] In matrix form, the threat index for banks who are in default can be computed by $(\mathbb{I}_{D\times D} - \boldsymbol{\pi}_{D\times D})\,\boldsymbol{\mu}_D = \mathbf{1}_D$. This is a homogeneous linear equation system where $\boldsymbol{\pi}_{D\times D}$ is the relative liability matrix only involving in the banks in set D; $\mathbb{I}_{D\times D}$ and $\mathbf{1}_D$ represent the $|D| \times |D|$ dimension identity matrix and the $|D|$ dimension identity vector, respectively, while $\boldsymbol{\mu}_D$ is the vector of threat indices for banks in default.

3.2 Computing and Approximating Optimal Outcomes

In this section, we present algorithmic and complexity results with respect to computing optimal cash injection (Section 3.2.1) and debt removal (Section 3.2.2) policies. Note that we omit to refer to default costs in our statements for those results that hold when $\alpha = \beta = 1$.

3.2.1 Optimal cash injections

We begin with a positive result about computing the optimal cash injection policy when default costs do not apply.

Theorem 3.1. *Computing the optimal cash injection policy can be solved in polynomial time.*

Proof. The proof follows by solving a linear program that computes the optimal cash injections and accompanying payments.

$$
\begin{aligned}
\text{maximize} \quad & \sum_i \sum_j p_{ij} & \\
\text{subject to} \quad & \textstyle\sum_i x_i \leq M, & \forall i, j\\
& p_{ij} \leq l_{ij}, & \forall i, j\\
& p_{ij} \leq (x_i + e_i + \textstyle\sum_k p_{ki}) \cdot \frac{l_{ij}}{L_i}, & \forall i, j\\
& x_i \geq 0, & \forall i\\
& p_{ij} \geq 0, & \forall i, j
\end{aligned}
$$

We denote by x_i the cash injection to bank i and by p_{ij} the payment from i to j. We aim to maximize the total liquidity, i.e., the total payments, subject to satisfying the limited liability and absolute priority principles. Recall that M is the budget, l_{ij} is the liability of i to j, e_i is the external assets of bank i, and L_i is the total liabilities of i.

The first constraint corresponds to the budget constraint, while the second and third sets of constraints guarantee that no bank pays more than her total assets or more than a given liability; hence, the limited liability principle is satisfied. It remains to argue about the absolute priority principle, i.e., a bank can pay strictly less than her total assets only if she fully repays all outstanding liabilities.

Consider the optimal solution corresponding to a vector of cash injections and payments p_{ij}; we will show that this solution satisfies the absolute priority principle as well. We distinguish between two cases depending on whether a bank is solvent or in default. In the first case, consider a solvent bank i, i.e., $x_i + e_i + \sum_j p_{ji} \geq L_i$, for which $p_{ik} < l_{ik}$ for some bank k. By replacing p_{ik} with $p'_{ik} = l_{ik}$, we obtain another feasible solution that strictly increases the objective function; a contradiction to the optimality of the starting solution. Similarly, consider a bank i with $x_i + e_i + \sum_j p_{ji} < L_i$ for which $\sum_j p_{ij} < x_i + e_i + \sum_j p_{ji}$. Then, there necessarily exists a bank k for which $p_{ik} < (x_i + e_i + \sum_j p_{ji}) \cdot \frac{l_{ik}}{L_i}$ and it suffices to replace p_{ik} with $p'_{ik} = (x_i + e_i + \sum_j p_{ji}) \cdot \frac{l_{ik}}{L_i}$ to obtain another feasible solution that, again, strictly increases the objective function. Hence, we have proven that the optimal solution to the linear program satisfies the absolute priority principle and the claim follows by providing each bank i a cash injection of x_i. □

Note that the optimal policy does not satisfy certain desirable properties. In particular, as observed in [43], cash injections are not monotone with respect to the budget. To see that, consider the financial network in Figure 3.1 and note that when $M = 0.5$, the optimal policy would give all available budget to bank v_3, while under an increased budget of 1.6, the entire budget would be allocated to v_4, hence v_3 would get nothing. Furthermore, our LP-based algorithm crucially relies on knowledge of the available budget.

To alleviate these undesirable properties, we turn our attention to efficiently approximating the optimal cash injection policy by a natural and intuitive greedy algorithm, and compute its approximation ratio under a limited budget, when we care about the total *increased* liquidity.

Definition 1 (GREEDY and its approximation ratio)**.** According to GREEDY (the Greedy cash injection policy), banks receive their cash injections in sequence, so that i_k, for $k = 1, \dots, L$, is the bank with the highest threat index after the cash injection at round $k-1$ (round 0 is defined to be the starting configuration), while t_k is the minimum amount that would cause a change in the vector of threat indexes at the time it is transferred (it would lead to some previously defaulting bank to become solvent).[4] This process is repeated until the budget runs out.

The *approximation ratio* of GREEDY shows how smaller the total *increased* liquidity (or

[4]Without loss of generality we assume ties are broken in favor of the smallest index.

money flow) can be, compared to the optimal total increased liquidity, and is computed as

$$\mathcal{R}_{Greedy} = \min_{N,M} \frac{\Delta\mathcal{F}_{Greedy}}{\Delta\mathcal{F}_{OPT}},$$

where the minimum is computed over all possible networks and budgets.

Let us revisit the example in Figure 3.1, assuming a budget $M = 1.6$. Initially banks v_3 and v_4 have the highest threat index of $\mu_3 = \mu_4 = 2$ compared to $\mu_1 = \mu_5 = 0$, and $\mu_2 = 1$. We can assume[5] that bank v_3 would receive the first cash injection ($i_1 = v_3$) and in fact this will be equal to $t_1 = 0.8$. Indeed, a cash injection of 0.8 to v_3 will result in v_3 becoming solvent (notice that v_3 receives 1 from v_4), while a smaller cash injection would not impose any change on the threat index vector. At this stage, the threat index of each bank is as follows $\mu'_1 = \mu'_3 = \mu'_5 = 0$ and $\mu'_2 = \mu'_4 = 1$. At this round, $i_2 = v_2$ would receive the remaining budget of $t_2 = 0.8$. Hence, the *total increased liquidity* achieved by GREEDY at this instance is $\Delta\mathcal{F}_{Greedy} = 2.4$ (t_1 will traverse edges (v_3, v_2) and (v_2, v_1), while t_2 will traverse edge (v_2, v_1)). However, the optimal cash injection policy is to inject the entire budget $M = 1.6$ to bank v_4 resulting in $\Delta\mathcal{F}_{OPT} = 3.2$. Therefore, this instance reveals $\mathcal{R}_{Greedy} \leq \frac{2.4}{3.2} = 3/4$.

Theorem 3.2. GREEDY*'s approximation ratio is at most* $3/4$*. For inputs satisfying* $M \leq t_1 \frac{\mu_v}{\mu_v - 1}$*, this ratio is tight.*

Proof. The upper bound follows from the instance in Figure 3.1 and the discussion above; in the following, we argue about the lower bound when the total budget M satisfies the condition in the statement. We consider the following properties and claim that the worst approximation ratio of GREEDY is achieved at networks that satisfy both properties. To prove this claim we will show that starting from an arbitrary network N on which GREEDY approximates the optimal total increased liquidity by a factor of r, we can create a new network that satisfies properties (P1) and (P2), such that GREEDY approximates the optimal total increased liquidity in the new network by a factor of at most r. We can then bound the approximation ratio of GREEDY on the set of networks that satisfy these properties.

(P1) The total increased liquidity achieved by GREEDY is exactly $t_1\mu_{i_1} + (M - t_1)$.

[5] This is consistent to our tie-breaking assumption that favors the least index.

(P2) The optimal total increased liquidity is exactly $\mu_{i_1}M$.

Let the bank with the highest threat index in N be v and let μ_v be that threat index. If $\mu_v = 1$ or $M \leq t_1$ then the claim is true (GREEDY trivially achieves an approximation ratio of 1). Specifically, when $\mu_v = 1$, this implies all insolvent banks are with the threat index of 1, then any cash injection policy would achieve the total increased liquidity of M. Regarding the cases where $M \leq t_1$, any GREEDY would not change the vector of threat indexes, then returning the total increased liquidity of $\mu_v M$. So we henceforth assume that $\mu_v > 1$ and $M > t_1$. We create N' (see Figures 3.2 and 3.3) as follows. To describe our construction it is convenient to express the highest threat index as $\mu_v = x + \frac{a}{a+b}$ for some $a, b > 0$ that satisfy $a + b = t_1$ and x a fixed integer. If μ_v is an integer, then we set $x = \mu_v - 1$ (which implies $a = t_1$ and $b = 0$), while if μ_v is not an integer, we set $x = \lfloor \mu_v \rfloor$ and $a = (\mu_v - \lfloor \mu_v \rfloor)t_1$. Our network N' comprises $\lceil \mu_v \rceil + 4$ nodes, i.e., u, v, w, z and v_i, for $i = 1, \ldots, \lceil \mu_v \rceil$. There is a directed path of length $\lceil \mu_v \rceil$, including the following nodes in sequence v and v_i, for $i = 1, \ldots, \lceil \mu_v \rceil$, where the first $\lfloor \mu_v \rfloor - 1$ edges have liability t_1. The remaining edge(s) on that path has/have liability a and there is an edge $(v_{\lfloor \mu_v \rfloor - 1}, u)$ with liability b. Moreover, there are edges (w, v) and (w, z) such that $l_{wv} = t_1$ and $l_{wz} = \frac{t_1}{(\mu_v - 1)}$. Note that the liabilities of the two outgoing edges of w are selected so that w and v both have the highest threat index in N'; indeed, the threat index of v in N' is $\mu'_v = \mu_v$, and the threat index of w in N' is $\mu'_w = 1 + \frac{l_{wv}}{L_w}\mu_v = \mu_v$, where $L_w = l_{wv} + l_{wz}$. Immediate consequences of our construction are (i) $\mu_v = \frac{L_w}{l_{wz}}$ and (ii) $\frac{l_{wv}}{L_w} = \frac{\mu_v - 1}{\mu_v}$ which will be useful later.

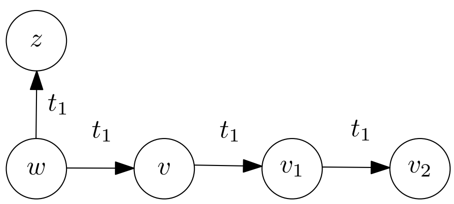


Figure 3.2: An example network used in the lower bound of the approximation ratio of GREEDY for $\mu_v = 2$ and $\mu_w = 1 + \frac{1}{2} \cdot 2 = 2$. The claim in the proof is that for any arbitrary network such that the first cash injection made by GREEDY is t_1, and the highest threat index is an integer, e.g. 2 in this case, the network in this figure achieves at most the same approximation ratio, while satisfying properties (P1) and (P2).

To see that N' satisfies property (P1), it suffices to consider that in the first step, GREEDY

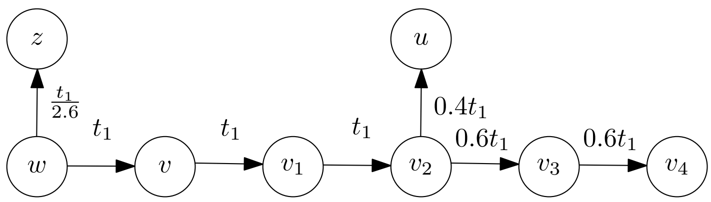


Figure 3.3: Similarly to Figure 3.2, for the case where the highest threat index is not an integer, e.g. $\mu_v = 3.6 = 1+(1+(1+\frac{0.6t_1}{t_1}\cdot 1))$ and $\mu_w = 1+\frac{t_1}{t_1+t_1/2.6}\cdot\mu_v = 1+\frac{2.6}{3.6}\cdot 3.6 = 3.6$.

offers a cash injection of t_1 to v in N'; w is then the only node in default and has threat index equal to 1. The total increased liquidity achieved by GREEDY in N' is exactly $t_1\mu_v+(M-t_1)$, while GREEDY achieves at least that total increased liquidity in N since each node in default has threat index at least equal to 1. Assume now that the regulator offers the entire budget to node w in N'; this would result to total increased liquidity of $\mu'_w M$ as required by (P2), and since, by construction $\mu'_w = \mu_v$ is the maximum total increased liquidity in N, it holds that $\mu'_w M$ is an upper bound on the optimal total increased liquidity in the original network too. We can conclude that it is without loss of generality to restrict attention to networks that satisfy properties (P1) and (P2) when proving a lower bound on the approximation ratio of GREEDY. Overall, it holds that the approximation ratio of GREEDY is lower bounded by

$$\begin{aligned}\mathcal{R}_{Greedy} &\geq \min_{\mathcal{N}^*,M} \frac{\Delta\mathcal{F}_{Greedy}}{\Delta\mathcal{F}_{OPT}}\\ &\geq \min_{\mathcal{N}^*,M} \frac{t_1\mu_v + (M-t_1)}{M\mu_v},\end{aligned}$$

where the minimum ranges over the set of networks $\mathcal{N}^*$ satisfying properties (P1) and (P2). Straightforward calculations after substituting $\mu_v = \frac{L_w}{l_{wz}}$ which holds by construction, leads to

$$\begin{aligned}\mathcal{R}_{Greedy} &\geq \min_{\mathcal{N}^*,M} \left\{1+\left(\frac{t_1}{M}-1\right)\frac{l_{wv}}{L_w}\right\}\\ &\geq \min_{\mathcal{N}^*} \left\{1+\left(\frac{\mu_v-1}{\mu_v}-1\right)\frac{\mu_v-1}{\mu_v}\right\}\\ &\geq 1-1/4\\ &= 3/4,\end{aligned}$$

where the second inequality holds by assumption that $M \leq t_1\frac{\mu_v}{\mu_v-1}$ and since, by construction

$\frac{l_{wv}}{L_w} = \frac{\mu_v - 1}{\mu_v}$, while the third inequality holds since $-1/4$ is a global minimum of function $f(x) = (x-1)x$. □

We conclude this section with some hardness results.

Theorem 3.3. *The following problems are NP-hard:*

a*) Compute the optimal cash injection policy under the constraint of integer payments.*

b*) Compute the optimal cash injection policy with default costs* $\alpha \in [0, 1)$ *and* $\beta \in [0, 1]$.

c*) Compute the minimum budget so that a given agent becomes solvent, with default costs* $\alpha \in [0, 1/2)$ *and* $\beta \in [0, 1]$.

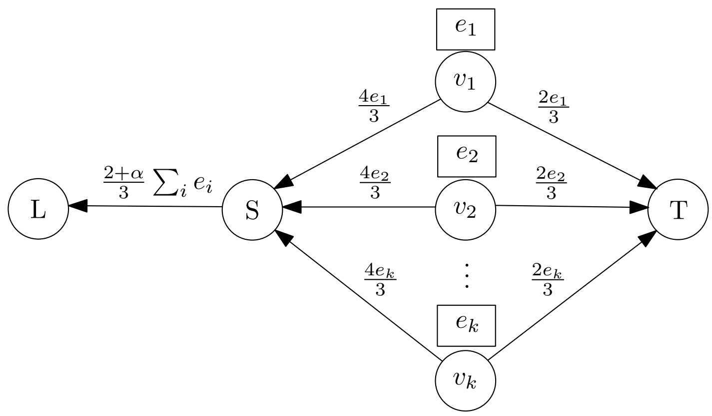


Figure 3.4: The reduction used to show hardness of computing the optimal cash injection policy when $\alpha < 1$.

Proof. We begin with the case where all proportional payments need to be integers, and then prove the cases where default costs apply.

Hardness of computing the optimal cash injection policy under integer payments. We warm up with a rather simple reduction from EXACT COVER BY 3-SETS (X3C), a well-known NP-complete problem. An instance of X3C consists of a set X of $3k$ elements together with a collection C of size-3 subsets of X. The question is whether there exists a subset $C' \subseteq C$ of size k such that each element in X appears exactly once in C'.

We build an instance of our problem as follows. We add an agent u_i for each $c_i \in C$ and an agent t_i for each element $x_i \in X$; we also add agent T. There are no external assets and the liabilities are as follows. Each agent u_i, corresponding to $c_i = \{a, b, c\}$ where a, b, c are elements in X, has liability 1 to each of the three agents t_a, t_b, t_c. Furthermore, each agent t_i has liability of 1 to agent T, while we assume that the budget equals $3k$.

Due to the integrality constraint and the fact that payments are proportional, a yes-instance for X3C, admitting a solution C', leads to a solution of liquidity $6k$, by injecting a payment of 3 to each u_i corresponding to the set $c_i \in C'$.

On the other direction, we argue that any solution with liquidity at least $6k$ leads to a solution for the instance of X3C. Indeed, any such solution must necessarily lead to the liquidity of at least $3k$ to the edges from the u_i agents to the t_i agents. Since the budget equals $3k$ this implies that exactly k of the u_i agents must receive a payment of 3 and these agents should cover the entire set of the t_i agents.

Hardness of computing the optimal cash injection policy with default costs. Our proof follows by a reduction from the PARTITION problem, an NP-complete problem. Recall that in PARTITION, an instance $\mathcal{I}$ consists of a set X of positive integers $\{x_1, x_2, \ldots, x_k\}$ and the question is whether there exists a subset X' of X such that $\sum_{i\in X'} x_i = \sum_{i\notin X'} x_i = \frac{1}{2}\sum_{i\in X} x_i$.

The reduction works as follows. Starting from $\mathcal{I}$, we build an instance $\mathcal{I}'$ by adding an agent v_i for each element $x_i \in X$ and allocating an external asset of $e_i = x_i$ to v_i; we also include three additional agents S, T and L. Each agent v_i has liability equal to $\frac{4e_i}{3}$ to S and equal to $\frac{2e_i}{3}$ to T, while S has liability $\frac{2+\alpha}{3}\sum_i e_i$ to L; see also Figure 3.4. We assume the presence of default costs $\alpha \in [0, 1)$, and $\beta \in [0, 1]$, while the budget is $M = \frac{1}{2}\sum_i e_i$; clearly, the reduction requires polynomial-time.

We first show that if $\mathcal{I}$ is a yes-instance for PARTITION, then the total liquidity is $\mathcal{F} = \frac{5\alpha+10}{6}\sum_i e_i$. Indeed, consider a solution X' for instance $\mathcal{I}$ satisfying $\sum_{i\in X'} x_i = \frac{1}{2}\sum_{i\in X} x_i$, and let the set B contain agents v_i where $x_i \in X'$. Then, since $M = \frac{1}{2}\sum_i e_i = \sum_{i\in B} e_i$, we choose to inject an amount of e_i to any agent $v_i \in B$. The total assets of agent S are $a_S = \frac{4}{3}\sum_{i\in B} e_i + \frac{2\alpha}{3}\sum_{i\notin B} e_i = \frac{2+\alpha}{3}\sum_i e_i$ hence, S is solvent. The total liquidity in this case is $\mathcal{F} = 2\sum_{i\in B} e_i + \alpha\sum_{i\notin B} e_i + \frac{2+\alpha}{3}\sum_i e_i = \frac{5\alpha+10}{6}\sum_i e_i$, as desired.

We now show that any cash injection policy that leads to total liquidity of at least $\frac{5\alpha+10}{6}\sum_i e_i$ leads to a solution for instance $\mathcal{I}$ of PARTITION. Assume any such cash injection policy and let B be the set of v_i agents that become solvent by it. Denote by $\mathcal{F}_v$ the liquidity arising solely from the payments made by the agents v_i, for $i = 1, \ldots, k$, to their direct neighbors, and we denote by t_S the amount of cash injected to S, we get that

$$
\begin{aligned}
\mathcal{F}_v &= 2\sum_{i\in B} e_i + \left(\sum_{i\notin B} e_i + \frac{\sum_i e_i}{2} - \sum_{i\in B} e_i - t_S\right)\cdot\alpha \\
&= 2\sum_{i\in B} e_i + \left(\frac{3}{2}\sum_{i} e_i - 2\sum_{i\in B} e_i - t_S\right)\cdot\alpha \\
&\leq 2(1-\alpha)\sum_{i\in B} e_i + \frac{3\alpha}{2}\sum_{i} e_i. \qquad (3.1)
\end{aligned}
$$

It also holds that

$$
\mathcal{F}_v \geq \left(1+\frac{\alpha}{2}\right)\sum_i e_i \qquad (3.2)
$$

since, by assumption, the policy under consideration leads to a total liquidity of at least $\frac{5\alpha+10}{6}\sum_i e_i$ and the payment from S to L can be at most $\frac{2+\alpha}{3}\sum_i e_i$.

Combining inequalities 3.1 and 3.2, we get that $2(1-\alpha)\sum_{i\in B} e_i + \frac{3\alpha}{2}\sum_i e_i \geq \left(1+\frac{\alpha}{2}\right)\sum_i e_i$, which implies that $2(1-\alpha)\sum_{i\in B} e_i \geq (1-\alpha)\sum_i e_i$ and then $\sum_{i\in B} e_i \geq \frac{1}{2}\sum_i e_i$ for $a<1$.

Moreover, note that each agent $v_i \in B$ needs (at least) an extra e_i to become solvent and, hence, it must be $\sum_{i\in B} e_i \leq \frac{1}{2}\sum_i e_i$, due to the budget constraint. We can conclude that $\sum_{i\in B} e_i = \frac{1}{2}\sum_i e_i$ and, hence, we can obtain a solution to instance $\mathcal{I}$ of PARTITION, as desired.

Hardness of computing the minimum budget that makes an agent solvent. The proof follows by the same reduction as in the previous case. We will prove that computing the minimum budget necessary to make agent S solvent corresponds to solving an instance from PARTITION. As before, whenever instance $\mathcal{I}$ admits a solution X', we inject an amount of e_i to each agent v_i such that $x_i \in X'$, and, as in the previous case, we obtain that a budget of $\frac{1}{2}\sum_i e_i$ suffices to make S solvent. We now argue that any cash injection policy with a budget of $\frac{1}{2}\sum_i e_i$ that can make agent S solvent leads to a solution for instance $\mathcal{I}$ when the default costs are $\alpha \in [0,1/2), \beta \in [0,1]$.

Let t_i be the cash injected at agent v_i and let t_S be the cash injected directly at agent S; clearly, $t_S + \sum_i t_i \leq \frac{\sum_i e_i}{2}$. As before, let B be the set of v_i agents that become solvent by the cash injection policy. Clearly, if $\sum_{i\in B} e_i = \frac{1}{2}\sum_i e_i$, we immediately obtain a solution to the PARTITION instance. Otherwise, $\sum_{i\in B} e_i < \frac{1}{2}\sum_i e_i$ and the total assets of S are

$$
\begin{aligned}
a_S &= \frac{4}{3}\sum_{i\in B} e_i + \frac{2\alpha}{3}\sum_{i\notin B}(e_i+t_i) + t_S \le \frac{4}{3}\sum_{i\in B} e_i + \frac{2\alpha}{3}\sum_{i\notin B} e_i + t_S + \sum_{i\notin B} t_i\\
&= \frac{4}{3}\sum_{i\in B} e_i + \frac{2\alpha}{3}\sum_{i\notin B} e_i + \frac{\sum_i e_i}{2} - \sum_{i\in B} e_i = \frac{1}{3}\sum_{i\in B} e_i + \frac{1}{2}\sum_{i} e_i + \frac{2\alpha}{3}\sum_{i\notin B} e_i\\
&= \frac{1}{3}\sum_{i\in B} e_i + \frac{1}{2}\sum_{i} e_i + \frac{2\alpha}{3}\left(\sum_i e_i - \sum_{i\in B} e_i\right) = \left(\frac{1}{2}+\frac{2\alpha}{3}\right)\sum_i e_i + \frac{1-2\alpha}{3}\sum_{i\in B} e_i\\
&< \left(\frac{1}{2}+\frac{2\alpha}{3}\right)\sum_i e_i + \frac{1-2\alpha}{3}\left(\frac{\sum_i e_i}{2}\right) = \frac{2}{3}\sum_i e_i + \frac{\alpha}{3}\sum_i e_i.
\end{aligned}
\tag{3.3}
$$

where the second equality holds due to the budget constraint and the strict inequality holds since $\alpha < 1/2$ and $\sum_{i\in B} e_i < \frac{1}{2}\sum_i e_i$; the claim follows. □

3.2.2 Optimal debt removals

In this section, we focus on maximizing systemic liquidity by appropriately removing edges/debts. As an example, consider again Figure 3.1, where the central authority can increase systemic liquidity by removing the edge between v_4 and v_5.

Theorem 3.4. *The problem of computing an edge set whose removal maximizes systemic liquidity is NP-hard.*

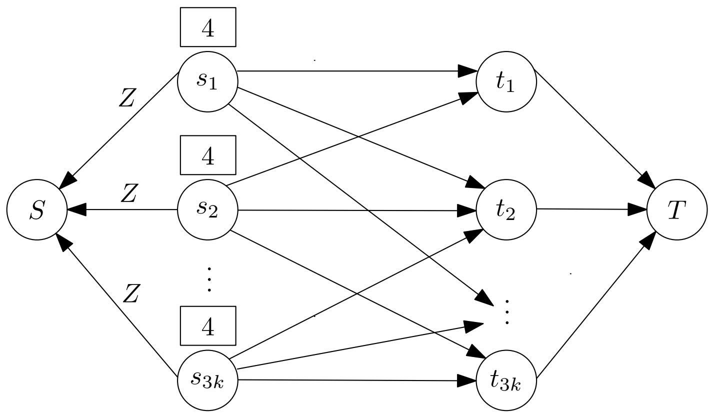


Figure 3.5: The reduction used to show the hardness of computing an edge-removal policy that maximizes systemic liquidity. All edges with missing labels correspond to liability 1.

Proof. The proof relies on a reduction from the NP-complete problem RXC3 [63], a variant of EXACT COVER BY 3-SETS (X3C). In RXC3, we are given an element set X, with $|X| =$

$3k$ for an integer k, and a collection C of subsets of X where each such subset contains exactly three elements. Furthermore, each element in X appears in exactly three subsets in C, that is $|C| = |X| = 3k$. The question is if there exists a subset $C' \subseteq C$ of size k that contains each element of X exactly once. Given an instance $\mathcal{I}$ of RXC3, we construct an instance $\mathcal{I}'$ as follows. We add an agent t_i for each element i of X and an agent s_i for each subset i in C, as well as two additional agents, S and T. Each s_i, corresponding to set $(x, y, z) \in C$, has external assets $e_i = 4$ and liability 1 to the three agents t_x, t_y, t_z corresponding to the three elements $x, y, z \in X$. In addition, each s_i has liability Z to agent S, where Z is a large integer. Finally, each agent t_i has liability 1 to agent T; see also Figure 3.5. Note that this construction requires polynomial time.

When instance $\mathcal{I}$ is a yes-instance with a solution C', we claim that $\mathcal{I}'$ admits a solution with systemic liquidity $14k$. Indeed, it suffices to remove all edges from the s_i agents, with $i \in C'$, towards S. Then, the liquidity due to agents s_i, with $i \in C'$, equals $6k$, while each of the $2k$ agents whose edge towards S was preserved generates a liquidity of 4.

It suffices to show that any solution that generates liquidity at least $14k$ can lead to a solution for instance $\mathcal{I}$. First, observe that it is never strictly better for the financial authority to remove an edge from some agent s_i towards an agent t_j. Let S_k and S_r be the subsets of agents s_i whose edges towards S are kept and removed, respectively, and let $\chi = |S_r|$. The liquidity traveling from agents in S_k towards their direct neighbors is exactly $4(3k-\chi)$, while the liquidity traveling from agents in S_r towards their direct neighbors is exactly 3χ. The maximum liquidity traveling from agents t_j towards T is at most $\min\{3k, 3\chi+(3k-\chi)\frac{12}{Z+3}\}$. We conclude that the maximum liquidity is bounded by $12k-\chi+\min\{3k, 3\chi+(3k-\chi)\frac{12}{Z+3}\}$.

Note that, whenever $\chi > k$, then the maximum liquidity is at bounded by $15k - \chi < 14k$. Similarly, whenever $\chi < k$ and since Z is arbitrarily large, the maximum liquidity is bounded by $12k + 2\chi + (3k - \chi)\frac{12}{Z+3} < 14k$. It remains to show that whenever $\chi = k$, a liquidity of at least $14k$ necessarily leads to a solution in $\mathcal{I}$. Indeed, by the discussion above, any such solution must have liquidity equal to $3k$ traveling from agents t_j towards T, i.e., all these liabilities are fully repaid. This, in turn, can only happen if each of the t_j agents receives a payment of at least 1 from the s_i agents. Using the assumptions that i) $\chi = k$, ii) Z is arbitrarily large, iii) payments are proportional to liabilities, and iv) each t_j has exactly three neighboring s_i agents, this property holds only when the neighbors of the χ agents in S_r are disjoint. This directly translates to a solution for instance $\mathcal{I}$ and the RXC3 problem. □

We note that the objective of systemic solvency, i.e., guaranteeing that all agents are solvent, can be trivially achieved by removing all edges. However, adding a liquidity target, makes this problem more challenging.

Theorem 3.5. *In networks with default costs, the following problems are NP-hard:*

a*) Compute an edge set whose removal ensures systemic solvency and maximizes systemic liquidity.*

b*) Compute an edge set whose removal ensures systemic solvency and minimizes the amount of deleted liabilities.*

c*) Compute an edge set whose removal guarantees that a given agent is no longer in default and minimizes the amount of deleted liabilities.*

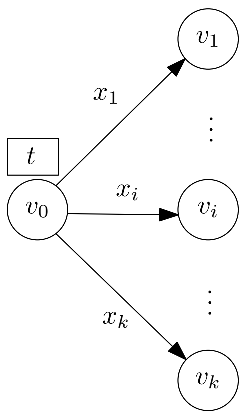


Figure 3.6: An example of the reduction in the proof of Theorem 3.5.

Proof. The proof for all these claims relies on a reduction from the SUBSET SUM problem, where the input consists of a set X of integers $\{x_1, x_2, \ldots, x_k\}$ and a target t and the question is whether there exists a subset $X' \subseteq X$ such that $\sum_{i \in X} x_i = t$. Given an instance $\mathcal{I}$ of SUBSET SUM, we construct an instance $\mathcal{I}'$ by adding an agent v_i for each integer x_i, adding an extra agent v_0 having $e_0 = t$ and setting the liability of v_0 to v_i to be equal to x_i; see also Figure 3.6.

Since v_0 is in default, the goal becomes to remove an edge set so that the remaining liability of v_0 is at most t. Whenever instance $\mathcal{I}$ is a yes-instance for SUBSET SUM admitting a solution X', we remove edges from v_0 to agents corresponding to integers not in X'. Then, v_0 is solvent and the systemic liquidity equals t. Otherwise, if $\mathcal{I}$ is a no-instance, no edge set removal that leaves v_0 solvent can lead to systemic liquidity of (at least) t. □

3.3 Edge-Removal Games

In this section, we consider the case of strategic agents who have the option to forgive debt. Consider a financial network N of n banks who act strategically. The strategy set of a bank is the power set of its incoming edges and a strategy denotes which of its incoming edges that bank will remove, thus erasing the corresponding debt owed to itself. The edge-removal game can be defined with and without cash injections. A given strategy vector will result in realized payments through maximal clearing payments including possible cash injections through a predetermined cash injection policy. Our results hold for both the optimal policy and GREEDY.

A bank is assumed to strategize over its incoming edges in order to maximize its utility, i.e., its total assets, where we remark that a possible cash injection can be seen as increasing one's external assets. The objective of the financial authority is to maximize the total liquidity of the system, i.e., social welfare is the sum of money flows that traverse the network.

We investigate properties of Nash equilibria in the edge-removal game with respect to their existence and quality, while we also address computational complexity questions under different assumptions on whether default costs and/or cash injections apply. Our results on the Effect of Anarchy of edge removal games imply that, rather surprisingly, in the presence of default costs even the worst Nash equilibrium can be arbitrarily better than the original network in terms of liquidity. However, the situation is reversed in the absence of default costs, where we observe that the original network can be considerably better in terms of liquidity than the worst equilibrium; in line with similar Price of Anarchy results. We begin with some results for the basic case, that is, without default costs; recall that we do not refer to default costs in the statements for results holding for $\alpha = \beta = 1$.

Our first result exploits the fact that for edge-removal games without cash injections, the strategy profile where all edges are preserved is a (not necessarily unique) Nash equilibrium.

Theorem 3.6. *Edge-removal games without cash injections always admit Nash equilibria.*

Proof. We claim that it is a Nash equilibrium if no bank removes an incoming edge. Consider a financial network and an arbitrary debt relation in that network represented by an edge e from v_i to v_j. Since there are no default costs, removing e and keeping everything else unchanged, will result in v_i instantly having p_{ij} additional assets to pay to its lenders (excluding v_j). This, however can lead to at most p_{ij} additional assets to reach v_j through indirect paths starting from v_i so it can not lead to more total payments from v_i to v_j.[6] Hence, v_j is better off not removing e. Since v_j is an arbitrary bank and e represents an arbitrary edge, our proof is complete. □

Theorem 3.7. *In edge-removal games without cash injections, the Effect of Anarchy is unbounded and the Effect of Stability is at most* 1.

Proof. The result on the Effect of Stability follows directly from the proof of Theorem 3.6. For the Effect of Anarchy, consider a simple network with two banks, each having a liability of 1 to the other. In the original network, the total liquidity is 2, while it is not hard to see that the network with both edges removed is a Nash equilibrium with a total liquidity of zero. □

Our next result shows that Nash equilibria may not exist once we allow for cash injections.

Theorem 3.8. *There is an edge-removal game with cash injections that does not admit Nash equilibria.*

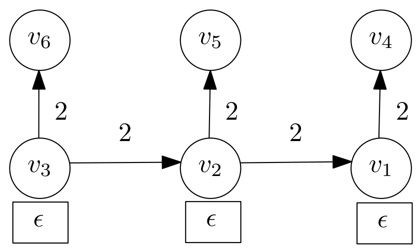


Figure 3.7: An edge-removal game without Nash equilibria in the case without default costs and with budget $M = 2 - 3\epsilon$, where ϵ is an arbitrarily small positive constant.

Proof. Consider the network shown in Figure 3.7, and a budget equal to $M = 2 - 3\epsilon$, where ϵ is an arbitrarily small constant.

[6] This wouldn't be true if additional money was inserted to the network in the form of a cash injection.

We begin by observing that v_4, v_5 and v_6 will never remove their incoming edges, since each of these agents has a single incoming edge, originating from some agent with positive externals. Hence, it suffices to consider the strategic actions of banks v_1 and v_2 regarding the possible removal of edges (v_2, v_1) and (v_3, v_2), respectively. There are four possible cases:

A) *both edges are present.* In this case, v_3 has the highest threat index of $7/4$, thus receiving the entire budget; it is not hard to verify that this is the optimal policy as well. The payments are, then, $p_{32} = 1 - \epsilon, p_{21} = 1/2$, resulting in total assets $a_1 = 1/2 + \epsilon, a_2 = 1$.

B) (v_2, v_1) *is present,* (v_3, v_2) *is removed.* In this case, v_2 receives the budget and the following payments are realized $p_{32} = 0, p_{21} = 1 - \epsilon$. This leads to total assets $a_1 = 1$ and $a_2 = 2 - 2\epsilon$.

C) *both edges are removed.* In this case, there is a tie on the maximum threat index (v_1, v_2 and v_3 have threat index 1), so, assuming that the banks with a lower index are prioritized, v_1 receives all the budget; again, it is not hard to verify that this is an optimal policy as well. This results to payments $p_{32} = 0$ and $p_{21} = 0$ with $a_1 = 2 - 2\epsilon, a_2 = \epsilon$.

D) (v_2, v_1) *is removed, while* (v_3, v_2) *is present.* In this case, v_3 receives the budget. The payments are $p_{32} = 1 - \epsilon$ and $p_{21} = 0$ (due to the removal), which implies $a_1 = \epsilon$ and $a_2 = 1$ respectively.

One can now easily check that the best response dynamics cycle, as starting from Case A, v_2 has an incentive to remove its incoming edge (v_3, v_2) and we reach Case B. Then, v_1 has an incentive to remove its incoming edge (v_2, v_1) (we are now in Case C), which leads v_2 to have an incentive to reinstate its incoming edge (v_3, v_2) (thus, reaching Case D). Finally, in Case D, v_1 has an incentive to reinstate (v_2, v_1), leading to Case A again. $\square$

Theorem 3.9. *The Price of Stability in edge-removal games (with or without cash injections) is unbounded.*

Proof. We present the proof for the case without cash injections and note that the proof carries over for the case of a limited budget regardless of who receives it. Moreover, note

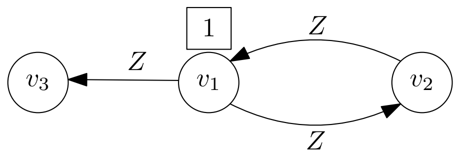


Figure 3.8: A financial network with no default costs that admits unbounded Price of Stability.

that the proof does not assume default costs but the result immediately applies to that (more general) case as well, i.e., $\alpha = \beta = 1$ is a special case of default costs.

Consider the network shown in Figure 3.8, where Z is an arbitrarily large constant. Since each bank has exactly one incoming edge, no edge removals occur at the unique Nash equilibrium. By the proportionality principle it holds that the three payments have to be equal (to 1), that is $p_{13} = p_{12} = p_{21} = 1$, which results in the systemic liquidity of 3. However, systemic liquidity of $2Z$ can be achieved when v_3 removes its incoming edge. Indeed, each remaining payment can be equal to Z. We conclude that PoS $\geq 2Z/3$, so the Price of Stability can be arbitrarily large for appropriately large values of Z, as desired. □

Theorem 3.10. *The Effect of Anarchy in edge-removal games with cash injections is at least* $n - 1$.

Proof. Consider the network shown in Figure 3.9 and assume a budget $M = 1$. If there are

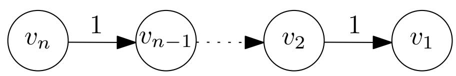


Figure 3.9: A network that yields EoA $\geq n - 1$ for budget $M = 1$.

no edge removals, then bank v_n will receive the entire budget, thus achieving total liquidity equal to $n - 1$. However, we claim that the state where every bank except v_1 removes their incoming edge is an equilibrium with total liquidity 1. Indeed, in this case v_2 receives the budget and this leads to total assets $a_1 = a_2 = 1$. Under the assumption that ties are broken in favor of the lowest index, the only edge addition that would change the recipient of the cash injection is if v_2 decides to reinstate edge (v_3, v_2), however, this will lead to exactly the same total assets for v_2 since v_3 will receive the budget. Clearly, v_1 does not have an incentive to remove its incoming edge. The proof is complete. □

Theorem 3.11. *In edge-removal games with cash injections, there exists a network such that the social welfare of the initial state is $\Omega(n)$ times greater than that of even the best equilibrium.*

Proof. Consider the network in Figure 3.10 where the budget $M = 1$, $k = n/2$ and H is arbitrarily larger than k. We start by noticing that $\mu_1 = 1$, while for $i = 2, \ldots, k$, it holds that

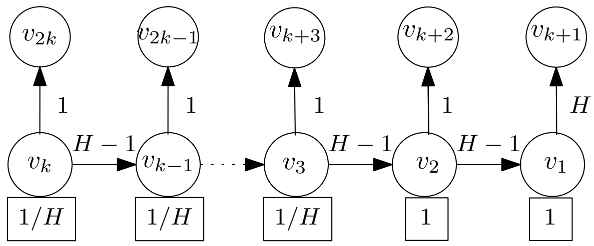


Figure 3.10: A network that yields EoS $= \Omega(n)$ for budget $M = 1$, $n = 2k$, and arbitrarily large H.

$\mu_i = 1 + \frac{H-1}{H}\mu_{i-1} \approx 1 + \mu_{i-1}$, for sufficiently large H; all other banks are solvent. Hence, the optimal total liquidity is achieved when v_k receives the entire budget of $M = 1$ as a cash injection, and is roughly $kM = n/2$, when H is sufficiently large.

