Chapter 1

THE NEW EVOLUTIONARY COMPUTATIONAL PARADIGM OF COMPLEX ADAPTIVE SYSTEMS

CHALLENGES AND PROSPECTS FOR ECONOMICS AND FINANCE

Sheri M. Markose
Economics Department and Institute of Studies in Finance (ISF)
University of Essex, UK.
scher@essex.ac.uk

Abstract The new evolutionary computational paradigm of market systems views these as complex adaptive systems. The major premise of 18^{th} century classical political economy was that order in market systems is spontaneous or emergent, in that it is the result of 'human action but not of human design'. This early observation on the disjunction between system wide outcomes and capabilities of micro level rational calculation marks the provenance of modern evolutionary thought. However, it will take a powerful confluence of *two* 20^{th} century epochal developments for the new evolutionary computational paradigm to rise to the challenge of providing long awaited explanations of what has remained anomalies or outside the ambit of traditional economic analysis. The first of these is the Gödel-Turing-Post results on incompleteness and algorithmically unsolvable problems that delimit formalist calculation or deductive methods. The second is the Anderson-Holland-Arthur heterogeneous adaptive agent theory and models for inductive search, emergence and self-organized criticality which can crucially show and explicitly study the processes underpinning the emergence of ordered complexity. Multi agent model simulation of asset price formation and the innovation based structure changing dynamics of capitalist growth are singled out for analysis of this disjunction between non-anticipating global outcomes and computational micro rationality.

Keywords: Complex adaptive systems; Emergence; Self-organized criticality; Algorithmic unsolvability; Inductive search; Innovation; Market efficiency.

Introduction

As with the contents of this book and with growing contributions of the thinkers of the Sante Fe Institute (SFI) and others, notably, Nicolis and Prigogine (1987), Chen and Day (1993), Dosi and Nelson (1994), Axtell and Epstein (1996), Krugman (1996), Albin (1998) and Velupillai (2000), there is clearly a resurgence of interest among economists in a world view that market systems are *complex adaptive* systems. However, as pointed out by Arthur (1993a), thought habits of economists dominated by a deductive/formalist methodology have in part been a barrier to understanding the significance of the development of inductive/evolutionary models of economic systems. In his strongest diatribe, Simon (1981) is dismissive of the neoclassical economists' lack of concern with the procedural lacunae of rationality: "rules of substantive rationality that are not backed by executable algorithms are worthless currency" (*Ibid.*, p.43).

In so far as an intuitive notion of an algorithm or calculation can be formalized, one of the major 20^{th} century intellectual achievements is the postulation in the Church-Turing thesis. By this thesis all finitely encodable sets of instructions defining algorithms implementable in a number of equivalent ways, including that of the Turing Machine, can be formalized by the class of general recursive functions. The units of modern adaptive models is what Arthur (1991) describes as "parametrized decision algorithms" or units whose behaviour is algorithmic and hence brought about by finitely codifiable programs. However, in Church's Theorem we have the basic result on algorithmic unsolvability generically referred to as the halting problem (see, Cutland, 1980). In other words, dynamical system outcomes produced by algorithmic agents need not be computable. Typically, as the set of all (countable infinite) partial recursive functions is co-extensive with the set of all Turing Machines, problems for which no partial recursive function[1] or Universal Turing Machine guarantees a solution are called undecidable or incomplete.

The purpose of this review is to highlight that a schism between the so called formalist/deductive school and the inductive/evolutionary schools

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[1] Note, a partial recursive function is number theoretic function, f: $\mathbf{N} \rightarrow \mathbf{N}$, that is not defined on the full domain of the set of all integers, $\mathbf{N}$. In other words, on some n $\in \mathbf{N}$ which is its input, the computation being implemented by the partial recursive function will not halt. Total recursive functions are defined on the full domain on $\mathbf{N}$. The set of all total computable functions is uncountably infinite and hence it is not recursively enumerable by any Turing Machine (see, Cutland, 1980).

is an outmoded framework of scientific discourse which has unfortunately prevailed far too long. In ***Table 1***, I have summarized the framework of discourse to assess the challenges and prospects of the new evolutionary computational paradigm for economics. I will argue that a powerful confluence of the 'new logic' of the limits of formalistic calculation (column II, ***Table 1***) with the adaptive algorithms of inductive search (column III, ***Table 1***) is required for the new evolutionary paradigm to rise to the challenge of providing long awaited explanations of what have remained anomalies or outside the ambit of traditional economics analysis. Indeed, in a discipline where its elites give pride of place to axiomatic and formal analysis, it is unfortunate that they remain for the most part ignorant of the epochal results in formalist mathematics of Gödel (1931), Turing (1936) and Post (1944) on the limits of the formalist/deductive methodology.

In Gödel (1931) we have for the first time, a proof of an impossibility result that strongly self-referential system wide properties such as a formal requirement of order in terms of internal consistency is not one that can be established by an algorithmic decision procedure by an internal observer who operates on codifiable information[2]. The undecidable proposition which is known to be true to an internal observer is without any unique recursive procedure in its derivation and hence contradictory inferences can be drawn from the same information. This also clearly sets a limit to knowledge that can be transferred in a codifiable form. Gödel (1931) axiomatically derived a class of algorithmically unsolvable decision problems, viz. the diophantine equations or polynomial equations with integer solutions. With the formal conditions for the incompleteness of predicate calculus also being identical for geometrical patterns arising in tiling problems, Roger Penrose (1989), one of the active proponents of a theory of patterns that do not have any recursive implementation, has emphasized that non-recursive outcomes rather than being a curiosum of science have an objective and pervasive existence that theoretical and empirical investigators should explicitly take on board as the probable explanation for many anomalies in the domain of their sciences.

Thus, a formalist cognizant of the limits of formalist calculation will be open to the necessity of trial and error style inductive inference while in turn the latter can be fully justified by the existence of decision prob-

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[2] Informally and in popular language, Gödel's Second Incompleteness Theorem implies that the price for logical consistency is incompleteness or the algorithmic unsolvability of a decision problem, see Binmore (1987).

Table 1.1. The New Framework of Scientific Discourse

I . Formalist / Deductive Inference	**II. The New Logic**	**III. Inductive Inference**
♯Axiomatic Proof Theory ♯Model Theory	♯Gödel(1931): Self-reference Undecidability and Incompleteness ♯Church-Turing-Post Algorithmic Unsolvability And Limits of Formalist Calculus	♯Theory of Emergence And Self-Organizing Complex Systems ♯Non-recursive Tiling Problems
Formalistic Methods	**Computability Methods**	**Methods of Adaptive Computing**
*Predicate and Propositional Calculus *Classical methods of optimization * Classical Probability Models	*Recursion Function/ Computability Theory *Algorithmic and Stochastic Complexity Theory	*Cellular Automata *Classifier Systems *Genetic Algorithms and Genetic Programs *Neural Networks *Numerical multi-agent simulations

lems that are algorithmically unsolvable[3]. When search has to proceed in domains that cannot be recursively enumerable or when patterns emerge for which no recursive implementation exists, the mathematical and the methodological framework given in column III is radically different to

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[3] In 1998, when I put it to the recently departed master, Herbert Simon, as to how algorithmically unsolvable problems are resolved, the following is the verbatim text of his email reply. " The world is full of problems that have no guaranteed solution method. .. With regard to mechanisms for problems where there is no algorithm that guarantees solutions, I would urge you to look hard at the word problem for semi-groups. (*My addition*, see, Penrose, 1988, pp. 509-513).) The point here is that, although there is no partial recursive function that guarantees solutions for all possible word problems, a particular partial recursive function that you happen to execute may solve the word problem before you. The way we (people and computers) get through life is by executing partial computing functions that sometimes solve the problem. No guarantee though; sooner or later we miss and are dead."

traditional methods in column I of ***Table 1***. The efficacy of classical optimization algorithms requires a recursive bijective mapping between actions and outcomes and this fails when the outcomes cannot be enumerated in advance[4]. For algorithmically unsolvable problems, in the absence of an unique decision procedure, the hallmark of inductive inference is that a multiplicity of decision procedures have to be considered, with issues of algorithmic unsolvability governing the search for which decision procedures to include, how to alter existing procedures and when to stop searching and so on.