We now claim that under any Nash equilibrium, v_2 will receive the budget and all edges (v_i, v_{i-1}) for $i \in \{3, \ldots, k\}$ are removed. This would complete the proof, as the total liquidity would be at most $\frac{1}{H}(k-2) + 2 + 2 \leq 5$. We now prove this claim. Consider any equilibrium and observe that v_i, for $i = k+1, \ldots, 2k$, must have their unique incoming edge present. Now, assume for a contradiction that some bank v_i with $i \in \{3, \ldots, k\}$ gets a cash injection; this implies that the edge (v_i, v_{i-1}) is present as, otherwise, the result holds trivially. Then, bank v_{i-1} has total assets $1 - 1/H^2 + e_{i-1}$, but can increase them to $1 + e_{i-1}$ by strategically removing its incoming edge. So, under any Nash equilibrium, either v_2 or v_1 receives a cash injection. In the former case, where edge (v_2, v_1) is present, $a_1 = 3 - 2/H$, while the assets of v_1 would be 2 if it removed its incoming edge and received the cash injection.

It remains to show no other edge (v_i, v_{i-1}) for $i \in \{3, \ldots, k\}$ exists in a Nash equilibrium. Now, observe that if such an edge exists, then neighboring edges on the horizontal path cannot exist as that would contradict that v_2 gets the cash injection. Then, when $i > 4$, bank v_{i-2} would have an incentive to add edge (v_{i-1}, v_{i-2}), thus, making bank v_i the recipient of the budget (for both optimal and greedy) and strictly increase its own total assets. The cases $i \in \{3, 4\}$ can be easily ruled out as well. Our proof is complete. □

We now present a series of results for the case where default costs exist, but cash injections are not allowed. Contrary to the case with neither default costs nor cash injections, we show that a Nash equilibrium is not guaranteed to exist; the next result is complementary to Theorem 3.8.

Theorem 3.12. *There is an edge-removal game with default costs but without cash injections that does not admit Nash equilibria.*

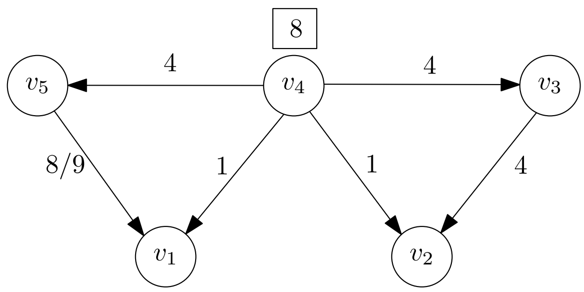


Figure 3.11: An edge-removal game without Nash equilibria, when $\alpha = \beta = 1/4$.

Proof. We present the proof for the case without cash injections and note that the proof carries over for the case of a limited budget regardless of who receives it, as a sufficiently small budget will not alter the players' incentives. Consider the financial network in Figure 3.11 and assume that default costs $\alpha = \beta = 1/4$ are applied.

We begin by claiming that v_5 (and, by symmetry v_3) will never remove its incoming edge. Indeed, since v_4 has positive external assets, then v_5 will receive a positive payment from v_4 if the edge between them remains, however v_5 will have zero incoming payments if the corresponding edge is removed. Similarly, v_1 (and, by symmetry v_2) will never remove the edge from v_5 (respectively v_3). Therefore, it suffices to consider the possible removal of edges (v_4, v_1) and (v_4, v_2). We will prove that none of the following states are at equilibrium: A) no edge is removed, B) (v_4, v_1) is removed but (v_4, v_2) remains, C) both edges are removed, and D) (v_4, v_1) remains but (v_4, v_2) is removed.

Note that in Case A v_4 is in default, hence, due to the default costs, its payments are broken down as $\mathbf{p}_4 = (1/5, 1/5, 4/5, 0, 4/5)$. But then, v_3 and v_5 are also in default, and

$p_{51} = p_{32} = 1/5$ in this case. So, the utility of both v_1 and v_2 is $2/5$. By removing (v_4, v_1) in Case A, and moving to Case B, v_1 would increase its utility to $8/9$. Indeed, in this case, $\mathbf{p}_4 = (0, 2/9, 8/9, 0, 8/9)$, while $p_{51} = 8/9$ and $p_{32} = 2/9$, so the utility of v_1 is $8/9$ and the utility of v_2 is $4/9$. But, then, it is beneficial for v_2 to remove (v_4, v_2). In this case (Case C), v_4 is now solvent and all existing debts are paid for, thus giving utility $8/9$ to v_1 and utility 4 to v_2. However, if v_1 then decides to reinstate its incoming edge (v_4, v_1), its utility in this case (Case D) is increased to $10/9$, while v_2's utility is $2/9$, since $\mathbf{p}_4 = (2/9, 0, 8/9, 0, 8/9)$, $p_{51} = 8/9$ and $p_{32} = 2/9$. Finally, v_2 prefers its utility in Case A to its utility in Case D so will decide to reinstate (v_4, v_2) when in Case D, thus defining a cycle between the four possible states. $\square$

For some restricted topologies, however, the existence of Nash equilibria is guaranteed; in particular, keeping all edges is a Nash equilibrium.

Theorem 3.13. *Edge-removal games with default costs but without cash injections always admit Nash equilibria if the financial network is a tree or a cycle.*

Proof. We present the proof for the case without cash injections and note that the proof carries over for the case of a limited budget regardless of who receives it. Consider a financial network that is a tree (similar argument works in case of a directed cycle) and an arbitrary debt relation in that network represented by a directed edge (v_i, v_j). By definition of the tree structure, it holds that there is no other path in the network from v_i to v_j. Hence, v_j cannot benefit by removing that edge, even if the removal affects (increases) v_i's available assets. $\square$

The following result demonstrates that the positive impact of (individually benefiting) edge removals dominates the negative impact of reducing the number of edges through which money can flow, hence, edge removals are in line with the regulator's best interest too.

Lemma 3.14. *Edge-removal games with default costs but no cash injections satisfy the following: given any network and any strategy profile, any unilateral removal of any edge(s) that weakly improves the total assets of the corresponding bank, also weakly improves the total assets of every other bank in the network. As a result, the total liquidity of the system is increased.*

Proof. Consider a network $N = (V, E)$ and a strategy profile $\mathrm{s} = (s_1, \ldots, s_n)$, under which banks have total assets according to $\mathbf{a} = (a_1, \ldots, a_n)$. Fix a bank i and let $\mathrm{s}' = (s'_i, \mathrm{s}_{-i})$ be the strategy profile that is derived by s if bank i changes its strategy from s_i to s'_i, where s'_i is derived by s_i by the removal of an edge $e = (v_j, v_i)$ (the argument can be applied repeatedly to prove the claim for more than one edge removals). By assumption, the total assets of bank i under s$'$, a'_i, satisfy $a'_i \geq a_i$. It holds that any bank reachable by i or j (the two endpoints of the edge that was removed) through a directed path will have at least the same total assets under s$'$ than with s, since there will be at least the same amount of money available to leave i and j and traverse these paths. The assets of banks not reachable by i or j will, clearly, not be affected by the removal of e. Hence, the assets of each bank in N are weakly higher under s$'$ than under s. The increase in the total liquidity follows since the total assets, by definition, equal external assets plus payments. □

In fact, the systemic liquidity of even the worst Nash equilibrium can be arbitrarily higher than at the original network. To see this, consider the network in the proof of Theorem 3.15, which admits a unique Nash equilibrium with arbitrarily higher total liquidity than that of the original network.

Theorem 3.15. *When default costs apply but there are no cash injections, the Effect of Stability is arbitrarily close to* 0*.*

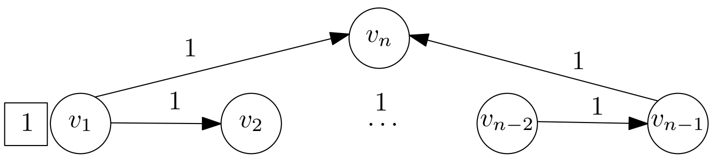


Figure 3.12: A financial network with $\alpha = \beta = \epsilon$, for an arbitrarily small positive ϵ, that admits effect of stability close to 0.

Proof. Consider the network in Figure 3.12 where default costs are $\alpha = \beta = \epsilon$ for some arbitrary small positive constant ϵ. If no edge is removed, then all banks except v_n are in default and the following payments are realized: $p_{12} = p_{1n} = \epsilon/2$ and $p_{i,i+1} = \epsilon \cdot p_{i-1,i} = \epsilon^i/2$. The systemic liquidity is then $\mathcal{F}_N = \epsilon/2 + \sum_{i=1}^{n-1} \frac{\epsilon^i}{2} < \frac{\epsilon}{1-\epsilon}$.

On the other hand, the unique Nash equilibrium is achieved when v_n removes the edge pointing from v_1 to itself. The systemic liquidity in this case is $n - 1$, and the proof follows.

□

We conclude with our results on computational complexity for the setting with default costs.

Theorem 3.16. *In edge-removal games with default costs, the following problems are NP-hard:*

a) *Decide whether a Nash equilibrium exists or not.*

b) *Compute a Nash equilibrium, when it is guaranteed to exist.*

c) *Compute a best-response strategy.*

d) *Compute a strategy profile that maximizes systemic liquidity.*

Proof. We begin by proving that the problem of computing a Nash equilibrium (even when its existence is guaranteed) is NP-hard.

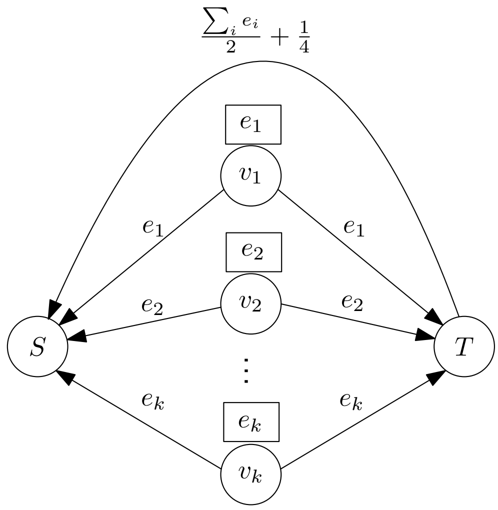


Figure 3.13: The instance arising from the reduction used in Theorem 3.16b.

Hardness of computing a Nash equilibrium. Our proof follows by a reduction from the PARTITION problem. Recall that in PARTITION, an instance $\mathcal{I}$ consists of a set X of positive integers $\{x_1, x_2, \dots, x_k\}$ and the question is whether there exists a subset X' of X such that $\sum_{i \in X'} x_i = \sum_{i \notin X'} x_i = \frac{\sum_{i \in X} x_i}{2}$.

The reduction works as follows. Starting from $\mathcal{I}$, we build an instance $\mathcal{I}'$ by adding an agent v_i for each element $x_i \in X$ and allocating an external asset of $e_i = x_i$ to v_i; we also include two additional agents S and T. Each agent v_i has liability equal to e_i to each of S and T, while T has liability $\frac{\sum_i e_i}{2} + \frac{1}{4}$ to S; see also Figure 3.13. We furthermore set default costs $\alpha = \beta = 1/\sum_i e_i$; clearly, the reduction requires polynomial time.

Observe that, in any Nash equilibrium, agent T keeps all its incoming edges. Indeed, removing an edge from agent v_i will decrease T's total assets, as there is no alternative path for payments originating at v_i to reach T. Similarly, agent S keeps its incoming edge from T at any Nash equilibrium, as deleting it will reduce S's total assets. Therefore, the only strategic choice in this financial network is by agent S about which edges from agents v_i to keep and which to remove. We denote by S_k and S_r the set of agents whose edges towards S are kept and removed, respectively, and observe that agents in S_k are in default while agents in S_r are not. Clearly, as S is essentially the only strategic agent, any best-response strategy by S forms a Nash equilibrium. This guarantees the existence of a Nash equilibrium in $\mathcal{I}'$. We will show that in instance $\mathcal{I}'$, agent S can compute her best response, and hence we can compute a Nash equilibrium, if and only if instance $\mathcal{I}$ of PARTITION is a yes-instance.

We first show that if $\mathcal{I}$ is a yes-instance for PARTITION, then agent S has total assets $a_S = \frac{\sum_i e_i}{2} + \frac{1}{2}$. Indeed, consider the subset X' in $\mathcal{I}$ with $\sum_{i \in X'} x_i = \frac{\sum_{i \in X} x_i}{2}$ and let S_k contain agents v_i where $x_i \in X'$, while S_r agents v_i with $x_i \notin X'$. Note that T obtains total assets

$$\begin{aligned} a_T &= \sum_{i \in S_r} e_i + \frac{1}{\sum_i e_i} \sum_{i \in S_k} \frac{e_i}{2} \\ &= \frac{\sum_i e_i}{2} + \frac{1}{4}, \end{aligned}$$

and is therefore solvent, while S obtains

$$\begin{aligned} a_S &= \frac{1}{\sum_i e_i} \sum_{i \in S_k} \frac{e_i}{2} + \frac{\sum_i e_i}{2} + \frac{1}{4} \\ &= \frac{\sum_i e_i}{2} + \frac{1}{2}. \end{aligned}$$

We will now show that if $\mathcal{I}$ is a no-instance, then S's total assets in $\mathcal{I}'$ are strictly less than $\frac{\sum_i e_i}{2} + \frac{1}{2}$; this suffices to prove the claim. Consider any subset $X' \subseteq X$ in $\mathcal{I}$ and the corresponding strategy profile in $\mathcal{I}'$ where S keeps incoming edges from agents in S_k while removes edges from agents in S_r. Let $\chi = \sum_{i \in S_r} e_i$ and observe that, as elements in X are integers, it holds either $\chi \leq \frac{\sum_i e_i}{2} - \frac{1}{2}$ or $\chi \geq \frac{\sum_i e_i}{2} + \frac{1}{2}$.

In the first case, when $\chi \leq \frac{\sum_i e_i}{2} - \frac{1}{2}$, we claim that T is in default. Indeed, T collects a total payment of χ from the agents in S_r and a total payment of $\frac{1}{\sum_i e_i} \sum_{i \in S_k} \frac{e_i}{2} = \frac{1}{\sum_i e_i} \frac{\sum_i e_i - \chi}{2} = \frac{1}{2} - \frac{\chi}{2\sum_i e_i}$ from the agents in S_k. So,

$$
\begin{aligned}
a_T &= \chi + \frac{1}{2} - \frac{\chi}{2\sum_i e_i} \\
&= \chi(1 - \frac{1}{2\sum_i e_i}) + \frac{1}{2} \\
&\leq \left(\frac{\sum_i e_i}{2} - \frac{1}{2}\right)\left(1 - \frac{1}{2\sum_i e_i}\right) + \frac{1}{2} \\
&< \frac{\sum_i e_i}{2},
\end{aligned}
$$

i.e., less than T's liability to S; the first inequality follows by the assumption on χ. Therefore, S can collect $\frac{1}{\sum_i e_i} \frac{\sum_i e_i - \chi}{2}$ from the agents in S_k and strictly less than $\frac{1}{2}$ from T. We conclude that $a_S < \frac{\sum_i e_i}{2} + \frac{1}{2}$ in this case.

In the second case, when $\chi \geq \frac{\sum_i e_i}{2} + \frac{1}{2}$, S obtains a total payment of $\frac{1}{\sum_i e_i} \sum_{i \in S_k} \frac{e_i}{2} = \frac{1}{2\sum_i e_i}(\sum_i e_i - \chi) \leq \frac{1}{2\sum_i e_i}(\frac{\sum_i e_i}{2} - \frac{1}{2}) = \frac{1}{4} - \frac{1}{4\sum_i e_i}$ from agents in S_k and a payment of at most $\frac{\sum_i e_i}{2} + \frac{1}{4}$ from T, i.e., $a_S < \frac{\sum_i e_i}{2} + \frac{1}{2}$ again.

Since in any case, a no-instance $\mathcal{I}$ for PARTITION leads to an instance $\mathcal{I}'$ where $a_S < \frac{\sum_i e_i}{2} + \frac{1}{2}$, the claim follows, as we cannot compute a best-response strategy for S, and, by the discussion above, a Nash equilibrium.

Hardness of computing a best-response strategy. The proof was given as in the previous case.

Hardness of maximizing systemic liquidity. The proof follows by the reduction from PARTITION described in the previous case by adding a path of ℓ agents as shown in Figure 3.14. By the discussion above, starting from a yes-instance in PARTITION, we have $a_S = \frac{\sum_i e_i}{2} + \frac{1}{2}$, and S as well as any agent u_i are solvent. On the contrary, starting from a no-instance for PARTITION, agent S is in default and the payments traveling to the u_i agents get reduced by a factor of $\sum_i e_i$ at each edge. By selecting ℓ to be large enough, the claim follows.

Hardness of deciding the existence of Nash equilibria. Again, the proof follows by the reduction from PARTITION used in Theorem 3.16b by adding five agents, ℓ, m, r, x and y with liabilities and external assets are shown in Figure 3.15. Recall that the default costs are $\alpha = \beta = \frac{1}{\sum_i e_i}$ and, without loss of generality, we assume that $\sum_i e_i \geq 3$.

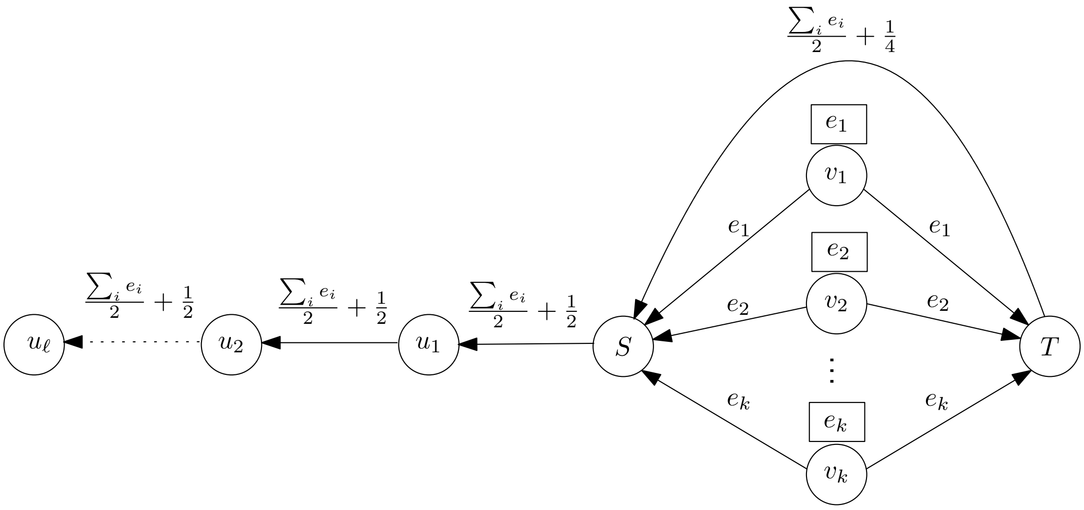


Figure 3.14: The modified instance in the proof of Theorem 3.16d.

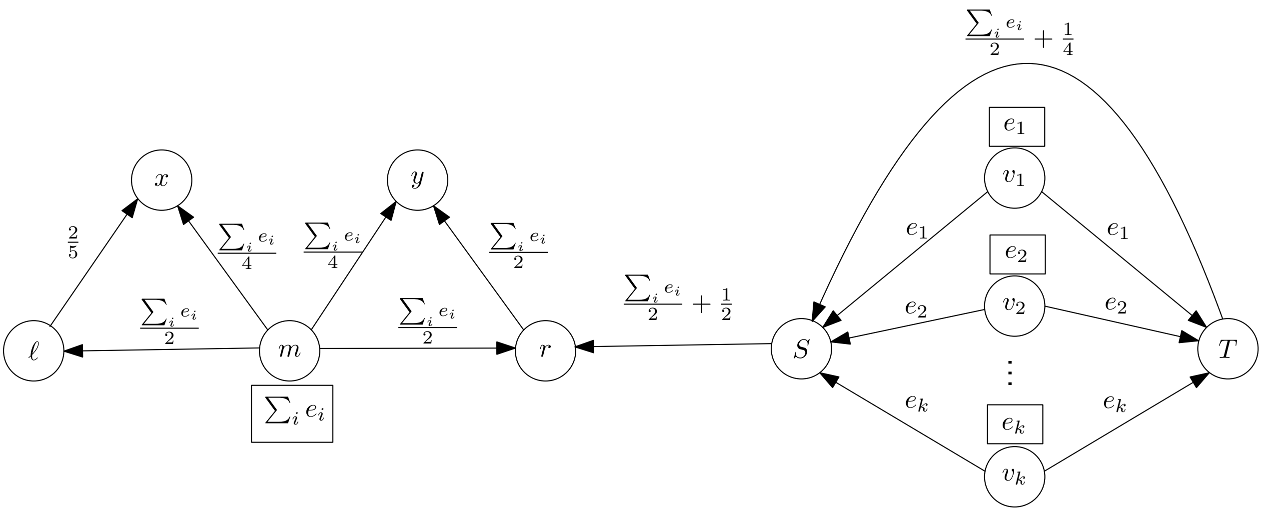


Figure 3.15: The instance used in the proof of Theorem 3.16a.

As argued earlier, T always keeps all its incoming edges, while S always keeps the edge from T. Similarly, agents ℓ and r always keep their incoming edges, as by removing any incoming edge their total assets strictly decrease. As agents v_i and m do not have incoming edges, the only strategic agents are x, y (with respect to the edges originating from m) and S (with respect to edges from the v_i agents). We first show that, if instance $\mathcal{I}$ of PARTITION is a yes-instance, then there is a Nash equilibrium. Indeed, we argued earlier that in this case $a_S = \frac{\sum_i e_i}{2} + \frac{1}{2}$ and, hence, S is solvent. So, r is solvent as well and, therefore, y always keeps the edge from m as the liability from r is always fully paid. This implies that m is necessarily in default and agent x should keep the edge from m as well. To conclude, the strategy profile where x, y keep all edges is a Nash equilibrium.

On the other hand, when instance $\mathcal{I}$ is a no-instance, we have shown that S is in default. Then, r collects a payment of $\chi \leq \frac{1}{2} + \frac{1}{2\sum_i e_i}$ from S and is, hence, necessarily in default. When both x and y keep their edges from m, their total assets are $a_x = \frac{1}{3\sum_i e_i} + \frac{1}{6}$ and $a_y = \frac{1}{6} + (\frac{1}{3} + \chi)\frac{1}{\sum_i e_i}$. When x removes the edge and y keeps it, it is $a_x = \frac{2}{5}$ and $a_y = \frac{1}{5} + (\frac{2}{5} + \chi)\frac{1}{\sum_i e_i}$, i.e., x improves compared to the previous case. When they both remove these edges, we have $a_x = \frac{2}{5}$ and $a_y = \frac{\sum_i e_i}{2}$, i.e., y improves compared to the previous case. Similarly, when x keeps the edge and y removes it, the total assets are $a_x = \frac{3}{5}$ and $a_y = (\frac{2}{5} + \chi)\frac{1}{\sum_i e_i}$, i.e., x improves compared to the previous case. The claim follows by observing that, in the last case, y improves by keeping the edge. $\square$

3.4 Conclusions

We considered problems arising in financial networks when a financial authority wishes to maximize the total liquidity in the financial network, either by injecting cash or by removing debt and we proved that most of the related optimization problems are computationally hard. We also studied the setting where banks are self-interested strategic agents that might prefer to forgive some debt contract if this leads to greater utility, and we analyzed the corresponding games with respect to the properties of Nash equilibria. In that context, we also introduced the notion of the Effect of Anarchy (Effect of Stability, respectively) that compares the liquidity in the initial financial network to that of the worst (best, respectively) Nash equilibrium.

Chapter 4

Debt Transfers in Debt-Only Networks

Loan assignments are means by which a lender can transfer its interest in a loan to another lender. For example, if someone has a right, e.g., to claim damages, against someone else, they can transfer that right to a third party. In this chapter, we consider loan assignments used to cancel out other debts and study their potential to improve the well-being of financial systems.

In a financial network, we use the term *debt transfer* to refer to an operation where a bank can choose to transfer the right it has to claim a debt to one of its lenders, if that would alleviate its own debt to that lender. In particular, if a bank B is owed a certain amount and at the same time B owes the same amount to some other bank, then B can decide to replace these two loans with a single loan from its borrower to its lender. Loan assignments are governed by law and have been considered extensively in the legal literature (see e.g., [1, 33]). To the best of our knowledge, loan assignments have not been considered in the scientific literature. Motivating examples can be encountered commonly in the retail/supply chain market where purchasing with credits is very common: if A buys something from B with credit and then B buys something from C with credit, then these purchasing activities can result in a single payment obligation from A to C (assuming that both purchases have the same monetary value). Another example is using vouchers to repay a debt: Consider for example an Amazon voucher as a right to claim products of a certain amount from Amazon. In other words, an Amazon voucher can be thought of as a debt obligation from Amazon to some party B that has purchased the voucher. Now if B has another debt towards some third party C, they can repay that debt by offering the voucher, i.e., B transfers to their lender C, their right to claim from Amazon.

There are several cases where debt transfers can be beneficial to the corresponding bank. This might seem counterintuitive, as a debt transfer directly reduces a bank's income and, at best, the bank will just write off an equivalent debt. It is, however, true that transferring one's debt can improve the well-being of the entire network, which may lead to fewer banks in default incurring associated costs. This may lead to an increased cash flow through the network that will benefit many banks, including the one that made the original debt transfer. In this paper, we aim to get a better understanding of the potential of debt transfer operations towards improving the well-being of a financial system.

Debt transfer operations can be meaningful from a regulator's perspective who can potentially enforce them in order to achieve some objectives. In this context the goal would be to compute a collection of debt transfers that achieve certain objectives related to the financial well-being of the system. Natural metrics of the system's financial well-being include the sum of total assets (equivalent to sum of total payments also known as total liquidity) and total equity, where the equity of a bank reveals the assets that it has available after making payments, if any (and is equal to zero for banks in default). Debt transfer operations can also be meaningful from a game theoretic perspective as banks can be strategic about transferring their debt claims. Our work considers both the centralized and distributed approach and performs a theoretical and empirical analysis of related questions.

4.1 The Model and Definitions

Recall that, following the preliminaries in Chapter 2, once a bank in default, then it can only use an α fraction of its external assets and a β fraction of its incoming payments due to *default costs*. However, in order to reduce the number of hyper-parameters and then simplify the model, we only focus on the case where $\alpha = \beta$ in this chapter, which is also a common assumption in literature (see, e.g., [97, 47, 45]). Therefore, under the principle of absolute priority, limited liability as well as proportionality, the clearing payments must satisfy $\mathbf{P} = \Phi(\mathbf{P})$, where

$$\Phi(\mathbf{x})_{ij} = \begin{cases} l_{ij}, & \text{if } L_i \leq e_i + \sum_{j=1}^{n} x_{ji} \\ \alpha \cdot \left(e_i + \sum_{j=1}^{n} x_{ji}\right) \cdot \pi_{ij}, & \text{otherwise.} \end{cases} \tag{4.1}$$

4.1.1 Debt transfer

A *debt transfer* $<v_j, v_i, v_k>$ is an operation, involving three banks, v_i (the broker), v_j (the borrower), and v_k (the lender) with $l_{ji} = l_{ik}$, where liabilities from v_j to v_i and from v_i to v_k are replaced by a single liability (of equal claim) from v_j to v_k.

4.1.2 An example

An example is presented in Figure 4.1. Let $\alpha = 1$, observe that $l_{12} = l_{23}$ and consider the financial network arising when these liabilities are replaced by a new one between v_1 and v_3. The clearing payments in the initial network (before the debt transfer) are $p_{12} = 1$, $p_{23} = 2$, $p_{24} = 1$, $p_{42} = 1$ with total assets $a_1 = 1$, $a_2 = 3$, $a_3 = 2$, $a_4 = 1$ and equities $E_1 = E_2 = E_4 = 0$, and $E_3 = 2$. After the debt transfer, we have $p'_{13} = 1$, $p'_{24} = 4$, $p'_{42} = 7/2$ with $a'_1 = 1$, $a'_2 = 9/2$, $a'_3 = 1$, $a'_4 = 4$ and $E'_1 = 0$, $E'_2 = 1/2$, $E'_3 = 1$ and $E'_4 = 1/2$. We conclude that v_2 is better off after the debt transfer in terms of both total assets and equity.

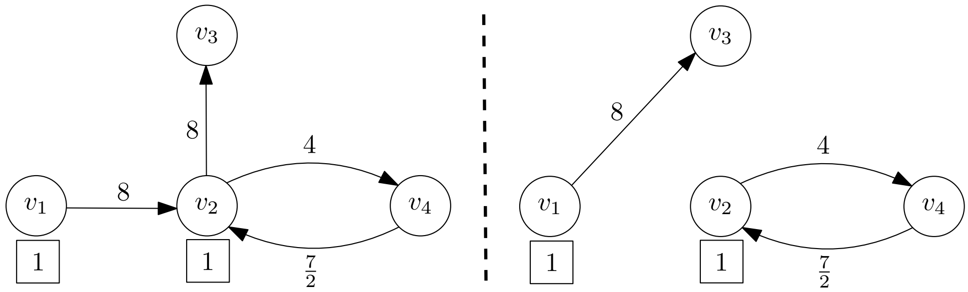


Figure 4.1: The left subfigure shows the initial network, while the right subfigure shows the network after the debt transfer by v_2. Nodes correspond to banks, edges are labeled with the respective liabilities, while external assets appear in a rectangle near the relevant bank.

Note that a debt transfer $<v_j, v_i, v_k>$ either creates a new liability between v_j and v_k or increases the existing liability. The latter might lead to new possible debt transfers involving v_j, v_k and another bank, where v_k would now be the broker.

4.2 Computing Optimal Debt Transfers

In this section, we study how a financial authority can exploit debt transfers to affect financial networks. In particular, we are interested in how a suitable collection of debt transfers can lead to *systemic solvency* (i.e., all banks are solvent), or to an increased total liquidity.

We begin with the objective of achieving systemic solvency. Although a series of debt transfers could reduce significantly the number of banks in default (see Figure 4.2), our first result states that the financial authority cannot use debt transfers to transform a financial network, with at least one bank in default, so that it becomes systemic solvent.

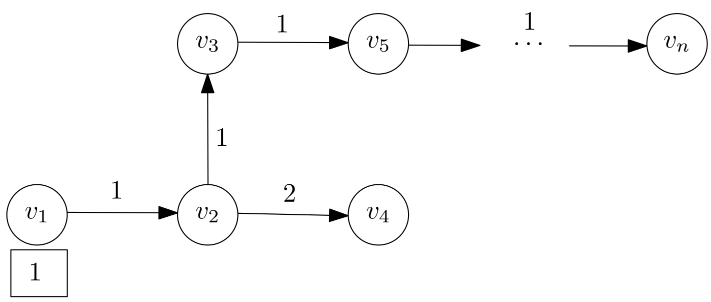


Figure 4.2: The number of defaulted banks is reduced from $n-2$ to 1 after v_2's debt transfer

Theorem 4.1. *A financial network with at least one bank in default cannot be made systemic solvent by debt transfers.*

Proof. Consider a network N with at least one bank in default, and let N' be the network that arises from N by transferring some debts. We will reach a contradiction by proving that, if N' is systemic solvent, so must N. To do so, we start reversing one-by-one the debt transfers that led to network N' and argue that, given that the network before reversing a debt transfer had no banks in default, so must the network arising after reversing the debt transfer.

Let v_i, v_j and v_k be the banks involved in the debt transfer whose reversal we are considering, so that v_i transferred its debt claim from v_j to v_k; note that this implies $l_{ji} = l_{ik}$. Since the network is systemic solvent before reversing the debt transfer, these three banks are solvent as well and, hence, v_j fully paid its liabilities to v_k. We first consider the network arising by increasing v_j's external assets by l_{ji} and adding liabilities of l_{ji} from v_j to v_i as well as from v_i to v_k. It is not hard to see that the network remains systemic solvent after this transformation. Finally, we remove the liability of v_j to v_k together with the additional external assets given to v_j in the previous step; see also Figure 4.3 for an example of this process.

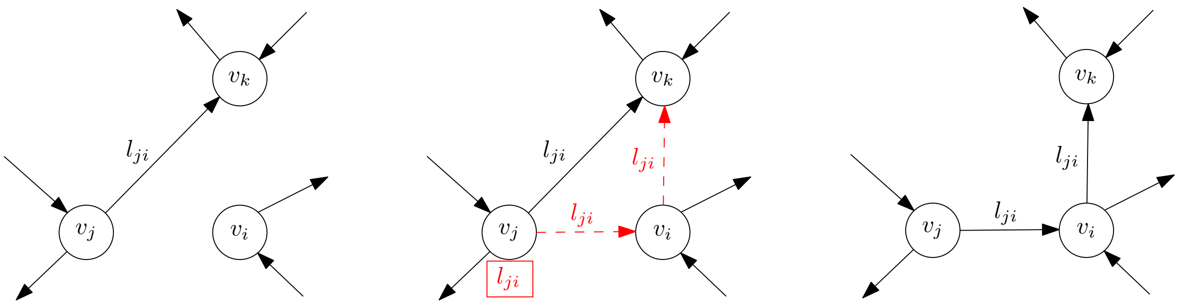


Figure 4.3: The leftmost sub-figure corresponds to the network before reversing a debt transfer, the middle sub-figure corresponds to the network after including external assets and liabilities, while the rightmost corresponds to the network after reversing the debt transfer.

Again, the network remains solvent and we have reversed the debt transfer involving v_i, v_j, and v_k.

By repeatedly applying this transformation to all debt transfers that led from N to N', we obtain that N must be systemic solvent; a contradiction. The proof is complete. □

We now focus on increasing total liquidity, i.e., the sum of clearing payments, and prove that computing an optimal collection of debt transfers is NP-hard, even when there are effectively no default costs. Note that this implies the hardness of maximizing the sum of total assets as well, as the latter equals total liquidity plus the (fixed) sum of external assets.

Theorem 4.2. *In networks without default costs, i.e., $\alpha = 1$, computing a collection of debt transfers that maximizes total liquidity is NP-hard.*

Proof. The proof relies on a reduction from the NP-complete problem Restricted Exact Cover by 3-Sets (RXC3) [63], a variant of Exact Cover by 3-Sets (X3C). In RXC3, we are given an element set X, with $|X| = 3k$ for an integer k, and a collection C of subsets of X where each such subset contains exactly three elements. Furthermore, each element in X appears in exactly three subsets in C, that is $|C| = |X| = 3k$. The question is if there exists a subset $C' \subseteq C$ of size k that contains each element of X exactly once.

Given an instance $\mathcal{I}$ of RXC3, we construct an instance $\mathcal{I}'$ as follows. We add bank t_i for each element i of X, banks v_i, v_i' and v_i'' for each subset i in C, as well as another bank T. Each bank v_i, corresponding to set $(x, y, z) \in C$, has external assets $e_i = 3$ and liability 1 to each of the three banks t_x, t_y, and t_z corresponding to the three elements $x, y, z \in X$,

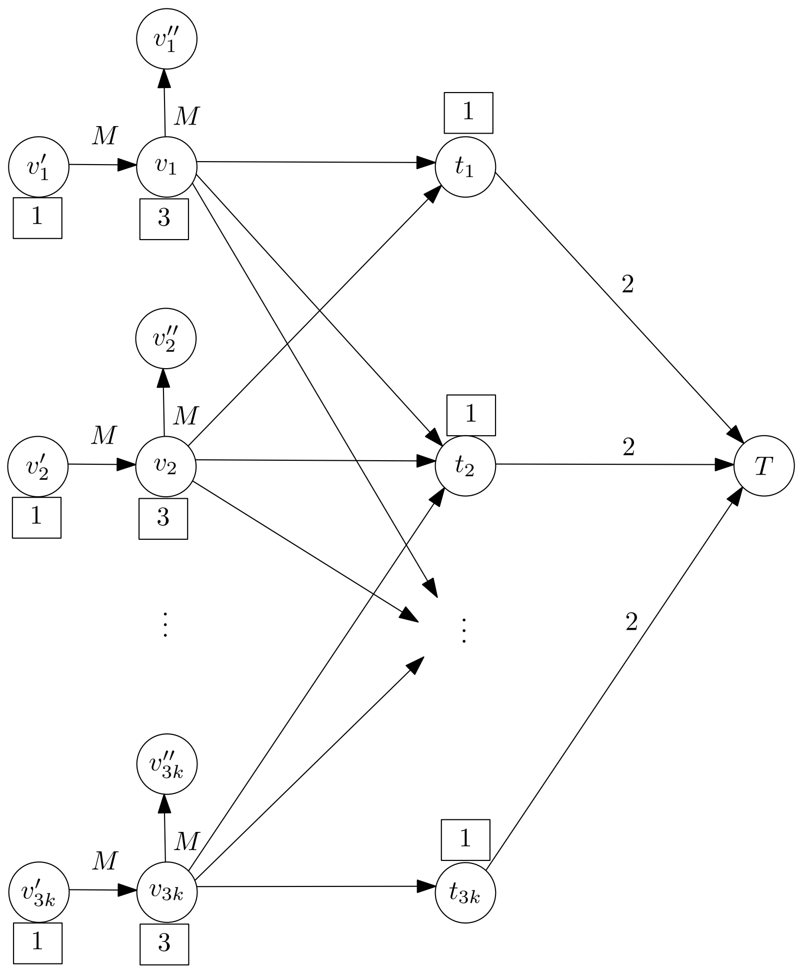


Figure 4.4: The reduction used in the proof of Theorem 4.2. All edges with missing labels correspond to liability 1.

as well as liability M to v_i'', where M is an arbitrarily large number. Furthermore, each v_i' has an external asset of 1 and liability of M to v_i, while all t_i's have liability 1 to T; see also Figure 4.4. Note that this construction requires polynomial time.

We first argue that, when systemic liquidity is maximized, no t_i, for $i \in \{1, 2, \dots, k\}$, makes any debt transfers, as keeping its own debt claim unchanged is weakly better in terms of systemic liquidity. Next, we show that the maximal systemic liquidity of $20k$ can be achieved if and only if instance $\mathcal{I}$ is a 'yes'-instance for problem RXC3.

Let instance $\mathcal{I}$ be a 'yes'-instance for RXC3 and let C' be the solution to $\mathcal{I}$. We claim that $\mathcal{I}'$ admits a solution with systemic liquidity $20k$. Indeed, it suffices to let all v_i's with $i \in C'$ make the debt transfer from v_i' to v_i'', while all other v_i's keep their debt claims unchanged. This choice makes each edge (t_i, T) for $i = \{1, 2, \dots, 3k\}$ saturated with the following payments. We have $\sum_{i \in C'} p_{v_i', v_i''} = k$ due to the debt transfers, while each v_i, with $i = (x, y, z) \in C'$, has total outgoing payments of 3 to t_x, t_y and t_z, which, taking into account also the external assets in t_x, t_y and t_z, lead to a total payment of 6 to T. Overall, the liquidity emanating from these k v_i's with $i \in C'$ is $9k$. Finally, the payments to and from banks v_i that do not transfer their debt claims are $10k$, as each of the $2k$ such banks receives a payment of 1 and pays 4 to its direct neighbors; hence, the systemic liquidity is $20k$.

It suffices to show that any collection of debt transfers that generates liquidity of at least $20k$ can lead to a solution for instance $\mathcal{I}$. Let χ be the number of agents v_i whose debt claim from v_i' is transferred. We first show that if the liquidity is at least $20k$, then it must be $\chi = k$.

Note that the total liquidity starting from the v_i''s to their neighbors is $3k$, while the total liquidity from all v_i's to their direct neighbors is $3\chi + 4(3k - \chi) = 12k - \chi$. When $\chi < k$, note that the total payments from v_i's to t_i's equal $3\chi + 4 \cdot \frac{3}{M+3}(3k - \chi) < 3\chi + 1$ as M is arbitrarily large. Therefore, the total liquidity from t_i's to T is at most $3\chi + 1 + 3k$ and the systemic liquidity is at most $3k + 12k - \chi + 3\chi + 1 + 3k = 18k + 2\chi + 1 < 20k$ as $\chi < k$. Similarly, when $\chi > k$, the systemic liquidity is at most $3k + 12k - \chi + 6k = 21k - \chi < 20k$ where $3k$ and $12k - \chi$ are the exact liquidity from v_i''s and v_i's to their own outgoing neighbors respectively, while the liquidity from t_i's to T is at most $6k$.

It remains to argue about the case $\chi = k$. If these k banks can cover all t_i's, then we obtain a solution to RXC3; a contradiction. So, there exists at least one bank t_i that receives payment from at least two v_i's and the total liquidity from the t_i's to T would be at most

$6k-1$. The total liquidity in that case would be at most $3k+11k+6k-1<20k$. The proof is complete. □

We make use of the following lemma while focusing on maximizing the total equity.

Lemma 4.3 ([79, 121])**.** *In any financial network without default costs, the total equity, after clearing, equals the sum of external assets, that is,* $\sum_i E_i = \sum_i e_i$.

As, according to Lemma 4.3, the sum of equities is always equal to the sum of external assets when there are no default costs, it holds that any collection of debt transfers maximizes the total equity.

Corollary 4.4. *Given a financial network without default costs, any collection of debt transfers maximizes total equity.*

The situation changes drastically, however, when non-trivial default costs apply.

Theorem 4.5. *In financial networks with default costs* $\alpha \in (0,1)$*, the following problems are NP-hard:*

a*) computing a collection of debt transfers maximizing the total equity;*

b*) computing a collection of debt transfers minimizing the number of banks in default;*

c*) computing a collection of debt transfers that guarantees that a given bank is no longer in default and minimizes the amount of debt claims transferred.*

Proof. Our proofs follow by reductions from PARTITION. Recall that in PARTITION, an instance $\mathcal{I}$ consists of a set X of positive integers $\{x_1, x_2, \ldots, x_k\}$ and the question is whether there exists a subset X' of X such that $\sum_{i\in X'} x_i = \frac{\sum_{i\in X} x_i}{2}$.

Hardness of computing a collection of debt transfers maximizing the total equity. Starting from $\mathcal{I}$, we create an instance $\mathcal{I}'$ as follows. First, we add banks v_i and v_i' for each element $x_i \in X$ and we allocate an external asset of x_i to v_i'. We also include four additional banks S, S', T and T' and allocate an external asset of $\frac{\sum_{i\in X} x_i}{2}$ to both S and T. Each bank v_i has a liability of x_i to S and T, while S and T have a liability of $\sum_{i\in X} x_i$ to S' and T' respectively; see also Figure 4.5. Clearly, the reduction requires polynomial time.

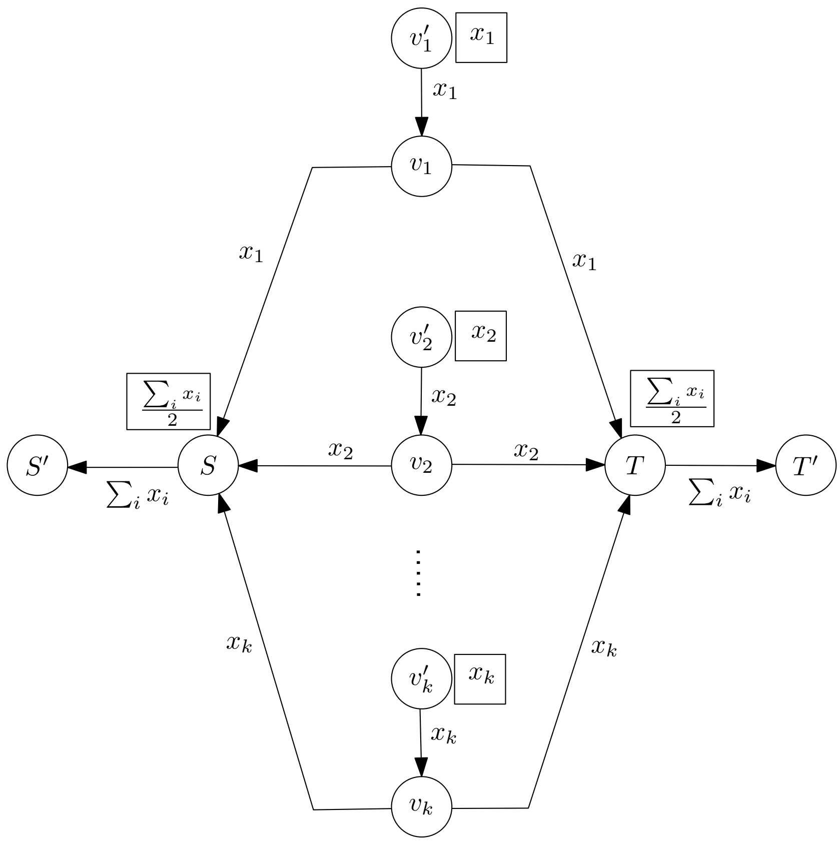


Figure 4.5: An example of the reduction used in the proof of Theorem 4.5a.

Observe that all v_i''s are exactly solvent, while all v_i's are in default. Due to default costs, the total payments from all v_i's to each of S and T is $\alpha\frac{\sum_{i\in X} x_i}{2} < \frac{\sum_{i\in X} x_i}{2}$, thus both S and T are in default. Hence, the equity, due to banks S' and T', is $\alpha\sum_{i\in X} x_i + \alpha^2\sum_{i\in X} x_i$.

In order to avoid the default costs as much as possible by transferring debts, we should make sure that all payments travel only through solvent banks among the v_i's and reach S and T so that both banks can be (exactly) solvent, thereby further avoiding default costs.

Now, we will show that in instance $\mathcal{I}'$, a total equity of at least $2\sum_{i\in X} x_i$ can be achieved if and only if instance $\mathcal{I}$ of PARTITION is a 'yes'-instance. If $\mathcal{I}$ is indeed a 'yes'-instance, i.e., there exists a subset $X' \subseteq X$ with $\sum_{i\in X'} x_i = \frac{\sum_{i\in X} x_i}{2}$, then we have each v_i with $x_i \in X'$ directly transfer its debt from v_i' to S, while each v_i with $x_i \notin X'$ transfers its debt to T. This leads to total payments from all v_i''s to each of S and T to be exactly $\frac{\sum_{i\in X} x_i}{2}$; this, in turn, ensures that S and T are solvent (albeit with equity of 0), resulting in $E_{S'} = E_{T'} = \sum_{i\in X} x_i$ and a total equity of $2\sum_{i\in X} x_i$.

Note that, by Lemma 4.3, it is not hard to see that the total equity cannot be more than $2\sum_{i\in X} x_i$ and this value can be obtained only when there are no equities lost. Hence, if there exists at least one bank with positive total assets that is in default, then the total equity is strictly less than $2\sum_{i\in X} e_i$. We will exploit this property to show that the total equity would be strictly less than $2\sum_{i\in X} x_i$ if instance $\mathcal{I}$ is a 'no'-instance.

Let X_S and X_T be the set of v_i's who decide to make a debt transfer from v_i' to S and T, respectively. If $X/(X_S \cup X_T) \neq \emptyset$, then each $v_i \in X/(X_S \cup X_T)$ is in default; since such a v_i has strictly positive total assets, we obtain that the total equity is below $2\sum_{i\in X} x_i$. Otherwise, if $X/(X_S \cup X_T) = \emptyset$, then $X_S \cup X_T = X$. Since instance $\mathcal{I}$ admits no solution X' with $\sum_{i\in X'} x_i = \frac{\sum_{i\in X} x_i}{2}$, we can assume, w.l.o.g, that $\sum_{i\in X_S} x_i > \frac{\sum_{i\in X} x_i}{2} > \sum_{i\in X_T} x_i$, which implies that the total incoming payment in T is strictly less than $\frac{\sum_{i\in X} x_i}{2}$ and bank T is in default; again, the total equity will be strictly less than $2\sum_{i\in X} x_i$. The proof is complete.

Hardness of computing a debt transfer combination minimizing the number of banks in default. The proof is, in essence, identical to that of Theorem 4.5a. As claimed in that proof, all v_i''s as well as S' and T' will be solvent, while all v_i's are necessarily in default; the uncertainty is only about banks S and T. That is, at most two banks can be saved from insolvency, which can be achieved only when there exists a solution to instance $\mathcal{I}$ of PARTITION; otherwise, the number of saved banks is at most one.

Hardness of computing a collection of debt transfers that guarantees that a given bank is no longer in default and minimizes the amount of debt claims transferred. This proof also follows by reducing an instance $\mathcal{I}$ of PARTITION to instance $\mathcal{I}'$. For each $x_i \in X$, we add banks v_i and v_i' and allocate external assets of $\frac{x_i}{\alpha}$ to v_i'. We also include two additional banks, S and T. Each v_i' and v_i, for $i \in \{1, 2, \ldots, k\}$, have liabilities of $\frac{x_i}{\alpha} + x_i$ to v_i and S, respectively, while bank S has a liability of $\frac{1+\alpha}{2} \cdot \sum_{i \in X} x_i$ to T, see also Figure 4.6. Furthermore, we are interested in bank S being solvent. Again, the reduction requires polynomial time.

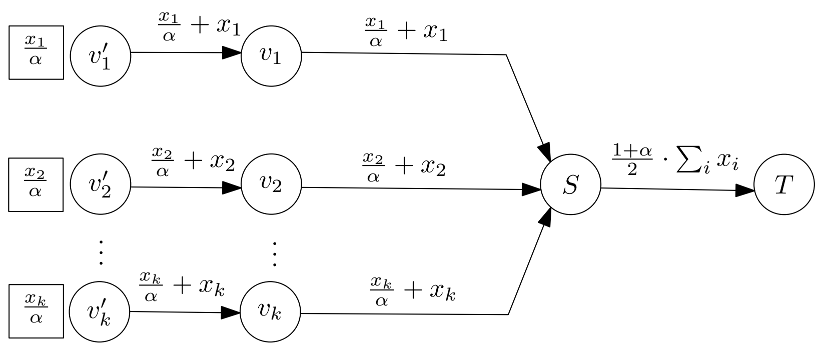


Figure 4.6: An example of the reduction used in the proof of Theorem 4.5c.

We will show that there is a collection of debt transfers that guarantees that S is not in default and minimizes the amount of the transferred debt if and only if instance $\mathcal{I}$ of PARTITION is a 'yes'-instance. If $\mathcal{I}$ is indeed a 'yes'-instance and there exists a subset $X' \subseteq X$ with $\sum_{i \in X'} x_i = \frac{\sum_{i \in X} x_i}{2}$, then we let each v_i with $x_i \in X'$ directly transfer its debt from v_i' to S, resulting in $a_S = \sum_{i \in X'} (\frac{x_i}{\alpha} \cdot \alpha) + \sum_{i \in X/X'} (\frac{x_i}{\alpha} \cdot \alpha^2) = \frac{1+\alpha}{2} \cdot \sum_{i \in X} x_i$, i.e., S is (barely) solvent, and the total amount of debts transferred is $\sum_{i \in X'} (\frac{x_i}{\alpha} + x_i) = \frac{1+\alpha}{2\alpha} \sum_{i \in X} x_i$.

Otherwise, if $\mathcal{I}$ is a 'no'-instance, we claim that the amount of debts transferred required to make S solvent is strictly more than $\frac{1+\alpha}{2\alpha} \sum_{i \in X} x_i$. Let A be the set of v_i's who make debt transfers from v_i' to S. For the total assets of S, we have

$$\begin{aligned} a_S &= \sum_{i \in A} \left(\frac{x_i}{\alpha} \cdot \alpha \right) + \sum_{i \in X/A} \left(\frac{x_i}{\alpha} \cdot \alpha^2 \right) \\ &= \sum_{i \in A} x_i + \sum_{i \in X/A} \alpha \cdot x_i \\ &= \sum_{i \in X} \alpha \cdot x_i + (1 - \alpha) \sum_{i \in A} x_i. \end{aligned}$$

As we require S to be solvent, it must hold

$$\sum_{i\in X}\alpha\cdot x_i+(1-\alpha)\sum_{i\in A}x_i\geq\frac{1+\alpha}{2}\sum_{i\in X}x_i \implies \sum_{i\in A}x_i\geq\frac{1}{2}\cdot\sum_{i\in X}x_i.$$

If $\sum_{i\in A}x_i=\frac{\sum_{i\in X}x_i}{2}$, we obtain a contradiction to our assumption that there is no solution to instance $\mathcal{I}$. When $\sum_{i\in A}x_i>\frac{\sum_{i\in X}x_i}{2}$, however, the amount of debts transferred is $\sum_{i\in A}\left(\frac{x_i}{\alpha}+x_i\right)>\frac{1+\alpha}{2\alpha}\sum_{i\in X}x_i$. The proof is complete. □

4.3 Debt Transfer Games

Given a financial network N, a bank can select to transfer some debt claims to maximize its utility (either total assets or equity). That is, bank v_i can transfer a debt claim from bank v_j to another bank v_k provided that $l_{ji}=l_{ik}$. Given v_i's possible debt transfers, its *strategy* s_i consists in selecting which debt claims to transfer and which to preserve.

In this section we consider the game-theoretic variant, where each bank may decide to transfer some of its debt claims, if applicable, in order to increase its utility. We first consider the case where banks care about their total assets and then we consider the equity; recall that, as the example in Figure 4.1 shows, a debt transfer can indeed lead to increased utility.

4.3.1 Maximizing total assets

We consider the utility function of total assets and observe that, when non-trivial default costs apply, there exist debt transfer games that do not admit Nash equilibria.

Theorem 4.6. *There exists a debt transfer game with default costs $\alpha\in(0,1)$ that does not admit Nash equilibria, when banks wish to maximize their total assets.*

Proof. Fix the default costs $\alpha\in(0,1)$ and let $\sigma\in[0,1)$. Consider the financial network in Figure 4.7, where $c=\frac{2}{\alpha^2(1-\alpha)}$ and $M\geq\frac{l_{71}+l_{82}+2}{1-\sigma}$.

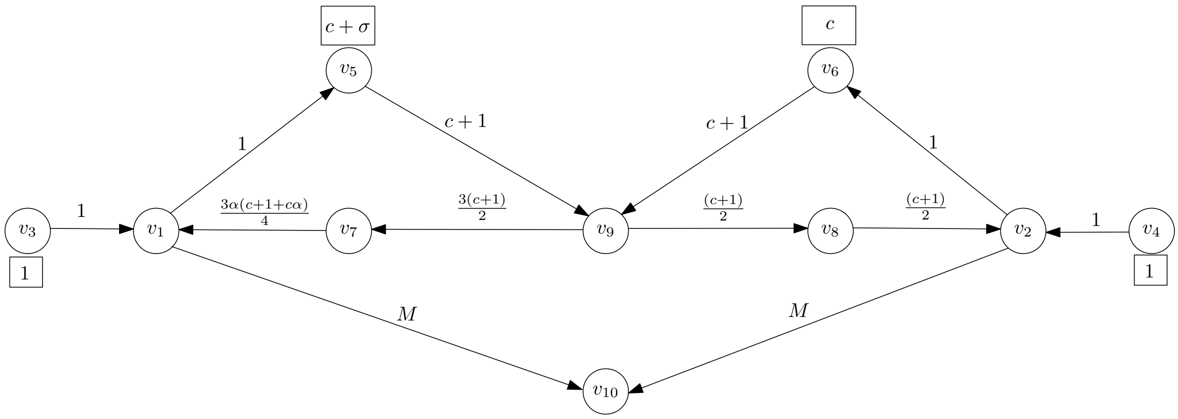


Figure 4.7: A debt transfer game with default costs $\alpha \in (0,1)$ that does not admit Nash equilibria when banks wish to maximize their total assets. $c = \frac{2}{\alpha^2(1-\alpha)}$, σ can take any fixed value in $[0,1)$, and $M \geq \frac{l_{71}+l_{82}+2}{1-\sigma}$.

In this financial network, only v_1, v_2 and v_8 have an incoming and an outgoing edge with the same liability, hence are eligible to make a debt transfer. In contrast to the first two, v_8 has a single incoming edge so its utility would be zero if it transferred a debt claim. Note that v_8 can get strictly positive utility if it does not transfer the debt claim, regardless of the strategies of v_1 and v_2; indeed, part of the external assets of v_5 and v_6 will reach v_8 through v_9. Since v_8 has a dominant strategy not to transfer its debt claim, it is without loss of generality that we consider the game between two banks v_1 and v_2. Thus, there are four possible strategy profiles where both, neither, or exactly one of v_1 and v_2 transfer their debt claim. We show that none of them are at equilibrium.

The following table represents the game in normal form. We present the detailed calculations of the utilities immediately afterward.

As $M \geq \frac{l_{71}+l_{82}+2}{1-\sigma}$, we have that M satisfies $(l_{71}+l_{82}+2)\frac{1}{M+1} < 1-\sigma$. The purpose of this choice is that whenever v_1 (respectively, v_2) does not transfer its debt, then its payment to v_5 (respectively, v_6) is negligibly small, i.e. $p_{15} \leq (l_{71}+1)\frac{1}{M+1}$ (respectively, $p_{26} \leq (l_{82}+1)\frac{1}{M+1}$), such that $p_{15}+p_{26}+\sigma < 1$.

First, consider the clearing state of the original network, when neither v_1 nor v_2 make a debt transfer. This results in payments p_{15} and p_{26}, which satisfy $p_{15}+p_{26}+\sigma < 1$. As a result both v_5 and v_6 are in default. It holds that $p_{59} = (c+\sigma+p_{15})\alpha$ and $p_{69} = (c+p_{26})\alpha$, so the incoming payments of v_9 satisfy $2c \cdot \alpha < a_9 < 2(c+1)\alpha$, and v_9 and v_8 are also in

v_1 \ v_2	NO Transfer	Transfer
NO Transfer	$a_1 < \frac{1+2\alpha}{1-\alpha} + \frac{3\alpha^3}{2}, \quad a_2 > \frac{1}{1-a}$	$1 + \frac{3(1+\alpha)}{2\alpha(1-\alpha)} + \frac{3\alpha}{4}, \quad a_2^d < \frac{1}{2(1-\alpha)} + \frac{\alpha^2(1+\alpha)}{4}$
Transfer	$\frac{3(1+\alpha)}{2\alpha(1-\alpha)} + \frac{3\alpha}{4}, \quad a_2^b < \frac{3-\alpha}{2(1-\alpha)} + \frac{\alpha^2(1+\alpha)}{4}$	$\frac{3(1+\alpha)}{2\alpha(1-\alpha)} + \frac{3\alpha}{4}, \quad \frac{1}{\alpha^2(1-\alpha)} + \frac{1}{2}$

Table 4.1: The table of utilities for the network in Figure 4.7

default. It holds that $p_{97} = \frac{3}{4}a_9 \cdot \alpha < \frac{3}{2}(c+1)\alpha^2$ and we observe that v_7 is also in default as $p_{97} < l_{71}$ so $p_{71} = p_{97} \cdot \alpha$. We also have that $p_{98} = \frac{1}{4}a_9 \cdot \alpha > \frac{1}{2}c \cdot \alpha^2 = \frac{1}{1-\alpha}$ and $p_{82} = p_{98} \cdot \alpha > \frac{\alpha}{1-\alpha}$. We conclude that the total assets of banks v_1 and v_2 in this case satisfy $a_1 = 1 + p_{71} < \frac{1+2\alpha}{1-\alpha} + \frac{3\alpha^3}{2}$ and $a_2 = 1 + p_{82} > \frac{1}{1-a}$.

Now consider the case where v_1 transfers its debt and v_2 does not, call that case b. In this case, v_5 would be solvent while v_6 is in default. In particular, $p_{59}^b = c+1$, $p_{69}^b = (c + p_{26}^b)\alpha$, so the incoming payments of v_9 satisfy $c + 1 + c\alpha < a_9^b < (c+1)(1+\alpha)$ so v_9 and v_8 are also in default. Since $a_9^b > c + 1 + c\alpha$, it holds $p_{97}^b > \frac{3}{4}(c + 1 + c\alpha)\alpha = l_{71}$ so v_7 is solvent and $a_1^b = p_{71}^b = p_{97}^b = \frac{3(1+\alpha)}{2\alpha(1-\alpha)} + \frac{3\alpha}{4}$. We also have $p_{98}^b = \frac{1}{4}a_9 < \frac{1}{4}(c+1)(1+\alpha)\alpha$, and $p_{82}^b = p_{98}^b \cdot \alpha$. Hence, $a_2^b = 1 + p_{82}^b < \frac{3-\alpha}{2(1-\alpha)} + \frac{\alpha^2(1+\alpha)}{4}$.

In case c, we assume that both v_1 and v_2 transfer their debts. All banks except v_1 and v_2 are solvent so $a_1^c = 1 + p_{71}^c = 1 + l_{71} = \frac{3(1+\alpha)}{2\alpha(1-\alpha)} + \frac{3\alpha}{4}$ and $a_2^c = 1 + p_{82}^c = 1 + l_{82} = \frac{3-\alpha}{2(1-\alpha)} + \frac{\alpha^2}{4}$.

Finally, consider the case where v_2 transfers its debt and v_1 does not, call that case d. In this case, v_6 is solvent but v_5 is in default. In particular, $p_{59}^d = (c + \sigma + p_{15}^d)\alpha$, $p_{69}^d = c+1$, so the incoming payments of v_9 satisfy $c + 1 + c\alpha < a_9^d < (c+1)(1+\alpha)$ so v_9 and v_8 are also in default. Since $a_9^d > c + 1 + c\alpha$, it holds $p_{97}^d > \frac{3}{4}(c + 1 + c\alpha)\alpha = l_{71}$ so v_7 is solvent and $a_1^d = 1 + p_{71}^d = 1 + p_{97}^d = 1 + \frac{3(1+\alpha)}{2\alpha(1-\alpha)} + \frac{3\alpha}{4}$. We also have $p_{98}^d = \frac{1}{4}a_9 < \frac{1}{4}((c+1)(1+\alpha))\alpha$, and $p_{82}^d = p_{98}^b \cdot \alpha$. Hence, $a_2^d = p_{82}^b < \frac{1}{2(1-\alpha)} + \frac{\alpha^2(1+\alpha)}{4}$.

Straightforward calculations imply that $a_1^b > a_1$ (Inequality (4.2)), $a_2^c > a_2^b$ (Inequality

(4.3)) and $a_2 > a_2^d$ (Inequality (4.4)). Is it also not hard to see that $a_1^d = 1 + a_1^c > a_1^c$, which implies that v_1 can increase its total assets by switching to NO Transfer (case d) from case c where both v_1 and v_2 did a debt transfer.

Indeed, it holds that bank v_1 can increase its total assets by transferring its claim and moving to case b from the state where no bank transferred debt claims (original network).

$$\begin{aligned} a_1^b - a_1 &> \left(\frac{3(1+\alpha)}{2\alpha(1-\alpha)} + \frac{3\alpha}{4}\right) - \left(\frac{1+2\alpha}{1-\alpha} + \frac{3\alpha^3}{4}\right) \\ &> \frac{3(1+\alpha)}{2\alpha(1-\alpha)} - \frac{1+2\alpha}{1-\alpha} \\ &> \frac{3(1+\alpha)}{2\alpha} - (1+2\alpha) \\ &> 3 - (1+2\alpha) \\ &> 0. \end{aligned} \tag{4.2}$$

It holds that v_2 can increase its total assets by transferring its debt claim and moving to case c from case b where v_1 made a debt transfer and v_2 did not.

$$\begin{aligned} a_2^c - a_2^b &> \left(\frac{1}{\alpha^2(1-\alpha)} + \frac{1}{2}\right) - \left(\frac{3-\alpha}{2(1-\alpha)} + \frac{\alpha^2(1+\alpha)}{4}\right) \\ &> \frac{1}{\alpha^2(1-\alpha)} - \frac{3-\alpha}{2(1-\alpha)} \\ &= \frac{2-3\alpha^2+\alpha^3}{2\alpha^2(1-\alpha)} \\ &\geq 0, \end{aligned} \tag{4.3}$$

where the first and last inequalities hold since $\alpha^2(1+\alpha) \leq 2$ and $3\alpha^2 - \alpha^3 \leq 2$, for $\alpha \in (0,1)$.

Finally, it holds that v_2 can increase its total assets by switching to NO Transfer (original network) from case d where v_2 did a debt transfer and v_1 did not.

$$\begin{aligned} a_2 - a_2^d &> \left(\frac{1}{1-\alpha}\right) - \left(\frac{1}{2(1-\alpha)} + \frac{\alpha^2(1+\alpha)}{4}\right) \\ &= \frac{1}{2(1-\alpha)} - \frac{\alpha^2(1+\alpha)}{4} \\ &> \frac{1}{2} - \frac{2\alpha^2}{4} \\ &> 0, \end{aligned} \tag{4.4}$$

since $\alpha \in (0,1)$.

So, we conclude that the game dynamics lead to a cycle between all possible states/strategy profiles which implies that no equilibrium exists. □

We now investigate the quality of equilibria. Although, the social welfare at an equilibrium could be arbitrarily lower than the optimal one, we find that the quality could be much better than that in the initial network in terms of social welfare.

Theorem 4.7. *The Price of Stability in debt transfer games with default costs $\alpha \in [0,1]$ where banks wish to maximize their total assets is unbounded.*

Proof. Consider the financial network N in Figure 4.8, where M is an arbitrarily large constant.

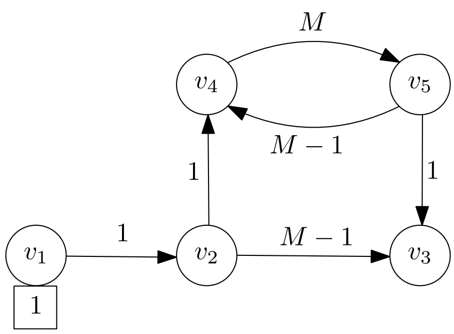


Figure 4.8: A debt transfer game with default costs $\alpha \in [0,1]$ with unbounded Price of Stability when players wish to maximize their total assets. M is an arbitrarily large constant.

Consider v_2 and notice that it is the only bank that can make a debt transfer in N. However, it is a dominant strategy for v_2 to not transfer its only debt claim, as this would result in utility 0, while it would always have positive total assets by keeping its debt claim; so, at the unique Nash equilibrium, v_2 keeps its debt claim. By assuming no default costs, we will show an upper bound of $SW_N \leq 5$ on the sum of total assets in the network; default costs $\alpha \in [0,1)$ can only result in smaller payments. Indeed, when $\alpha = 1$, the clearing state satisfies that each bank has total assets equal to 1 according to the following payments $p_{12} = 1, p_{23} = 1 - 1/M, p_{24} = 1/M, p_{45} = 1, p_{53} = 1/M$ and $p_{54} = 1 - 1/M$. However, if v_2 does transfer its debt claim, it will be the only bank in default making zero payment to v_3, resulting in total assets $a'_1 = 1, a'_2 = 0, a'_3 = 1, a'_4 = M$ and $a'_5 = M$ and a social welfare equal to $2M + 2$, regardless of the value of α; so, we have $OPT \geq 2(M+1)$.

We conclude that the Price of Stability is $\frac{OPT}{SW_N} \geq \frac{2(M+1)}{5}$ and the claim follows since M is arbitrarily large. $\square$

Theorem 4.8. *The Effect of Stability in debt transfer games with default costs $\alpha \in [0, 1]$ where banks wish to maximize their total assets is arbitrarily close to* 0.

Proof. Consider the network N represented in Figure 4.9 and observe that v_2 is the only bank that can transfer a debt claim.

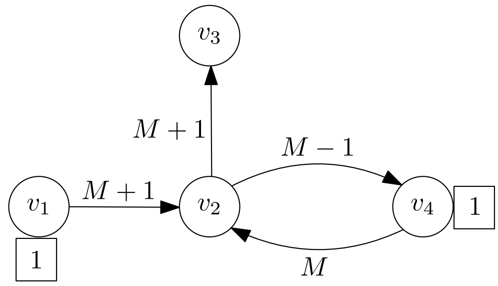


Figure 4.9: A debt transfer game with default costs $\alpha \in [0, 1]$ that yields Effect of Stability arbitrarily close to 0 when players wish to maximize their total assets.

In the clearing state of the original network, when v_2 has not transferred its debt claim, it is in default as the payment it receives from v_1 is α. In effect, v_4 will also be in default so the payments will satisfy $p_{12} = \alpha$, $p_{42} = \alpha(p_{24} + 1)$, $p_{23} = \alpha(p_{12} + p_{42})\frac{M+1}{2M}$ and $p_{24} = \alpha(p_{12} + p_{42})\frac{M-1}{2M}$. In particular, the payments are $p_{12} = \alpha$, $p_{42} = \frac{(M-1)\alpha^3+2M\alpha}{2M-(M-1)\alpha^2}$, $p_{23} = \frac{2(M+1)\alpha^2}{2M-(M-1)\alpha^2}$ and $p_{24} = \frac{2\alpha^2(M-1)}{2M-(M-1)\alpha^2}$. The sum of total assets equals

$$\begin{aligned} SW_N &= \frac{2M\alpha + (M-1)\alpha^3 + 4M\alpha^2}{2M - (M-1)\alpha^2} + 2 + \alpha \\ &\leq \frac{7M-1}{M+1} + 3 \\ &< 10, \end{aligned}$$

where the first inequality holds since the expression is an increasing function of $\alpha \in [0, 1]$.

If v_2 transfers its debt claim, then we get clearing payments $p_{13} = \alpha$, $p_{24} = M - 1$ and $p_{42} = M$, yielding a sum of total assets equal to $2M + 1 + \alpha$. As this is the unique Nash equilibrium, we obtain that the Effect of Stability is at most $\frac{10}{2M+1+\alpha}$, i.e., it can become arbitrarily close to 0 for sufficiently large M. $\square$

We now consider further questions regarding the complexity of computing equilibria and of deciding on their existence.

Theorem 4.9. *In debt transfer games with default costs $\alpha \in (0,1)$, where banks wish to maximize their total assets, the following problems are NP-hard:*

a) *computing a Nash equilibrium when one is guaranteed to exist;*

b) *computing the best response;*

c) *deciding if there exists a pure Nash equilibrium.*

Proof. Our proofs follow by reductions from PARTITION. Recall that in PARTITION an instance $\mathcal{I}$ consists of a set X of positive integers $\{x_1, x_2, \ldots, x_k\}$ and the question is whether there exists a subset X' of X such that $\sum_{i \in X'} x_i = \frac{\sum_{i \in X} x_i}{2}$.

Hardness of computing a Nash equilibrium when one is guaranteed to exist. Starting from $\mathcal{I}$, we create an instance $\mathcal{I}'$ as follows. We first add banks v_i and u_i, for each element x_i, and allocate external assets of $\frac{x_i}{\alpha}$ to v_i. Furthermore, we include three additional banks S, T and G, and allocate external assets of $\left(\frac{1}{2(1-\alpha)} - \frac{1+\alpha^2}{2}\right) \sum_{i \in X} x_i + \epsilon$ to T, where $\epsilon > 0$ is arbitrarily small. Each bank v_i has a liability of $M \cdot x_i/\alpha$ to S, where M is arbitrarily large, while each u_i has liability x_i to T. Additionally, bank S owes $M \cdot \frac{x_i}{\alpha}$ to each u_i and owe $\frac{M}{2\alpha(1-\alpha)} \sum_{i \in X} x_i$ to bank G, while, finally, bank T has a liability of $\frac{1}{2(1-\alpha)} \sum_{i \in X} x_i$ to S; see also Figure 4.10. Clearly, the reduction requires polynomial time.

Note that the only strategic player is S as no other bank can transfer debts; this guarantees the existence of a Nash equilibrium in $\mathcal{I}'$. We will show that we can compute a Nash equilibrium where in fact $a_S = \frac{2-\alpha}{2(1-\alpha)} \sum_{i \in X} x_i$, if and only if there is a solution to instance $\mathcal{I}$.

First, we prove that if $\mathcal{I}$ is a 'yes'-instance, then bank S obtains $a_S = \frac{2-\alpha}{2(1-\alpha)} \sum_{i \in X} x_i$. Indeed, consider the subset $X' \subseteq X$ in $\mathcal{I}$ with $\sum_{i \in X'} x_i = \frac{\sum_{i \in X} x_i}{2}$ and let only agents v_i, where $i \in X'$, transfer their debts from v_i to u_i.

Consider the following payments and observe that all v_i's as well as S are necessarily in default. For every $i \in X'$, $p_{v_i,u_i} = x_i$, and, hence, $\sum_{i \in X'} p_{v_i,u_i} = \sum_{i \in X} x_i/2$. Similarly, for every $i \notin X'$, $p_{v_i,S} = x_i$ and, hence, $\sum_{i \notin X'} p_{v_i,S} = \sum_{i \in X} x_i/2$.

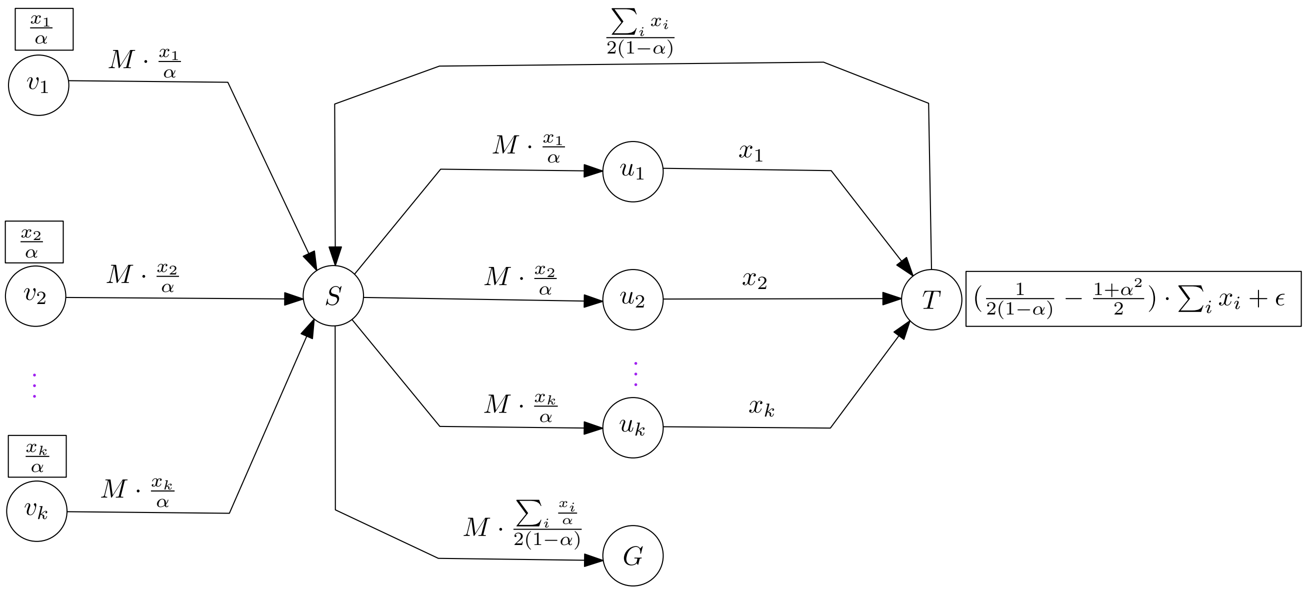


Figure 4.10: An example of the reduction used in the proof of Theorem 4.9.

Furthermore, we set $p_{S,u_i} = \alpha \cdot x_i$, for every $i \notin X'$; therefore, $\sum_{i \notin X'} p_{S,u_i} = \alpha \cdot \sum_{i\in X} x_i/2$. Since, under such payments, each u_i with $i \notin X'$, is in default, we set $p_{u_i,T} = \alpha^2 \cdot x_i$ for each such u_i and we have $\sum_{i \notin X'} p_{u_i,T} = \alpha^2 \cdot \sum_{i\in X} x_i/2$. As each u_i, with $i \in X'$, is solvent, we have $p_{u_i,T} = x_i$ for each such u_i and $\sum_{i\in X'} p_{u_i,T} = \sum_{i\in X} x_i/2$.

Finally, we set $p_{T,S} = \frac{1}{2(1-\alpha)} \cdot \sum_{i\in X} x_i$ and $p_{S,G} = \frac{\alpha}{2(1-\alpha)} \sum_{i\in X} x_i$. It is not hard to verify that these are clearing payments and we obtain $a_S = \frac{2-\alpha}{2(1-\alpha)} \sum_{i\in X} x_i$ as desired; note also that $a_T = \frac{1}{2(1-\alpha)} \cdot \sum_{i\in X} x_i + \epsilon$.

Reversely, we now prove that if $\mathcal{I}$ is a 'no'-instance, then S's total assets in $\mathcal{I}'$ are strictly less than $\frac{2-\alpha}{2(1-\alpha)} \sum_{i\in X} x_i$; this suffices to prove the claim. Consider any subset $X' \subseteq X$ in $\mathcal{I}$ and the corresponding setting in $\mathcal{I}'$ where S transfers its debt claims from banks v_i with $i \in X'$ while maintaining the claims from banks v_i corresponding to $i \notin X'$. Let $\chi = \sum_{i\notin X'} x_i$ and note that $\chi = \sum_{i \notin X'} p_{v_i,S}$. As elements in X are integers and $\mathcal{I}$ is a 'no'-instance, we have either $\chi \leq \frac{1}{2}\sum_{i\in X} x_i - \frac{1}{2}$ or $\chi \geq \frac{1}{2}\sum_{i\in X} x_i + \frac{1}{2}$. The total assets of S are

$$a_S = \sum_{i \notin X'} p_{v_i,S} + p_{T,S} \leq \chi + \frac{1}{2(1-\alpha)} \sum_{i\in X} x_i. \tag{4.5}$$

Clearly, when $\chi \leq \frac{1}{2}\sum_{i\in X} x_i - \frac{1}{2}$, by (4.5) we obtain $a_S < \frac{2-\alpha}{2(1-\alpha)} \sum_{i\in X} x_i$. When $\chi \geq \frac{1}{2}\sum_{i\in X} x_i + \frac{1}{2}$, we first show that banks u_i with $i \notin X'$ as well as T would be in default, and then we prove that $a_S < \frac{2-\alpha}{2(1-\alpha)} \cdot \sum_i x_i$.

Note that the proportion ρ of the total liabilities of S to u_i's, with $i \notin X'$, to its total liability is $\rho = \frac{\chi \cdot \frac{M}{\alpha}}{(\frac{\sum_{i\in X} x_i}{2(1-\alpha)}+\chi)\cdot\frac{M}{\alpha}} = \frac{\chi}{\frac{\sum_{i\in X} x_i}{2(1-\alpha)}+\chi}$. Since bank S is necessarily in default, it holds that

$$\begin{aligned}
\sum_{i\notin X'} p_{S,u_i} &= \rho \cdot a_S \cdot \alpha \\
&= \rho\left(\chi + p_{T,S}\right)\alpha \\
&\leq \rho\left(\chi + \frac{1}{2(1-\alpha)} \cdot \sum_{i\in X} x_i\right)\alpha \\
&= \chi\alpha \\
&< \chi,
\end{aligned}$$

where the last equality holds by the definition of ρ; this implies that u_i with $i \notin X'$ is insolvent.