Thus, as ***Table 1*** shows the paradigm shift from traditional constraint optimization methods of column I to adaptive methods of inductive inference in column III is symbiotically related to the powerful axiomatic limitative results on deduction and calculation given in column II. In other words, the full recognition that there are problems, indeed all non-trivial market related problems may be such, for which methods in column I in **Table 1** are of limited use, is slow in coming. The whole thrust of adaptive computing methods in the form of Classifier Systems, Genetic Algorithms (**GA**s) and Genetic Programs(**GP**s) pioneered by John Holland (1975, 1998) and John Koza (1992), respectively, is to evolve computer programs that can solve a problem from elementary bit strings or units of several different programs. In the classical methods, known solution algorithms are tried out sequentially or even in parallel but they cannot be grown over time from simpler units as in the adaptive methods such as **GA**s and **GP**s where the programs coevolve as the problems change. In an evolutionary framework, the domain of decision rules contain implementable solution structures or objects which are a subset of an uncountably infinite set of total computable functions. As this set has no algorithmic decision procedure, the outcome of an inductive search may often be far from a global optimum. Holland (1975) pioneered the evolutionary principle for the selection of decision rules. The latter are selected in proportion to their fitness relative to the average fitness of the population of decision rules. In multi-species environments, the rates of cross-over and mutation operators that can improve fitness of decision rules may arise endogenously. As we will see, oppositional structures that arise between species can encourage strategic innovation and prevent entrapment in local optima.

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[4] It is increasingly being understood that the standard Savage model of choice under uncertainty is inadequate in the bigger scheme of things. See, Easley and Rustichini (1999) who develop a framework of rational choice in a complex environment and attempt to relax the assumption of a one-to-one mapping between actions and outcomes.

The adaptive computing method of neural networks (see ***Table 1***, column III) involve universal function approximators. Thus, while it is the case that for any function there is a neural network able to approximate it, there is, however, no general way in which such a neural network can be approximated in terms of the weights of the links and the threshold values. Again the networks have to learn to recognize patterns in a supervised or unsupervised way, by trial and error, Hertz *et. al.* (1991). But, the complexity of the task that can be learnt is unlimited and can exceed the human capacity to specify the task in logical terms. The theory of validation in adaptive inductive inference when the data generating process is unknown is still in its infancy. Algorithmic and stochastic complexity theory of column II in ***Table 1*** has developed some necessary principles, while in column III, characteristic features are now well known for macro systems with large numbers of interacting agents at a micro level.

Building on the cellular automata[5] with the same recursive power of a Turing Machine, John von Neumann pioneered the theory of self-organizing and complex systems, Burks (1957). It is now well known from the Wolfram-Chomsky scheme (see, Wolfram, 1984, Dawid, 1999, Foley, in Albin, 1998, pp. 42-55, Markose, 2001a) that on varying the computational capabilities of agents, different system wide or global dynamics can be generated. Finite automata produce **Type 1** dynamics with unique limit points; push down automata produce **Type 2** dynamics with limit cycles; linear bounded automata produce **Type 3** chaotic output trajectories with strange attractors. However, it takes agents with full powers of Turing Machines capable of simulating other Turing Machines[6] , a property called *computational universality*, to produce the **Type 4** irregular innovation based structure changing dynamics associated with capitalist growth. While in general, the computational agents

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[5] These results show that the notion of symbol sequences associated with Turing computability and formalism is not essential to computation theory. When conditions of pragmatic implementation are at stake, it is useful to consider number theoretic computable functions that have inputs and outputs that are integer encoded descriptions of finite objects. However, as there are computable continuous functions with no computable solutions in the class of ordinary differential equations (see, Pour-el and Richards, 1989) what is important is the nature of the problem that brings about noncomputability rather than whether the function is continuous or discrete. I'm grateful to Ken Burdett for pressing for this clarification.

[6] In other words, computationally universal agents have the full powers of deductive inference, including that of self-reference, based on codifiable information. In section 3, I will sketch an analytical proof as to why such full powers of deduction are necessary for innovation to arise from rational strategic necessity in the Nash equilibrium of a game rather than from random mutation.

associated with higher types of dynamics can compute/learn lower type of dynamics, **Type 3** and **Type 4** dynamics pose computational problems. **Type 3** dynamics in the Wolfram-Chomsky schema though algorithmically solvable, in principle, could pose problems of computational intractability. In the latter case, the computational cost of solving a problem optimally may place a barrier to the determination of global optima leading to the decision rule being based on arbitrary local or past conditions and/or being temporally myopic. **Type 4** dynamics is algorithmically unsolvable. The central issues that arise here are many. We need to understand how computational agents produce such complex global dynamics which pose either computational intractability or algorithmic unsolvability in terms of their individual decision making. How is order then brought about? How does such complex global dynamics impinge on agents' capability to learn, cope and operate effectively? How do agents acquire computational universality, Langton (1992, p.69) ? The modern theory of emergent and complex phenomena deals with this.

The 1977 Nobel laureate in Physics, Phillip Anderson, is considered to be the father of emergent phenomenon. The seminal contributions of Holland (1975,1992), Holland et. al. (1987), Koza (1992), Goldberg (1989), Arthur (1993b), Kaufmann (1993) and others have made it possible to simulate heterogeneous computationally intelligent **(CI)** agents in adaptive settings to give a material counterpart in computer or virtual environments of the otherwise elusive phenomena of emergence and self-organization. It is increasingly becoming a methodological tool that will be used as a means of understanding complex social and other environments. The theory of emergent phenomena for intelligent adaptive agents identifies at least five following characteristics of complexity.

(i). First, there are adaptive agents. Adaptive agents have algorithmic capabilities that are not pre-programmed to respond in a fixed fashion to changes in global states but have a non-linear feedback loop that enable them to change rules of behaviour which in turn change global properties to produce coevolving local and global systems.

(ii). The global properties of a heterogeneous adaptive set of agents is not the scaled up version of the purposive behaviour or program of any agent or group of agents in the system.

(iii). Knowledge of the local programs of agents gives no clue to the global outcomes. The non-anticipating nature of global outcomes causes and retains heterogeneity in agents despite local self-organizing tendencies and homogeneity characterized by convergence to attractors. It must be noted that self-organized order as in the classic Schelling (1978) result on racially segregated neighbourhoods does not necessarily correlate with what many regard to be desirable. Self-organizing[7] and emergent systems produce typical statistical 'signatures' in the macro dynamical data such as the power laws, logistic curves, self-similar structures or fractals and chains of interrelations that manifest long memory.

(iv). With adaptive learning in complex environments canalization and lock ins can stabilize certain categories of behaviour within what may be well adapted operational or shared schemes (see, Ackley and Littman, 1992, Kaufmann, 1993). With the emergence of the latter, learning becomes instinctive or habitual and atrophied into skilful behaviour. The Ackley and Littman (1992) Artifical Life simulation where agents have neural network brains show that when the adaptive operational schema emerge they free up neural networks to do other things.

(v). As first postulated in the Wolfram-Chomsky schema, the **Type 4** irregular structure changing dynamics, over and above those characterized solely by convergence to attractors, arise only in the case of agents with powers of a Universal Turing Machine. Langton (1992) makes an important observation that physical systems capable of complexity, experience a critical slowing down at the phase transition between order (analogous to halting computations) and chaos, as they are in principle involved in a non-terminating computational loop that characterizes an undecidable global ordering problem at that juncture. Langton concludes that physical dynamical systems "are bound by the same *in principle* limitations as computing devices" (*ibid.* p.82). In system simulations with computational agents it has been found that cooperative and competitive structures develop. Competitors soon learn, in what can only be described as paradoxical conditions, the advantages of adopting surprise or innovative strategies that their rivals cannot predict (see, Ray, 1992, Hillis, 1992). This can move the system in unpredictable structure changing directions which are error driven and highly dissippative of the old order.

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[7] Bak (1996) style self organized criticality refers to order in large systems with many interacting agents that is poised at the edge of chaos or at a phase transition. At the point of criticality, minor disturbances can cause such a system to dissipate and to reorganize new patterns.

The above characterization of emergent complex system (**ECS**, for short) adaptive agent models is not meant to be exhaustive of all the work already done in this area. It should only to be taken as a working hypothesis.

The properties of emergence that can be generated in computer simulated environments can aid in a deeper understanding of the nexus between the non-anticipating nature of global outcomes and individual agent programs or rationality. The **ACE** (Adaptive Computational Economist, see, Testfasion, 1998) is the species that has evolved after decades of methodological entrainment of neoclassical economics that simply failed to see that the domain of economic analysis cannot be restricted to column I of ***Table 1***. However, my preamble here aims to guard against a cavalier attitude that emergent phenomena is what ever that comes out of an **ACE** computer simulation or that non-trivial mathematical issues of non-anticipating or 'surprise' outcomes remain unaddressed. It is clear that the mathematics of algorithmic unsolvability and incompleteness along with algorithmic and stochastic complexity theory of column II in ***Table 1*** can give formal underpinnings for adaptive learning in domains in which outcomes can be modelled as non-anticipating or surprises[8]. Likewise, studies (see, Lux and Marchesi, 1999, Solomon, 2000, Solomon and Levy, 2001, and others) based on techniques of column III in ***Table 1***, suggest that power laws in macro dynamical data such as in asset returns and investment wealth distribution are a manifestation of self-organized complexity in micro activity. Chen and Yeh (2000), Arthur *et. al.*(1997), Challet and Zhang (1998), Savit *et. al.* (1999) also raise some of these issues within the context of economic dynamics from computer intelligent adaptive agent models of stock markets. However, in terms of substantive details of economic models regarding issues such as autonomy, innovation and decentralization, learning Rational Expectations Equilibria (**REE**) and the Efficient Market Hypothesis (**EMH**), this area is still in its infancy and what constitutes emergent outcomes is still not beyond controversy.

Some of the recent advances in the mathematics of self-organized complexity come from the study of physical systems rather than from adaptively intelligent micro agents. Do the principles of emergent outcomes

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[8] See, Casti (1994, pp.143-149) for an informal discussion on the equivalence between the theorems of formal systems which are the outputs of Universal Turing Machines and the attractor set of a dynamical process. Casti (1994) makes the intriguing connection between complexity and surprises with the latter formally corresponding to algorithmic unsolvability and Gödel Incompleteness. See, also the Langton (1992) thesis on noncomputability and complexity.

that apply to inanimate particles such as in cloud formation and sand piles need to differ in any way for systems with intelligent agents? The main objective of this essay, is to convey the view that the necessity of inductive inference and problems of self-reference that arise in emergent outcomes with intelligent micro agents has no explicit counterpart in large ensembles of inanimate particles[9]. There are as yet unexploited benefits from cross-fertilization of methods in columns II and III of ***Table 1*** in understanding market systems as complex adaptive systems.

The rest of this chapter is organized as follows. Section 1 briefly outlines the classical 18^{th} century provenance of evolutionary theories of market systems and the thesis of spontaneous order or order without design. I only highlight some aspects of this tradition in classical political economy to show how the challenges of an evolutionary agenda for economics remain perennial and wide ranging even though the domain of scientific discourse has altered greatly. As decentralized heterogeneous agents are crucial in the agenda of **ECS** adaptive agent models, in Section 1.2, I briefly review the well articulated thesis on this from classical political economy and assess also the logical/methodological impasse that the neoclassical theory of decentralization ran into in the 1970s. It is interesting to contrast the latter framework with one where the determination of the degree of autonomy and decentralization of agents in markets can fruitfully be viewed and modelled as emergent phenomena. Section 2 focuses on price formation in stock/asset markets. It is an important area where the algorithmic unsolvability of rational expectations equilibria, Spear (1989), the necessity of heterogeneous inductive models for trading and the above cited properties of complex systems can throw light on ongoing discussions (Friedman and Rust, 1993, Arthur *et.al.* 1997, Chen and Yeh 2000, Challet and Zhang, 1998, Solomon and Levy, 2001, and others) on the nexus between non-anticipating global nature of asset market prices and individual rationality. Section 3 surveys the ubiquitous rivalrous structure of capitalism that has produced unprecedented novelty to the system. However, it is error ridden. In no other area has the classical methods of constraint optimization (column I, ***Table 1***) failed economic analysis more than in the explanation of irregular structure changing innovative growth in market systems. Again

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[9] Indeed, one might say that with **ACE** agents being computer programs, there is already the potential for dynamics of information as opposed to the pure dynamics of energy that characterize simple physical systems, Langton (1992, p.42). Hence, the conditions leading to complexity defined as those under which "we expect a dynamics of information to emerge spontaneously and come to dominate the behaviour of the physical system" (*ibid*) are less problematic.

I hope to show how point (v) on **ECS** on the emergence of surprises or novelty from complex system simulations with adaptive computational agents has an analogue in formally undecidable systems. This is followed by a brief concluding section.

1. The Classical and other Precursors of Spontaneous or Emergent Order: The Challenges Outlined

1.1. The Classical Legacy

The provenance of evolutionary theories of society that goes back to the classical forebears of Economics with the Scottish Enlightenment is known to predate even Darwin's evolutionary thesis on the origin of species, Hodgson (1993). On observing the unfolding of early western capitalism and the libertarian market forms, the classical Scottish thinkers were led to conclude that the elaborate structures of society ranging from language, civil society, monetary exchange, *laissez faire* and economic progress did not appear as if they were the product of human execution of a human plan. These outcomes were not produced by intentional design. Spontaneity of the order or pattern, therefore, refers to the absence of direct intentionality of a designing mind in the emergence of such observable outcomes. The following epigrams were coined: nations "*stumble* on establishments that are the result of *human action and not the execution of any human design*[10]" (italics added); or civil society is "the unintended consequence" of actions of individuals pursuing some other proximate objective which in the hands of Bernard Mandeville may seem like private vices. Adam Smith's famous "invisible hand" explanation refers to the elusive nature of the ordering principle manifesting entirely as observable outcomes of individuals' actions especially in the case of equilibrium in multiple markets, rather than the proximate objective of anybody within the system implemented by rational calculation[11].