Then, the total assets of bank T are

$$\begin{aligned}
a_T =& \left(\sum_{i\notin X'} p_{v_i,S} + p_{T,S}\right) \cdot \rho \cdot \alpha^2 + \sum_{i\in X'} p_{v_i,u_i} + \left(\frac{1}{2(1-\alpha)} - \frac{1+\alpha^2}{2}\right) \cdot \sum_{i\in X} x_i + \epsilon \\
\leq& \left(\chi + \frac{1}{2(1-\alpha)} \sum_{i\in X} x_i\right) \cdot \rho \cdot \alpha^2 + \sum_{i\in X} x_i - \chi + \left(\frac{1}{2(1-\alpha)} - \frac{1+\alpha^2}{2}\right) \cdot \sum_{i} x_i + \epsilon \\
=& \chi \cdot \alpha^2 + \sum_{i\in X} x_i - \chi + \left(\frac{1}{2(1-\alpha)} - \frac{1+\alpha^2}{2}\right) \cdot \sum_{i} x_i + \epsilon \\
=& (\alpha^2 - 1) \cdot \chi + \left(\frac{1}{2(1-\alpha)} - \frac{1+\alpha^2}{2} + 1\right) \cdot \sum_{i} x_i + \epsilon \\
\leq& (\alpha^2 - 1) \cdot \left(\frac{1}{2}\sum_{i\in X} x_i + \frac{1}{2}\right) + \left(\frac{1}{2(1-\alpha)} - \frac{1+\alpha^2}{2} + 1\right) \cdot \sum_{i} x_i + \epsilon \\
<& \frac{1}{2(1-\alpha)} \sum_{i\in X} x_i,
\end{aligned}$$

where the first inequality holds since $p_{T,S} \leq l_{S,T}$ and by the definition of χ, the second equality holds by the definition of ρ, while the last two inequalities hold since $\alpha^2 - 1 < 0$, $\chi \geq \frac{1}{2}\sum_{i\in X} x_i + \frac{1}{2}$ and ϵ is arbitrarily small. We conclude that bank T is in default, as desired.

Now, we move into the last step of the proof and show that $a_S < \frac{2-\alpha}{2(1-\alpha)} \cdot \sum_i x_i$. Indeed, as T is in default, we have $a_S = \sum_{i\notin X'} p_{v_i,S} + p_{T,S} < \sum_{i\in X} x_i + \alpha \cdot \frac{1}{2(1-\alpha)} \sum_{i\in X} x_i = \frac{2-\alpha}{2(1-\alpha)} \cdot \sum_i x_i$. This completes the proof.

Hardness of computing the best response. The proof of this claim follows by the previous proof. Indeed, since bank S is the only bank that can strategize, we have that finding the best response of S is equivalent to computing a pure Nash equilibrium.

Hardness of deciding if there exists a pure Nash equilibrium.

The proof is based Theorems 4.6 and 4.9b by appropriately combining the corresponding gadgets. In particular, we use bank w_{11} to bridge the two gadgets and we also set $\sigma = 0$ for w_5; see also Figure 4.11. Note that the payment from w_{11} to w_5 acts as a substitute for σ as it increases w_5's total assets.

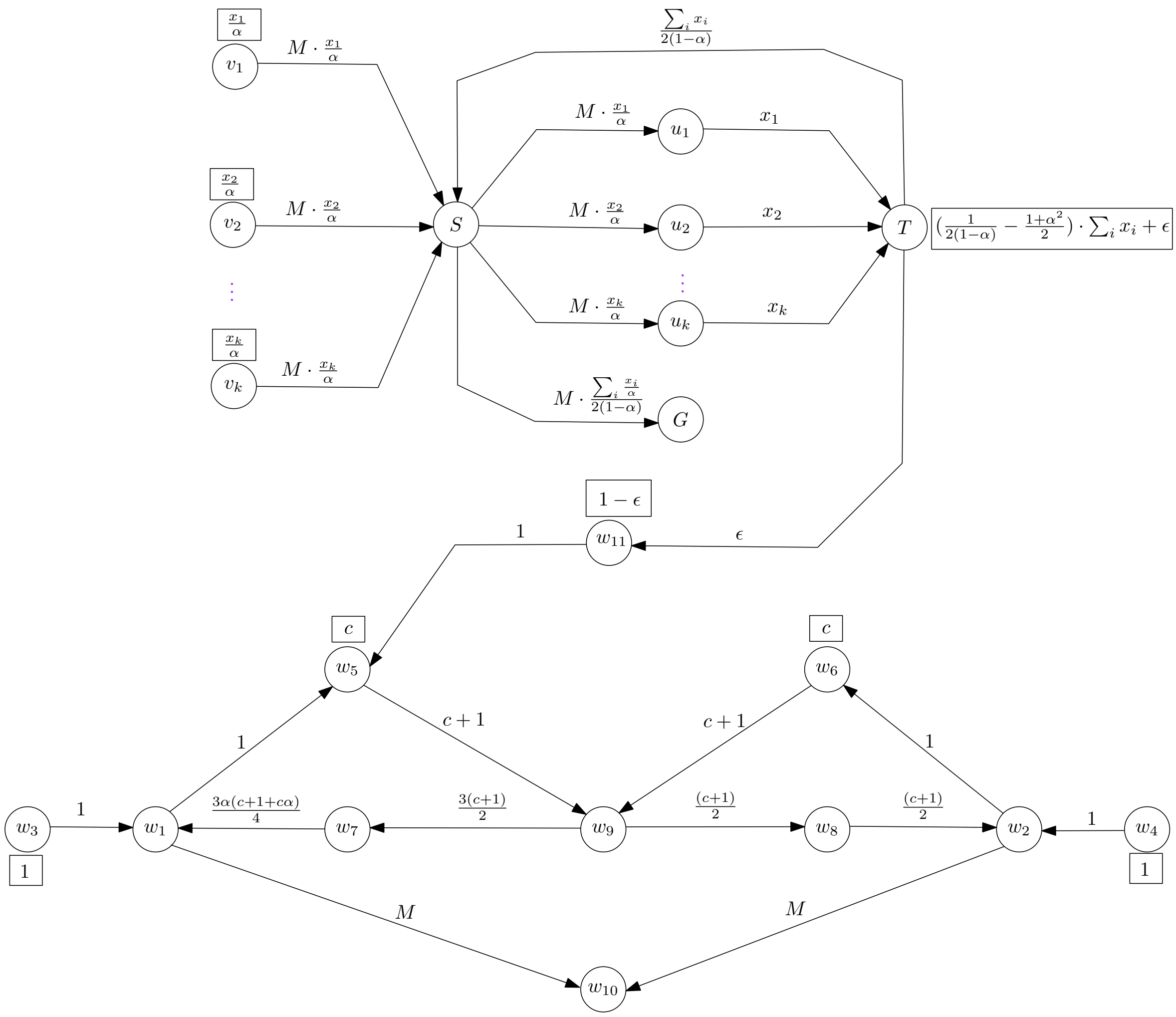


Figure 4.11: An example of the reduction used in the proof of Theorem 4.9c.

We denote by N_w the subnetwork induced by all w_i's and all their liabilities, while $N \setminus N_w$ is the subnetwork used also in the proof of Theorem 4.9b. Note that there is a single directed edge connecting $N \setminus N_w$ and N_w, i.e., the liability from bank T to w_{11}. Hence, any deviation by some w_i does not affect the payoff of bank S.

By the proof of Theorem 4.6, we know that subnetwork N_w cannot reach a Nash equilibrium if $a_{w_5} < c + 1$, i.e., if $p_{w_{11},w_5} < 1$. On the contrary, note that w_5 is always solvent when $p_{w_{11},w_5} = 1$, as this implies that the edge (w_5, w_9) is saturated. This, in turn, implies that maintaining the claim is a dominant strategy for w_1. To see that, observe that w_1's incoming payment from w_7 does not depend on w_1's strategy, since (w_5, w_9) is saturated in any case, while the incoming payment from w_3 does so. So, N_w admits an equilibrium if and only if $p_{w_{11},w_5} = 1$, i.e., if $p_{T,w_{11}} = \epsilon$.

We have proven (in Theorem 4.9b) that bank S's best response leads to $a_S = \frac{2-\alpha}{2(1-\alpha)} \sum_{i \in X} x_i$ and $a_T = \frac{1}{2(1-\alpha)} \cdot \sum_{i \in X} x_i + \epsilon$ if and only if there is solution to the instance of PARTITION. By the discussion above, since $p_{T,w_{11}} = \epsilon$ if and only if T is solvent, this suffices for our claim to hold. $\square$

4.3.2 Maximizing equity

We now shift our focus on the utility function being the bank's equity; we begin by proving the existence of Nash equilibria for the setting without default costs.

Theorem 4.10. *In debt transfer games without default costs, where banks wish to maximize their equity, the strategy profile where all banks transfer their debt claims is a Nash equilibrium.*

Proof. Consider a debt transfer game on a financial network S with banks v_q for $q = 1, \ldots, n$. Let S^* denote a state where all eligible debt claims are transferred.[1] Assume for a contradiction that some bank v_i can increase its equity by deviating from S^* to a strategy where v_i does not transfer the set of debts $\mathcal{C}$, which contains at least one of its debt claims that is transferred under S^*; denote the resulting state by S^*_{-i}.

Our proof uses two auxiliary networks, A^* and A^*_{-i}, that are constructed as follows; Figures 4.12 and 4.13 show how we can adapt S^* and S^*_{-i} to get A^* and A^*_{-i} respectively.

[1] Note that the set of identified eligible debt transfers at the original network S might not be identical to the set of debt transfers that are implemented in S^*, as it is possible that transferring a debt might make another one ineligible or might create a new eligible one. This does not contradict the fact that S^* is the network that will be reached starting from S, if the strategy of each bank is to transfer all eligible debts.

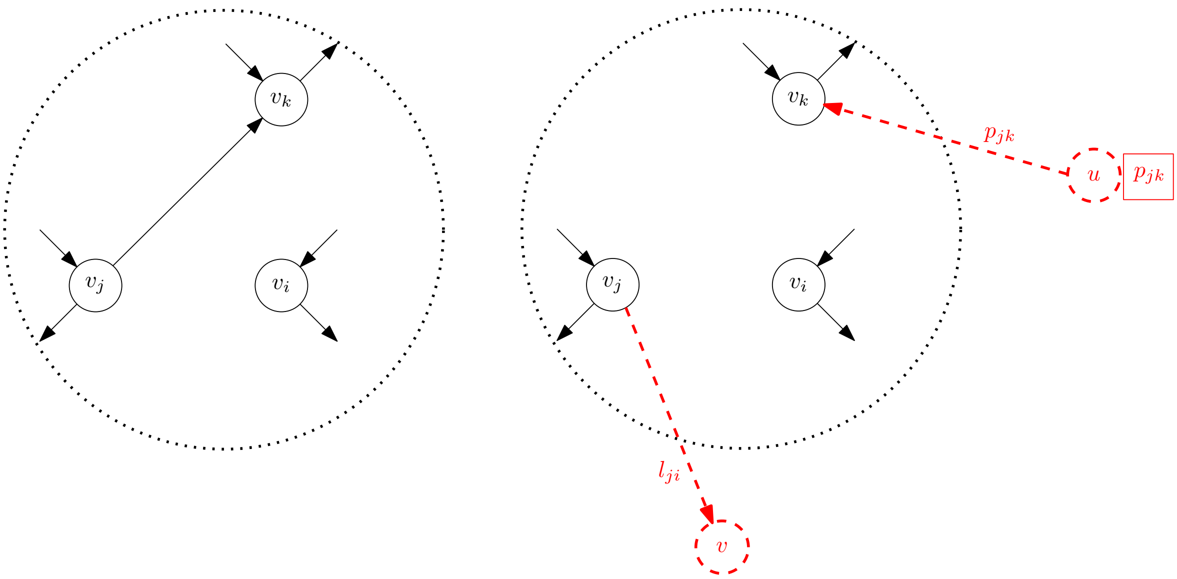


Figure 4.12: The figure shows how to construct auxiliary network A^* from S^* by focusing on the relevant part.

A^* is constructed by S^* as follows. For each debt $<v_j, v_i, v_k> \in \mathcal{C}$ (i.e., that v_i decides not to transfer), we remove edge (v_j, v_k) and add banks u and v with one edge from v_j to v having liability $l_{v_j v} = l_{ji}$ as well as one edge from u to v_k having liability $l_{uv_k} = p_{jk}$, where p_{jk} is the payment from v_j to v_k in the clearing state of S^*. The new banks have external assets $e_u = l_{uv_k} = p_{jk}$ and $e_v = 0$. Observe that by construction, under the clearing state of A^*, each bank v_q, $q = 1, \ldots, n$, will have the exact same equity as they did under S^*.

Our second auxiliary network A^*_{-i} is constructed by S^*_{-i} if for each debt claim $<v_j, v_i, v_k> \in \mathcal{C}$ we do the following. We remove edges (v_j, v_i) and (v_i, v_k) and add banks U and V with one edge from v_j to V having liability $l_{v_j V} = l_{ji}$, one edge from v_i to V having liability $l_{v_i V} = l_{ik}$, one edge from U to v_i having liability $l_{Uv_i} = p'_{ji}$, as well as one edge from U to v_k having liability $l_{Uv_k} = p'_{ik}$, where p'_{ji} and p'_{ik} are the corresponding payments in the clearing state of S^*_{-i}. The new banks have external assets $e_U = l_{Uv_i} + l_{Uv_k} = p'_{ji} + p'_{ik}$ and $e_V = 0$. Since, by assumption, bank v_i's deviation from S^* to S^*_{-i} is profitable, it holds that $E_i(S^*_{-i}) > E_i(S^*) \geq 0$, hence v_i fully repays its obligations at the clearing state of S^*_{-i} and $p'_{ik} = l_{ik}$ for each v_k appearing in $\mathcal{C}$. Similarly to before, we observe that by construction, under the clearing state of A^*_{-i}, each bank v_q, $q = 1, \ldots, n$, will have the exact same equity as they did under S^*_{-i}.

By Lemma 4.3, and since no default costs apply, we know that the total equity under

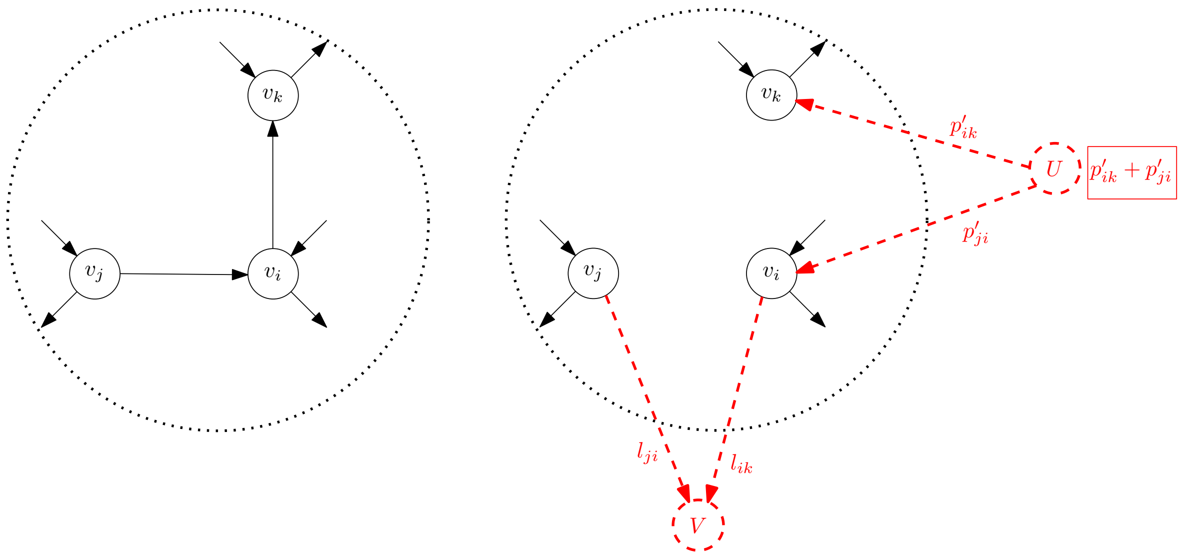


Figure 4.13: The figure shows how to construct auxiliary network A^*_{-i} from S^*_{-i} by focusing on the relevant part.

the clearing state of both networks S^* and S^*_{-i} equals the sum of the corresponding external assets, hence it is the same. By construction of the two auxiliary networks we conclude that the sum of equities of all banks v_q, $q = 1, \ldots, n$, in A^* and in A^*_{-i} is also the same, i.e.,

$$\sum_q E_q(A^*_{-i}) = \sum_q E_q(A^*) = \sum_q e_q. \tag{4.6}$$

By assumption of the profitable deviation of v_i from S^* to S^*_{-i}, i.e., $E_i(S^*_{-i}) > E_i(S^*)$, and the equivalence between the equities of respective banks between S^* and A^* as well as S^*_{-i} and A^*_{-i} we have that $E_i(A^*_{-i}) > E_i(A^*) \geq 0$. This implies that v_i has positive equity and, thus, is solvent and can repay all its liabilities ($p'_{ik} = l_{ik} \geq p_{jk}$). So, the incoming payments in A^*_{-i} of each v_k that appears in $\mathcal{C}$ are at least equal to the ones in A^*. By considering the propagation of the assets of v_i and the assets of each v_j that appears in $\mathcal{C}$, to the otherwise equivalent networks A^*_{-i} and A^*, we can conclude that $E_q(A^*_{-i}) \geq E_q(A^*)$ for each $q = 1, \ldots, n$ (recall that the inequality is strict for $q = i$). This implies that $\sum_q E_q(A^*_{-i}) > \sum_q E_q(A^*)$; a contradiction to Equality (4.6). We conclude that no bank can benefit by deviating from S^*, so the strategy profile where all banks transfer their debt claims is a pure Nash equilibrium as desired. $\square$

Recall that, by Lemma 4.4, when $\alpha = 1$ the sum of equities is independent of the bank strategies and, hence, the Price of Anarchy and Stability, as well as the Effect of Anarchy and Stability, is 1. When $\alpha < 1$, however, we obtain results on the quality of equilibria that are similar to those when maximizing total assets.

Theorem 4.11. *The Price of Stability in debt transfer games with default costs* $\alpha \in [0, 1)$ *where banks wish to maximize their equity is unbounded.*

Proof. Consider the network as shown in Figure 4.14. Note that v_1 is always in default, while v_2 is the only bank that can transfer a debt claim. When v_2 preserves the debt claim, its equity is α, as v_1 is necessarily in default, while when it transfers it, its equity is 2; hence, in the unique equilibrium, v_2 transfers the debt claim.

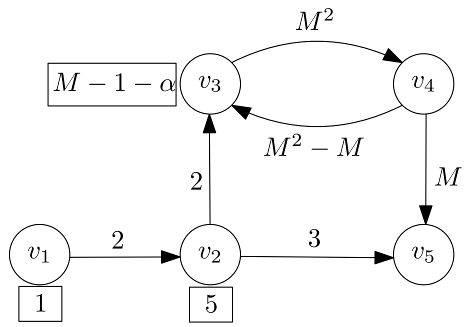


Figure 4.14: The network leading to unbounded Price of Stability.

In the original network, we have that only v_1 is in default and the payments are $p_{12} = \alpha$, $p_{23} = 2$, $p_{25} = 3$, $p_{34} = M^2$, $p_{43} = M^2 - M$, and $p_{45} = M$. Therefore, the social welfare equals $M + 4$.

In the unique equilibrium, where v_2 transfers the debt claim, all banks except for v_2 and v_5 are in default. We obtain payments $p'_{13} = \alpha$, $p'_{25} = 3$, $p'_{34} = \frac{M(M-1)\alpha}{M-(M-1)\alpha^2}$, $p'_{43} = \frac{(M-1)^2\alpha^2}{M-(M-1)\alpha^2}$, and $p'_{45} = \frac{(M-1)\alpha^2}{M-(M-1)\alpha^2}$, for a social welfare of $5 + \frac{(M-1)\alpha^2}{M-(M-1)\alpha^2} < 5 + \frac{\alpha^2}{1-\alpha^2}$. The claim follows since M is arbitrarily large.

□

Theorem 4.12. *The Effect of Stability in debt transfer games with default costs* $\alpha \in [0, 1)$ *where banks wish to maximize their equity is arbitrarily close to* 0.

Proof. Consider the network as shown in Figure 4.15, where M is arbitrarily large and $c = \frac{M\alpha}{M-(M-1)\alpha^2}$. Note that, unless $\alpha = 0$, it holds $\alpha < c$, and also that v_1 is in default while v_2 is the only bank that can transfer a debt claim.

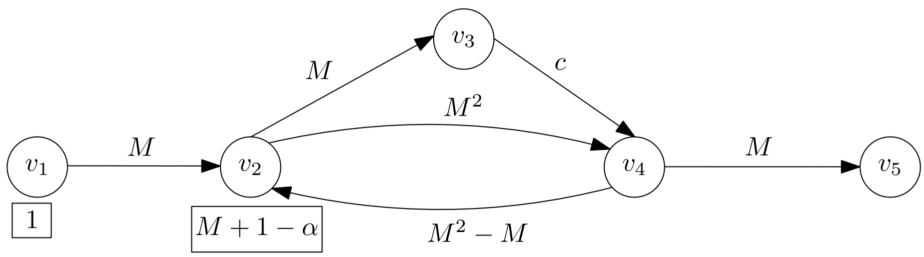


Figure 4.15: The network leading to an Effect of Stability arbitrarily close to 0.

In the original network, where v_2 does not transfer its debt, v_2's total assets are $a_2 \leq M^2+1$, i.e., less than its total liabilities of M^2+M, and v_2 is in default. Therefore, we obtain $p_{12}=\alpha, p_{23}=c, p_{24}=M\cdot c, p_{34}=c, p_{42}=\frac{(M^2-1)\alpha}{M}\cdot c$ and $p_{45}=\frac{(M+1)\alpha}{M}\cdot c$. Hence, all banks, except for v_5 have an equity of 0 and the social welfare equals the equity of v_5. That is, $SW_N=E_5=\frac{(M+1)\alpha}{M}\cdot c=\frac{(M+1)\alpha^2}{M-(M-1)\alpha^2}<\frac{2\alpha^2}{1-\alpha^2}$.

However, when v_2 transfers its debt, we have payments $p'_{13}=\alpha$, $p'_{34}=\alpha^2$, $p'_{24}=M^2$, $p'_{42}=M^2-M$, and $p'_{45}=M$. Note that $p'_{34}=\alpha^2$ as v_3 is in default when $\alpha>0$, while it holds trivially when $\alpha=0$. Therefore, we have $E'_2=1-\alpha$, $E'_4=\alpha^2$, and $E'_5=M$ for a social welfare of $M+1-\alpha+\alpha^2$ in the unique equilibrium. The proof is complete. □

4.4 Empirical Analysis

We present below the full details of our empirical analysis.

4.4.1 Empirical analysis of the centralized case

In this section, we examine the performance of different algorithms for computing debt transfer combinations on synthetic networks. Recall that, in Section 4.2 we saw that it is NP-hard to compute collections of debt transfers that maximize the sum of total assets (or, equivalently, the total liquidity, Theorem 4.2) even in the case without default costs, while for the case with default costs it is NP-hard to compute collections of debt transfers that maximize the total equity or that minimize the number of banks in default (Theorem 4.5). We here check how a rather straightforward approach performs on a set of randomly generated networks, in terms of all the aforementioned objectives and for different values of default costs.

4.4.1.1 Experimental setup for the centralized case

Network generation. Our choice of parameters regarding generating a set of random networks is shown in Table 4.2. As is common in the literature (see, e.g., [92]), and for simplicity reasons, we have chosen to work with the uniform distribution in various ranges. We construct 1000 networks of $N = 25$ nodes each, corresponding to banks. We consider each of these networks for each of the following default costs values: $\alpha \in \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$. For each bank, we select its number of outgoing edges (that correspond to liabilities) from a uniform distribution in $[0, 24]$ and create outgoing edges to that many randomly picked other banks. To capture heterogeneity among debts and, in particular, the existence of small, medium and large loans in a financial network, we follow the approach in [92] and randomly pick among ranges $[0, 10]$, $[0, 20]$, and $[0, 35]$ for each liability; we then select a number from the corresponding range as the amount of the corresponding liability. Regarding the external assets each bank holds, we pick a number s uniformly at random from the range $[0, 25]$ and set the externals of s (randomly picked) banks to zero. The amount of external assets of each remaining bank in the network is drawn from a uniform distribution in the range $U(0, 100)$. This approach simulates an external shock to s banks after the creation of the network, as is common in the literature (see, e.g., [92]); it helps us keep realistic aspects like random external assets at financially weak networks.

Table 4.2: Experimental setup for the centralized case.

	Parameters
Number of Banks	25
Outgoing Edge Degree	$U(0, 24)$
External Assets	$U(0, 100)$
Liabilities	$U(0, x), x \in \{10, 20, 35\}$
Number of Banks with no External Assets	$U(0, 25)$

Our algorithms. We consider the following algorithms/heuristics. We note that the distinction between the two classes of algorithms below is that the latter only allows initially insolvent banks (banks in default) to perform debt transfers.

RANDOM BANKS (RB$_x$): Identify all banks with eligible debt transfers. Randomly pick a fraction of $x\%$ of them and execute their debt transfers in an arbitrary order. If a debt transfer is no longer eligible after the execution of previous ones, we skip it. This is defined for $x \in \{25, 50, 75, 100\}$.

RANDOM INSOLVENT BANKS (RIB$_x$): Identify all insolvent banks with eligible debt transfers. Randomly pick a fraction of $x\%$ of them and execute their debt transfers in an arbitrary order. If a debt transfer is no longer eligible after the execution of previous ones, we skip it. This is defined for $x \in \{25, 50, 75, 100\}$.

ALL INSOLVENT BANKS (AIB): Run RIB$_{100}$ repeatedly until no eligible debt transfer among insolvent banks exists. Keep the order in which the banks are considered consistent across different rounds.

Evaluation metrics. We evaluate the performance of the above algorithms according to four different criteria, and by comparing their value at the original network N, and the network N' that emerges after the execution of the algorithm. In particular,

(i) With respect to the total liquidity (sum of total payments), we consider the ratio

$$\mathcal{R}_{TL} = \frac{\sum_j \sum_i p_{ij}(N)}{\sum_j \sum_i p_{ij}(N')}.$$

(ii) With respect to the total equity, we consider the ratio

$$\mathcal{R}_{TE} = \frac{\sum_i E_i(N)}{\sum_i E_i(N')}.$$

(iii) With respect to the number of insolvent banks, $|D(\cdot)|$, we consider the ratio

$$\mathcal{R}_{IB} = \frac{|D(N')|}{|D(N)|}.$$

(iv) With respect to the total recovery rate (see also [106, 121]), we consider the ratio

$$\mathcal{R}_{RR} = \frac{\sum_i r_i(N)}{\sum_i r_i(N')},$$

where the recovery rate of bank v_i with $i \in [n]$ captures the share of its liabilities it is able to pay and is formally defined as follows

$$r_i = \begin{cases} 1, & \text{if } L_i \leq e_i + \sum_{j=1}^n p_{ji} \\ \alpha \frac{a_i(\mathbf{P})}{L_i}, & \text{otherwise.} \end{cases} \tag{4.7}$$

4.4.1.2 Results

In our results, we display the trimmed mean of corresponding datasets, that is, we calculate the mean of the data after discarding outliers. This widely-used approach (e.g., see [98, 44]) allows us to measure the average level of data with eliminating the influence of outliers. Regarding our definition of outliers, we follow the standard approach (e.g., Boxplot) where we calculate the interquartile range (IQR) between the first (Q_1) and third (Q_3) quartile, and all data outside of the range $[Q_1 - 1.5 \cdot IQR, Q_3 + 1.5 \cdot IQR]$ are considered as outliers.

Figure 4.16 displays our results on total liquidity, total equity, number of insolvent banks and total recovery rate for each of the algorithms RB$_x$ and RIB$_x$, for $x \in \{25, 50, 75, 100\}$, and for each value of default costs $\alpha \in \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$.

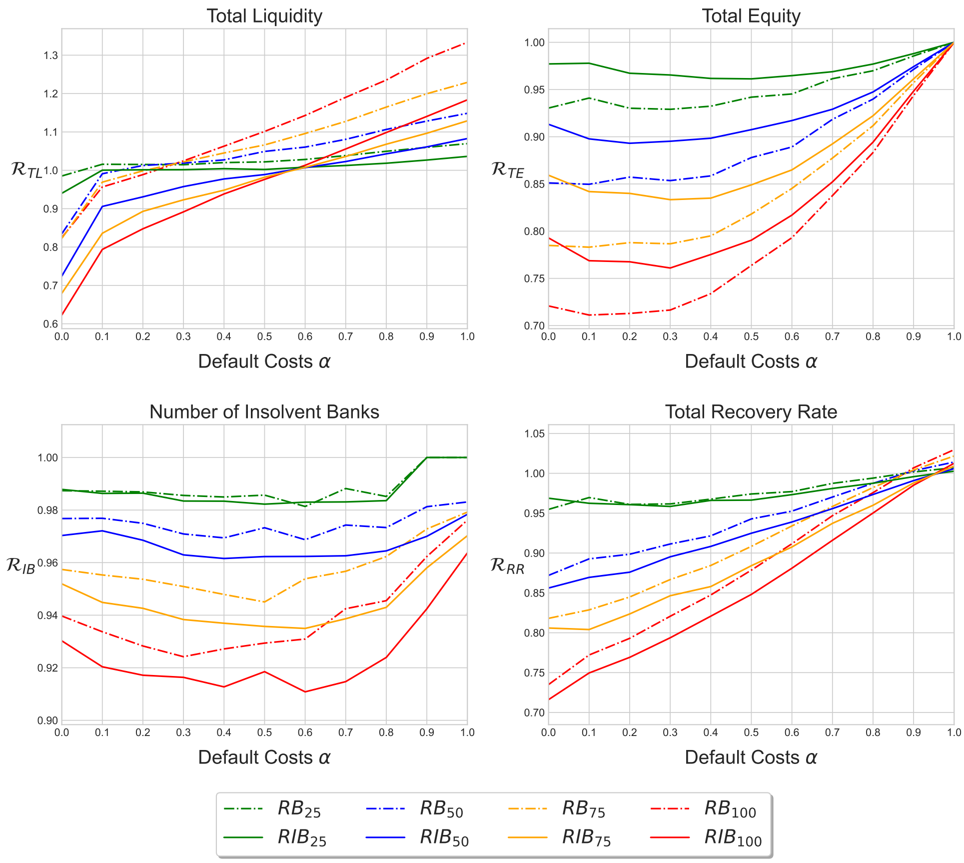


Figure 4.16: Performance comparison between RB$_x$ and RIB$_x$, for $x \in \{25, 50, 75, 100\}$.

Recall that our metrics are expressed as ratios, so values below 1 represent an improvement compared to the original network, and the lower the value, the bigger the improvement.

With respect to the total liquidity we notice that all algorithms result in networks with a higher total liquidity than the original one for small values of α. But, as α increases (i.e., the costs of default decrease) the algorithms appear to return a smaller total liquidity than that of the original network. Specifically, the RB_x algorithms, $x \in \{25, 50, 75, 100\}$, that consider all banks, only achieve an improvement for $\alpha < 0.3$, while RIB_x algorithms, which focus on insolvent banks, achieve an improved performance for a wider range of default costs, i.e., $\alpha \leq 0.6$. Overall, a better performance from RIB_x algorithms, i.e., that focus on insolvent banks, is evident, especially when $\alpha \leq 0.6$. We also observe that at the range where an improvement is achieved, the higher the percentage of (insolvent) banks doing a debt transfer the better the improvement. This evidence supports the use of RIB_{100} over the alternatives, especially when the costs of default are relatively high (α is low).

With respect to the total equity we observe an improvement for all values of α and all algorithms. We observe a similar behavior with respect to α for all algorithms, i.e., the plot seems to be decreasing from low values of α and then becomes increasing until it reaches 1 for $\alpha = 1$. In the absence of default costs ($\alpha = 1$) the total equity of the original network and the one after the execution of the algorithm are equal as expected, since the total equity is equal to the sum of external assets (see Lemma 4.3). Moreover, we observe that, for a given $x \in \{25, 50, 75, 100\}$ the corresponding RB_x algorithm performs better than its RIB_x counterpart, with higher percentages yielding a better performance overall.

Regarding the number of banks in default, we again observe that debt transfers can be useful, as all algorithms show an improvement for all values of α. Similarly to the case of total equity, there is a similar trend in all algorithms, however, in this case, the algorithms focusing on insolvent banks seem to perform better than their counterparts. That is, for a given $x \in \{25, 50, 75, 100\}$ the corresponding RIB_x algorithm performs better than its RB_x counterpart. Higher percentages yield a better performance overall in this case as well.

An improvement is also evident with respect to the total recovery rate, for a wide range of α values for all algorithms. For low values of α the improvement is more intense and its effect monotonically decreases with α but the effect of debt transfers remains positive for all values $\alpha \leq 0.85$. Algorithms focusing on insolvent banks perform better and the higher the percentage of banks doing debt transfers the better the outcome with respect to the total recovery rate.

Overall, our findings imply that debt transfers can effectively improve the well-being of a financial system regarding total liquidity, total equity, number of insolvent banks and total recovery rate, for a wide range of α values; total liquidity improves for $\alpha \leq 0.6$ but each other metric improves for almost all values of α. By comparing the various algorithms, it seems that RIB$_{100}$, which performs all eligible debt transfers of insolvent banks, outperforms the others; Figure 4.17 summarizes its performance. Moreover, with the exception of the number of banks in default, the other plots seem to demonstrate an upward trend as α increases which implies that debt transfers have a better effect in systems with high costs of default (α is low), where there is less money flow through the network.

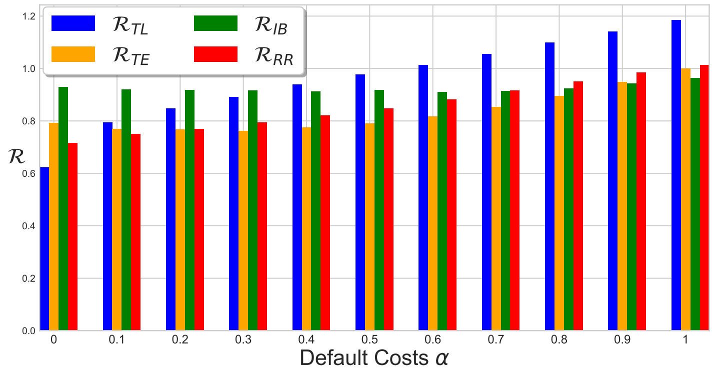


Figure 4.17: The performance of RIB$_{100}$.

We now consider a repeated variant of RIB$_{100}$, which was the best performing algorithm among RB$_x$ and RIB$_x$ for $x \in \{25, 50, 75, 100\}$. In particular, AIB executes all debt transfers of all eligible insolvent banks and then checks again for eligible insolvent banks and repeats the process until no other insolvent bank has eligible debt transfers. Figure 4.18 compares the (one-round) RIB$_{100}$ with its repeated variant, AIB, in terms of total liquidity, total equity, number of insolvent banks and total recovery rate for each value of default cost values $\alpha \in \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$. Overall, AIB shows a better performance which supports even further the assumption that debt transfers of insolvent banks can improve the well-being of a financial system in several aspects.

4.4.2 Empirical analysis of debt transfer games

In this section, we examine debt transfer games in practice. We are interested in observing how fast an initial network N will converge to a "stable" network, i.e., one with no more eligible debt transfers, if banks are allowed to transfer their debt claims strategically. We

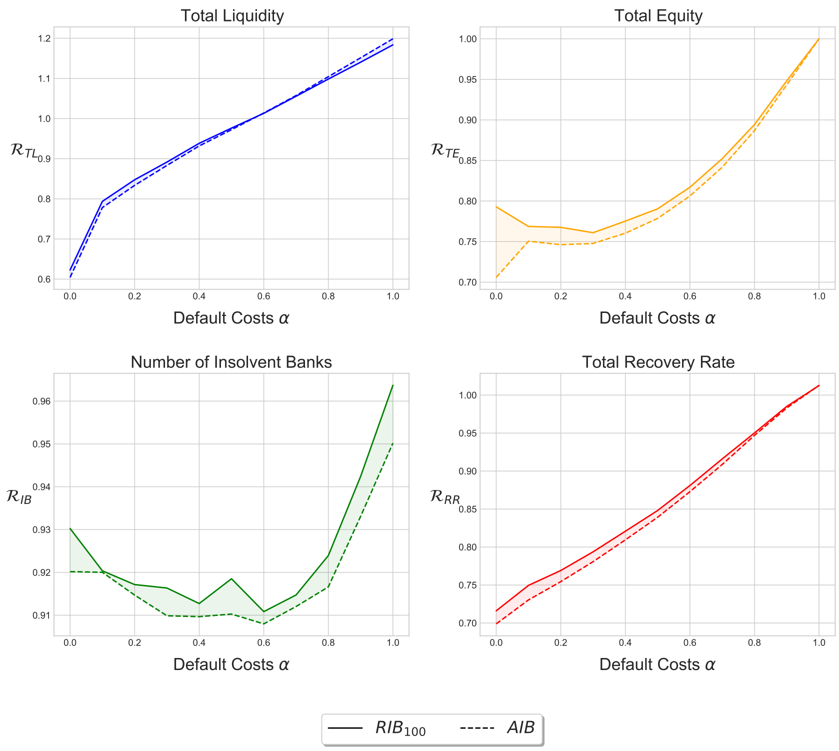


Figure 4.18: Performance comparison between RIB_{100} (solid line) and AIB (dashed line).

examine whether such observed outcomes demonstrate improved properties compared to N with respect to total liquidity, total equity, number of insolvent banks and total recovery rate, and also compare them to the best outcomes that emerge in our experiment.

4.4.2.1 Experimental setup for debt transfer games

Network generation. Our choice of parameters regarding generating a set of random networks is shown in Table 4.3. Our network generation setup is similar to the one for the centralized case in principle, however, as the calculations required in this computer experiment are much more intensive, we consider smaller networks (and adjust relevant aspects accordingly); please refer to paragraph Network Generation in Section 4.4.1.1 for justifications on our choices. We construct 1000 networks of $N = 10$ nodes each, that correspond to banks. We consider each of these networks for each of the following default costs values: $\alpha \in \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$. For each bank, we select its number of outgoing edges (that correspond to liabilities) from a uniform distribution in $[0, 9]$ and create outgoing edges to that many randomly picked other banks. For each liability, we randomly pick among ranges $[0, 10]$, $[0, 20]$, and $[0, 35]$ and select a number from the corresponding range as the amount of the corresponding liability. Regarding the external assets each bank holds, we pick a number s uniformly at random from the range $[0, 10]$ and set the externals of s (randomly picked) banks to zero. The amount of external assets of each remaining bank in the network is drawn from a uniform distribution in the range $U(0, 40)$.

Table 4.3: Debt-transfer game setup.

	Parameters
Number of Banks	10
Outgoing Edge Degree	$U(0, 9)$
External Assets	$U(0, 40)$
Liabilities	$U(0, x), x \in \{10, 20, 35\}$
Number of Banks with no External Assets	$U(0, 10)$

Game details. We consider the game where the banks in the aforementioned network structures behave strategically about transferring their debt claims. Each bank/player selects between two strategies: transfer all eligible debt claims or do not perform any debt transfers at all. We consider two game variations with respect to how the utility of the players is calculated, namely the individual utility is equal to the equity or the total assets of the corresponding bank. In each case, we consider the game played in rounds. In Round 1, we consider the initial network N and identify banks that have eligible debt transfers. Using the *game_theory* module in *QuantEcon.py*[2], we determine all "stable" outcomes and corresponding networks, where the banks that have been identified in this round are at equilibrium, and randomly pick one of them, call that N_1. If a bank v_i transfers its debt claim from v_j to v_k in N_1 and edge (v_j, v_k) already existed in N, then its liability is increased by l_{ji} as a result of the debt transfer; note that this might create additional eligible debt transfers. Each following round $i \geq 2$ repeats round $i - 1$ while considering N_{i-1} in place of the initial network. The process stops when the network under consideration at a given round has no additional eligible debt transfers. Note that we exclude the games that do not have "stable" outcomes

[2]https://github.com/QuantEcon/QuantEcon.py

within a certain round.