There is a direct parallel here on the spontaneous development of a system of legal and moral rules governing cooperation and competition in interactions between individuals capable of producing stable outcomes

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[10] This is attributed to Adam Fergusson, 1767.

[11] On the emergence of division of labour which he sees as the engine of progress, Adam Smith (1976, p.25) states that it is not "originally the effect of any human wisdom, which foresees and intends the general opulence to which it gives occasion. It is the necessary though very slow and gradual consequence of a certain propensity to truck, barter and exchange one thing for another."

in society with what is found in Smith's arbitrageur in the economic realm. It is in the work of David Hume that we have the clearest statements on the non-constructivist view of reason's role in the development of the moral and legal rules of liberty and just society. Thus, "the rules of morality are not the conclusions of our reason," Hume (1888). Further, Hume couches the non-consequentialist nature of the abstract and general structure of the rules of justice with their independence from satisfying any particular desires of agents": these rules are not derived from any utility or advantage which either the particular person or public may reap from his enjoyment of any particular good", Hume (1888). Kant (1965) is known to have given a formal characterization of the rules of just society in that coercively applied rules are end neutral and by their operation do not bring about predetermined outcomes in society. Absenting knowledge of the utility that rules produce in particular instances is seen by many a liberal theorist (see, O'Neill, 1989) as a useful methodological ploy often called the 'veil of ignorance" in the construction of rules of justice so that they are not instruments satisfying the invidious interests of elites or special groups[12]. Nevertheless, it is still not fully understood how or why libertarian market systems evolved rules which possess the formal quality of their end neutrality and specifically why their capacity to produce non-anticipating outcomes in society is upheld as an important normative property in liberal Kantian political philosophy. Likewise, the Humean position that no *a priori* rationalism, can succeed in designing these system of rules, *de novo*, that would produce the desired outcome of liberty, to this day, remains one of the most baffling tenets of classical liberalism (see, Suzumura, 1990).

Modern theorists of emergent phenomena will no doubt see parallel between the late 20^{th} century theory and the classical thesis on spontaneous order. Thus, the classical theorists saw that system outcomes though the result of agents' actions is removed from the agents' objectives and hence the global outcomes is non-anticipative in terms of individual rationality. Just as market clearing, rules of morality, law and other system wide operating schemas or institutions, the classical thinkers viewed as the product of evolution. Hence, the issues covered in an evolutionary agenda on spontaneous order are very far reaching indeed: it includes explanation of autonomy, decentralization, market institutions and even libertarian morality in its ambit.

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[12] It has been suggested in Markose-Cherian (1991) that the classical Kantian view that rules of just society satisfy no predetermined outcomes may formally mean that the dynamical outcomes of such rules of engagement are formally undecidable.

However, despite its venerable origins, the classical tradition of spontaneous order in the two centuries that followed, more often than not, fell prey to the ultra-rationalist riposte that the patterns of spontaneous order "look to be the product of someone's intentional design", Nozick (1974, p.19) or "appear to be a product of some omniscient designing mind", Barry (1982, p.8). As the feeble foundations of the evolutionary tradition made arguments to the contrary seem theoretically unconvincing (see, Ullmann-Margalit, 1978), the 19^{th} and early 20^{th} century political economy was dominated by experiments to exert rational and centralized control on society. In this context, Ullmann-Margalit's statement that "it took the powerful minds - and all the logical arsenal at their disposal - of Hume and Kant, as well as the works of Darwin and Mill, to explode the logic of this Argument from Design"(*ibid*, p.268) as if to say that the task at hand was successfully accomplished by the said luminaries, is hopelessly optimistic. In other words, it was not well into the 20^{th} century that the twin theoretical pillars for an evolutionary thesis on markets were in place. The first pillar is the *contra* Argument from Design which requires an impossibility result on why no agent or agents in a system can bring about determinate systemic outcomes when all agents have computational capabilities of simulating other such agents. Indeed, not till Gödel (1931) and undecidability of the Church-Turing halting problem could the mathematical principle of non-recursiveness or algorithmic unsolvability, namely a logical impossibility result on the limits of finitary procedures, be brought to bear on the anti-creationist principle at the heart of evolutionary systems[13]. As observed earlier, Penrose (1988) can be credited to be the first to have explicitly mentioned the connection between non-recursive or non-algorithmic implementation as the *sine qua non* of emergent patterns in nature and in artificial settings.

The second major pillar of the evolutionary thesis is the demonstration of emergent phenomena in the absence of a blueprint or of an encoding for system wide outcomes. Not till the seminal work of von Neumann in the 1950's on cellular automata and then of Holland, Koza and thinkers of the SFI, did we have the necessary tools to pin down the elusive and central tenet of evolution that is of its spontaneous and emergent properties specifically in terms of orderliness and patterns. The system has

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[13] To understand that there is indeed a non-trivial mathematical problem in the modelling of complex adaptive intelligent systems, see, the account in Sugden (1989) on what is spontaneous order which makes no mention of the methodological developments in columns II and III of ***Table 1***.

to run its course as there is a lack of algorithmic predetermination of outcomes. To be in position at the end of the 20^{th} century to recreate dynamical systems of heterogenous agents with computational intelligence of varying degrees that can evolve complexity and self-organize in virtual environments of the computer is an outstanding achievement.

1.2. F.A Hayek: Decentralization and Autonomy As Emergent Phenomena

For most part in equilibrium theory of decentralized markets, decentralization refers to both the units of decision making in terms of their autonomy of action and the informational setting that guides their decisions. Adam Smith's invisible hand argument was seen by Arrow and Hahn (1971) as the "poetic expression" of the fact observed by early 18^{th} century economists of the absence of central direction in the resource allocation process with regard to what, when and how much to produce and whom to distribute it to in terms of demand. In the formal equilibrium theory of markets, it was thought that the burden of proof placed by the invisible hand type argument involved the establishment of a *formal possibility or an existence result* on an equilibrium in a decentralized economy where individuals motivated by self interest and guided by price signals alone will result in a resource allocation that not only satisfies the consistency of the economic plans of the individuals but also one that is most efficient. It was never in the domain of discourse that the emergence of decentralized market systems with the dynamical properties described by the classical economists also placed a burden of proof along the lines of what John Rust (1987) has called *computational decentralization*, viz. the impossibility[14] or the non-existence of an overarching program that can control programs of all agents in the system in the determination of system wide outcomes.

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[14] Rust (1997) only hints at the logical necessity of the impossibility result on computation that underpins evolutionary and emergent phenomena with decentralized decision units. Lewis (1985,1988) was the first to rigorously prove that the two most basic problems of economic theory viz. micro rational choice and Walrasian general equilibrium outcomes are *not* Turing computable. Indeed, Lewis has asserted that his results have "serious consequences for the foundations of neoclassical mathematical economics......that the foundations of neoclassical economics are hopelessly non-effective computationally and therefore must be considered *irrational* from the standpoint of *computational viability*" (Lewis, 1985,p.46, p.72, italics in the original). How do we come to terms with the growing number of theoretical results that state that virtually all of what is considered to be of fundamental interest to economics is *outside* the domain of Turing computability or uniform procedures ? The answer is, ofcourse, to make a laborious intellectual journey from column I to column III of ***Table 1***.