Evaluation metrics. We evaluate the performance of the outcomes of the game defined above according to four different criteria, namely total liquidity (sum of total payments), total equity, number of insolvent banks, and total recovery rate. We first compare the value of each criterion at the original network N and at the network N' that emerges after the execution of the algorithm in order to check if an improvement or deterioration with respect to the particular criterion is evident. We also show a more detailed quantitative comparison of N and N' in terms of total equity by considering the ratio of the total equity they admit.

To get some Price of Anarchy/Stability- and the Effect of Anarchy/Stability-type bounds we narrow our attention to all "stable" networks of Round 1 and compare the ones with the lowest (highest, respectively) social welfare (total liquidity or total equity) with the highest observed social welfare so far (in all networks, not necessarily "stable" ones) and with the social welfare of the corresponding initial network. In particular,

(i) The *Price-of-Round1Anarchy* (PoA_1) of a network N is calculated as the ratio of the maximum social welfare observed among all networks of Round 1 of the debt transfer game (note that N is included), where N is the initial network over the minimum social welfare identified among all Round 1 "stable" networks. The Price-of-Round1Anarchy of our experiment is calculated as the average among the Price-of-Round1Anarchy of all 1000 initial networks considered.

(ii) The *Price-of-Round1Stability* (PoS_1) of a network N is calculated as the ratio of the maximum social welfare observed among all networks of Round 1 of the debt transfer game, also including N, where N is the initial network over the maximum social welfare identified among all Round 1 "stable" networks. The Price-of-Round1Stability of our experiment is calculated as the average among the Price-of-Round1Stability of all 1000 initial networks considered.

(iii) The *Effect-of-Round1Anarchy* (EoA_1) of a network N is calculated as the ratio of the social welfare of N over the minimum social welfare observed among the "stable" networks identified in Round 1 of the debt transfer game where N is the initial network. The Effect-of-Round1Anarchy of our experiment is calculated as the average among the Effect-of-Round1Anarchy of all 1000 initial networks considered.

(iv) The *Effect-of-Round1Stability* (EoS_1) of a network N is calculated as the ratio of the social welfare of N over the maximum social welfare observed among the "stable" networks identified in Round 1 of the debt transfer game where N is the initial network. The Effect-of-Round1Anarchy of our experiment is calculated as the average among the Effect-of-Round1Anarchy of all 1000 initial networks considered.

4.4.2.2 Results

Similarly to the centralised case, in our results, the trimmed mean of corresponding datasets is displayed each time, where we calculate the mean of the data after discarding outliers; see beginning of Section 4.4.1.2 for justification and exact definition of outliers.

The data on Table 4.4 demonstrate the number of rounds required for the debt transfer game to terminate for each of the initial networks, when banks wish to maximize their total assets. Termination at Round 0 means that the corresponding initial network had no eligi-

Table 4.4: Number of rounds required for the debt transfer game to terminate for each of the initial networks, when banks wish to maximize their total assets. 1 counts the games whose initial network corresponds to the unique "stable" outcome of Round 1.

Round No.	Number of Games that Terminate
0	77
1	8030
(1)	7051
2	2772
3	121

ble debt transfers and 7 out of the 1000 networks considered satisfy this property (each of these networks is considered in 11 games for each of the different α values). The games that terminate in Round 1 do have eligible debt transfers in their initial network, but a "stable" outcome with no additional eligible debt transfers is selected in Round 1. 8030 out of 11000 games considered (1000 networks, each for 11 different α values) satisfy this property. By observing closely these 8030 games, we identified that for 7051 of them the initial network N is identified as the unique "stable" outcome of Round 1. This means that the games that terminate in Round 1 do have eligible debt transfers in their initial network, but banks strategically chose not to execute them. This is not surprising as doing a claim transfer implies giving up an incoming debt which directly reduces a bank's total assets. Compensating for that loss in utility through the network would be uncommon, so in most cases a bank would prefer to not transfer its claim.

In what follows, we turn our attention to debt transfer games where each bank wants to maximize its equity and the social welfare is defined as the total equity of all banks. Intuitively, banks are more likely to benefit by transferring their claims in order to maximize their equity, because the loss of the actual incoming payment they are foregoing is always no more than the nominal liability they will avoid.

The data on Table 4.5 demonstrate the number of rounds required for the debt transfer game to terminate for each of the initial networks, when banks wish to maximize their equity.

Table 4.5: Number of rounds required for the debt transfer game to terminate for each of the initial networks, when banks wish to maximize their equity.

Round No.	Number of Games that Terminate
0	88
1	8107
2	2684
3	110
4	11

Figure 4.19 summarizes for all games considered, the data on the ratio of the total equity at the final "stable" outcome compared to that of the initial network via box-plots demonstrating the following statistics of the corresponding dataset: minimum, Q_1, Q_2, Q_3, maximum, and mean (triangle), all excluding outliers, see the beginning of Section 4.4.1.2.

As illustrated in Figure 4.19, although the "stable" outcome of the game in the final

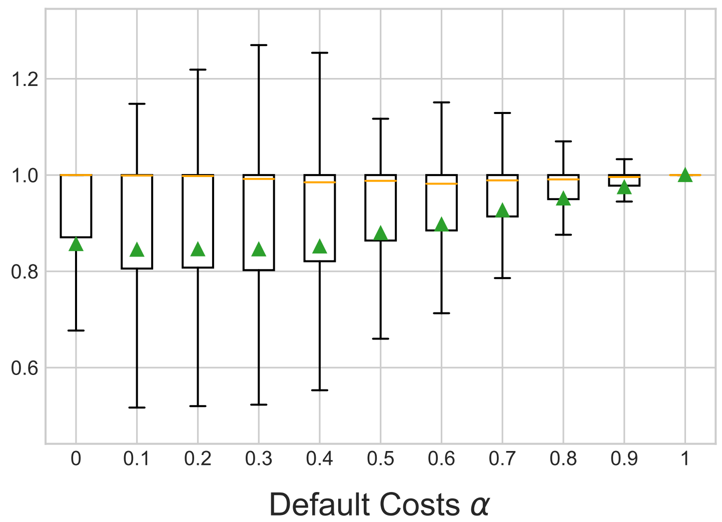


Figure 4.19: The ratio of the total equity at the final "stable" outcome compared to that of the initial network for different values of α.

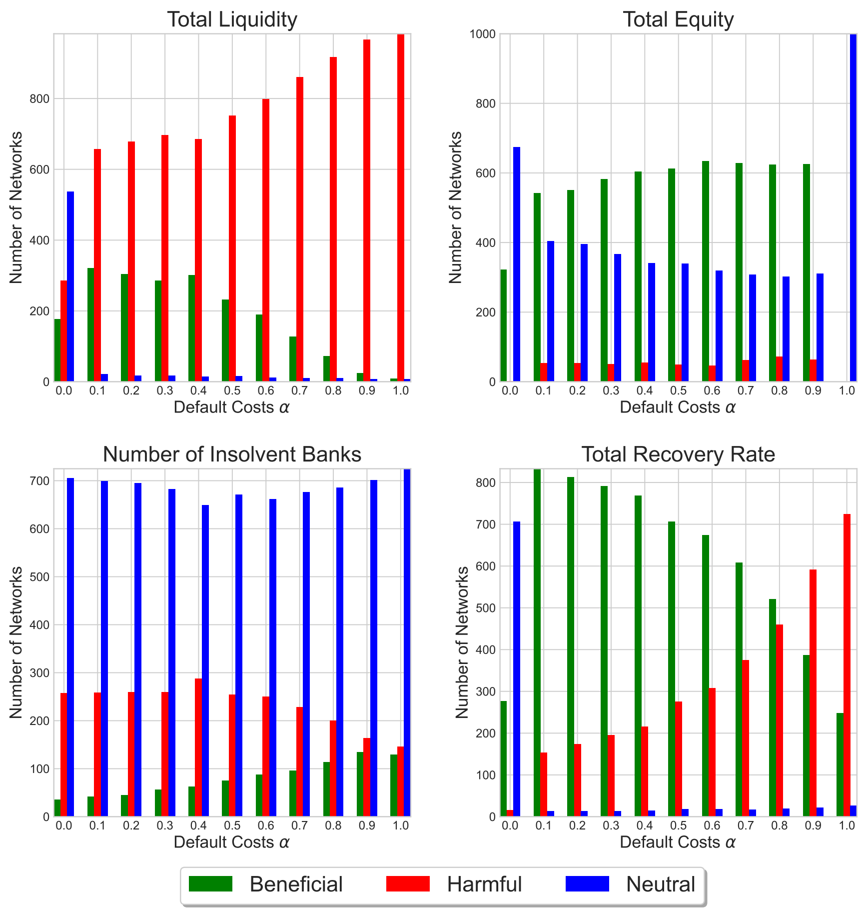


Figure 4.20: Effect of strategic debt transfers when banks want to maximize their equity.

round is not always strictly better than the initial network in terms of total equity, it holds that approximately 75% of all networks are at least as good as the initial network in terms of total equity. We also observe that the average ratio increases as α increases.

Figure 4.20 compares the change identified in the total liquidity, the total equity, the number of insolvent banks, and the total recovery rate, between the initial network and the final "stable" outcome of the game.

With respect to total liquidity, most final "stable" outcomes demonstrate a worse total liquidity compared to the initial network. This is not surprising, as debt transfers reduce the number of edges compared to the initial network, which in turn implicitly reduces the total payments traversing the network. This negative effect seems to increase with α which is justified by the fact that a higher value of α means more total payments traversing the network, so more payments lost if edges are deleted because of debt transfers.

Regarding total equity, for various values of α, the final "stable" outcomes of the majority of networks demonstrate an improvement with respect to the original network. All the networks are neutral when α is 1 as expected by Lemma 4.3.

Most initial networks have the same number of insolvent banks as the corresponding final "stable" outcome. The number of networks for which an improvement is observed in this respect is increasing with α, although it only occupies a small fraction of all networks.

With respect to the total recovery rate, the number of networks for which an improvement is observed decreases with α, contrary to the number of networks for which a deterioration in the total recovery rate is observed.

Recalling Table 4.5, the fact that the debt transfer game, for a big majority of the networks considered, terminated at Round 1 implies that Price-of-Round1Anarchy/Round1Stability and Effect-of-Round1Anarchy/Round1Stability results can offer some intuition with respect to the quality of "stable" outcomes. Relevant results are summarized in Figure 4.21.

As illustrated in Figure 4.21, the best Round 1 "stable" outcomes are essentially the outcomes demonstrating the maximum total equity (not necessarily "stable"). The efficiency of the worst Round 1 "stable" outcome attains its worst value for $\alpha \approx 0.4$ on average, with a monotonic behavior for smaller and larger α values. Our experimental data are consistent with Lemma 4.3 stating that all outcomes have the same total equity under no default costs, i.e. $\alpha = 1$. With respect to the Effect of Round1Anarchy (Round1Stability, respectively),

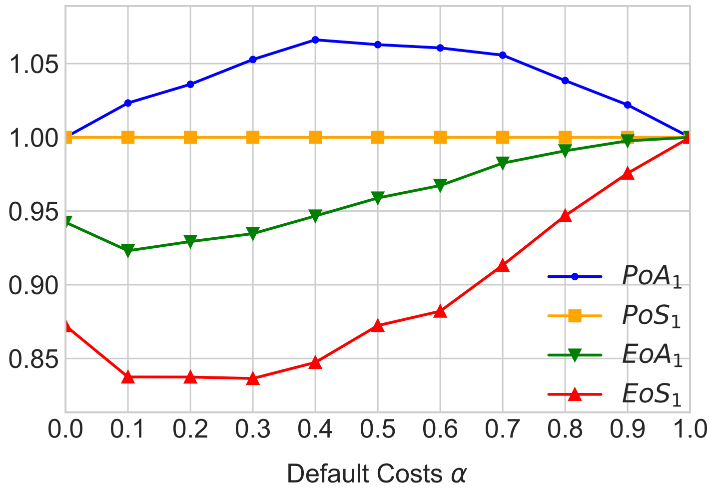


Figure 4.21: Quality of Round 1 “stable” outcomes.

although the gap between the worst (respectively, best) Round 1 “stable” outcome and the initial network in terms of total equity tends to be smaller as α increases, the Round 1 “stable” outcomes are always better than the initial network.

Overall, our results support the assumption that debt transfers can be a beneficial operation to both individual banks as well as the entire network.

4.5 Conclusions

We considered the impact of debt transfers on financial networks. Our results indicate that it is computationally hard to identify the optimal collection of debt transfers to maximize systemic liquidity, or to achieve similar objectives related to the viability of financial networks. Furthermore, we studied the strategic games arising from such operations and focused on the existence, computation, and quality of Nash equilibria. Our theoretical investigations were complemented by an experimental study on synthetic networks, both for the centralized and distributed setting.

Chapter 5

Priority-Proportional Payments in Networks with CDS

Payments with priorities are simple to express as well as quite common and very well-motivated in the financial world. Indeed, bankruptcy codes allow for assigning priorities to the payout to different creditors, in case an entity is not able to repay all its obligations. Such a distribution of payments can be part of a reorganization plan ordered by the court [16]. Priority classes in bankruptcy law have also been considered in [50, 78], among others. Moving beyond regulated financial contexts, similar behavior is common in everyday transactions between individuals with pairwise debt relations.

In this chapter, we define a new payment scheme that allows allocations of creditors to priority classes independently of the available assets. We refer to this type of payments as *priority-proportional payments*, and the corresponding strategies as *priority-proportional strategies*. We focus on the setting where each bank in financial networks can strategically make priority-proportional payments which we define here to maximize its own utility. Surprisingly, even if all banks' strategies are fixed, the existence of a unique payment profile is not guaranteed. So, we first investigate the existence and computation of valid payment profiles for fixed payment strategies. Specifically, we propose a polynomial-time algorithm to compute maximal *proper*[1] clearing payment profile for fixed payment strategies, which is a necessary step before performing our game-theoretic analysis as it allows us to argue about

[1] The differences between the maximal clearing payments with or without *proper* requirement are demonstrated by Example 1 in Section 5.1.3

well-defined deviations by considering clearing payments consistently among different strategy profiles.

Our game-theoretic analysis, as in Chapter 4, also considers two different definitions of utility motivated by the financial literature, namely *total assets*, computed as the sum of external assets and incoming payments, or *equity*, respectively. In addition to the network models considered in the previous chapters (Chapter 3 and 4), we further enhance the model by also considering financial features commonly arising in practice, such as Credit Default Swap (CDS) contracts [121], and negative external assets [43] (definitions appear in Section 5.1) in this chapter.

5.1 The Model and Definitions

Recall that we only focus on debt-only networks in previous chapters where all debt contracts are unconditional. However, in this chapter, we enrich the model by considering conditional obligations i.e., *Credit Default Swap (CDS)*. Moreover, a new strategic financial operation called *priority-proportional payments* is studied in this chapter. Thus, the following presents the additional definitions and concepts needed within this chapter.

In contrast to the certain debt contract in the debt-only network, a *Credit Default Swap (CDS)* describes a conditional payment obligation of a debtor v_i to a creditor v_j, but this payment obligation is subjected to the default of a third party v_k, called *reference entity*. In particular, v_i is only required to pay this amount to v_j in case if v_k in default, and the amount of the conditional liability is $(1-r_k) \cdot l_{ij}^k$ where r_k stands for a recovery rate of v_k, i.e., either the fraction of its total liabilities that v_k can fulfill when it is in default, or 1 otherwise, while l_{ij}^k represents the conditional debt contract of v_i to v_j. Note that l_{ij}^0 stands for the certain debt contract of v_i to v_j. Overall, in the financial system with both debt contracts and CDSs, the total liability of bank v_i to bank v_j is as follows.

$$l_{ij} = l_{ij}^0 + \sum_{k\in[n]} (1-r_k) \cdot l_{ij}^k.$$

5.1.1 Priority-proportional payments

Recall that under the scheme of *proportional payments*, as in Chapter 3 and 4, each p_{ij} must satisfy $p_{ij} = \min\{l_{ij}, (e_i + \sum_{k\in[n]} p_{ki})\frac{l_{ij}}{L_i}\}$ for any given clearing payments $\mathbf{P}$ (see Equation (2.1)). In words, solvent banks pay their liabilities in full, while banks in default split their total assets among their creditors, proportionally, relative to their respective liabilities.

When constrained to use proportional payments, there is no strategic decision making involved. In this chapter, we release the principle of *proportion* and focus on a new payment scheme *priority-proportional payments*, where a bank's strategy is independent of its total assets and consists of a complete ordering of its creditors allowing for ties. Creditors of higher priority must be fully repaid before any payments are made towards creditors of lower priority, while creditors of equal priority are treated as in proportional payments. For example, a bank v_i having banks $a, b, c,$ and d as creditors may select strategy $s_i = (a, b|c|d)$ that has banks a, b in the top priority class, followed by c and, finally, with d in the lowest priority class.

Let $L_i^{(m)}$ denote the total liability of bank v_i to banks in its m-th priority class. We use parameters k_{ij} to imply that bank v_j is in the k_{ij}-th priority class of bank v_i and denote by π'_{ij} the relative liability of v_i towards v_j in the corresponding priority class, i.e., $\pi'_{ij} = \frac{l_{ij}}{L_i^{(k_{ij})}}$ if $L_i^{(k_{ij})} > 0$, and $\pi'_{ij} = 0$ if $L_i^{(k_{ij})} = 0$. For given priority-proportional strategies for all banks, the clearing payments $\mathbf{P}$ (see Equation (2.1)) must also satisfy

$$p_{ij} = \min\left\{\max\left\{\left(p_i - \sum_{m=1}^{k_{ij}-1} L_i^{(m)}\right) \cdot \pi'_{ij}, 0\right\}, l_{ij}\right\}. \tag{5.1}$$

That is, the payment of v_i to a creditor in priority class $L_i^{(k_{ij})}$ occurs only after all payments to creditors of higher priority have been guaranteed. Then, payments to creditors in $L_i^{(k_{ij})}$ are made proportionally to their claims in that priority class. Finally, we also have $0 \le p_{ij} \le l_{ij}$.

It is worth noting that proportional payments can be regarded as a specific instance of priority-proportional payments, provided that all banks opt to make payments that are proportional to their respective liabilities.

5.1.2 Proper clearing payments

As mentioned above, payments $\mathbf{P}$ that satisfy $0 \leq p_{ij} \leq l_{ij}$ an Equation 5.1 are called *clearing payments*[2] under the priority-proportional payment scheme. Now, we further define the notion of *proper* clearing payments, which are clearing payments where all the money circulating in the financial network have originated from some bank with positive external assets. The specific difference between the clearing payments with and without proper constraint is presented in Example 1. In this Chapter 5, our focus is exclusively directed toward payments that are both proper and maximal in nature. For the sake of expediency, however, we omit the aforementioned modifiers and refer simply to "clearing payments" in the remainder of this chapter.

5.1.3 Examples

We represent a financial network by a graph as follows. Nodes correspond to banks and black solid edges correspond to debt-liabilities; a directed edge from node v_i to node v_j with label l_{ij}^0 implies that bank v_i owes bank v_j an amount of money equal to l_{ij}^0. Nodes are also labeled, their label appears in a rectangle and denotes their external assets; we omit these labels for banks with external assets equal to 0. A pair of red edges (one solid and one dotted) represents a CDS contract: a solid directed edge from node v_i to node v_j with label l_{ij}^k and a dotted undirected edge connecting this edge with a third node v_k implies that bank v_i owes v_j an amount of money equal to $(1 - r_k)l_{ij}^k$.

Example 1 Consider a debt-only network with two banks, v_1 and v_2, as well as two edges, (v_1, v_2) and (v_2, v_1). Both edges are with unit liability, while neither v_1 nor v_2 have positive external assets; see also Figure 5.1. Clearly, both v_1 and v_2 have only one single liability to each other, then they have no strategic choices but to pay the other side only. Therefore, any payments satisfying $0 \leq p_{12} = p_{21} \leq 1$ are clearing payments. In particular,

- Under the maximal but not proper clearing payments, the actual payment on both edges should be 1. That is, $p_{12} = p_{21} = 1$.

[2] Clearing payments are not necessarily unique.

- However, under the maximal proper clearing payments, both edges have zero payment, i.e., $p'_{12} = p'_{21} = 0$, because there is no money circulating have originated from some bank with positive external assets.

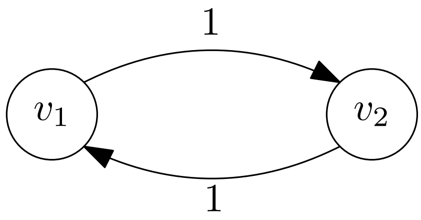


Figure 5.1: An example to show the difference between maximal clearing payments with or without *proper* requirements.

Example 2 Figure 5.2 depicts a financial network with five banks having external assets $e_1 = e_4 = 1$ and $e_2 = e_3 = e_5 = 0$. There exist four debt contracts, i.e., bank v_1 owes v_2 and v_3 two coins and one coin, respectively; v_2 owes v_1 one coin, and v_3 owes v_5 one coin. There is also a CDS contract between v_4, v_5, and v_3, with nominal liability $l^3_{45} = 1$, with v_4 being the debtor, v_5 the creditor, and v_3 the reference entity. v_4 will only need to pay v_5 if v_3 is in default; the amount owed would be equal to $1(1 - r_3)$.

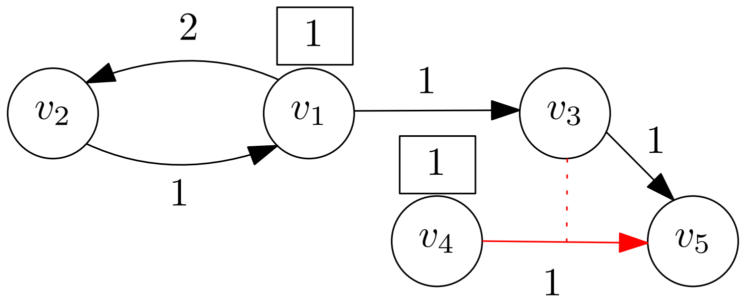


Figure 5.2: An example of a financial network with a CDS.

Clearly, only v_1 can strategize about its payments, hence we focus just on v_1's strategy:

- Let v_1 select the priority-proportional strategy $s_1 = (v_2|v_3)$. Then, the payment vector would be $\mathbf{p} = (2, 1, 0, 1, 0)$ with $\mathbf{p}_1 = (0, 2, 0, 0, 0)$. Note that this is valid since $r_3 = \frac{p_3}{L_3} = \frac{p_{35}}{l_{35}} = \frac{0}{1} = 0$ and $p_{45} = (1 - r_3)l^3_{45} = 1$. The total assets of the banks are $a_1(\mathbf{P}) = 2$, $a_2(\mathbf{P}) = 2$, $a_3(\mathbf{P}) = 0$, $a_4(\mathbf{P}) = 1$ and $a_5(\mathbf{P}) = 1$, and the social welfare is $SW(\mathbf{P}) = 6$.

- If v_1's strategy is $s'_1 = (v_3|v_2)$, the only consistent payment vector would be $\mathbf{p}' = (1,0,1,0,0)$ with $\mathbf{p}'_1 = (0,0,1,0,0)$. Then, $a_1(\mathbf{P}') = 1$, $a_2(\mathbf{P}') = 0$, $a_3(\mathbf{P}') = 1$, $a_4(\mathbf{P}') = 1$ and $a_5(\mathbf{P}') = 1$, resulting in $SW(\mathbf{p}') = 4$.

- If v_1 decides to pay proportionally, that is $s''_1 = (v_2, v_3)$, the payment vector would be $\mathbf{p}'' = (2, 1, 2/3, 1/3, 0)$ with $\mathbf{p}''_1 = (0, 4/3, 2/3, 0, 0)$; note that $p''_{45} = l^3_{45}(1 - r_3) = 1(1 - \frac{p_{35}}{l_{35}}) = 1/3$. Then, $a_1(\mathbf{P}'') = 2$, $a_2(\mathbf{P}'') = 4/3$, $a_3(\mathbf{P}'') = 2/3$, $a_4(\mathbf{P}'') = 1$ and $a_5(\mathbf{P}'') = 1$, resulting in $SW(\mathbf{P}'') = 6$.

Comparing the total assets under the different strategies discussed above, v_1 would select either strategy $(v_2|v_3)$ or (v_2, v_3), returning the maximum possible total assets (i.e., 2). Therefore, any strategy profile s where either $s_1 = (v_2|v_3)$ or $s_1 = (v_2, v_3)$ is a Nash equilibrium.

5.2 Existence and Computation of Clearing Payments

This section contains our results relating to the existence and the properties of (proper) clearing payments in priority-proportional games. We begin by arguing that, given a strategy profile, clearing payments always exist even in presence of default costs. Furthermore, in case there are multiple clearing payments, there exist maximal payments, i.e., ones that pointwise maximize all corresponding payments, and we provide a polynomial-time algorithm that computes them. Note that this result is in a non-strategic context, but it is necessary in order to perform our game-theoretic analysis as it allows us to argue about well-defined deviations by considering clearing payments consistently among different strategy profiles.

Lemma 5.1. *In priority-proportional games with default costs, there always exist maximal clearing payments under a given strategy profile.*

Proof. The proof follows by Tarski's fixed-point theorem, along similar lines to [48, 116]. We first focus on the case of not necessarily proper payments. Indeed, the set of payments forms a complete lattice. Any payment p_{ij} is lower-bounded by 0 and upper-bounded by l_{ij} and for any two clearing payments $\mathbf{P}$ and $\mathbf{P}'$ such that $\mathbf{P} \geq \mathbf{P}'$ ($\mathbf{P}$ is pointwise at least as big as $\mathbf{P}'$) it holds that $\Phi(\mathbf{P}) \geq \Phi(\mathbf{P}')$, where Φ is defined in (2.1). Therefore, $\Phi(\cdot)$ has a greatest fixed-point and a least fixed-point.

We now claim that the existence of maximal clearing payments implies the existence of maximal proper clearing ones. Indeed, consider some maximal clearing payments $\mathbf{P}$ and the resulting proper payments $\mathbf{P}'$ obtained by $\mathbf{P}$ when ignoring all payments that do not originate from some bank with positive external assets. We note that $\mathbf{P}'$ are clearing payments since the payments that are deleted, in the first place only reached banks whose outgoing payments are decreased to zero as well. Clearly, if $\mathbf{P}'$ are not maximal proper clearing payments, then there exist proper clearing payments $\tilde{\mathbf{P}}$ with $p'_{ij} < \tilde{p}_{ij}$ for some banks v_i, v_j. If $p'_{ij} = p_{ij}$ we obtain a contradiction to the maximality of $\mathbf{P}$, otherwise if $p'_{ij} < p_{ij}$, then $\tilde{\mathbf{P}}$ cannot be proper. □

We now show how to compute such maximal clearing payments. Given a strategy profile in a priority-proportional game, Algorithm 1 below, that extends related algorithms in [48, 116], computes the maximal (proper) clearing payments in polynomial time. In particular, and since the strategy profile is fixed, we will argue about the payment vector consisting of the total outgoing payments for each bank; the detailed payments then follow by the strategy profile.

Lemma 5.2. *The payment vectors computed in each round of Algorithm 1 are pointwise non-increasing, i.e.,* $\mathbf{p}^{(\mu)} \leq \mathbf{p}^{(\mu-1)}$ *for any round* $\mu \geq 0$.

Proof. We prove the lemma by induction. The base of our induction is $\mathbf{p}^{(0)} \leq \mathbf{p}^{(-1)} = \mathbf{l}$. It holds that $p_i^{(0)} = L_i$ if $v_i \in \mathcal{N} \setminus \mathcal{D}_0$, so it suffices to compute $p_i^{(0)}$ and show that $p_i^{(0)} \leq L_i$ for $v_i \in \mathcal{D}_0$.

We wish to find the solution $\mathbf{x}$ to the following system of equations

$$\begin{aligned} x_i &= \alpha e_i + \beta \left(\textstyle\sum_{j \in \mathcal{D}_0} x_{ji} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_0} l_{ji} \right), && \forall v_i \in \mathcal{D}_0, \\ x_i &= L_i, && \forall v_i \in \mathcal{N} \setminus \mathcal{D}_0. \end{aligned} \tag{5.2}$$

We compute $\mathbf{x}$ using a recursive method and starting from $\mathbf{x}^{(0)} = \mathbf{p}^{(-1)} = \mathbf{l}$. We define $x^{(k)}$, $k \geq 1$, recursively by

$$x_i^{(k+1)} = \alpha e_i + \beta \left(\sum_{j \in \mathcal{D}_0} x_{ji}^{(k)} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_0} l_{ji} \right).$$

Now for $v_i \in \mathcal{D}_0$, we have

Algorithm 1: MCP

/* The algorithm assumes given strategies (priority classes). By abusing notation and for ease of exposition, we denote by $p_i^{(\kappa)}$ the total outgoing payments of bank v_i at round κ and by $\mathbf{p}^{(\kappa)}$ the vector of total outgoing payments at round κ. */

1 Set $\mu = 0, \mathbf{p}^{(-1)} = \mathbf{l}$ and $\mathcal{D}_{-1} = \emptyset$;

2 Compute $E_i^{(\mu)} := e_i + \sum_{j \in [n]} p_{ji}^{(\mu-1)} - L_i$, for $i = 1, \ldots, n$;

3 $\mathcal{D}_\mu = \left\{ v_i : E_i^{(\mu)} < 0 \right\}$;

4 **if** $\mathcal{D}_\mu \neq \mathcal{D}_{\mu-1}$ **then**

5 Compute $\mathbf{p}^{(\mu)}$ that is consistent with Equation (5.1) and satisfies

$$p_i^{(\mu)} = \begin{cases} \alpha e_i + \beta \left(\sum_{j \in \mathcal{D}_\mu} p_{ji}^{(\mu)} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_\mu} l_{ji} \right), & \forall v_i \in \mathcal{D}_\mu \\ L_i & \forall v_i \in \mathcal{N} \setminus \mathcal{D}_\mu. \end{cases};$$

6 Set $\mu = \mu + 1$;

7 **go to** Line 2;

8 **else**

9 Run PROPER($\mathbf{p}^{(\mu-1)}$)

Algorithm 2: PROPER($\mathbf{x}$)

/* The algorithm takes in input payments $\mathbf{x}$ and returns proper payments. */

Set MARKED $= \{v_i : e_i > 0\}$ and CHECKED $= \emptyset$;

while MARKED $\neq \emptyset$ **do**

  Pick $v_i \in$ MARKED;

  for $v_j \notin$ MARKED $\cup$ CHECKED *with* $x_{ij} > 0$ **do**

    MARKED $=$ MARKED $\cup \{v_j\}$;

  MARKED $=$ MARKED $\setminus \{v_i\}$ and CHECKED $=$ CHECKED $\cup \{v_i\}$;

for $v_i \notin$ CHECKED **do**

  Set all outgoing payments from v_i in **x** to 0;

Return $\mathbf{x}$

$$x_i^{(1)} = \alpha e_i + \beta \left(\sum_{j \in \mathcal{D}_0} x_{ji}^{(0)} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_0} l_{ji} \right) \leq e_i + \sum_{j \in \mathcal{D}_0} x_{ji}^{(0)} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_0} l_{ji} < L_i = x_i^{(0)},$$

where the first inequality holds since $\alpha, \beta \leq 1$ and the second inequality holds by our assumption that $v_i \in \mathcal{D}_0$. Hence, sequence $\mathbf{x}^{(k)}$ is decreasing. Since the solution to Equation (5.2) is non-negative, $\mathbf{x}$ can be computed as $\mathbf{x} = \lim_{k \to \infty} \mathbf{x}^{(k)}$, completing the base of our induction.

Now assume that $\mathbf{p}^{(\mu)} \leq \mathbf{p}^{(\mu-1)}$ for some $\mu \geq 0$. We will prove that $\mathbf{p}^{(\mu+1)} \leq \mathbf{p}^{(\mu)}$. Similarly to before, $p_i^{(\mu+1)} = p_i^{(\mu)} = L_i$ if $v_i \in \mathcal{N} \setminus \mathcal{D}_{\mu+1}$, so it suffices to compute $p_i^{(\mu+1)}$ and show that $p_i^{(\mu+1)} \leq p_i^{(\mu)}$ for $v_i \in \mathcal{D}_{\mu+1}$.

The desired $p_i^{(\mu+1)}$ is the solution $\mathbf{x}$ to the following system of equations

$$x_i = \begin{cases} \alpha e_i + \beta \left(\sum_{j \in \mathcal{D}_{\mu+1}} x_{ji} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_{\mu+1}} l_{ji} \right), & \forall i \in \mathcal{D}_{\mu+1}, \\ L_i, & \forall v_i \in \mathcal{N} \setminus \mathcal{D}_{\mu+1}. \end{cases} \tag{5.3}$$

We compute $\mathbf{x}$ recursively starting with $\mathbf{x}^{(0)} = \mathbf{p}^{(\mu)}$. We define $x^{(k)}$, $k \geq 1$, recursively by

$$x_i^{(k+1)} = \alpha e_i + \beta \left(\sum_{j \in \mathcal{D}_{\mu+1}} x_{ji}^{(k)} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_{\mu+1}} l_{ji} \right). \tag{5.4}$$

For $v_i \in \mathcal{D}_{\mu+1}$, we have

$$\begin{aligned}
x_i^{(1)} &= \alpha e_i + \beta \left(\sum_{j \in \mathcal{D}_{\mu+1}} x_{ji}^{(0)} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_{\mu+1}} l_{ji} \right) \\
&= \alpha e_i + \beta \left(\sum_{j \in \mathcal{D}_{\mu+1}} p_{ji}^{(\mu)} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_{\mu+1}} l_{ji} \right) \\
&= \alpha e_i + \beta \left(\left(\sum_{j \in \mathcal{D}_{\mu}} p_{ji}^{(\mu)} + \sum_{j \in \mathcal{D}_{\mu+1} \setminus \mathcal{D}_{\mu}} p_{ji}^{(\mu)} \right) + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_{\mu+1}} l_{ji} \right) \\
&= \alpha e_i + \beta \left(\left(\sum_{j \in \mathcal{D}_{\mu}} p_{ji}^{(\mu)} + \sum_{j \in \mathcal{D}_{\mu+1} \setminus \mathcal{D}_{\mu}} l_{ji} \right) + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_{\mu+1}} l_{ji} \right) \\
&= \alpha e_i + \beta \left(\sum_{j \in \mathcal{D}_{\mu}} p_{ji}^{(\mu)} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_{\mu}} l_{ji} \right),
\end{aligned}$$

where we note that our assumption $\mathbf{p}^{(\mu)} \leq \mathbf{p}^{(\mu-1)}$ implies that $\mathcal{D}_{\mu+1} \supseteq \mathcal{D}_{\mu}$. Now we can split the set $\mathcal{D}_{\mu+1}$ into $\mathcal{D}_{\mu}$ and $\mathcal{D}_{\mu+1} \setminus \mathcal{D}_{\mu}$. For $v_i \in \mathcal{D}_{\mu}$ we have $x_i^{(1)} = \mathbf{p}^{(\mu)} = x_i^{(0)}$. For $v_i \in \mathcal{D}_{\mu+1} \setminus \mathcal{D}_{\mu}$ we have

$$\begin{aligned}
x_i^{(1)} &= \alpha e_i + \beta \left(\sum_{j \in \mathcal{D}_{\mu}} p_{ji}^{(\mu)} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_{\mu}} l_{ji} \right) \\
&\leq e_i + \sum_{j \in \mathcal{D}_{\mu}} p_{ji}^{(\mu)} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_{\mu}} l_{ji} \\
&= e_i + \sum_{j \in [n]} p_{ji}^{(\mu)} \\
&< L_i = p_i^{(\mu)} = x_i^{(0)},
\end{aligned}$$

which implies that the sequence $x^{(k)}$ is decreasing. Since the solution to Equation (5.3) is non-negative, $\mathbf{x} = \mathbf{p}^{\mu+1}$ can be computed as $\mathbf{x} = \lim_{k \to \infty} \mathbf{x}^{(k)}$, which completes our claim that $\mathbf{p}^{(\mu)} \leq \mathbf{p}^{(\mu-1)}$ for any round $\mu \geq 0$. □

We now present the main result of this section.

Theorem 5.3. *Algorithm 1 computes the maximal clearing payments under priority-proportional strategies in polynomial time.*

Proof. The algorithm proceeds in rounds. In each round μ, tentative vectors of payments, $\mathbf{p}^{(\mu)} = (p_1^{(\mu)}, \ldots, p_n^{(\mu)})$, and effective equities, $E^{(\mu)}$, are computed. At the beginning of round 0, all banks are marked as tentatively solvent, which we denote by $\mathcal{D}_{-1} = \emptyset$; $\mathcal{D}_\mu$ is used to denote the banks in default after the μ-th round of the algorithm. The algorithm works so that once a bank is in default in some round, then it remains in default until the termination of the algorithm. Indeed, by Lemma 5.2 the vectors of payments are non-increasing between rounds and the strategies are fixed. Algorithm PROPER is called when $\mathcal{D}_\mu = \mathcal{D}_{\mu-1}$, which requires at most n rounds; clearly, each round requires polynomial time. The running time of PROPER is also polynomial. Indeed, note that each bank can enter set MARKED at most once and will leave MARKED to join set CHECKED after each other bank is examined at most once.

Regarding the correctness of the Algorithm, we start by proving by induction that the payment vector provided as input to Algorithm 2 is at least equal to the maximal clearing vector $\mathbf{p}^*$. As a base of our induction, it is easy to see that $\mathbf{p}^{(-1)} = \mathbf{l} \geq \mathbf{p}^*$. Now assume that $\mathbf{p}^{(\mu-1)} \geq \mathbf{p}^*$ for some $\mu \geq 0$; we will prove that $\mathbf{p}^{(\mu)} \geq \mathbf{p}^*$. We denote by $\mathcal{D}_*$ the banks in default under the maximal clearing vector $\mathbf{p}^*$, i.e., $\mathcal{D}_* = \left\{v_i : e_i + \sum_{j\in[n]} p^*_{ji} < L_i\right\}$. Our inductive hypothesis $\mathbf{p}^{(\mu-1)} \geq \mathbf{p}^*$ implies $\mathcal{D}_\mu \subseteq \mathcal{D}_*$. Hence, for banks $v_i \in \mathcal{N} \setminus \mathcal{D}_\mu$, we have $\mathbf{p}_i^{(\mu)} = L_i \geq p_i^*$. For $v_i \in \mathcal{D}_\mu$ we refer to the proof of Lemma 5.2 above and consider Equation (5.4) again while starting the recursive solution with $x_i^{(0)} = p_i^{(\mu)}$. For $v_i \in \mathcal{D}_\mu$, we observe

$$
\begin{aligned}
x_i^{(1)} &= \alpha e_i + \beta \left(\sum_{j\in\mathcal{D}_\mu} p_{ji}^{(\mu)} + \sum_{j\in\mathcal{N}\setminus\mathcal{D}_\mu} l_{ji} \right) \geq \alpha \cdot e_i + \beta \cdot \left(\sum_{j\in\mathcal{D}_\mu} p^*_{ji} + \sum_{j\in\mathcal{N}\setminus\mathcal{D}_\mu} l_{ji} \right) \\
&\geq \alpha e_i + \beta \left(\sum_{j\in[n]} p^*_{ji} \right) = L_i^{(0)}.
\end{aligned}
$$

Recursion (5.4) then implies that $x^{(k)} \geq \mathbf{p}^*$ for all k and hence $\mathbf{p}^{(\mu+1)} = \mathbf{x} = \lim_{k\to\infty} \mathbf{x}^k \geq \mathbf{p}^*$.