The main bridgehead between the classical thesis on spontaneous or emergent order and the modern one is F.A Hayek. In a pair of papers, Hayek (1937, 1945) cites the two most commonplace but singularly intractable informational constraints in society as being part of the rationale for decentralized systems. First, information in society is found in a dispersed form subject to time and place matrices and it is perceived by individuals in a subjective fashion. Second, it is impossible to centralize all information by communication alone as the knowledge needed to make decisions is tacit and not in a codifiable form. Hayek's much quoted observation on this is that:"We cannot expect that this problem will be solved by first communicating all this information to a central board which, after integrating all knowledge, issues its orders *the problem is to show how a solution is produced by the interaction of people each of whom has partial knowledge* ", Hayek (1945, italics added).

Hayek's *third* postulate on markets is remarkable in that he had well before 1950 made the connection that many economists have yet to do so, that is, market institutions that have coevolved with human reason enable us to solve problems that is impossible to do so by direct rational calculation. He called the latter the limits of constructivist reason and on why we failed for so long to acknowledge these limits he relates back to Cartesian rationalism, Hayek (1967).

Hayek's fortuitous Viennese connection with Kurt Gödel led him to see that there was a logical impossibility result on why algorithmic centralized control is impossible. There is ample evidence that Hayek's rampant evolutionary thinking on morals to markets to price formation and human reason itself arose as he traversed the modern Gödelian route from column II to III in ***Table 1***. There is explicit reference to the Gödel incompleteness result and its implications for his work on *The Theory of Complex Phenomena*, Hayek (1982) and also in his work on cognition and the brain as complex and incomplete phenomena in *The Sensory Order* (see, Hayek, 1953, 1982, Weimar, 1982)[15]. Hayek saw

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[15] Needless to say, many of Hayek's commentators have failed to see the Gödelian input in Hayek's evolutionary position. This is despite the above explicit references by Hayek to the Gödel incompleteness result, and his informal exposition of it in terms of Cantor's proof of the uncountable, Hayek (1967). Hayek's methodological position on the limits of 'constructivist' reason and hence his view on evolutionary solutions constitutes a *formal* Gödelian one rather than an empirical one that for instance, Suzumura (1990) is prepared to accept : "as an *empirical* observation on the historical evolution of social orders and/or institutions, Hayek's negative verdict on constructivist rationalism is quite convincing" (italics, added). Unfortunately, in an otherwise appreciative account, Vriend (2000) perpetuates the erroneous view that "Hayek's work was firmly rooted in the "antirationalistic" approach of the English

that complexity and incompleteness are two sides of the same coin and that as such domains make problem solving by direct rational calculation impossible he was led to the necessity of evolutionary solutions.

Not withstanding problems of localized information, Hayek clearly saw that autonomy of action is also necessary on account of tacit knowledge or cognitive incompleteness. The latter which is a by product of evolved complexity of the brain (Hayek, 1953) permits the agent no algorithmic access to rules of inference which prompt action. Much tacit knowledge will be lost to the world unless it can be directly precipitated in action in a manner that warrants no verbal or algorithmic justification. Thus, Hayek's work (Hayek, 1967, 1953) does lend itself to the interpretation that classes of tacit knowledge can fall into two respective categories of (i) emergent phenomena and (ii) the products of canalization from the evolution of the species or from learnt behaviour in the process of socialization. Groups become dominant from selective pressure on account of individuals being carriers of successful norms or rules of engagement which in turn select successive institutions. Thus, Hayek has provided a cogent bridgehead to the classical liberal tradition and particularly to the Kantian normative injunctions on the end neutrality of coercive rules of the state which then give scope to emergent phenomena with autonomous actors. Interestingly, no thinker is more struck by the irony of evolutionary selection especially in reference to libertarian values on which the West owes its unprecedented material success : these are best served when they operate as category (iv) in the **ECS** schema (see, Introduction) as habitual and atrophied skilful behaviour. Thus, as we will see, Hayek's observations on the economy of information and calculation that the market price setting institutions permit in the emergence of efficient allocations also correspond to category (iv) in the **ECS** (schema). With these provisos in place, Hayek may well qualify to be an **ACE**, as Vriend (2000) puts it.

Decentralization in society, thus, necessarily presupposes the granting of some degree of autonomy to the individual decision maker. Economists (e.g. Marschak, 1959, Arrow and Hurwicz, 1977) were quick to incorporate Hayek's postulate on the local nature of information which

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(*sic*) individualism as known, for example, from Adam Smith's Invisible Hand..". Hayek's thesis on the limits of constructive rationalism was far from "antirationalistic" in that he more than any economist to date was well aware of the Gödelian mountebank who ignorant of the limits of his rational capabilities attempts proof of his own logicality and becomes culpable of antirationalism and illogicality (see, Markose-Cherian, 1991, for an early discussion of this).

constrains the resource allocation problems in an economy. In the mainstream literature on decentralized decision making initiated by Marschak (1959), the optimal level of decentralization determining the extent or degree to which decision making entities should be given autonomy to make decisions on the basis of private information, requires direct and rational calculations of the relative speeds and costs involved in the alternative systems of communication and control. The logical and analytical impasse such an approach to decentralization posed was soon detected by Hurwicz (1960). Hurwicz's seminal development of the notion of incentive compatibility suggests that the reporting protocol was open to abuse as agents find it in their interest to misrepresent their endowments and preferences. He concluded that "it is the characteristic of the current state of the literature on decentralization, that one may be provided with a definition of what it means to have a more or less centralized *command* (italics added) economy"; virtually nothing is known about the decentralizing processes of the market system. A consistent paradigm for the existence of decentralized systems requires in particular that the process that determines the level of decentralization in the system should itself conform not only with the informational constraint but with what Rust (1997) calls decentralized computational constraints. Thus, Athans (1975) has indicated the potential analytical/ mathematical impasse inherent in viewing decentralized systems of control as an extension of the centralized paradigm of control in classical control theory, viz. the 'top down' approach. Within the latter framework as Athans (1965) notes that a mere imposition of the first, but not the second and I might add the third of the Hayekian postulates on decentralization in markets, may still lead to the mathematical result on the non-existence of decentralized decisions: it is optimal for the centre to cancel local decisions. In other words, the operative non-redundant aspect of decentralized decision making is the algorithmic unsolvability by agent or agents of the global system wide outcomes. How do market systems actually put in place a self-enforcing structure of rules that brings about non-anticipating global outcomes ?

In Markose-Cherian (1991) the political economy of expanding market societies of Europe is studied to glean insights into how in fact a larger market order is formed. In the creation of a larger market order with the European Economic Community, end neutral or end independent rules can be observed to emerge from a negative selection process of rule elimination. Through a decentralized litigious process initiated by individual litigants who challenge rules for their inability for general implementation, rules are progressively eliminated as unjust as they cannot be 'universalized' over what is now a larger territory and peoples. In Section 3, an even more powerful and commonplace hypothesis is put forward for the development of legal rules that favour *laissez faire* type non-anticipating systemic outcomes. Regulation that aims to bring about specific predetermined outcomes can be rendered dead letter by regulatory arbitrage when private individuals find it profitable for them to contravene these rules. As individuals rule break and attempt to exit from given regulatory systems, it is rationally strategic for them to innovate as they step out. Many policy rules whose outcomes are predictable may in fact suffer elimination as they fail to be Nash implementable in the face of such contrarian strategic behaviour on part of regulatees. This is part of a very large literature on the regulatory dialectic that has recently enjoyed great resurgence of interest as a process of institutional innovation (e.g., Miller, 1986, Schanze, 1995, and others). However, what is interesting, for our purposes, is that we can relate the above issues on autonomy and decentralization in markets to the **Type 4** structure changing dynamics in the Wolfram-Chomsky schema produced only by agents with computational universality or capability of self-referential mappings.