We have proved that the input to Algorithm 2 is at least equal (pointwise) to the maximal clearing vector $\mathbf{p}^*$. However, by Lemma 5.2 we know that $\mathbf{p}^{(\mu)} \leq \mathbf{p}^{(\mu-1)}$ for all $\mu \geq 0$. It holds by design that the input of Algorithm 2 is a clearing vector, so $\mathbf{p}^*$ is the only possible such input. By the arguments in the proof of Lemma 5.1, Algorithm 2 with input the maximal clearing payments computes the maximal proper clearing payments, and the claim follows. □

5.3 Priority-Proportional Payment Games

We will now define the notion of *Nash equilibrium* in a financial network game. First, let us stress that a strategy profile has consistent clearing payments that are not necessarily unique. It is standard practice (see, e.g. [48, 116]) to focus attention to maximal clearing payments (such payments point-wise maximize all corresponding payments) to avoid this ambiguity. So, we say that a strategy profile $\mathbf{s}$ is a *Nash equilibrium* if no bank can increase her utility by deviating to another payment strategy. We only consider pure Nash equilibria and clarify that the utility of a bank for a given strategy profile is computed based on the assumption that the maximal clearing payments will be realized every time. In Section 5.2, we show how we can compute such payments efficiently. Extending strategy deviations to coalitions and joint deviations, we are interested in *strong equilibria* where no coalition can cooperatively deviate so that all coalition members obtain strictly greater utility.

5.3.1 Maximizing total assets

We now turn our attention to financial network games under priority-proportional strategies when the utility is defined as the total assets. We note that in this case, the maximal clearing payments, computed in Section 5.2 are weakly preferred by all banks among all clearing payments of the given strategy profile; indeed, the utility is computed as the sum between a fixed term (external assets) and the incoming payments which are by definition maximized. So, in case of various clearing payments, it is reasonable to limit our attention to the (unique) maximal clearing payments computed in Section 5.2.

We begin with a negative result regarding the existence of Nash equilibria.

Theorem 5.4. *Nash equilibria are not guaranteed to exist when banks aim to maximize their total assets.*

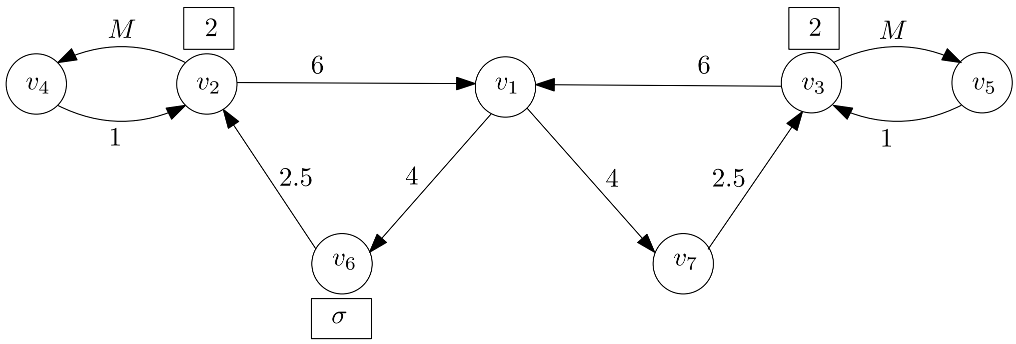


Figure 5.3: A game that does not admit Nash equilibria.

Proof. Consider the financial network depicted in Figure 5.3 where M is an arbitrarily large integer and σ is a positive constant strictly less than $\frac{1}{4} - \frac{15\epsilon}{4}$. Only banks v_1, v_2 and v_3 have more than one available strategies and, hence, it suffices to argue about them. The instance is inspired by an equivalent result in [25] regarding edge-ranking strategies, however, the instance used for that result admits an equilibrium under priority-proportional strategies.

Observe that since M is large, whenever v_2 has v_4 in its top priority class, either alone or together with v_1, then the payment towards v_1 is at most $(5.5 + \sigma)\varepsilon$, where $\varepsilon = \frac{6}{M+6}$, i.e., a very small payment. Similarly, even v_3 prioritizes the payment towards v_1 alone, the payment from v_3 to v_1 is at most 5.5ε. Furthermore, v_2 and v_3 can never fully repay their liabilities to any of their creditors; this implies that any (non-proportional) strategy that has a single creditor in the topmost priority class will never allow for payments to the second creditor.

If none of v_2 and v_3 has v_1 as their single topmost priority creditor, then, by the remark above about payments towards v_1, at least one of them has an incentive to deviate and set v_1 as their top priority creditor. Indeed, each of v_2, v_3 has utility at most $\frac{3+\sigma+3\varepsilon}{1-\varepsilon}$, and at least one of them would receive utility at least 4 by deviating. Furthermore, it cannot be that both v_2 and v_3 have v_1 as their single top priority creditor, as at least one of them is in the top priority class of v_1. This bank, then, wishes to deviate and follow a proportional strategy so as to receive also the payment from its other debtor.

It remains to consider the case where one of v_2, v_3, let it be v_2, has v_1 as the single top priority creditor and the remaining bank, let it be v_3, either has a proportional strategy or has v_5 as its single top priority creditor. In this setting, if v_1 follows a proportional strategy, v_1 wishes to deviate and select v_2 as its single top priority creditor. Otherwise, if v_1 has v_2 as its top priority creditor, then v_3 deviates and sets v_1 as its top creditor. Finally, if v_1 has v_3 as its single top priority creditor, then v_3 deviates to a proportional strategy.

The full table of utilities appears in Table 5.1.

$\square$

$s_1 = (v_6 \vert v_7)$			
	$s_3 = (v_1 \vert v_5)$	$s_3 = (v_1, v_5)$	$s_3 = (v_5 \vert v_1)$
$s_2 = (v_1 \vert v_4)$	$9, 4.5, 4.5$	$\frac{4.5-\varepsilon}{1-\varepsilon}, 4.5, \frac{3.5}{1-\varepsilon}$	$4.5, 4.5, 3.5$
$s_2 = (v_1, v_4)$	$2 + \frac{(5+\sigma)\varepsilon}{1-\varepsilon}, \frac{5+\sigma}{1-\varepsilon}, 2$	$\frac{(6+\sigma)\varepsilon}{1-\varepsilon}, \frac{3+3\varepsilon+\sigma}{1-\varepsilon}, 3$	$\frac{(3+\sigma)\varepsilon}{1-\varepsilon}, \frac{3+\sigma}{1-\varepsilon}, 3$
$s_2 = (v_4 \vert v_1)$	$2, 5+\sigma, 2$	$3\varepsilon, 3+3\varepsilon+\sigma, 3$	$0, 3, 3$
$s_1 = (v_7 \vert v_6)$			
	$s_3 = (v_1 \vert v_5)$	$s_3 = (v_1, v_5)$	$s_3 = (v_5 \vert v_1)$
$s_2 = (v_1 \vert v_4)$	$9, 4.5, 4.5$	$2 + \frac{(5\varepsilon+\sigma)}{1-\varepsilon}, 2+\sigma, \frac{(5+\sigma)}{1-\varepsilon}$	$2+\sigma, 2+\sigma, 5+\sigma$
$s_2 = (v_1, v_4)$	$\frac{4.5-1.5\varepsilon+\sigma\varepsilon}{1-\varepsilon}, \frac{3.5+\sigma}{1-\varepsilon}, 4.5$	$\frac{(6+\sigma)\varepsilon}{1-\varepsilon}, 3+\sigma, \frac{3+3\varepsilon+\sigma\varepsilon}{1-\varepsilon}$	$3\varepsilon+3\sigma, 3+\sigma, 3+3\varepsilon+\sigma\varepsilon$
$s_2 = (v_4 \vert v_1)$	$4.5, 3.5+\sigma, 4.5$	$\frac{3\varepsilon}{1-\varepsilon}, 3+\sigma, \frac{3}{1-\varepsilon}$	$0, 3+\sigma, 3$
$s_1 = (v_6, v_7)$			
	$s_3 = (v_1 \vert v_5)$	$s_3 = (v_1, v_5)$	$s_3 = (v_5 \vert v_1)$
$s_2 = (v_1 \vert v_4)$	$9, 4.5, 4.5$	$\frac{4+6\varepsilon+2\sigma}{1-\varepsilon}, \frac{4+\varepsilon+2\sigma-\sigma\varepsilon}{1-\varepsilon}, \frac{5+\sigma}{1-\varepsilon}$	$4+2\sigma, 4+2\sigma, 5+\sigma$
$s_2 = (v_1, v_4)$	$\frac{4+6\varepsilon+2\sigma}{1-\varepsilon}, \frac{5+\sigma}{1-\varepsilon}, \frac{4+\varepsilon+\sigma\varepsilon}{1-\varepsilon}$	$\frac{6\varepsilon+\sigma\varepsilon}{1-\varepsilon}, \frac{6+\sigma\varepsilon}{2(1-\varepsilon)} + \sigma, \frac{6+\sigma\varepsilon}{2(1-\varepsilon)}$	$\frac{6\varepsilon+2\sigma\varepsilon}{2-\varepsilon}, \frac{6+2\sigma}{2-\varepsilon}, \frac{6+\sigma\varepsilon}{2-\varepsilon}$
$s_2 = (v_4 \vert v_1)$	$4, 5+\sigma, 5$	$\frac{6\varepsilon}{2-\varepsilon}, \frac{6\varepsilon}{2-\varepsilon} + \sigma, \frac{6\varepsilon}{2-\varepsilon}$	$0, 3+\sigma, 3$

Table 5.1: The table of utilities for the network without Nash equilibria; note that $\varepsilon = \frac{6}{M+6}$ and $\sigma < \frac{1}{4} - \frac{15\varepsilon}{4}$. Each cell entry contains the utilities of v_1, v_2, v_3 in that order. Each 3×3 subtable corresponds to a fixed strategy for v_1.

Remark 1. Note that the network as shown in Figure 5.3 does not admit Nash equilibrium if $\sigma < \frac{1}{4} - \frac{15\epsilon}{4}$. However, once this condition does not hold, i.e., $\sigma \geq \frac{1}{4} - \frac{15\epsilon}{4}$, then the strategy profile where v_1 and v_3 pay proportionally while v_2 plays $(v_1|v_4)$ forms a Nash equilibrium since nobody has incentive to deviate unilaterally.

We now quantify the social welfare loss in Nash equilibria when each bank aims to maximize its total assets. While the focus is on financial network games under priority-proportional payments, we warm-up by considering the well-studied case of proportional payments, where we show that these may lead to outcomes where the social welfare can be far from optimal.

Theorem 5.5. *Proportional payments can lead to arbitrarily bad social welfare loss with respect to total assets. In acyclic financial networks, the social welfare loss is at most a factor of* $n/2$.

Proof. Consider the financial network between banks v_1, v_2, and v_3 that is shown in Figure 5.4, where M is arbitrarily large. Observe that paying proportionally leads to clearing

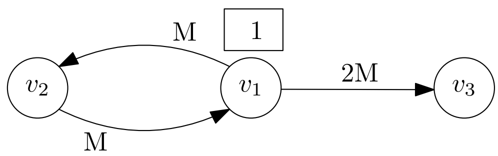


Figure 5.4: A financial network where proportional payments lead to low social welfare with respect to total assets. M is arbitrarily large.

payments $\mathbf{p}_1 = (0, 1/2, 1), \mathbf{p}_2 = (1/2, 0, 0), \mathbf{p}_3 = (0, 0, 0)$. Hence, the total assets are $a_1(\mathbf{P}) = 3/2$, $a_2(\mathbf{P}) = 1/2$, and $a_3(\mathbf{P}) = 1$, and, therefore, $SW(\mathbf{P}) = 3$. However, if bank v_1 chooses to pay bank v_2, the resulting clearing payments would be $\mathbf{p}'_1 = (0, M, 1), \mathbf{p}'_2 = (M, 0, 0), \mathbf{p}'_3 = (0, 0, 0)$ with total assets $a_1(\mathbf{P}') = 1 + M$, $a_2(\mathbf{P}') = M$, and $a_3(\mathbf{P}') = 1$, that sum up to $SW(\mathbf{P}') = 2M + 2$. Since $OPT \geq SW(\mathbf{P}')$ and M can be very large, we conclude that the social welfare achieved by proportional payments can be arbitrarily smaller than the optima.

For the case of acyclic financial networks, let ξ_1 and ξ_2 denote the total external assets of non-leaf and leaf nodes, respectively; note that a leaf node has no creditors in the network. Clearly, since there are no default costs, a non-leaf bank v_i with external assets e_i and total liabilities L_i will generate additional revenue of at least $\min\{e_i, L_i\}$ through payments to its neighboring banks. Hence, for any clearing payments $\mathbf{P}$ we have that $SW(\mathbf{P}) \geq \xi_1 + \sum_i \min\{e_i, L_i\} + \xi_2$. In the optimal clearing payments, each non-leaf bank v_i with external assets e_i may generate additional revenue of at most $(n-1) \cdot \min\{e_i, L_i\}$, since the network is acyclic, and, therefore, we have that $OPT \leq \xi_1 + (n-1) \sum_i \min\{e_i, L_i\} + \xi_2$, i.e., the social welfare loss is at most a factor of $n/2$ as the ratio is maximized when $e_i = L_i$ for each v_i.

To see that this social welfare loss factor is almost tight, consider an acyclic financial network where bank v_1 with external asset $e_1 = 1$ has two creditors, v_2 and v_3, with $l_{12} = l^0_{12} = M$ and $l_{13} = l^0_{13} = 1$, where M is an arbitrarily large integer. bank v_3 is, then, the first bank along a path from v_3 to v_n, where for $i \in \{4, \dots, n\}$ v_i is a creditor of v_{i-1} and all liabilities equal 1. Under proportional payments, we obtain that $\mathbf{p}_1 = (0, M/(M+1), 1/(M+1), \dots, 0), \mathbf{p}_2 = \mathbf{p}_n = (0, \dots, 0)$ and for any $i \in \{3, \dots, n-1\}$, $\mathbf{p}_i = (0, \dots, 1/(M+1), \dots, 0)$ where $1/(M+1)$ is the $(i+1)^{th}$ entry, thus $SW(\mathbf{P}) = 2 + (n-3)/(M+1)$. In the optimal clearing payments, bank v_1 fully repays its liability towards v_3 and this payment propagates along the path from v_3 to v_n resulting to $OPT = n-1$, leading to a social welfare loss factor of $(n-1)/2 - \varepsilon$, where ε goes to zero as M tends to infinity. □

We remark that, given clearing payments with proportional payments, a bank may have incentives to deviate unilaterally.

Remark 2. Proportional payments may not form a Nash equilibrium when banks aim to maximize their total assets.

Proof. Consider the instance shown in Figure 5.4 and the proof of Theorem 5.5; bank v_1 has more total assets when playing strategy $(v_2|v_3)$ than using the proportional strategy. (v_2, v_3). □

We now turn our attention to priority-proportional strategies. To avoid text repetitions, we omit referring to priority-proportional games in our statements. We start with a positive result on the quality of equilibria when allowing for default costs in the (extreme) case $\alpha = \beta = 0$.

Theorem 5.6. *The Price of Stability is* 1 *if default costs* $\alpha = \beta = 0$ *apply and banks aim to maximize their total assets.*

Proof. Consider the equilibrium resulting to the optimal social welfare. Clearly, each solvent bank pays all its liabilities to its creditors, hence, its strategy is irrelevant. Furthermore, each defaulted bank cannot make any payments toward its creditors, due to the default costs, therefore its strategy is irrelevant as well. Since no bank can increase its total assets, any strategy profile resulting in optimal social welfare is a Nash equilibrium and the theorem follows. □

In the more general case, however, the Price of Stability may be unbounded, as the following result suggests. Recall that the setting without default costs corresponds to $\alpha = \beta = 1$.

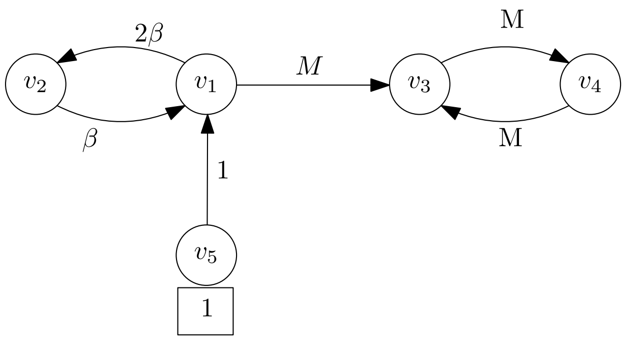
Theorem 5.7. *The Price of Stability is unbounded if default costs $\alpha > 0$ or $\beta > 0$ apply when banks aim to maximize their total assets.*

Proof. We begin with the case where $\beta > 0$ and consider the financial network shown in Figure 5.5(a) where M is arbitrarily large. bank v_1 is the only bank that can strategize about its payments. Its strategy set comprises $(v_2|v_3)$, $(v_3|v_2)$, and (v_2, v_3), which result in utility $1 + \beta$, 1, and $1 + \frac{2\beta^3}{M+2\beta-2\beta^3}$, respectively; note that, unless v_1 selects strategy $(v_2|v_3)$, v_2 is also in default as M is arbitrarily large. For sufficiently large M, we observe that any Nash equilibrium must have v_1 choosing strategy $(v_2|v_3)$, leading to clearing payments $\mathbf{p}_1 = (0, \beta^2+\beta, 0, 0, 0), \mathbf{p}_2 = (\beta, 0, 0, 0, 0), \mathbf{p}_3 = \mathbf{p}_4 = (0, 0, 0, 0, 0)$, and $\mathbf{p}_5 = (1, 0, 0, 0, 0)$ with $SW(\mathbf{P}) = 2 + \beta^2 + 2\beta$. Now, when v_1 chooses strategy $(v_3|v_2)$, we obtain clearing payments $\mathbf{p}'_1 = (0, 0, \beta, 0, 0), \mathbf{p}'_2 = (0, 0, 0, 0, 0), \mathbf{p}'_3 = (0, 0, 0, M, 0), \mathbf{p}'_4 = (0, 0, M, 0, 0)$, and $\mathbf{p}'_5 = (1, 0, 0, 0, 0)$ with $SW(\mathbf{P}') = 2M + 2 + \beta$. The claim follows since $OPT \geq SW(\mathbf{P}')$.

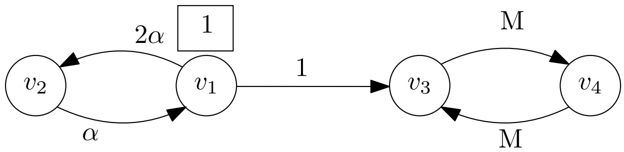
Now, let us assume that $\alpha > 0$ and $\beta = 0$ and consider the financial network shown in Figure 5.5(b). Again, bank v_1 is the only bank that can strategize about its payments. v_1's total assets when choosing strategies $(v_2|v_3)$, $(v_3|v_2)$, and (v_2, v_3) are $1+\alpha$, 1, and 1, respectively; note that when v_1 chooses strategy (v_2, v_3), bank v_2 is also in default as it receives a payment of $\frac{2\alpha^2}{1+2\alpha}$ which is strictly less than α for any $\alpha > 0$. Hence, in any Nash equilibrium v_1 chooses strategy $(v_2|v_3)$, resulting in clearing payments $\mathbf{p}_1 = (0, \alpha, 0, 0), \mathbf{p}_2 = (\alpha, 0, 0, 0), \mathbf{p}_3 = \mathbf{p}_4 = (0, 0, 0, 0)$ with social welfare $SW(\mathbf{P}) = 1 + 2\alpha$. The optimal social welfare, however, is achieved when v_1 chooses strategy $(v_3|v_2)$, resulting in clearing payments $\mathbf{p}'_1 = (0, 0, \alpha, 0), \mathbf{p}'_2 = (0, 0, 0, 0), \mathbf{p}'_3 = (0, 0, 0, M)$, and $\mathbf{p}'_4 = (0, 0, M, 0)$ with $OPT = 2M + 1 + \alpha$. □

We now show that the Price of Stability may also be unbounded in the absence of default costs, if negative external assets are allowed.

Theorem 5.8. *The Price of Stability is unbounded if negative external assets are allowed and banks aim to maximize their total assets.*



(a) The case where $\beta > 0$



(b) The case where $\alpha > 0$ and $\beta = 0$

Figure 5.5: The instances used in the proof of Theorem 5.7.

Proof. Consider the financial network shown in Figure 5.6.

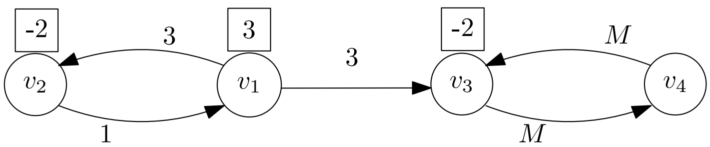


Figure 5.6: A financial network with negative external assets admits an unbounded price of stability.

Only bank v_1 can strategize about its payments. Clearly, v_1's total assets equal 3, unless v_2 pays its debt (even partially). This is only possible when v_1 chooses strategy $(v_2|v_3)$ and prioritizes the payment of v_2 (note v_2's negative external assets will "absorb" any payment that is at most 2). Therefore, the resulting Nash equilibrium leads to the clearing payments $\mathbf{p}_1 = (0, 3, 1, 0), \mathbf{p}_2 = (1, 0, 0, 0), \mathbf{p}_3 = (0, 0, 0, 0)$, and $\mathbf{p}_4 = (0, 0, 0, 0)$ with social welfare $SW(\mathbf{P}) = 4$. However, when v_1 chooses strategy $(v_3|v_2)$ we obtain the clearing payments $\mathbf{p}'_1 = (0, 0, 3, 0), \mathbf{p}'_2 = (0, 0, 0, 0), \mathbf{p}'_3 = (0, 0, 0, M)$, and $\mathbf{p}'_4 = (0, 0, M, 0)$ with social welfare $SW(\mathbf{P}') = 2M + 2$. Hence, since $OPT \geq SW(\mathbf{P}')$ we obtain the theorem. $\square$

The proof of Theorem 5.8 in fact holds for any type of strategies as v_1 always prefers to pay in full its liability to v_2. This includes the case of a very general payment strategy scheme, namely coin-ranking strategies [25] that are known to have a Price of Stability of 1 with non-negative external assets.

Next, we show that the Price of Anarchy can be unbounded even in the absence of default costs, CDS contracts, and negative externals. Bertschinger et al. [25] have shown a similar result for coin-ranking strategies, albeit for a network that has no external assets; our result extends to the case of coin-ranking strategies and strengthens the result of [25] to capture the case of proper clearing payments.

Theorem 5.9. *The Price of Anarchy is unbounded when banks aim to maximize their total assets.*

Proof. Consider the financial network between banks v_i, $i \in [4]$, that is shown in Figure 5.7, where M is arbitrarily large. Clearly, bank v_1 is the only bank that can strategize about

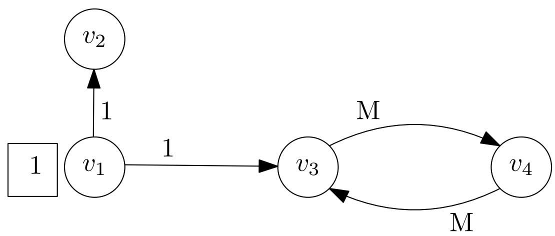


Figure 5.7: A financial network with an unbounded Price of Anarchy with respect to total assets.

its payments, and observe that $a_1 = 1$ regardless of v_1's strategy. Hence, any strategy profile admits a Nash equilibrium. Consider the clearing payments $\mathbf{p}_1 = (0, 1, 0, 0), \mathbf{p}_2 = \mathbf{p}_3 = \mathbf{p}_4 = (0, 0, 0, 0)$ that are obtained when v_1's strategy is $s_1 = (v_2|v_3)$ and note that $SW(\mathbf{P}) = 2$. If, however, v_1 selects strategy $s'_1 = (v_3|v_2)$, we end up with the clearing payment $\mathbf{p}'_1 = (0, 0, 1, 0), \mathbf{p}'_2 = (0, 0, 0, 0), \mathbf{p}'_3 = (0, 0, 0, M)$, and $\mathbf{p}'_4 = (0, 0, M, 0)$ and we obtain $SW(\mathbf{P}') = 2M + 2$. Hence, PoA $\geq \frac{2M+2}{2} = M + 1$ which can become arbitrarily large. $\square$

5.3.2 Maximizing equity

In this section we consider the case of equities. Similarly, the social welfare is defined as the sum of equities. We present interesting properties of clearing payments and observe that Nash equilibria always exist in such games, contrary to the case of total assets.

We warm up with a known statement in absence of default costs; the short proof is included here for completeness. In particular, each bank obtains the same equity under all clearing payments, so it does not have a preference; this provides additional justification to our assumption to limit our attention to maximal clearing payments computed in Section 5.2, in case of various clearing payments.

Lemma 5.10 ([65])**.** *Each bank obtains the same equity under different clearing payments, given a strategy profile. That is, given the banks' strategies, for any two different (not necessarily maximal) clearing payments* $\mathbf{P}$ *and* $\mathbf{P}'$*, it holds* $E_i(\mathbf{P}) = E_i(\mathbf{P}')$ *for each bank* v_i*.*

Proof. Let $\mathbf{P}^*$ be the maximal clearing payments and let $\mathbf{P}$ be any other clearing payments. The corresponding equities for each bank v_i are

$$E_i(\mathbf{P}^*) = \max(0, e_i + \sum_{j\in[n]} p^*_{ji} - L_i) = e_i + \sum_{j\in[n]} p^*_{ji} - \sum_{j\in[n]} p^*_{ij} \tag{5.5}$$

and

$$E_i(\mathbf{P}) = \max(0, e_i + \sum_{j\in[n]} p_{ji} - L_i) = e_i + \sum_{j\in[n]} p_{ji} - \sum_{j\in[n]} p_{ij}, \tag{5.6}$$

respectively. The rightmost equalities above hold, since each bank either pays all its liabilities if it is solvent, or uses all its total assets to pay part of its liabilities if it is in default.

Using (5.5) and (5.6) and by summing over all banks, we obtain

$$\sum_{i\in[n]} E_i(\mathbf{P}^*) - \sum_{i\in[n]} E_i(\mathbf{P}) = \sum_{i\in[n]}\sum_{j\in[n]} \left(p^*_{ji} - p^*_{ij}\right) - \sum_{i\in[n]}\sum_{j\in[n]} (p_{ji} - p_{ij}) = 0,$$

as, for any clearing payments, the total incoming payments equal the total outgoing payments.

We remark that, since $\mathbf{P}^*$ are maximal clearing payments we get that $\sum_{j\in[n]} p^*_{ij} \geq \sum_{j\in[n]} p_{ij}$ for each bank v_i and, hence, by (5.5) and (5.6) it can only be that $E_i(\mathbf{P}^*) \geq E_i(\mathbf{P})$. Therefore, since we have shown that $\sum_{i\in[n]} E_i(\mathbf{P}^*) = \sum_{i\in[n]} E_i(\mathbf{P})$, we conclude that $E_i(\mathbf{P}^*) = E_i(\mathbf{P})$ for each bank v_i. □

Lemma 5.10 also indicates that, for any given strategy profile, any bank is always either solvent or in default in all resulting clearing payments. We exploit this property to obtain the following result; this extends a result by Papp and Wattenhofer (Theorem 7 in [106]) which holds for the maximal clearing payments.

Theorem 5.11. *Even with CDS contracts, any strategy profile is a Nash equilibrium, when banks aim to maximize their equity. This holds even if the clearing payments that are realized are not maximal.*

Proof. Assume otherwise that there exists a strategy profile which is not a Nash equilibrium. Let v_i be a bank that wishes to deviate and s_i be its strategy. Clearly, if v_i is solvent it can repay all its liabilities in full and, hence, the payment priorities are irrelevant. So, let us assume that v_i is in default. If, by deviating, v_i remains in default, then its equity remains 0. Therefore, assume that v_i deviates to another strategy s'_i where it is solvent. In that case, however, v_i could fully repay its liabilities regardless of the payment priorities and, therefore, it can still repay its liabilities when playing s_i; a contradiction. □

Note that Lemma 5.10 no longer holds once default costs are introduced; see e.g., Example 3.3 in [116] where both banks, each having a singleton strategy set, may be in default or solvent depending on the clearing payments. The next result extends Theorem 7 in [106] to the setting with default costs, and guarantees the existence of Nash equilibria (and actually strong ones) when banks wish to maximize their equity.

Theorem 5.12. *Even with default costs and negative external assets, any strategy profile is a strong equilibrium when banks aim to maximize their equity.*

Proof. We begin by transforming an instance $\mathcal{I}$ with negative external assets into another instance $\mathcal{I}'$ without negative external assets, albeit with a slightly restricted strategy space for each bank. In particular, we add an auxiliary bank t and define liabilities and assets as follows. For any pair of banks v_i, v_j which does not include t, we set $l'^0_{ij} = l^0_{ij}$. For each bank v_i with $e_i < 0$, we set $e'_i = 0$ and set liability $l'^0_{it} = e_i$, while for any other bank i' we set $e'_{i'} = e_{i'}$ and $l'^0_{i't} = 0$. Furthermore, we restrict the strategy space of each bank in $\mathcal{I}'$ so that their topmost priority class includes only bank t. An example of this process is shown in Figure 5.8.

Given a strategy profile for instance $\mathcal{I}$, we create the corresponding strategy profile for instance $\mathcal{I}'$ by having bank t as the single topmost priority creditor and, then, append the initial strategy profile. It holds that, for any given strategy profile, the maximal clearing payments for instance $\mathcal{I}$ corresponds to maximal clearing payments in $\mathcal{I}'$ for the new strategy

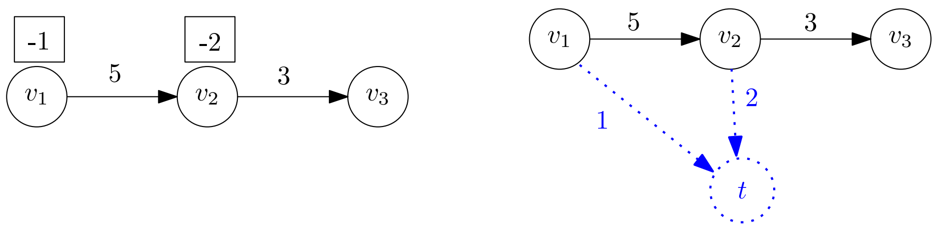


Figure 5.8: Transforming an instance with negative external assets into an instance without negative external assets.

profile; this can be easily proved by contradiction. We can now proceed with the proof by assuming non-negative external assets, without loss of generality.

Consider maximal clearing payments $\mathbf{P}^*$ and the associated strategy profile s. Let us assume that there is a coalition of banks $C = \{v_{C_1}, \dots, v_{C_k}\}$ where each member of the coalition can strictly increase its equity after a joint deviation. In particular, let v_{C_i}, for $i = 1, \dots, k$ change its strategy from s_{C_i} to s'_{C_i}. Clearly, each v_{C_i} must have a strictly positive equity in the resulting new maximal clearing payments $\mathbf{P}'$, since its equity was 0 before. But then, each v_{C_i} should remain solvent under strategy s_{C_i} as well, and therefore v_{C_i}'s actual payment priorities are irrelevant, for $i = 1, \dots, k$. Note that $\mathbf{P}'$ should also be maximal clearing payments under the initial strategy profile; a contradiction to the maximality of $\mathbf{P}^*$. $\square$

We start by noting that Lemma 5.10 together with Theorem 5.11 imply the following, which we note holds for any payment scheme.

Corollary 5.13. *The Price of Anarchy in financial network games with CDS contracts is* 1 *when banks aim to maximize their equity.*

The above positive result, however, no longer holds when default costs or negative external assets exist. For these cases we derive the following results.

Theorem 5.14. *The Price of Anarchy with default costs* (α, β) *when banks aim to maximize their equity, is*

a*)* 1 *when* $\alpha = \beta = 0$*,*

b*) unbounded when: i)* $\beta \in (0, 1)$*, ii)* $\beta = 0$ *and* $\alpha \in (0, 1]$*, or iii)* $\beta = 1$ *and* $\alpha = 0$*,*

c) *at least* $1/\alpha - \varepsilon$*, if* $\beta = 1$ *and* $\alpha \in (0,1)$ *for any* $\varepsilon > 0$.

Proof. We begin with the case $\alpha = \beta = 0$. We claim that all strategy profiles correspond to the same clearing payments hence admit the same social welfare. It suffices to observe that neither the strategy of a solvent bank, nor the strategy of a bank in default, affect the set of banks in default and consequently the clearing payments. Consider two strategy profiles $\mathbf{s}$ and $\mathbf{t}$. A bank i that is solvent under $\mathbf{s}$, will be solvent and continue to make payments $p_i = L_i$ under any strategy (given the strategies of everyone else), i.e., will be solvent at the clearing payments consistent with strategy vector $(t_i, \mathbf{s}_{-i})$, derived by $\mathbf{s}$ if bank i alone changes her strategy from s_i to t_i. On the other hand, a bank i that is in default under $\mathbf{s}$, will similarly remain in default under $(t_i, \mathbf{s}_{-i})$ and continue to make 0 payments since $\alpha = \beta = 0$, thus not affecting the set of banks in default. The claim follows by considering the individual deviations from s_i to t_i of all banks i sequentially and observing that the set of banks in default and, hence, the clearing payments are unaffected at each step.

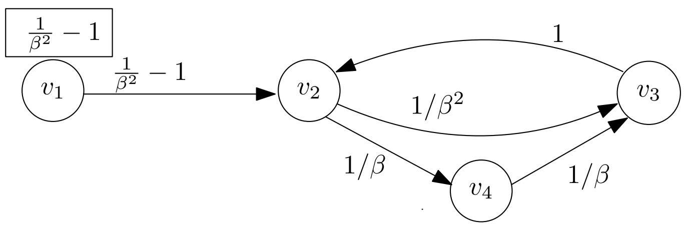


Figure 5.9: The instance in the proof of Theorem 5.14, where we assume $\beta \in (0,1)$.

Regarding the case $\beta \in (0,1)$, consider the financial network in Figure 5.9. Clearly, only bank v_2 can strategize and observe that it is always in default irrespective of its strategy. When v_2 selects strategy $s_2 = (v_4|v_3)$, we obtain the clearing payments $\mathbf{p}_1 = (0, \frac{1}{\beta^2} - 1, 0, 0)$, $\mathbf{p}_2 = (0, 0, 0, \frac{1+\beta}{\beta(\beta^2+\beta+1)})$, $\mathbf{p}_3 = (0, \frac{\beta(1+\beta)}{\beta^2+\beta+1}, 0, 0)$, and $\mathbf{p}_4 = (0, 0, \frac{1+\beta}{\beta^2+\beta+1}, 0)$. Notice that v_2, v_3 and v_4 are in default and we have $SW(\mathbf{P}) = 0$.

When, however, v_2 selects strategy $s_2' = (v_3|v_4)$, we obtain the clearing payments $\mathbf{p}_1' = (0, \frac{1}{\beta^2} - 1, 0, 0)$, $\mathbf{p}_2' = (0, 0, \frac{1}{\beta}, 0)$, $\mathbf{p}_3' = (0, 1, 0, 0)$, and $\mathbf{p}_4' = (0, 0, 0, 0)$. Now, v_2 and v_4 are in default and we have $SW(\mathbf{P}') = 1/\beta - 1$. Since $OPT \geq SW(\mathbf{P}')$, the claim follows.

Regarding the case $\beta = 0$ and $\alpha \in (0,1]$, consider the financial network in Figure 5.10 and note that v_2 is always in default irrespective of its strategy. If v_2 selects strategy $s_2 = (v_6|v_3)$, then v_6 ends up having equity α. However, strategy $(v_3|v_6)$ is also an equilibrium strategy for v_2, but it results in each bank having equity 0.

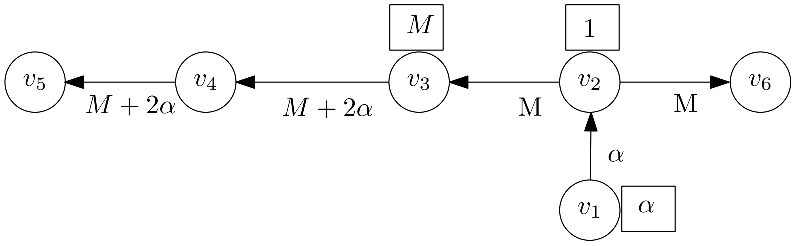


Figure 5.10: The instance in the proof of Theorem 5.14, when $\beta \in \{0, 1\}$ and $\alpha \neq \beta$.

Regarding the remaining case, i.e., $\beta = 1$ and $\alpha \in [0, 1)$, consider again the financial network in Figure 5.10. As before, v_2 is always in default. If v_2 selects strategy $(v_3|v_6)$, then v_5 ends up having equity $M + 2\alpha$. However, $(v_6|v_3)$ is also an equilibrium strategy for v_2, but it results in equity 2α for v_6, equity αM for v_5 and equity 0 for the remaining banks. The theorem follows by straightforward calculations. □

We complement this result with a tight upper bound for the case $\beta = 1$.

Theorem 5.15. *The Price of Anarchy with default costs* $(\alpha, 1)$ *when banks aim to maximize their equity is at most* $1/\alpha$*. This holds even if the clearing payments that are realized are not maximal.*

Proof. Consider any clearing payments $\mathbf{P}$ and let $S(\mathbf{P})$ and $D(\mathbf{P})$ be the set of solvent and in default banks under $\mathbf{P}$. Recall that for any bank v_i, we have $E_i(\mathbf{P}) = \max\{0, a_i(\mathbf{P}) - L_i\}$. For any bank $v_i \in S(\mathbf{P})$, it holds that $E_i(\mathbf{P}) = e_i + \sum_{j\in[n]} p_{ji} - \sum_{j\in[n]} p_{ij}$, as $\sum_{j\in[n]} p_{ij} = \min\{e_i + \sum_{j\in[n]} p_{ji}, L_i\}$. Similarly, for any $v_i \in D(\mathbf{P})$, we have $E_i(\mathbf{P}) = \alpha e_i + \sum_{j\in[n]} p_{ji} - \sum_{j\in[n]} p_{ij}$, since $\beta = 1$. By summing over all banks, we get

$$\begin{aligned} E(\mathbf{P}) &= \sum_{i:v_i\in S(\mathbf{P})} E_i(\mathbf{P}) + \sum_{i:v_i\in D(\mathbf{P})} E_i(\mathbf{P}) \\ &= \sum_{i\in[n]} \left(e_i + \sum_{j\in[n]} p_{ji} - \sum_{j\in[n]} p_{ij} \right) - (1-\alpha) \sum_{i:v_i\in D(\mathbf{P})} e_i \\ &= \sum_{i\in[n]} e_i - (1-\alpha) \sum_{i:v_i\in D(\mathbf{P})} e_i. \end{aligned}$$

Clearly, the social welfare is maximized when $D(\mathbf{P}) = \emptyset$, while it is minimized when $D(\mathbf{P})$ includes all banks, i.e., $E(\mathbf{P}) \geq \alpha \sum_{i\in[n]} e_i$; this completes the proof. □

Extending the setting to allow for negative external assets again may lead to unbounded social welfare loss.

Theorem 5.16. *The Price of Anarchy is unbounded when negative external assets are allowed and banks aim to maximize their equity.*

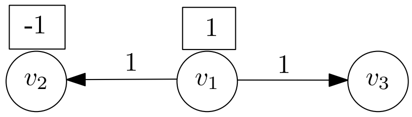


Figure 5.11: A financial network with negative external assets and unbounded Price of Anarchy.

Proof. Consider the financial network in Figure 5.11. Clearly, only v_1 can strategize but, irrespective of its strategy, v_1 is always in default and $E_1 = 0$; hence, any strategy profile is a Nash equilibrium. When v_1 selects strategy $s_1 = (v_2|v_3)$ we obtain the clearing payments $\mathbf{p}_1 = (0, 1, 0), \mathbf{p}_2 = (0, 0, 0), \mathbf{p}_3 = (0, 0, 0)$. However, when v_1 selects strategy $s_1' = (v_3|v_2)$ we obtain the clearing payments $\mathbf{p}_1' = (0, 0, 1), \mathbf{p}_2' = (0, 0, 0), \mathbf{p}_3' = (0, 0, 0)$ with $SW(\mathbf{P}') = 1$. Since $OPT \geq SW(\mathbf{P}')$, the claim follows. □

Still, in the presence of default costs or negative external assets, Theorem 5.12 leads to the next positive result.

Corollary 5.17. *The strong Price of Stability is* 1 *even with default costs and negative external assets, when banks aim to maximize their equity.*

If we relax the stability notion and consider the Price of Stability in super-strong equilibria, where a coalition of banks deviates if at least one strictly improves its utility and no bank suffers a decrease in utility, then we obtain a negative result.

Theorem 5.18. *The Price of Stability in super-strong equilibria is unbounded when negative external assets are allowed and banks aim to maximize their equity.*

Proof. Consider the financial network as shown in Figure 5.12, where $\varepsilon > 0$ is a small constant. Observe that v_1 is always in default since even its maximal possible total asset $a_1 = 2 - \varepsilon$ is still less than its total liabilities $L_1 = 3$, which means $E_1 = 0$ in any scenario. If v_1's strategy is other than $s_1 = (v_2|v_3)$, then v_2 is always in default no matter what strategies it chooses, that is $E_2 = 0$. Thus, any strategy profile where v_1 does not prioritize the payment

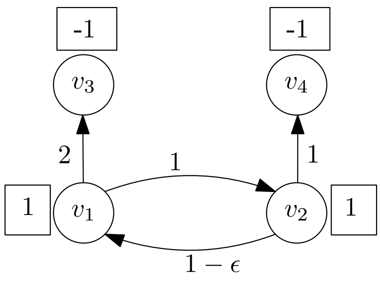


Figure 5.12: A financial network with negative external assets and unbounded Price of Stability in super-strong equilibria.

of v_2 is a Nash equilibrium with $E_1 = E_2 = 0$. However, if v_1 and v_2 form a coalition, then the only superstrong equilibrium occurs when v_1 and v_2 prioritize the payment of each other, resulting in the clearing payments $\mathbf{p}_1 = (0, 1, 1-\epsilon, 0), \mathbf{p}_2 = (1-\epsilon, 0, 0, 1), \mathbf{p}_3 = \mathbf{p}_4 = (0, 0, 0, 0)$ with $SW(\mathbf{P}) = \varepsilon$. Furthermore, when v_1 follows the strategy $s_1' = (v_3|v_2)$ and v_2 follows strategy $s_2' = (v_1|v_4)$, we obtain the clearing payments $\mathbf{p}_1' = (0, 0, 2-\epsilon, 0), \mathbf{p}_2' = (1-\epsilon, 0, 0, \epsilon), \mathbf{p}_3' = \mathbf{p}_4' = (0, 0, 0, 0)$ with $SW(\mathbf{P}') = 1-\varepsilon$. Since $OPT \geq SW(\mathbf{P}')$, we obtain that the superstrong Price of Stability is at least $\frac{1-\varepsilon}{\varepsilon}$ and the theorem follows since ε can be arbitrarily small. □

5.4 Computational Complexity

In the following, we turn our attention to the computational aspects in both centralized and decentralized settings. In particular, we provide some hardness results on addressing equilibria-related problems, and computing payment profiles that maximize social welfare, respectively.

5.4.1 Decentralized case

Theorem 5.19. *The following problems are NP-hard:*

a*) computing a Nash equilibrium when one is guaranteed to exist.*

b*) computing the best response strategy.*

Proof. The proof relies on a reduction from the NP-complete problem RXC3 [63], a variant of EXACT COVER BY 3-SETS (X3C). In RXC3, we are given an element set X, with $|X| =$

$3k$ for an integer k, and a collection C of subsets of X where each such subset contains exactly three elements. Furthermore, each element in X appears in exactly three subsets in C, that is $|C| = |X| = 3k$. The question is if there exists a subset $C' \subseteq C$ of size k that contains each element of X exactly once.

Given an instance $\mathcal{I}$ of RXC3, we construct an instance $\mathcal{I}'$ as follows. There is a central bank v, having external assets of $3k$, and we add a bank t_i for each element i of X, as well as two banks u_i, g_i for each element i in C; all banks except v have zero external assets. bank v has a liability of M to each u_i, where M is an arbitrarily large integer. Each u_i, corresponding to set $(x, y, z) \in C$, has liability M to the three banks t_x, t_y, t_z respectively corresponding to the three elements $x, y, z \in X$. Also, u_i has liability 6 towards g_i, while g_i has liability 3 towards u_i. Finally, each bank t_i has liability 1 towards bank v. Note that this construction requires polynomial time; see also Figure 5.13.

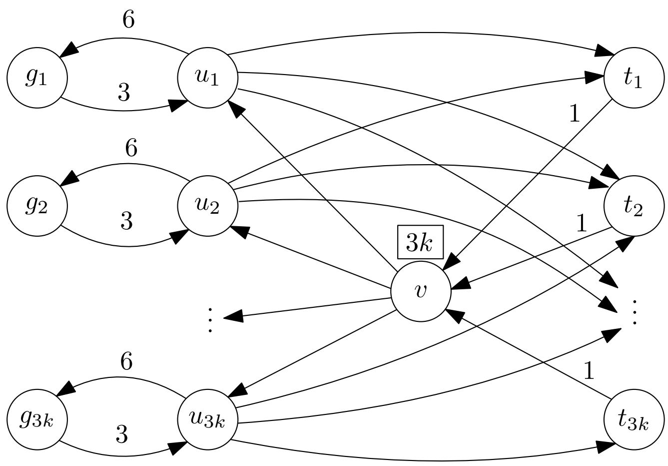


Figure 5.13: The reduction used to prove the hardness of computing a Nash equilibrium when one is guaranteed to exist. All edges with missing labels correspond to liability M where M is an arbitrarily large constant.

In our proof, we will first show the existence of Nash equilibria and, then, we will identify a condition on bank v's external assets in an equilibrium that will allow us to decide instance $\mathcal{I}$ of RXC3. For any fixed strategy profile, we denote by $\mathcal{T}_v$ the set of u_i banks belonging to v's top priority class. We exploit the fact that, in any equilibrium, any u_i in $\mathcal{T}_v$ has just g_i in its top priority class. We prove this claim below.

To prove that an equilibrium exists, observe that any bank t_i or g_i, for $i \in \{1, \ldots, 3k\}$, has only one outgoing edge in the financial network, so it cannot strategize about its payments.

Since M is arbitrarily large and $a_v \leq 6k$, any u_i not in $\mathcal{T}_v$ has zero incoming payments and, hence, its strategy is irrelevant. The equilibrium existence follows by arguing that the strategy profile maximizing a_v is a Nash equilibrium. To see that, observe that if this strategy profile is not an equilibrium, then some u_i from $\mathcal{T}_v$ can deviate profitably. By our claim above, any such u_i has only g_i in its top priority class and, since $a_v \geq 3k$ it holds that $p_{g_i,u_i} = 3$. Therefore, any increase in u_i's total assets must necessarily occur due to an increase in a_v; a contradiction to our choice of strategy profile.

Now, consider any Nash equilibrium; we claim that there exists a solution to instance $\mathcal{I}'$ of RXC3 if and only if $a_v = 6k$ in the equilibrium. Let us assume that there exists a solution to $\mathcal{I}'$, which means that there exists $C' \subseteq C$ of size k that contains each element of X exactly once. In this case, the strategy profile where the top priority class $\mathcal{T}_v$ of bank v includes the u_i's corresponding to $i \in C'$, while any u_i chooses strategy $(g_i|t_x, t_y, t_z)$, forms a Nash equilibrium where $a_v = 6k$. Indeed, observe that, under this strategy profile, each t_i receives a payment of 1 from each neighboring u_i, while, similarly, v receives a payment of 1 from each t_i; hence, $a_v = 6k$. Clearly, v cannot improve its total assets, while g_i's and t_i's cannot strategize. It remains to argue about the u_i's. Note that for those u_i's corresponding to $i \in C/C'$, their strategy is irrelevant as they receive no payment from v, and hence from g_i as well. For those u_i's corresponding to $i \in C'$, it suffices to note that, since v already receives the maximum possible incoming payments, any deviating strategy for u_i cannot increase u_i's incoming payments.

On the other direction, we now show that any Nash equilibrium with $a_v = 6k$ leads to a solution for instance $\mathcal{I}'$. We begin by showing that, in order for $a_v = 6k$ to hold, it must be that $|\mathcal{T}_v| = k$. Indeed, if $|\mathcal{T}_v| < k$, then at most $3 \cdot |\mathcal{T}_v| < 3k$ t_i's receive any payment from the u_i's and, hence, it cannot be the case that v receives a total payment of $3k$ from the t_i's; a contradiction to $a_v = 6k$. Otherwise, when $|\mathcal{T}_v| > k$, since each u_i in $\mathcal{T}_v$ strictly prioritizes g_i, it holds that the payment from each such u_i to its neighboring t_i's is $\max\{0, a_{u_i} - 6\} = \max\{0, \frac{a_v}{|\mathcal{T}_v|} + 3 - 6\}$, as each such u_i receives the same payment from v. So, the total payments from the u_i's in $\mathcal{T}_v$ to the t_i's equals $\sum_{i \in \mathcal{T}_v} \max\left\{0, \frac{a_v}{|\mathcal{T}_v|} - 3\right\} = \max\{0, a_v - 3 \cdot |\mathcal{T}_v|\} < a_v - 3k = 3k$, as $|\mathcal{T}_v| > k$. That is, the total payments received by the t_i's are strictly less than $3k$ and, therefore, so are the payments received by v; a contradiction to $a_v = 6k$.

It suffices to prove our claim that, in any equilibrium, any u_i in $|\mathcal{T}_v|$ has just g_i in its topmost priority class, i.e., this choice maximizes a_{u_i}. Assume that there exists a strategy profile s where this does not hold for at least one u_i in $\mathcal{T}_v$. We will prove that s is not an equilibrium. In particular, we will first treat the case where, under s, g_i is not in $\mathcal{T}_{u_i}$ and, then, the case where g_i is not the only bank in $\mathcal{T}_{u_i}$.

Recall that only bank v and the u_i's can strategize about their payments. For any u_i, we denote by $\mathcal{O}_{u_i}$ the three neighbors of u_i among the t_i's.

Case 1. When g_i is not in $\mathcal{T}_{u_i}$, then $p_{u_i,g_i} = p_{g_i,u_i} = 0$, since M is an arbitrarily large integer. So, $a_{u_i}(\mathrm{s}) = \frac{a_v(\mathrm{s})}{|\mathcal{T}_v|}$. We now consider the deviation of u_i to a strategy having g_i as the only bank in $\mathcal{T}_{u_i}$, regardless of how the remaining priority class are formed; let s$'$ be the resulting profile. In that case, since u_i is in $\mathcal{T}_v$, we have $p_{g_i,u_i} = 3$, as any incoming payment from v to u_i in $\mathcal{T}_v$ will necessarily travel along the cycle with g_i until the liability of g_i to u_i is fully paid. So, the total assets of u_i under its new strategy is $a_{u_i}(\mathrm{s}') = \frac{a_v(\mathrm{s}')}{|\mathcal{T}_v|} + 3$ and it remains to argue about $a_v(\mathrm{s}')$. Observe that the maximum incoming payment to v from banks in $\mathcal{O}_{u_i}$ is at most 3 and this bounds the impact on v's total assets caused by u_i's deviation, i.e., $a_v(\mathrm{s}) - a_v(\mathrm{s}') \leq 3$.

Therefore, we can get

$$a_{u_i}(\mathrm{s}) = \frac{a_v(\mathrm{s})}{|\mathcal{T}_v|} \leq \frac{a_v(\mathrm{s}') + 3}{|\mathcal{T}_v|} = \frac{a_v(\mathrm{s}')}{|\mathcal{T}_v|} + \frac{3}{|\mathcal{T}_v|} < \frac{a_v(\mathrm{s}')}{|\mathcal{T}_v|} + 3 = a_{u_i}(\mathrm{s}').$$

Note that the last strict inequality clearly holds when $|\mathcal{T}_v| > 1$. When $|\mathcal{T}_v| = 1$, there is only bank in $\mathcal{T}_v$ and, then, $a_{u_i}(\mathrm{s}) \leq 3k + 3$ if g_i is not in $\mathcal{T}_{u_i}$. However, if g_i is in $\mathcal{T}_{u_i}$, then $a_{u_i}(\mathrm{s}') \geq 3k + 4 > 3k + 3 \geq a_{u_i}(\mathrm{s})$.

Case 2. When g_i is not the only bank in $\mathcal{T}_v$, due to the arbitrarily large liability M between u_i and the bank(s) from $\mathcal{O}_{u_i}$ belonging to $\mathcal{T}_{u_i}$, we have that $p_{u_i,g_i} = p_{g_i,u_i} < 3$.

Let $\lambda \in \{1, 2, 3\}$ be the number of banks from $\mathcal{O}_{u_i}$ that are in $\mathcal{T}_{u_i}$, together with g_i. Since $p_{u_i,g_i} = p_{g_i,u_i} < 3$ we obtain

$$p_{g_i,u_i} + \frac{a_v(\mathrm{s})}{|\mathcal{T}_v|} = p_{u_i,g_i} + \frac{\lambda M}{6} \cdot p_{u_i,g_i}. \tag{5.7}$$

As $p_{u_i,g_i} = p_{g_i,u_i}$, we get $p_{g_i,u_i} = \frac{6a_v(\mathrm{s})}{\lambda|\mathcal{T}_v| \cdot M}$, therefore the total assets of u_i are $a_{u_i}(\mathrm{s}) = \frac{a_v(\mathrm{s})}{|\mathcal{T}_v|} + \frac{6a_v(\mathrm{s})}{\lambda|\mathcal{T}_v| \cdot M}$. When u_i deviates to a strategy having g_i as the only bank in $\mathcal{T}_{u_i}$, we obtain the strategy profile s$'$ and u_i's total assets become

$$a_{u_i}(\mathbf{s}') = \frac{a_v(\mathbf{s}')}{|\mathcal{T}_v|} + 3.$$

As before, we have $a_v(\mathbf{s}) - a_v(\mathbf{s}') \leq 3$, and we obtain

$$\begin{aligned} a_{u_i}(\mathbf{s}) &= \frac{a_v(\mathbf{s})}{|\mathcal{T}_v|} + \frac{6a_v(\mathbf{s})}{\lambda|\mathcal{T}_v| \cdot M} \\ &\leq \frac{a_v(\mathbf{s}') + 3}{|\mathcal{T}_v|} + \frac{6a_v(\mathbf{s})}{\lambda|\mathcal{T}_v| \cdot M} \\ &= \frac{a_v(\mathbf{s}')}{|\mathcal{T}_v|} + \frac{3}{|\mathcal{T}_v|} + \frac{6a_v(\mathbf{s})}{\lambda|\mathcal{T}_v| \cdot M} \\ &< \frac{a_v(\mathbf{s}')}{|\mathcal{T}_v|} + 3 \\ &= a_{u_i}(\mathbf{s}'), \end{aligned} \tag{5.8}$$

where the last (strict) inequality holds since $|\mathcal{T}_v| > 1$ and since M is arbitrarily large.

Observe that when $|\mathcal{T}_v| = 1$, we have $a_{u_i}(\mathbf{s}) \leq 3k + 3 + \frac{18k+18}{\lambda M}$ when g_i shares the first priority with λ banks from $\mathcal{O}_{u_i}$. However, if u_i selects just g_i to be in $\mathcal{T}_{u_i}$, then $a_{u_i}(\mathbf{s}') \geq 3k + 4 > a_{u_i}(\mathbf{s})$.

This concludes the proof for our claim that, at any equilibrium, each u_i in $\mathcal{T}_v$ has g_i as the only bank in $\mathcal{T}_{u_i}$, and, hence, the first part of the theorem follows.

For our second part of the theorem, about computing best responses, the proof follows along (almost) identical lines to the previous one. Indeed, it suffices to fix the strategy of each u_i so that g_i is the only entry in $\mathcal{T}_{u_i}$ while all banks in $\mathcal{O}_{u_i}$ are in the second priority class. Then, using identical arguments as before, we can show that v has a strategy that leads to $a_v = 6k$ if and only if the instance in RXC3 admits a solution. □

Theorem 5.20. *Determining whether a Nash equilibrium exists is NP-hard.*

Proof. Our proof follows from a reduction from the RXC3 problem, similarly to the proof of Theorem 5.19 for the hardness of computing a Nash equilibrium, when one is guaranteed to exist. In fact, we modify the construction used in the proof of Theorem 5.19 and combine it with the network used in the proof of Theorem 5.4 that doesn't admit Nash Equilibria.

In RXC3, we are given an element set X, with $|X| = 3k$ for an integer k, and a collection C of subsets of X where each such subset contains exactly three elements. Furthermore, each

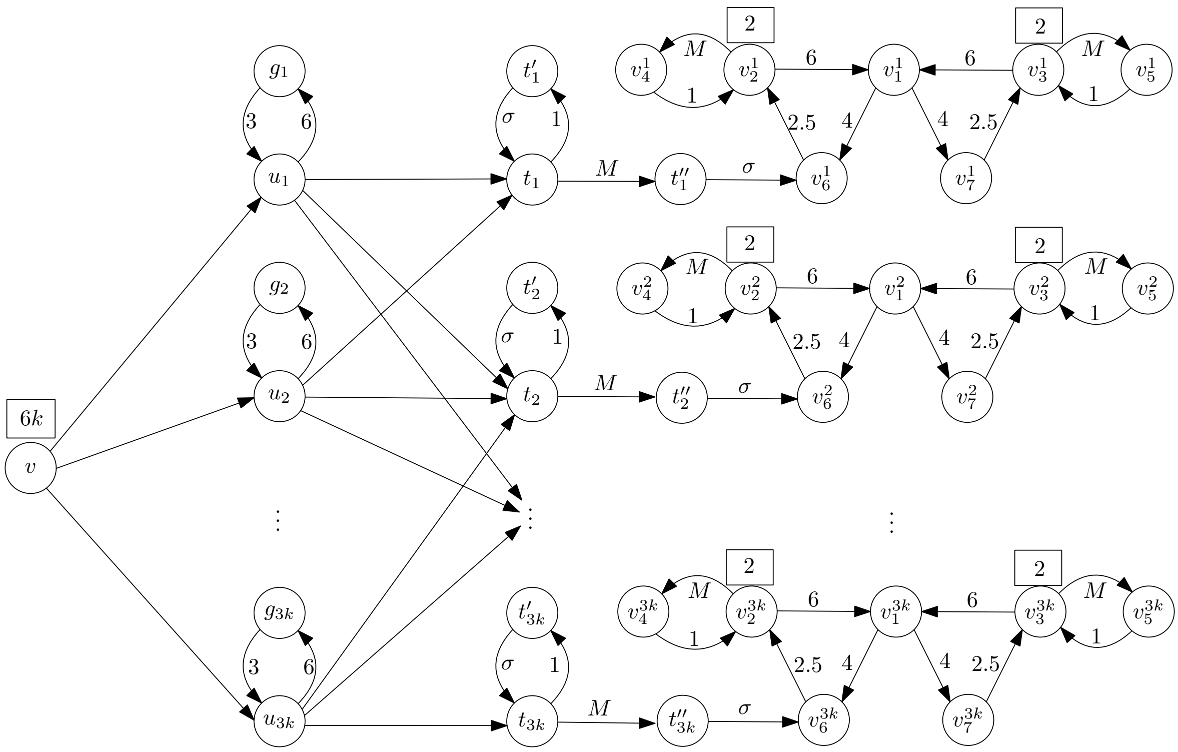


Figure 5.14: The reduction used to show hardness of determining if pure Nash equilibrium exists where M is an arbitrarily large constant, and $\varepsilon = \frac{6}{M+6}$ as well as $\sigma = \frac{1}{4} - \frac{15}{4}\varepsilon$.

element in X appears in exactly three subsets in C with $|C| = |X| = 3k$. The question is if there exists a subset $C' \subseteq C$ of size k that contains each element of X exactly once.

Given an instance $\mathcal{I}$ of RXC3, we construct an instance $\mathcal{I}'$ as follows. There is a central bank v, having external assets of $6k$, and we add banks t_i, t'_i and t''_i for each element i of X, as well as two banks u_i, g_i for each element i in C; all banks except v have zero external assets. bank v has a liability of M to each u_i, where M is an arbitrarily large integer. Each u_i, corresponding to set $(x, y, z) \in C$, has liability M to the three banks t_x, t_y, t_z respectively corresponding to the three elements $x, y, z \in X$. Also, u_i has liability 6 towards g_i, while g_i has liability 3 towards u_i. Furthermore, each t_i has liabilities of 1 and M towards t'_i and t''_i respectively, while bank t'_i has liability σ to t_i. Lastly, we add $3k$ copies of the instance depicted in Figure 5.3 and introduce a liability of σ from each t''_i to bank v_6^i that is the corresponding v_6 bank from the i-th copy; see also Figure 5.14.

As before, let $\mathcal{T}_j$ denote the top priority of bank j, while $\mathcal{O}_{u_i}$ stands for the set of those t_i's with an incoming edge from u_i. Note that, as in the proof of Theorem 5.19, the (weakly) dominant strategy of each u_i is to prioritize g_i alone, while the (weakly) dominant strategy of each t_i is to prioritize t_i' alone. Observe that any u_i in $\mathcal{T}_v$ that collects a payment of x from v will pay $\max\{0, x-3\}$ to the t_i's, while, similarly, any t_i that collects a payment of x from the u_i's will pay $\max\{0, x-(1-\sigma)\}$ to the t_i''.

According to Remark 1, an additional incoming payment of (at least) σ to v_6 in Figure 5.3 is a sufficient and necessary condition for a Nash equilibrium to exist. This implies that a necessary and sufficient condition for a Nash equilibrium to exist in Figure 5.14 is that each t_i'' makes a payment of at least σ to the corresponding v_6^i bank. This, in turn, is equivalent to each t_i receiving payments of at least 1 from the u_i's.

Now, we will claim that there exists a Nash equilibrium if and only if there is a solution to $\mathcal{I}'$ of RXC3. If there is a solution to $\mathcal{I}'$ of RXC3, then we let the top priority class $\mathcal{T}_v$ of bank v include just the u_i's corresponding to $i \in C'$, while any u_i chooses the strategy where g_i is alone in $\mathcal{T}_{u_i}$ while all banks in $\mathcal{O}_{u_i}$ are in the second priority class. This ensures that each t^i receives an incoming payment of 1 from the u_i's, as required for a Nash equilibrium to exist.

On the other hand, if there does not exist a solution to $\mathcal{I}'$ of RXC3, we show that at least one t_i receives a payment of strictly less than 1 and, hence, no equilibrium exists. To see that, we argue about $|\mathcal{T}_v|$. If $|\mathcal{T}_v| < k$, then at most $3|\mathcal{T}_v| < 3k$ t_i's can receive incoming payments from the u_i's, so at least one t_i bank receives less than 1. If $|\mathcal{T}_v| > k$, then the aggregate payment received by the t_i's from the u_i's is at most $6k - 3|\mathcal{T}_v| < 3k$, i.e., again at least one t_i receives less than 1. It remains to argue about the case where $|\mathcal{T}_v| = k$. Then, the only case where k u_i's can cover all $3k$ t_i's is when the RXC3 instance admits a solution; a contradiction. □

5.4.2 Centralized case

In the centralized case, we study how a financial authority (such as a regulator) can control a priority-proportional payment profile in order to maximize social welfare, i.e., the sum of individual utility.

Theorem 5.21. *Computing a strategy profile that maximizes the sum of individual total assets is NP-hard.*

Proof. Our proof follows by a reduction from the PARTITION problem. Recall that in

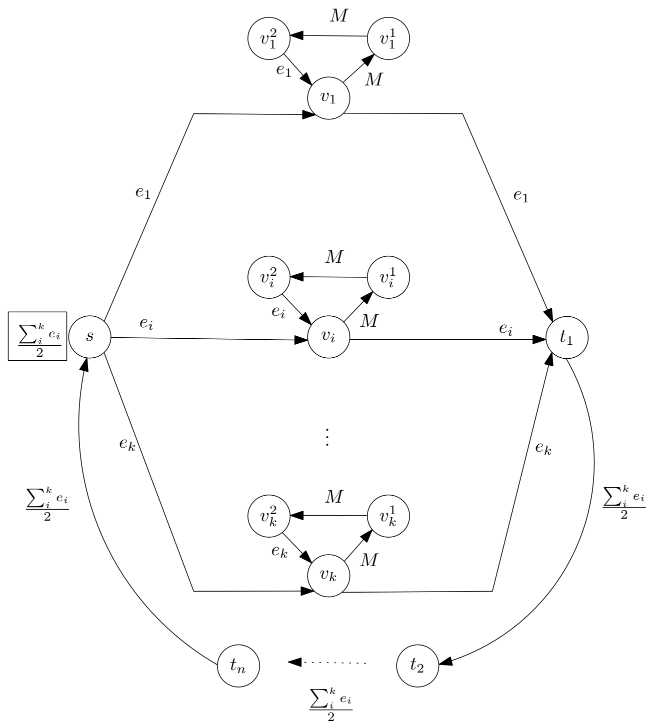


Figure 5.15: The reduction used to show the hardness of computing a strategy profile that maximizes the sum of individual total assets.

PARTITION, an instance $\mathcal{I}$ consists of a set X of positive integers $\{x_1, x_2, ..., x_k\}$ and the question is whether there exists a subset X^* of X such that $\sum_{i\in X^*} x_i = \frac{\sum_{i\in X} x_i}{2} = \frac{S}{2}$; we restrict attention to non-trivial instances where S is even. Starting from $\mathcal{I}$, we build an instance $\mathcal{I}'$ as follows. For each element $x_i \in X$, we add three banks v_i, v_i^1 and v_i^2; for each i, we add edges with liability $M = 3k \max_i {e_i}^2$ from v_i to v_i^1 and again from v_i^1 to v_i^2, while, we add an edge with liability x_i from v_i^2 to v_i. We add an extra bank s that has liability x_i towards each bank v_i, for $i = 1, \dots, k$. s has external assets equal to $\frac{S}{2}$. Moreover, we add a sequence of nodes $t_1, t_2, \dots, t_n$, where $n \geq 10$. We add edges of liability x_i from each v_i to t_1, as well as edges of liability $\frac{S}{2}$ from each t_i to t_{i+1}, for $i = 1, \dots, n-1$. Finaly, there is an edge of liability $\frac{S}{2}$ from t_n to s; see also Figure 5.15. Clearly, the reduction requires polynomial time.

We will prove that there exists a solution to instance $\mathcal{I}$ of PARTITION if and only if social welfare of $\frac{n+9}{2}S = (n+9)\frac{\sum_{i\in X} x_i}{2}$ can be achieved. Let us assume that instance $\mathcal{I}$ admits a solution, i.e. there exists a subset X^* in $\mathcal{I}$ with $\sum_{i\in X^*} x_i = \frac{S}{2}$. Let each v_i that corresponds to $x_i \in X^*$ prioritize payments towards t_1 (pay according to $(t_1|v_i^1)$), while the remaining

v_i's corresponding to $x_i \in X \setminus X'$ prioritize v_i^1 (pay according to $(v_i^1|t_1)$). This implies that the total assets of t_1 are $a_{t_1} \geq \frac{\sum_{i\in X} x_i}{2}$, since all edges in the path from t_1 to s, as well as all outgoing edges of s, are saturated. Overall, the social welfare of this assignment, measured as the sum of the total assets of all banks, satisfies

$$\begin{aligned}
\mathcal{SW}(\mathbf{s}_Y^{opt}) &\geq e_s + \sum_{i\in X} p(s,v_i) + \sum_{i\in X^*} p(v_i,t) + \sum_{i\in X\setminus X^*} \{p(v_i,v_i^1) + p(v_i^1,v_i^2) + p(v_i^2,v_i)\} \\
&+ \sum_{i=1}^{n-1} p(t_i,t_{i+1}) + p(t_n,s) \\
&= \frac{S}{2} + S + \sum_{i\in X^*} x_i + \sum_{i\in X\setminus X^*} \{2x_i + 2x_i + x_i\} + (n-1)\frac{S}{2} + \frac{S}{2} \\
&= (n+3)\frac{S}{2} + \sum_{i\in X^*} x_i + \sum_{i\in X^*} \{2x_i + 2x_i + x_i\} \\
&= (n+9)\frac{S}{2},
\end{aligned}$$

where the last two inequalities hold since X^* is a solution to instance $\mathcal{I}$. In fact, the inequality above is actually equality, as the flow on the edges that we seemingly ignore is always zero, but this is not important at this point.

We will now prove that if there doesn't exist a solution to instance $\mathcal{I}$ of PARTITION then the optimal social welfare is less than $\frac{n+9}{2}S = (n+9)\frac{\sum_{i\in X} x_i}{2}$. We note that each bank v_i has three possible strategies, namely to prioritize t_1, to prioritize v_i^1, and to pay proportionally. The strategy each of these banks, v_i, chooses, determines the payments between itself, v_i^1, v_i^2 and t_1, for a fixed incoming payment. Figure 5.16 shows the actual payments on the respective subgraph based on different strategies of the v_is, if we assume an incoming payment of e_i to v_i; which will be useful later in the proof.

Our proof relies on the following Claim 5.22.

Claim 5.22. *If* $\mathcal{SW}(\mathrm{s}) \geq (n+9)\frac{S}{2}$ *then*

a) *t_1 is solvent,*

b) *$\sum_{v_i\in T} e_i \geq \frac{S}{2}$, where T is the set of v_is prioritising payments to t_1, and*

c) *none of the v_is pay proportionally at the strategy profile that maximizes the sum of total assets.*

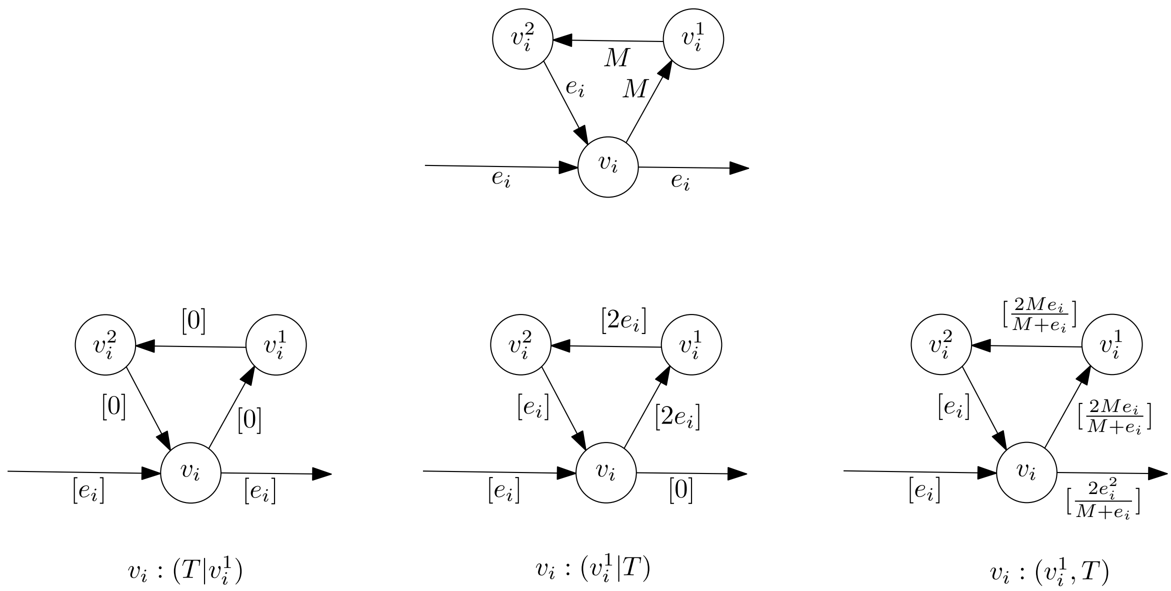


Figure 5.16: The network on the top is the original sub-network of Figure 5.15 which we focus on. The following three subgraphs show the different payments respectively depending on different strategies that v_i uses. The numbers on the square brackets are the corresponding clearing payments.

Proof. We begin with some definitions that will be useful in the proof. Let ξ_1 be the set of v_is that prioritize payments towards t_1 (pay according to $(t_1|v_i^1)$), let ξ_2 be the set of v_i's that prioritize payments towards v_i^1 (pay according to $(v_i^1|t_1)$), and let ξ_3 be the set of v_i's paying proportionally (pay according to (t_1, v_i^1)); clearly ξ_1, ξ_2 and ξ_3 form a partition of the v_is. We prove part (i) by showing that if t_1 is insolvent then the optimal social welfare is less than $\frac{n+9}{2}S = (n+9)\frac{\sum_{i \in X} x_i}{2}$. If we denote the actual payment through the edges on the path from t_1 to s by F, then the social welfare under any strategy profile s, satisfies the following (the calculations are shown in Figure 5.16).

$$\begin{aligned} \mathcal{SW}(\mathbf{s}) &\le e_s + n \cdot F + 2\sum_{v_i \in \xi_1} e_i + 6\sum_{v_i \in \xi_2} e_i + \sum_{v_i \in \xi_3}\left(6e_i - 2\frac{e_i^2}{M+e_i}\right) \\ &\le \frac{S}{2} + n \cdot F + 2\sum_{v_i \in \xi_1} e_i + 6\sum_{v_i \in \xi_2 \cup \xi_3} e_i - 2\sum_{v_i \in \xi_3}\frac{e_i^2}{M+e_i}. \end{aligned} \tag{5.9}$$

If we assume that t_1 is insolvent, then we can show that the above expression is strictly less than $(n+9)\frac{S}{2}$; we distinguish between two cases depending on the quantity $\sum_{\xi_2 \cup \xi_3} e_i$. If $\sum_{\xi_2 \cup \xi_3} e_i \le \frac{S}{2}$, then $\mathcal{SW}(\mathbf{s})$ is maximized for $\sum_{\xi_1} e_i = \sum_{\xi_2 \cup \xi_3} e_i = \frac{S}{2}$. To see this note that ξ_1, ξ_2 and ξ_3 form a partition of the v_is and $M = 5k \cdot (max\{e_1, \cdots e_k\})^2$ by definition. Since, by assumption, t_1 is insolvent, i.e. $F < \frac{S}{2}$, we get that $\mathcal{SW}(\mathbf{s}) < (n+9)\frac{S}{2}$. Now, if $\sum_{\xi_2 \cup \xi_3} e_i > \frac{S}{2}$, then we can conclude that $\sum_{\xi_2 \cup \xi_3} e_i = \frac{S}{2} + \lambda$ for some $\lambda \geq 1$, since S is even and the e_is are integers. Then $\sum_{\xi_1} e_i = \frac{S}{2} - \lambda$ and the total incoming payments of t_1 are $\frac{S}{2} - \lambda + 2\sum_{\xi_3} \frac{e_i^2}{M+e_i}$, implying F is upper-bounded by this quantity. Therefore, from (5.9), we get that

$$\begin{aligned}
\mathcal{SW}(\mathbf{s}) &\leq \frac{S}{2} + n \cdot \left(\frac{S}{2} - \lambda + 2\sum_{\xi_3} \frac{e_i^2}{M+e_i}\right) + 2 \cdot (\frac{S}{2} - \lambda) + 6 \cdot (\frac{S}{2} + \lambda) - 2\sum_{\xi_3} \frac{e_i^2}{M+e_i} \\
&\leq (n+9)\frac{S}{2} + 2 \cdot (n-1)\sum_{\xi_3} \frac{e_i^2}{M+e_i} - (n-4) \cdot \lambda \\
&< (n+9)\frac{S}{2} + 2 \cdot (n-1)k\frac{e_i^2}{3ke_i^2} - (n-4) \\
&\leq (n+9)\frac{S}{2} + \frac{2}{3} \cdot (n-1) - (n-4) \\
&\leq (n+9)\frac{S}{2} - \frac{n-10}{3} \\
&\leq (n+9)\frac{S}{2},
\end{aligned}$$

where the strict inequality holds since $|\xi_3| \leq k$, $e_i > 0$ and $M = 3k \max_i {e_i}^2$, while the last inequality holds since $n \geq 10$ by assumption. We can conclude that if $\mathcal{SW}(\mathbf{s}) \geq (n+9)\frac{S}{2}$, then t_1 is solvent.

Next, we show part (ii), i.e. that if $\mathcal{SW}(\mathbf{s}) \geq (n+9)\frac{S}{2}$ then $\sum_{\xi_1} e_i \geq \frac{S}{2}$. From Figure (5.16), we can see that the total assets (incoming payments) of t_1 satisfy the following

$$\begin{aligned}
a_{t_1} &= \sum_{\xi_1} e_i + 0 + \sum_{\xi_3} \frac{2e_i^2}{M+e_i} \\
&\leq \sum_{\xi_1} e_i + 0 + k\frac{2e_i^2}{3ke_i^2 + e_i} \\
&< \sum_{\xi_1} e_i + 0 + \frac{2}{3},
\end{aligned}$$

where the strict inequality holds since $e_i > 0$ and $M = 3k \max_i {e_i}^2$. By the integrality of the e_is we can conclude that $\sum_{\xi_1} e_i < \frac{S}{2}$ is equivalent to $\sum_{\xi_1} e_i \leq \frac{S}{2} - 1$, which would imply that $_{t_1} < \frac{S}{2}$. However, we have proved that our assumption that $\mathcal{SW}(\mathbf{s}) \geq (n+9)\frac{S}{2}$ implies that t_1 is solvent, hence reaching a contradiction. We can conclude that $\sum_{\xi_1} e_i \geq \frac{S}{2}$.

Finally, we prove part (iii), by showing that at the strategy profile that maximizes the social welfare, none of the v_is pay proportionally, under the assumptions that t_1 is solvent and $\sum_{\xi_1} e_i \geq \frac{S}{2}$, also using parts (i) and (ii). Since the payments from banks in ξ_1 to t_1 are enough to saturate the edges of the path starting from t_1 to s, then these, as well as the outgoing edges of s are all saturated regardless of the set of banks in ξ_2 and ξ_3. By looking at Figure 5.16, we can see that the total flow on the edges $(v_i, v_i^1), (v_i^1, v_i^2), (v_i^2, v_i)$ and (v_i, t_1) is higher for $v_i \in \xi_2$ as opposed to $v_i \in \xi_3$. We can conclude that at the strategy profile that maximizes social welfare, if t_1 is solvent and $\sum_{\xi_1} e_i \geq \frac{S}{2}$, then no bank v_i pays proportionaly, i.e. $\xi_3 = \emptyset$. □

We continue with the proof of our main claim and prove that if there doesn't exist a solution to instance $\mathcal{I}$ of PARTITION then the optimal social welfare is less than $\frac{n+9}{2}S = (n+9)\frac{\sum_{i\in X} x_i}{2}$. We assume otherwise, that there doesn't exist a solution to instance $\mathcal{I}$ of PARTITION but the optimal social welfare is greater than $\frac{n+9}{2}S = (n+9)\frac{\sum_{i\in X} x_i}{2}$. From Lemma 5.22(iii), we know that none of the v_i's pays proportionally at the strategy profile that maximizes social welfare, so we denote by X' be the set of v_i that prioritize payments towards t_1 (pay according to $(t_1|v_i^1)$), while the remaining v_i's corresponding to $x_i \in X \setminus X'$ prioritize payments towards v_i^1 (pay according to $(v_i^1|t_1)$). From Lemma 5.22(i) we know that the edges on the path between t_1 and s, as well as the outgoing edges of s are saturated. Overall, we get that the optimal social welfare $\mathcal{SW}(\mathrm{s}^{opt})$, under our assumptions, satisfies

$$\begin{aligned}
\mathcal{SW}(\mathbf{s}^{opt}) &= e_s + n \cdot \frac{S}{2} + S + \sum_{i\in X'} e_i + \sum_{i\in X\setminus X'} 5e_i \\
&= (n+3)\frac{S}{2} + \sum_{i\in X} e_i + 4\sum_{i\in X\setminus X'} e_i \\
&< (n+3)\frac{S}{2} + 3\sum_{i\in X} e_i \\
&= (n+3)\frac{S}{2} + 3S \\
&= (n+9)\frac{S}{2},
\end{aligned} \tag{5.10}$$

where the first strict inequality is derived by Lemma 5.22(ii) and our assumption that there doesn't exist a solution to instance $\mathcal{I}$. This implies that$\sum_{X'} e_i > \frac{S}{2}$, hence $\sum_{i\in X\setminus X'} e_i < \frac{S}{2}$. We have reached a contradiction to our assumption that the optimal social welfare is at least equal to $\frac{n+9}{2}S = (n+9)\frac{\sum_{i\in X} x_i}{2}$, as desired. The proof is complete. □

As we claimed before, in the network without default cost, the sum of equities (also called *market value* in finance terminology) remains unchanged and is equal to $\sum_i e_i$. However, once we take default costs into account, then things are much different.