2. Emergence of Efficiency In Asset Markets and Individual Rationality

In the previous section the emphasis was on the open ended structure of the constitutional rules of market systems which appears to permit the evolution of complexity and emergent outcomes. In this section, the focus is on a specific class of markets, viz. financial markets, in which controversy and hence a lack of understanding of the nexus between the non-anticipating nature of global outcomes relative to individual agent programs or rationality has featured in a big way. Here we will address two issues: (a) Why does asset price formation cause problems of inductive inference? (b) Is asset market efficiency an emergent outcome?

2.1. Inductive Rationality and Heterogeneous Beliefs

Let us start with the as yet unproved premise that the stock price is a strongly self-referential mapping of system wide information of market conditions[16]. It is emergent from trading activity of agents and hence manifests a non-anticipating quality with respect to individual agent ra-

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[16] Arthur *et. al.* (1997) refers to the problem of reflexivity or self-reference for deductive inference posed by the fact that prices are generated as the result of trader's expectations of

tionality or computing. The latter can, indeed, be interpreted to mean exactly what the Fama type assertion on efficient market hypothesis (EMH) states: prices, P, that contain all (publically) available information, H, follow a martingale such that price changes are random and not serially correlated. If the theory of emergence or algorithmic undecidability of strong reflexive encodings is not explicitly considered, it is easy to become a victim of the fallacy of composition. It can be held that the global outcome of non-anticipative prices *is* the consequence of rational agents who believe this to be the case. That is, if i indexes all N agents, the following states that all agents have homogenous beliefs that the asset price is a random walk

$$\forall E_i(P_{t+1} \mid H_t) = P_t,\ i = 1, 2, ., N.$$

In the case when assets are traded for purely speculative reasons, viz. for pecuniary gains based on price expectations, homogenous price expectations results in the non-existence of speculative trading. We have the paradox that with the cessation of trade, the price at t+1, P_{t+1} , never gets determined. If one is to take the Samuelson (1965)[17] view that 'proper' anticipation of prices implies taking conditional expectations,

$$\mathrm{E}(\mathrm{P}_{t+1} \mid \mathrm{H}_t),$$

it is clear that this is without unique procedural content. In Arthur *et. al.* (1997) they make a case for heterogeneous multi-agent models where each agent uses genetic algorithms to arrive at future price predictions. "Agents, in facing the problem of choosing appropriate predictive models, face the same problem that statisticians face when choosing appropriate predictive models given a specific data set, but *no objective means by which to choose a functional form.* The expectational models investors choose affect the price sequence, so that our statisticians very choices of model affect their data and so their choices of model"

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prices formed on the basis of anticipation of others' expectations of others' expectations of prices.

[17] The well known Samuelsonian tenet states 'properly' anticipated prices fluctuate randomly. Here by 'proper' anticipation is meant that conditional expectation, represented as $\mathrm{E}(P_{t+1}|\mathrm{H}_t$) are taken. However, as conditional expectations operators by the Tower/iterated property satisfies the martingale condition, random fluctuations in properly anticipated prices is satisfied by tautology. Thus, by the Tower property $\mathrm{E}(\mathrm{E}(\mathrm{P}_{t+1}|\mathrm{H}_t)|\mathrm{H}_{t-1}) = \mathrm{E}(\mathrm{P}_{t+1}|\mathrm{H}_{t-1})$ implying that $\mathrm{E}(\mathrm{P}_{t+1}|\mathrm{H}_t)$ - $\mathrm{E}(\mathrm{P}_{t+1}|\mathrm{H}_{t-1}) = \varepsilon_t$, where ε_t is white noise.

(*ibid.* p.305, italics added). The message here is that it is not tenable to justify **EMH** on truisms such as rational agents do not make systematic errors. In algorithmically unsolvable problems the domain of the decision problem is not a recursive one and agents have no systematic way of determining the fixed point mapping for the rational expectations equilibrium of the price[18]. Agents will make systematic prediction errors for the future price from the vantage of their respective misspecified models. Thus, in the framework of emergence as being set out here, one first needs a justification for the existence of agents with heterogenous models of the world. In other words, in the emergent efficient market hypothesis on asset prices (**EEMH**, to distinguish from the traditional view) what is being challenged is the necessity of micro behaviour of all agents to converge to that of homogenous martingale believers to necessitate unpredictable price dynamics.

To pin down the algorithmically unsolvable nature of a rational expectations price or the absence of an unique objective decision procedure for agents to compute the functional fixed point mapping, I will outline the issues on inductive inference learning first raised in Spear (1989). The model is tailored to suit an asset market equilibrium for a single asset that can be bought and sold in standardized units where the total quantity of the asset is Q. Time is discrete and denoted by $t = 0, 1, 2, ..., T$. There are N agents indexed by $i = 1, 2, 3, ..., N$. Agents can choose to be buyers (N_b) or sellers (N_s), $N_b + N_s \leq N$. In a one period ahead forecast horizon for agents, in the absence of any fundamental value determining factors for the asset price, we will consider the minority game studied in Challet and Zhang (1998) and Savit *et. al.* (1999). The prototype of this was first considered in the Arthur (1994) El Farol game.

The minority game is ideal to study pure speculative behaviour when spot price dynamics is entirely generated by endogenous uncertainty. Agents execute their trades at t, having information (to be specified) up to t. Their payoffs are related to the price at t+1, P_{t+1}. Performance is assessed period by period. In a minority game it pays buyers to be in the

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[18] For instance if we are to take the definition " noise traders form erroneous beliefs about the future distribution of returns on a risky asset", Schleifer(2000, p. 33) this must be the rule than an exception in decision problems that are on non-recursive domains. Schleifer goes on to say that this is may because agents are "subject to one of the behavioural biases in processing information and forecasting returns" (*ibid.* p.33). However, it is customary (see, Brock and Hommes, 1998) to refer to noise traders as those who use technical trading rules in contrast with the 'smart money' traders who believe that the price of an asset is determined solely by it fundamental value.

minority and likewise for the sellers of the asset. The price function as we will specify will enhance profitability or payoff monotonically with the smallness of the minority and hence reinforces learning in the direction of being contrarian. That is when agents compete to be in a minority, it pays to be first to break ranks and do the opposite of what others are doing especially when a majority is forming in the selling or the buying direction. Further, random walk believers will not trade and hence as only trading can net profits there is an inbuilt pressure for speculative agents not be 'rational' as in the no trade results. I will assume that agents can buy or sell only one unit of the asset. This is to abstract from features such as build up of speculative inventories by agents and also for the winning outcome to be directly ascertained by a simple condition based on the relative size of the numbers of buyers and sellers.

The payoff for those who trade is given by

$$\pi_i = \begin{cases} (\mathrm{r}\text{-}\mathrm{P}_{t+1}) > 0 & \text{if } \mathrm{N}_b < N_s \Rightarrow \text{Buyers win: 1} \qquad \text{(a)} \\ (\mathrm{r}\text{-}\mathrm{P}_{t+1}) < 0 & \text{if } \mathrm{N}_s < N_b \Rightarrow \text{Sellers win: 0} \qquad \text{(b)} \\ \mathrm{r}+\varepsilon,\ \varepsilon \sim \mathrm{N}(0,1) & \text{if } \mathrm{N}_s = N_b \Rightarrow \text{(1,0) with probability .5} \qquad \text{(c)} \end{cases} \tag{1}$$

Above, r is the reservation price which is the same for all agents. If r is taken to be the long run mean of the price, the above conditions that determine agents' payoffs and in turn their strategies can set in motion strong mean reverting properties. Condition (1.c) stipulates the random walk model when the price fluctuates around r by a white noise error, ε.

We will consider two classes of information sets. (i) All agents observe the past history of prices including the current price, $\mathrm{H}_t{}^* = \ \mathrm{P}_{t-1} \quad i = 0, 1, 2, ..t.$; (ii) All agents observe past history of prices and the total number of buyers and sellers from t_0 to t, $\mathrm{H}_t{}^{**} = \ \mathrm{P}_{t-1}$, (N_{bt-i}, N_{st-i}) , $i = 0, 1, 2, ..t$. As a result of (1.c), even though $\mathrm{H}_t{}^*$ is a bijection on (1,0) at each date, it is a noisy signal as to the actual numbers who play the winning strategy relative to the numbers who lose.

For computational agents, their decision procedures and forecast rules are computable functions. Each agent's prediction function of price at t+1 given $\mathrm{H}_t{}^*$ is given by $\hat{\mathrm{f}_i}$ is a mapping from the information set to set of spot prices for the asset,

$$\hat{f_i} : H_t^* \rightarrow \{P_{t+1}\}. \tag{2}$$

Note there are three pure strategies to buy , sell or not to trade, {b, s, ♯}. Agents' decision procedure in pure strategies is defined as

$$b_i : (H_t^*, \hat{f_i}) \rightarrow \begin{cases} \text{If } \hat{\mathrm{f}_i} - r < 0 & \text{then 1, buy, and } \hat{\mathrm{f}}_{i\,b} \quad \text{(a)} \\ \text{If } \hat{\mathrm{f}_i} - r > 0 & \text{then 0, sell, and } \hat{\mathrm{f}}_{i\,s} \quad \text{(b)} \\ \text{If } \hat{\mathrm{f}_i} = r & \text{then no trade, and } \mathrm{f}_i^\sharp. \quad \text{(c)} \end{cases} \tag{3}$$

The decision rule for speculation is simple: agent i buys (sells) if the spot price at t+1 is predicted to make i a winner and does not trade if the agent is a random walk believer, $f_i^\sharp$ in (3c), and the expected return is equal to r. In other words, it is plausible to assume that a speculator will not trade unless he is expects to win. The spot market prices are determined by a total computable function g which is mapping from agents' decision rules based on their forecast functions to the set of admissible spot prices,

$$g : (b_{it}, \hat{f_{it}}, \forall_i, i = 1, ...N) \rightarrow P_{t+1}.$$

The spot market price function g is given by

$$g = \begin{cases} (\{\mathrm{P}^0\}|P_{t+1} > r + \varepsilon) & \text{if } \sum\limits_i \hat{f_i}^b > \sum\limits_j \hat{f_j}^s, \mathrm{i}{\neq}\mathrm{j}, \text{ or } N_b > \frac{N-1}{2} \quad \text{(a)} \\ (\{\mathrm{P}^1\}|P_{t+1} < r + \varepsilon) & \text{if } \sum\limits_i \hat{f_i}^s > \sum\limits_j \hat{f_j}^b, i \text{ or } \forall_i \; \hat{f_i} = f_i^\sharp \text{ and } \mathrm{P}_{t+1} = 0 \quad \text{(b)} \\ (\{\mathrm{P}^\sharp\}|P_{t+1} = r + \varepsilon), & \text{if } \sum\limits_i \hat{f_i}^b = \sum\limits_j \hat{f_j}^s, \; i \neq j. \quad \text{(c)} \end{cases} \tag{4}$$

Here, ε is a white noise term, $\varepsilon \sim \mathrm{N}(0,1)$. The market price function g precisely determines the payoff for the minority speculative market game. In (4a) it yields the set of prices $\{\mathrm{P}^0\}$ which leads sellers to win given they are in the minority as stated on the **RHS** of (4). $\{\mathrm{P}^0\}$ is an increasing function of the size of the majority implying that larger the buying majority the more profitable is it for j to be a seller. Likewise, in (4b), the set $\{\mathrm{P}^1\}$ makes it a win situation for buyers given they are in a minority. The extreme case of $\mathrm{P}_{t+1} = 0$ is obtained when no trade occurs and when all agents become random walk believers.

Assumption 1: The spot market price function g is a total computable function given in (4) which given appropriate encoding of the domain and range of the function is a number theoretic function, g: **N** → **N**. By the Second Recursion Theorem (Cutland, 1980), for any total computable function g and for a fixed enumeration of partial computable functions ϕ_0,ϕ_1,ϕ_2, ..., g has a fixed point in the sense that there exists computable functions ϕ_a such that $\phi_{g(a)}= \phi_a$.

Definition 1 : In a rational expectations equilibrium (**REE**) there exists some computable forecast function $\hat{f} = \phi_a$[19] such that

$$\phi_{g_{(a)}} \cong \phi_a, \tag{5}$$

then a is a fixed point of the market price function[20].

Note, a is the encoding of the algorithm or program that computes that output of the market game when the market price function g that determines the outcome is consistent with agents' prediction functions for P_{t+1}. In the absence of perfect information on the population distribution of forecast rules, in principle an agent has to find a meta forecast rule as on the **RHS** of (5). That is, the agent has to identify a proper subset of the set of all partial computable functions, ϕ_0, ϕ_1,ϕ_2, ..., such that only the fixed points of the total computable function g are identified, viz.

$$\{m \mid \phi_{g_{(m)}} = \phi_m\}. \tag{6}$$

By Rice's Theorem (see, Spear, 1989) there is no recursive/ algorithmic procedure to identify the set of indices in (6) and hence to learn the **REE** of the market price function g. Inductive trial and error processes that begins search in an arbitrary subset of diverse forecast rules specified in Column III ***Table 1*** have to used to. Before a survey of these are undertaken in the next section, we will formalize why a homoge-

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[19] Note, each computable function is identified by the index or code of the program that computes it when operating on an input and producing an output if the function is defined or the calculation terminates at this point. Following a well known notational convention, Cutland (1980), we state this for a single valued computable function as follows $f(x) \cong \phi_a(x) = q$. That is, the value of a computable function f(x) when computed using the program/Turing Machine with index a is equal to an integer $\phi_a(x) = q$, if $\phi_a(x)$ is defined or halts (denoted as $\phi_a(x) \downarrow$) or the function f(x) is undefined (∼) when $\phi_a(x)$ does not halt (denoted as $\phi_a(x) \uparrow$).

[20] As stated in Spear (1989) it does not follow that g(a) = a. It is only required that both a and g(a) identify functions that produce identical outputs.

nous computable **REE** cannot exist in market games that resemble the minority game.

Definition 2: We say agent i has rational expectations of the spot price if $\hat{f_i} = \phi_a$ and a is the fixed point in (5).

Definition 3: The **REE** of the minority market game has a computable fixed point if (5) is computable and $\phi_{g(a)} = \phi_a = P_{t+1}$.
Then the **RHS**, ϕ_a, and the **LHS** $\phi_{g_{(a)}}$ of (5) must produce the same outcome here in the classes of $\{P^1\}$, $\{P^0\}$, $\{P^\sharp\}$ and hence which is the appropriate pure strategy for t+1 is predictable at t.

Definition 4: A homogenous **REE** (**HREE**) is one in which, $\forall_i, i = 1, ...N$, there exists a $\hat{f_i}$, such that $\hat{f_i} = \phi_a$.