Theorem 5.23. *In a network with default costs $\alpha \in (0,1)$ and $\beta \in [0,1)$, computing a strategy profile that maximizes the sum of equities is NP-hard.*

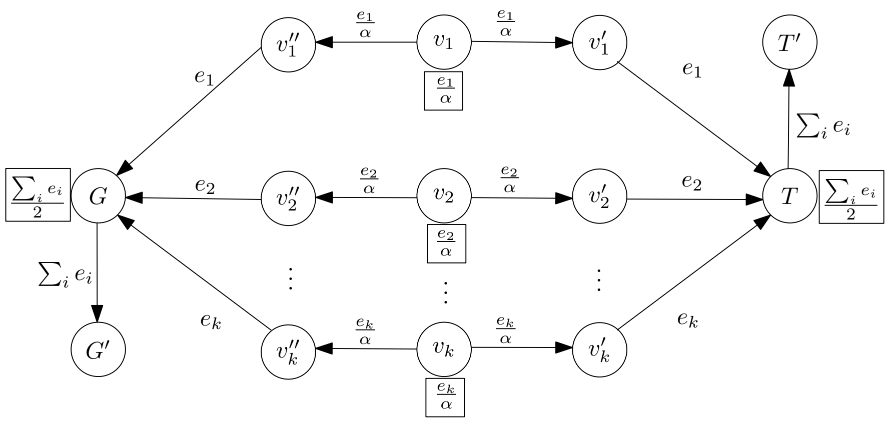


Figure 5.17: The reduction used to show the hardness of computing a strategy profile maximizing the sum of equities.

Proof. Our proof follows a reduction from the PARTITION problem. Recall that in PARTITION, an instance $\mathcal{I}$ consists of a set X of positive integers $\{x_1, x_2, ..., x_k\}$ and the question is whether there exists a subset X^* of X such that $\sum_{i\in X^*} x_i = \frac{\sum_{i\in X} x_i}{2}$; we restrict attention to non-trivial instances where the sum of x_i is even. Starting from $\mathcal{I}$, we build an instance $\mathcal{I}'$ as follows. For each element $x_i \in X$, we add three banks v_i, v_i' and v_i'', and allocate external assets of $\frac{e_i}{\alpha} = \frac{x_i}{\alpha}$ to each v_i for any given α. For each i, we add edges with liability $\frac{e_i}{\alpha}$ from v_i to v_i' and to v_i'' respectively. Furthermore, we add four extra banks G, G', T and T' where both G and T have external assets of $\frac{\sum_i e_i}{2}$. Moreover, for each i, v_i' and v_i'' have a liability of $e_i = x_i$ to G and T, respectively; while G and T have a liability of $\sum_i e_i$ to G' and T' respectively. See also Figure 5.17. Clearly, the reduction requires polynomial time.

Note that each v_i would be always in default since its total liabilities of $\frac{2e_i}{\alpha}$ are strictly larger than its total assets of $\frac{e_i}{\alpha}$, and the total outgoing payments from each single v_i is exactly equal to $\frac{e_i}{\alpha} \cdot \alpha = e_i$. Additionally, since $\frac{e_i}{\alpha} > e_i$, then the other v_i's creditor would receive nothing if v_i prioritizes the payment towards one of v_i' and v_i'' uniquely.

We will prove that there exists a solution to instance $\mathcal{I}$ of PARTITION if and only if the sum of equities of $2\sum_i e_i$ can be achieved. Let us assume that instance $\mathcal{I}$ admits a solution, i.e. there exists a subset X^* in $\mathcal{I}$ with $\sum_{i \in X^*} x_i$. Let each v_i that corresponds to $x_i \in X^*$ prioritize payments towards v_i' uniquely, while the remaining v_i's corresponding to $x_i \in X \setminus X^*$ prioritize v_i'' alone. In this case, all the v_i's corresponding to $x_i \in X^*$ are exactly solvent with sending out a payment of e_i to T, thereby making T exactly solvent and resulting with $E_{T'} = \sum_i e_i$. The similar thing occurs to the remaining v_i's corresponding to $X \setminus X^*$, which returns $E_{G'} = \sum_i e_i$. Now all the banks except G' and T' have zero equities, therefore, this strategy profile results in the sum of equities of $2 \sum_i e_i$.

We will now prove that if there doesn't exist a solution to instance $\mathcal{I}$ of PARTITION then the maximal sum of equities is less than $2 \sum_i e_i$. Before continuing the detailed proof, we first present Claim 5.24 which is necessary afterwards.

Claim 5.24. *If at least one v_i pays proportionally, then the sum of equities must be strictly less than $2 \sum_i e_i$. Namely, none of v_is would pay proportionally under the strategy profile that maximizes the sum of equities.*

Proof. Let V' and V'' be the set of v_is who uniquely prioritize v_i' and v_i'' respectively, while V^* stands for the remaining v_is who pay proportionally. As claimed before, regardless of what strategy that v_i plays, the total payments that v_i can make is exactly e_i, i.e., $p_{v_i, v_i'} + p_{v_i . v_i''} \equiv e_i$. According to the rule of proportionality, if v_i pays proportionally, then we have $p_{v_i, v_i'} = p_{v_i . v_i''} = \frac{e_i}{2} < e_i$, which would make corresponding v_i' and v_i'' insolvent and trigger the loss of equities. In particular, the total incoming payments from all v_i's to T is $\beta \cdot \sum_{v_i \in V*} p_{v_i, v_i'} + \sum_{v_i \in V'} p_{v_i, v_i'} = \beta \cdot \sum_{v_i \in V*} \frac{e_i}{2} + \sum_{v_i \in V'} e_i$, resulting in $a_T = \beta \cdot \sum_{v_i \in V*} \frac{e_i}{2} + \sum_{v_i \in V'} e_i + \frac{\sum_i e_i}{2}$. Similarly, we can achieve $a_G = \beta \cdot \sum_{v_i \in V*} \frac{e_i}{2} + \sum_{v_i \in V''} e_i + \frac{\sum_i e_i}{2}$. Since the sum of liabilities for G and T is $L_G + L_T = 2 \sum_i e_i$, but their total asset is

$$
\begin{aligned}
a_T + a_G &= (\beta \cdot \sum_{v_i \in V*} \frac{e_i}{2} + \sum_{v_i \in V'} e_i + \frac{\sum_i e_i}{2}) + (\beta \cdot \sum_{v_i \in V*} \frac{e_i}{2} + \sum_{v_i \in V''} e_i + \frac{\sum_i e_i}{2}) \\
&= \sum_i e_i + \beta \cdot \sum_{v_i \in V*} e_i + \sum_{v_i \in V'} e_i + \sum_{v_i \in V''} e_i \\
&< 2 \sum_i e_i = L_G + L_T.
\end{aligned}
$$

This implies that at least one bank between G and T would be in default. Without loss of generality, let's assume that G is in default. Then the sum of equities satisfies

$$\begin{aligned}\sum_i E_i &= E_{G'} + E_{T'} \\ &\leq \max\{\alpha, \beta\} \cdot a_G + a_T \\ &< a_T + a_G \\ &< 2\sum_i e_i.\end{aligned}$$

The strict inequalities above are due to both α and β are strictly less than 1. □

We continue with the proof of our main claim and prove that if there doesn't exist a solution to instance $\mathcal{I}$ of PARTITION then the optimal market value is less than $2\sum_i e_i$. We assume otherwise, that there doesn't exist a solution to instance $\mathcal{I}$ of PARTITION but the maximal sum of equities is greater than $2\sum_i e_i$. From Lemma 5.24, we know that v_i prioritizes either v_i' or v_i'' alone at the strategy profile that maximizes the sum of equities, so we denote by X' the set of v_i that prioritizes payments towards v_i', while the remaining v_i's corresponding to $x_i \in X \setminus X'$ prioritize payments towards v_i'' alone. Since $\sum_i p_{v_i,v_i'} + \sum_i p_{v_i,v_i''} = \sum_i e_i$ and there does not exist a solution to instance $\mathcal{I}$ of PARTITION, therefore, without loss of generality, let us assume that $\sum_i p_{v_i,v_i'} > \frac{\sum_i e_i}{2} > \sum_i p_{v_i,v_i''}$, which implies that bank G would be in default but T solvent. Hence, we get that the maximal sum of equities, under our assumptions, satisfies

$$\begin{aligned}\sum_i E_i &= E_{G'} + E_{T'} \\ &\leq \max\{\alpha, \beta\} \cdot a_G + a_T \\ &= \max\{\alpha, \beta\} \cdot (\sum_i p_{v_i,v_i''} + \frac{\sum_i e_i}{2}) + (\sum_i p_{v_i,v_i'} + \frac{\sum_i e_i}{2}) \\ &< \sum_i e_i + \left[\max\{\alpha, \beta\} \cdot \sum_i p_{v_i,v_i'} + \sum_i p_{v_i,v_i''}\right] \\ &< \sum_i e_i + \left[\sum_i p_{v_i,v_i'} + \sum_i p_{v_i,v_i''}\right] \\ &= \sum_i e_i + \sum_i e_i = 2\sum_i e_i.\end{aligned}$$

The proof is complete.

□

5.5 Conclusions

We have studied strategic payment games in financial networks with priority-proportional payments and presented an almost full picture with respect to the structural properties of clearing payments and the quality of Nash equilibria. In terms of computational aspects, we provided a series of hardness results in computing equilibria-related problems, e.g., finding out a Nash equilibrium when it exists, and achieving desirable societal objectives, e.g., computing the payment profiles that maximize social welfare, respectively. Note that we do realize that a part of our results, e.g., Theorem 5.3, Theorem 5.4, Theorem 5.11, and Theorem 5.16, respectively, can also hold in more general settings, i,e., under no-proper clearing payments; while there are still some theorems in this Chapter, such as Theorem 5.9, Theorem 5.19, and Theorem 5.21, strongly relying on the proper clearing payments.

Chapter 6

Conclusions and Future Work

This thesis studies the impact of a range of financial operations, i.e., cash injection, debt removal, debt transfer, and priority-proportional payments, respectively, in both centralized and decentralized manner.

In the context of centralized cases, we considered a scenario where a financial authority has the capacity to exercise control over the operation of each bank, with the objective of achieving desirable financial outcomes. In this regard, we demonstrated that a majority of optimization problems, such as maximizing liquidity by removing a combination of liabilities (as presented in Chapter 3), maximizing total equity by transferring a collection of debt (as presented in Chapter 4), and computing the payments profile with the highest total assets (as presented in Chapter 5), are generally computationally intractable. However, we found that the optimal cash injection in financial networks without default costs (as presented in Chapter 3) can be solved by linear programming in polynomial time. As a supplement, we introduced several algorithms that possess desirable properties and can also approximate the optimal algorithms effectively under specific assumptions, e.g., the greedy algorithm in Chapter 3. Furthermore, we investigated several heuristic algorithms and conducted experimental studies to verify their performance in synthetic financial networks.

With regard to the games formulated by the financial operations, we investigated the existence, quality, and computational aspects of the equilibria that arise from these games. In this regard, we presented a network that does not admit Nash equilibria in these games when each bank acts as total-asset-maximizing agents, while equilibria always exist when banks aimed to maximize their equity in both debt transfer (in Chapter 4) and priority-proportional payment games (in Chapter 5). Additionally, we provided an almost complete picture on the quality of equilibria and demonstrated the hardness results on equilibria for each game. These findings provide insights into the equilibrium behavior of financial networks and complement the results presented earlier regarding the computational challenges faced by financial authorities in centralized cases.

While our work has addressed several important aspects of financial networks, there remain a number of open questions that require further investigation.

Cash injections. Even though the optimal cash injection policy can be computed in polynomial time by solving linear programming when default costs do not apply, it does not satisfy certain desirable properties, e.g., monotonicity, which led us to study a monotone greedy algorithm. However, with respect to the greedy algorithm, we have proved that its approximation ratio is at least $\frac{3}{4}$ under the limited budget $M \leq t_1 \frac{\mu_v}{\mu_v - 1}$ (presented in Theorem 3.2), An interesting question is to generalize this ratio to any amount of budget that is less than the minimal amount of cash injections sufficient to guarantee systemic solvency. In addition, computing an optimal cash injection policy when default costs apply is NP-hard, and the greedy algorithm based on the threat index could perform arbitrarily bad compared to the optimal, thus this leaves open the possibility of designing some approximation algorithms with better guarantees.

Debt removals. Given the computational hardness of some optimization problems by removing debts, it makes sense to consider approximation algorithms. Furthermore, we only consider the discrete variant of debt removal, i.e., either removing an entire edge or not removing debt at all. One could extend to more realistic models where partial debt removals are allowed. The relaxation of this assumption is also more promising for constructing optimal debt removals in the network without default cost. From the angle of a mechanism designer, designing a strategy-proof mechanism ensuring that nobody has an incentive to remove its incoming edges seems to be a challenging task.

Debt transfers. Regarding the existence of equilibria for debt transfer games, we have shown a network without Nash equilibrium when banks try to maximize their total assets for default cost $\alpha \in (0, 1)$, while the case $\alpha = 0$ or $\alpha = 1$ is still open. In addition, we proved that Nash equilibria always exist when banks are equity-maximizing players in networks without default costs, thus the question of the existence of equilibria when $\alpha \in [0, 1)$ is also open. With respect to the empirical part, one could extend the number of banks in the debt transfer game, and also, it will be more realistic to assume that the action that each bank can take is to pick up a combination of debts to transfer, instead of either transferring all eligible debts or not performing at all.

Priority-proportional payments. One could consider addressing computational complexity questions on other optimization problems such as minimizing the number of defaulted banks. Empirically, one could make an experimental comparison between priority-proportional and proportional payment schemes in terms of social welfare, in order to check if this new payment scheme is more beneficial for the entire financial system on average.

In addition to the aforementioned unresolved questions, our work reveals several potential research directions for the future.

Donating External Assets. In very recent work, Papp and Wattenhofer [106] propose a novel strategic operation of donating external assets among banks. As a next step, one could consider this operation also in centralized and decentralized settings respectively.

Empirical Game-Theoretic Analysis. Another interesting direction could be reasoning about banks' strategic operations in larger-scale financial network games, i.e., with more strategies and bank players. Since the space of strategy profiles in most games expands exponentially with the number of players, an increasing number of players would come at the expense of huge processing times, so in practice we cannot explore the space exhaustively for huge games, as we did in Chapter 4. To overcome this, one alternative is through simulation and sampling [115, 131], an approach that has been termed *empirical game-theoretic analysis (EGTA)* [134]. For more details on the applications of EGTA in financial games, we refer the reader to these related works [96, 97, 132, 93].

Reinforcement Learning for Financial Network Games. A great research potential lies at the intersection between multi-agent reinforcement learning (MARL), game theory, and financial networks. Inspired by the work of Yu [135] who shows that MARL can solve the game theoretic dilemmas, i.e., reducing the inefficiency of equilibria, in the financial network game of donating external assets, one could apply this method to the games that we proposed, e.g., the edge-removal game. In addition, since financial networks are naturally characterized by a graph structure, combining Graph Neural Networks (GNN) and MARL to improve the efficiency of equilibria would be another interesting research area.

Bibliography

[1] Assignment of debt, 2021. `http://www.designingbuildings.co.uk/wiki/Assignment_of_debt`.

[2] Mariana Piaia Abreu, Rosanna Grassi, and Renata R Del-Vecchio. Structure of control in financial networks: An application to the brazilian stock market. *Physica A: Statistical Mechanics and Its Applications*, 522:302–314, 2019.

[3] Daron Acemoglu, Asuman Ozdaglar, and Alireza Tahbaz-Salehi. Systemic risk and stability in financial networks. *American Economic Review*, 105(2):564–608, 2015.

[4] Viral Acharya, Thomas Philippon, Matthew Richardson, and Nouriel Roubini. The financial crisis of 2007-2009: Causes and remedies. *Restoring financial stability: how to repair a failed system*, pages 1–56, 2009.

[5] Franklin Allen and Ana Babus. Networks in finance. *The network challenge: strategy, profit, and risk in an interlinked world*, 367, 2009.

[6] Franklin Allen and Elena Carletti. The role of liquidity in financial crises. *Available at SSRN 1268367*, 2008.

[7] Nizar Allouch and Maya Jalloul. Strategic default in financial networks. Technical report, Queen Mary University of London, 2018.

[8] Levent Altinoglu and Joseph E Stiglitz. Collective moral hazard and the interbank market. Technical report, National Bureau of Economic Research, 2022.

[9] Hamed Amini, Rama Cont, and Andreea Minca. Stress testing the resilience of financial networks. *International Journal of Theoretical and applied finance*, 15(01):1250006, 2012.

[10] Hamed Amini, Rama Cont, and Andreea Minca. Resilience to contagion in financial networks. *Mathematical finance*, 26(2):329–365, 2016.

[11] Hamed Amini, Damir Filipović, and Andreea Minca. To fully net or not to net: Adverse effects of partial multilateral netting. *Operations Research*, 64(5):1135–1142, 2016.

[12] Hamed Amini and Andreea Minca. Clearing financial networks: Impact on equilibrium asset prices and seniority of claims. In *Pushing the Boundaries: Frontiers in Impactful OR/OM Research*, pages 154–175. INFORMS, 2020.

[13] Kartik Anand, Ben Craig, and Goetz Von Peter. Filling in the blanks: Network structure and interbank contagion. *Quantitative Finance*, 15(4):625–636, 2015.

[14] Elliot Anshelevich, Anirban Dasgupta, Jon Kleinberg, Éva Tardos, Tom Wexler, and Tim Roughgarden. The price of stability for network design with fair cost allocation. *SIAM Journal on Computing*, 38(4):1602–1623, 2008.

[15] Zeta Avarikioti, Lioba Heimbach, Yuyi Wang, and Roger Wattenhofer. Ride the lightning: The game theory of payment channels. In *International Conference on Financial Cryptography and Data Security*, pages 264–283. Springer, 2020.

[16] Editors: John D. Ayer, Michael L. Bernstein, and Jonathan Friedland. Priorities. *Issues and Information for the Insolvency Professional Journal*, 11, 2004.

[17] Moshe Babaioff, Yoav Kolumbus, and Eyal Winter. Optimal collaterals in multi-enterprise investment networks. In *Proceedings of the ACM Web Conference 2022*, pages 79–89, 2022.

[18] Ana Babus. The formation of financial networks. *The RAND Journal of Economics*, 47(2):239–272, 2016.

[19] Tathagata Banerjee, Alex Bernstein, and Zachary Feinstein. Dynamic clearing and contagion in financial networks. *arXiv preprint arXiv:1801.02091*, 2018.

[20] Albert-László Barabási and Réka Albert. Emergence of scaling in random networks. *science*, 286(5439):509–512, 1999.

[21] Pallavi Baral. Strategic behavior and endogenous risk of contagion in a financial network: A network formation game, 2012.

[22] Stefano Battiston, Michelangelo Puliga, Rahul Kaushik, Paolo Tasca, and Guido Caldarelli. Debtrank: Too central to fail? financial networks, the fed and systemic risk. *Scientific reports*, 2(1):1–6, 2012.

[23] Benjamin Bernard, Agostino Capponi, and Joseph E Stiglitz. Bail-ins and bailouts: Incentives, connectivity, and systemic stability. *Journal of Political Economy*, 130(7):000–000, 2022.

[24] Nils Bertschinger, Martin Hoefer, Simon Krogmann, Pascal Lenzner, Steffen Schuldenzucker, and Lisa Wilhelmi. Equilibria and convergence in fire sale games. *arXiv preprint arXiv:2206.14429*, 2022.

[25] Nils Bertschinger, Martin Hoefer, and Daniel Schmand. Strategic payments in financial networks. In *Proceedings of the 11th Innovations in Theoretical Computer Science Conference (ITCS)*, pages 46:1–46:16, 2020.

[26] Monica Billio, Mila Getmansky, Andrew W Lo, and Loriana Pelizzon. Econometric measures of connectedness and systemic risk in the finance and insurance sectors. *Journal of financial economics*, 104(3):535–559, 2012.

[27] David Bindel, Jon Kleinberg, and Sigal Oren. How bad is forming your own opinion? *Games and Economic Behavior*, 92:248–265, 2015.

[28] Financial Stability Board. Progress and next steps towards ending "too-big-to-fail"(tbtf). *Report of the Financial Stability Board to the G*, 20(2), 2013.

[29] Giulio Bottazzi, Alessandro De Sanctis, and Fabio Vanni. Non-performing loans, systemic risk and resilience in financial networks. Technical report, LEM Working Paper Series, 2016.

[30] Ricardo J Caballero and Alp Simsek. Fire sales in a model of complexity. *The Journal of Finance*, 68(6):2549–2587, 2013.

[31] Giuseppe Calafiore, Giulia Fracastoro, and Anton V Proskurnikov. On optimal clearing payments in financial networks. In *2021 60th IEEE Conference on Decision and Control (CDC)*, pages 4804–4810. IEEE, 2021.

[32] Agostino Capponi, Peng-Chu Chen, and David D Yao. Liability concentration and systemic losses in financial networks. *Operations Research*, 64(5):1121–1134, 2016.

[33] Maria Căzănel. The effects of debt assignments. *Ovidius University Annals, Series Economic Sciences*, 16(2), 2016.

[34] Satyajit Chatterjee and Burcu Eyigungor. A seniority arrangement for sovereign debt. *American Economic Review*, 105(12):3740–65, 2015.

[35] Hong Chen, Tan Wang, and David D Yao. Financial network and systemic risk—a dynamic model. *Production and Operations Management*, 30(8):2441–2466, 2021.

[36] Lucy Chennells and Venetia Wingfield. Bank failure and bail-in: an introduction. *Bank of England Quarterly Bulletin*, page Q3, 2015.

[37] Rodrigo Cifuentes, Gianluigi Ferrucci, and Hyun Song Shin. Liquidity risk and contagion. *Journal of the European Economic association*, 3(2-3):556–566, 2005.

[38] Ben Craig and Goetz Von Peter. Interbank tiering and money center banks. *Journal of Financial Intermediation*, 23(3):322–347, 2014.

[39] Péter Csóka and P Jean-Jacques Herings. Liability games. *Games and Economic Behavior*, 116:260–268, 2019.

[40] Péter Csóka and P Jean-Jacques Herings. Decentralized clearing in financial networks. *Management Science*, 64(10):4681–4699, 2018.

[41] Pietro De Giovanni. Blockchain and smart contracts in supply chain management: A game theoretic model. *International Journal of Production Economics*, 228:107855, 2020.

[42] Victor J Defeo, Richard A Lambert, and David F Larcker. The executive compensation effects of equity-for-debt swaps. *Accounting Review*, pages 201–227, 1989.

[43] Gabrielle Demange. Contagion in financial networks: a threat index. *Management Science*, 64(2):955–970, 2018.

[44] Jim Dolmas. Trimmed mean pce inflation. *Federal Reserve Bank of Dallas Working Paper*, 506, 2005.

[45] Zhi-Long Dong, Jiming Peng, Fengmin Xu, and Yu-Hong Dai. On some extended mixed integer optimization models of the Eisenberg-Noe model in systemic risk management. *International Transactions in Operational Research*, 28(6):3014–3037, 2021.

[46] KJ Donnelly. Report by the comptroller and auditor general: Child maintenance service 2018-19. 2019.

[47] Beni Egressy and Roger Wattenhofer. Bailouts in financial networks. *arXiv:2106.12315*, 2021.

[48] Larry Eisenberg and Thomas H. Noe. Systemic risk in financial systems. *Management Science*, 47(2):236–249, 2001.

[49] Matthew Elliott, Benjamin Golub, and Matthew O Jackson. Financial networks and contagion. *American Economic Review*, 104(10):3115–53, 2014.

[50] Helmut Elsinger et al. *Financial networks, cross holdings, and limited liability*. Oesterreichische Nationalbank Austria, 2009.

[51] Helmut Elsinger, Alfred Lehar, and Martin Summer. Risk assessment for banking systems. *Management science*, 52(9):1301–1314, 2006.

[52] Paul Erdős, Alfréd Rényi, et al. On the evolution of random graphs. *Publ. Math. Inst. Hung. Acad. Sci*, 5(1):17–60, 1960.

[53] Selman Erol. Network hazard and bailouts. *Available at SSRN 3034406*, 2019.

[54] Eyal Even-Dar, Michael J Kearns, and Siddharth Suri. A network formation game for bipartite exchange economies. In *SODA*, pages 697–706, 2007.

[55] Karol Flores-Szwagrzak. Priority classes and weighted constrained equal awards rules for the claims problem. *Journal of Economic Theory*, 160:36–55, 2015.

[56] Henri Froese, Martin Hoefer, and Lisa Wilhelmi. The complexity of debt swapping. *arXiv preprint arXiv:2302.11250*, 2023.

[57] Edoardo Gaffeo, Lucio Gobbi, and Massimo Molinari. The economics of netting in financial networks. *Journal of Economic Interaction and Coordination*, 14(3):595–622, 2019.

[58] Edoardo Gaffeo, Lucio Gobbi, and Massimo Molinari. Liquidity contagion with a "first-in/first-out" seniority of claims. *Economics Bulletin*, 39(4):2572–2579, 2019.

[59] Prasanna Gai and Sujit Kapadia. Contagion in financial networks. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 466(2120):2401–2423, 2010.

[60] Andrea Galeotti and Christian Ghiglino. Cross-ownership and portfolio choice. *Journal of Economic Theory*, 192:105194, 2021.

[61] Paul Glasserman and H Peyton Young. How likely is contagion in financial networks? *Journal of Banking & Finance*, 50:383–399, 2015.

[62] Paul Glasserman and H Peyton Young. Contagion in financial networks. *Journal of Economic Literature*, 54(3):779–831, 2016.

[63] Teofilo F. Gonzalez. Clustering to minimize the maximum interclusted distance. *Theoretical Computer Science*, 38:293–306, 1985.

[64] Robin Greenwood, Augustin Landier, and David Thesmar. Vulnerable banks. *Journal of Financial Economics*, 115(3):471–485, 2015.

[65] Mirjam Groote Schaarsberg, Hans Reijnierse, and Peter Borm. On solving mutual liability problems. *Mathematical Methods of Operations Research*, 87:383–409, 2018.

[66] Stein Hansen. Debt for nature swaps—overview and discussion of key issues. *Ecological Economics*, 1(1):77–93, 1989.

[67] İlhan Helvacı. *Turkish Contract Law*. Springer, 2017.

[68] PÉTER CSÓKA-P JEAN-JACQUES HERINGS. An axiomatization of the pairwise netting proportional rule in financial networks. 2023.

[69] Martin Hoefer and Lisa Wilhelmi. Seniorities and minimal clearing in financial network games. In *Proceedings of the 15th International Symposium on Algorithmic Game Theory (SAGT)*, pages 187–204, 2022.

[70] Gur Huberman and Dominika Halka. Systematic liquidity. *Journal of Financial Research*, 24(2):161–178, 2001.

[71] Anne-Caroline Hüser, Grzegorz Hałaj, Christoffer Kok, Cristian Perales, and Anton van der Kraaij. The systemic implications of bail-in: a multi-layered network approach. *Journal of Financial Stability*, 38:81–97, 2018.

[72] Stavros Ioannidis, Bart De Keijzer, and Carmine Ventre. Financial networks with singleton liability priorities. In *Proceedings of the 15th International Symposium on Algorithmic Game Theory (SAGT)*, pages 205–222, 2022.

[73] Stavros D. Ioannidis, Bart de Keijzer, and Carmine Ventre. Strong approximations and irrationality in financial networks with financial derivatives. *arXiV*, abs/2109.06608, 2021.

[74] Mohammad T Irfan and Luis E Ortiz. Causal strategic inference in a game—theoretic model of multiplayer networked microfinance markets. *ACM Transactions on Economics and Computation (TEAC)*, 6(2):1–58, 2018.

[75] Matthew O. Jackson and Agathe Pernoud. Credit freezes, equilibrium multiplicity, and optimal bailouts in financial networks. *arXiv:2012.12861*, 2020.

[76] Rui Jing, Meng Wang, Hao Liang, Xiaonan Wang, Ning Li, Nilay Shah, and Yingru Zhao. Multi-objective optimization of a neighborhood-level urban energy network: Considering game-theory inspired multi-benefit allocation constraints. *Applied energy*, 231:534–548, 2018.

[77] Christos Kaklamanis, Panagiotis Kanellopoulos, and Sophia Tsokana. On network formation games with heterogeneous players and basic network creation games. *Theoretical Computer Science*, 717:62–72, 2018.

[78] Marek M. Kaminski. Hydraulic rationing. *Mathematical Social Sciences*, 40(2):131–155, 2000.

[79] Panagiotis Kanellopoulos, Maria Kyropoulou, and Hao Zhou. Financial network games. In *Proceedings of the 2nd ACM International Conference on AI in Finance (ICAIF)*, pages 1–9, 2021.

[80] Panagiotis Kanellopoulos, Maria Kyropoulou, and Hao Zhou. Forgiving debt in financial network games. In *Proceedings of the Thirty-First International Joint Conference on Artificial Intelligence (IJCAI)*, pages 335–341, 2022.

[81] Panagiotis Kanellopoulos, Maria Kyropoulou, and Hao Zhou. Debt transfer in financial networks: complexity and equilibria. In *Proceedings of the 22nd International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, 2023.

[82] Masayasu Kanno. Assessing systemic risk using interbank exposures in the global banking system. *Journal of Financial Stability*, 20:105–130, 2015.

[83] Ariah Klages-Mundt and Andreea Minca. Cascading losses in reinsurance networks. *Management Science*, 66(9):4246–4268, 2020.

[84] Peter Klimek, Sebastian Poledna, J Doyne Farmer, and Stefan Thurner. To bail-out or to bail-in? answers from an agent-based model. *Journal of Economic Dynamics and Control*, 50:144–154, 2015.

[85] Dániel Kondor, Márton Pósfai, István Csabai, and Gábor Vattay. Do the rich get richer? an empirical analysis of the bitcoin transaction network. *PloS one*, 9(2):e86197, 2014.

[86] Elias Koutsoupias and Christos Papadimitriou. Worst-case equilibria. *Comput. Sci. Rev.*, 3(2):65–69, 2009.

[87] Michael Kusnetsov and Luitgard Anna Maria Veraart. Interbank clearing in financial networks with multiple maturities. *SIAM Journal on Financial Mathematics*, 10(1):37–67, 2019.

[88] Bart M Lambrecht and Alex SL Tse. Liquidation, bailout, and bail-in: Insolvency resolution mechanisms and bank lending. *Forthcoming in the Journal of Financial and Quantitative Analysis*, 2021.

[89] Matt V Leduc, Sebastian Poledna, and Stefan Thurner. Systemic risk management in financial networks with credit default swaps. *arXiv preprint arXiv:1601.02156*, 2016.

[90] Seung Hwan Lee. Systemic liquidity shortages and interbank network structures. *Journal of Financial Stability*, 9(1):1–12, 2013.

[91] Marc Lelarge. Diffusion and cascading behavior in random networks. *Games and Economic Behavior*, 75(2):752–775, 2012.

[92] John Leventides, Kalliopi Loukaki, and Vassilios G Papavassiliou. Simulating financial contagion dynamics in random interbank networks. *Journal of Economic Behavior & Organization*, 158:500–525, 2019.

[93] Buhong Liu, Maria Polukarov, Carmine Ventre, Lingbo Li, Leslie Kanthan, Fan Wu, and Michail Basios. The spoofing resistance of frequent call markets. In *Proceedings of the 21st International Conference on Autonomous Agents and Multiagent Systems*, pages 825–832, 2022.

[94] Ziyao Liu, Nguyen Cong Luong, Wenbo Wang, Dusit Niyato, Ping Wang, Ying-Chang Liang, and Dong In Kim. A survey on applications of game theory in blockchain. *arXiv preprint arXiv:1902.10865*, 2019.

[95] Bruce Lyons. Competition policy, bailouts and the economic crisis. *Bailouts and the Economic Crisis (March 24, 2009)*, 2009.

[96] Katherine Mayo, Shaily Fozdar, and Michael P Wellman. An agent-based model of strategic adoption of real-time payments. In *Proceedings of the Second ACM International Conference on AI in Finance*, pages 1–9, 2021.

[97] Katherine Mayo and Michael P Wellman. A strategic analysis of portfolio compression. In *Proceedings of the 20th International Conference on Autonomous Agents and MultiAgent Systems*, pages 1599–1601, 2021.

[98] Brent Meyer and Guhan Venkatu. Trimmed-mean inflation statistics: just hit the one in the middle. *Federal Reserve Bank of Cleveland Working Paper*, 2014.

[99] Oskar Morgenstern and John Von Neumann. *Theory of games and economic behavior*. Princeton university press, 1953.

[100] Stephen Morris. Contagion. *The Review of Economic Studies*, 67(1):57–78, 2000.

[101] Stephen Morris and Hyun Song Shin. Global games: Theory and applications. 2001.

[102] Hervé Moulin. Priority rules and other asymmetric rationing methods. *Econometrica*, 68(3):643–684, 2000.

[103] R Myerson. Game theory: Analysis of conflict harvard univ. *Press, Cambridge*, 3, 1991.

[104] Zhang Nana, Wei Xiujian, and Zhang Zhongqiu. Game theory analysis on credit risk assessment in e-commerce. *Information Processing & Management*, 59(1):102763, 2022.

[105] Noam Nisan, Tim Roughgarden, Eva Tardos, and Vijay V Vazirani. *Algorithmic game theory*. Cambridge university press, 2007.

[106] Pál András Papp and Roger Wattenhofer. Network-aware strategies in financial systems. In *Proceedings of the 47th International Colloquium on Automata, Languages, and Programming (ICALP)*, pages 91:1–91:17, 2020.

[107] Pál András Papp and Roger Wattenhofer. Debt swapping for risk mitigation in financial networks. In *Proceedings of the 22nd ACM Conference on Economics and Computation (EC)*, pages 765–784, 2021.

[108] Pál András Papp and Roger Wattenhofer. Default ambiguity: finding the best solution to the clearing problem. In *Proceedings of the 17th Conference on Web and Internet Economics (WINE)*, 2021.

[109] Lasse Heje Pedersen. Game on: Social networks and markets. *Journal of Financial Economics*, 146(3):1097–1119, 2022.

[110] Marcello Pericoli and Massimo Sbracia. A primer on financial contagion. *Journal of economic surveys*, 17(4):571–608, 2003.

[111] Anastasios Petropoulos, Vasilis Siakoulis, Panagiotis Lazaris, and Sotirios Chatzis. Re-constructing the interbank links using machine learning techniques. an application to the greek interbank market. *Intelligent Systems with Applications*, 12:200055, 2021.

[112] Thomas Philippon and Aude Salord. Bail-ins and bank resolution in europe. *Geneva Reports on the World Economy Special Report*, 4, 2017.

[113] Matias Puig and Christoph Siebenbrunner. Network valuation under equity conversions. *Available at SSRN 3225943*, 2018.

[114] Youyang Qu, Shui Yu, Longxiang Gao, Wanlei Zhou, and Sancheng Peng. A hybrid privacy protection scheme in cyber-physical social networks. *IEEE Transactions on Computational Social Systems*, 5(3):773–784, 2018.

[115] Daniel M Reeves. *Generating trading agent strategies: Analytic and empirical methods for infinite and large games*. University of Michigan, 2005.

[116] Leonard CG Rogers and Luitgard Anna Maria Veraart. Failure and rescue in an interbank network. *Management Science*, 59(4):882–898, 2013.

[117] Tim Roughgarden. Algorithmic game theory. *Communications of the ACM*, 53(7):78–86, 2010.

[118] Steffen Schuldenzucker. *A formal analysis of complexity and systemic risk in financial networks with derivatives*. PhD thesis, University of Zurich, 2019.

[119] Steffen Schuldenzucker and Sven Seuken. Portfolio compression in financial networks: Incentives and systemic risk. *Available at SSRN 3135960*, 2021.

[120] Steffen Schuldenzucker, Sven Seuken, and Stefano Battiston. Finding clearing payments in financial networks with credit default swaps is PPAD-complete. In *Proceed-*

ings of the 8th Innovations in Theoretical Computer Science Conference (ITCS), pages 32:1–32:20, 2017.

[121] Steffen Schuldenzucker, Sven Seuken, and Stefano Battiston. Default ambiguity: Credit default swaps create new systemic risks in financial networks. *Management Science*, 66(5):1981–1998, 2020.

[122] Andreas S. Schulz and Nicolás Stier-Moses. On the performance of user equilibria in traffic networks. In *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 86–87, 2003.

[123] Steven L Schwarcz. Too big to fool: Moral hazard, bailouts, and corporate responsibility. *Minn. L. Rev.*, 102:761, 2017.

[124] Isaac Sonin and Konstantin Sonin. A continuous-time model of financial clearing. *University of Chicago, Becker Friedman Institute for Economics Working Paper*, (2020-101), 2020.

[125] John Stachurski. Systemic risk in financial systems: Properties of equilibria. *arXiv preprint arXiv:2202.11183*, 2022.

[126] Wayes Tushar, Tapan Kumar Saha, Chau Yuen, Thomas Morstyn, Malcolm D McCulloch, H Vincent Poor, and Kristin L Wood. A motivational game-theoretic approach for peer-to-peer energy trading in the smart grid. *Applied energy*, 243:10–20, 2019.

[127] Dr Kubi Udofia. Debt assignments and notice to debtors. *Available at SSRN 3593432*, 2020.

[128] Luitgard Anna Maria Veraart. Distress and default contagion in financial networks. *Mathematical Finance*, 30(3):705–737, 2020.

[129] Luitgard Anna Maria Veraart. When does portfolio compression reduce systemic risk? *Mathematical Finance*, 32(3):727–778, 2022.

[130] Stefania Vitali, James B Glattfelder, and Stefano Battiston. The network of global corporate control. *PloS one*, 6(10):e25995, 2011.

[131] William E Walsh, David C Parkes, and Rajarshi Das. Choosing samples to compute heuristic-strategy nash equilibrium. In *International Workshop on Agent-Mediated Electronic Commerce*, pages 109–123. Springer, 2003.

[132] Xintong Wang, Christopher Hoang, Yevgeniy Vorobeychik, and Michael P Wellman. Spoofing the limit order book: A strategic agent-based analysis. *Games*, 12(2):46, 2021.

[133] Stefan Weber and Kerstin Weske. The joint impact of bankruptcy costs, fire sales and cross-holdings on systemic risk in financial networks. *Probability, Uncertainty and Quantitative Risk*, 2(1):1–38, 2017.

[134] Michael P Wellman. Methods for empirical game-theoretic analysis. In *AAAI*, pages 1552–1556, 2006.

[135] Bryan Yu. Multi agent reinforcement learning for game theory: Financial graphs, 2021.

[136] Jiannan Yu and Jinlou Zhao. Prediction of systemic risk contagion based on a dynamic complex network model using machine learning algorithm. *Complexity*, 2020.