Theorem 1: Given that the total computable market price function g always has a fixed point, there is no computable homogenous **RE** equilibrium for the minority speculative market game in the three pure strategies buy, sell or no trade {b,s, $\sharp$}. There is no algorithmic decision procedure to determine optimal winning strategies in (3) or P_{t+1} is not predictable.

Proof : We consider two cases. Assume that (5) is computable and for $\forall_i, i = 1, ..., N$ there exists a $\hat{f_i}$ such that $\hat{f_i} = \phi_a$.

Case 1: *For the pure strategies to sell or buy*: By ϕ_a on the **RHS** of (5) let (4a) hold and $\{P^0\}$ be the predicted outcome. This, however, results in the decision rule (3b) to follow for all agents. Hence, all agents become sellers which results in $\phi_{g_{(a)}}$ to output $\{P^1\}$. As this leads to the **RHS** and the **LHS** in (5) to yield contradictory outcomes $\{P^0\}$ and $\{P^1\}$, we conclude that (5) is not computable.

Case 2: *For pure strategy not to trade* : By ϕ_a on the **RHS** of (5) let (4c) hold and $\{\mathrm{P}^\sharp\}$ be the predicted outcome which leads to agents' decision rule (3c) to follow for all agents. Thus, when all agents do not trade the **LHS** of (5) produces the outcome $\{\mathrm{P}^1\}$ as the minority outcome. Thus, a contradiction follows and there is no computable homogenous **REE** in pure strategies.

The upshot so far is the following : learning **REE** is in general inductive or non-recursive, viz. with no unique decision procedure. With no computable fixed point as above, homogeneity of agents' forecast rules is impossible in stock market models where contrarian actions are reinforced. Is there a stable Nash equilibrium in mixed strategies where in all agents randomly buy or sell with probability a half and with prices converging to a random walk? What are the conditions for self-organized criticality or of emergence of efficiency in asset markets? The evidence from a number of computational multi-agent stock market markets will surveyed in Section 2.3.

Thus, in the absence of a unique algorithmic decision procedure for the market price, agents resort to heterogeneous adaptive computing models for the price. This makes it tenable that even agents with the full powers of a Turing Machine must to agree to disagree with regard to price predictions. Likewise, an algorithmic unsolvable problem may prompt an evolutionary solution along the lines of the emergence of an adaptive operating schema or institution (see, category (iv) in the **ECS** schema in Introduction). Both these possibilities have been the subject of intense discussion in recent years.

2.2. Price Formation in Double Auction (DA) Markets: Why Zero Intelligence?

The rules of double auction that determine transactions prices in goods with standardized units are known to have been adopted in markets that we now call organized markets at least as early as the advent of the London Stock Exchange in the 17^{th} century. In continuous **DA** agents simultaneously post bids (the price at which they will buy) and offers (the price at which they will sell). The success of the **DA** in achieving transactions prices close to equilibrium market clearing prices rests on the simple institutional rule that traders can get their bids/offers accepted only if they are the best prices at a given time. The best price rule requires that the bid is the highest quoted and the ask is the lowest at time of transaction.

The problem as critiqued by papers in Friedman and Rust (1993) of the institution free analysis of price formation is that in the absence of institutions, agents have to show extra ordinary powers of computation to work out the Bayesian Nash equilibrium of a continuous double auction. The latter has no analytical solution and specifically "nobody has yet been able to calculate the exact timing and size of bids and offers" (*Ibid.* p.xxi). One of the important insights from multi-agent adaptive simulations such as by Ackley and Littman (1992) is the inverse relationship between the need for explicit calculation and learning and the evolutionary development of system wide well adapted operational schema or institutions. Once the latter are in place agents need to exercise very little intelligence to get things 'right'. What explicit learning was initially needed when fully adapted becomes instinct and atrophied into skilled behaviour. It is precisely and only in this framework that one must interpret Hayek's early observation (Hayek, 1945) on how market institutions in the context of equilibrium price setting facilitates the emergence of the latter with agents trading on the basis of limited information, calculation and explicit learning of their environment. The bounded rational behaviour observed in contexts such as of double auctions can be interpreted to be the product of evolution.

The recent experimental behavioural studies surrounding the variants of double auction show that the market in action can produce easy convergence to competitive market clearing prices with very few traders who use a minimum of strategic behaviour at that. Easley and Ledyard (1993) led the way by showing that the Bayesian game against nature strategies, played by agents who ignore the impact of their decisions on that of their opponents, successfully generate competitive equilibrium trajectories. In a now celebrated paper, Gode and Sunder (1993) show that continuous **DA** markets populated by *zero intelligent* agents are highly efficient in extracting gains form trade and price trajectories converge to competitive equilibrium prices. Zero intelligence corresponds to simple computer programs that generate random bids (or asks) subject to a no loss constraint. The latter means that traders cannot buy above their redemption values or sell below their costs and no attempt is made to maximize profits. This was sufficient to obtain 98% of the gains from trade. It is highly conceivable that successful market making in specialist markets bolstered by non-disclosure rules on bloc trades does not involve strategies that are more complicated than the zero intelligence one. The best price rule of execution is a simple but powerful device to obtain competitive outcomes with the great economies of computation and information that Hayek emphasized.

Carmerer and Weigelt (1993) introduce token valuations that are not set by the experimenter but is endogenously determined by traders who form expectations of the prices. Then, we no longer have an objective criteria of what constitutes gains from trade. Even with the rules of the double auction in operation, the situation becomes one where traders use heterogenous beliefs to guide their trades. As we will see, this leads us to as yet fully resolved issues on efficient market prices as emergent phenomena and on individual rationality.

2.3. Emergent Efficient Market Hypothesis (EEMH) in SFI Related Artificial Stock Markets and Minority Game Models

Robert Shiller (1981) marks the beginning of the end of an extensive period of methodological rigidity when informational efficiency of asset prices was asserted to be a property of an unspecified capability for rational calculation by agents. The simple random walk model of asset prices has given way to a current consensus which favours the following class for asset market prices. Stock market returns are serially uncorrelated[21] but their variance is long memory fractally integrated process rather than **GARCH** (Generalized Autoregressive Conditional Heteroscedasticity). The latter has an autocorrelation function that dies down far too fast at an exponential rate while long memory in volatility of stock returns shows persistance and follows a power law decay. The professed goal of the growing number of papers in the area of the multiagent simulated asset market price dynamics has been to see what type of microbehaviour can generate in the simulated asset prices the properties of fat tails, volatility clustering, crashes and the like that have been observed in traded asset market prices.

Following the pioneering work of Arifovic (1994) much attention was focused on how a population of agents with heterogenous **GA** forecasting rules produces a global/market price process that converges to the **REE**. It is also the case that forecasting rules of all agents also converge with the conclusion here being that there is inductive learning of the unique **REE** asset price. In contrast the formal arguments made in section 2.1, in asset markets where there are profits from contrarian strategies,

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[21] The fractal Pareto-Levy Stable distribution (with infinite variance) but independent increments with rich volatility related dynamics is a possible contender for asset price returns. It is interesting to note how Mandelbroit (1966) theoretically justifies the Pareto-Levy stable distribution in asset price dynamics in one of the early studies on how **EMH** is achieved by 'perfect' arbitrage. Starting with an assumption of stationarity (Gaussian) and finite variance for the unobserved non-arbitraged price process, if perfect arbitrage with infinite horizon least squares prediction is made, the distribution of the arbitraged price changes will be a known distribution rescaled by an infinite constant,viz. a Gaussian with divergent variance, *which is absurd* (*ibid.* p. 228). For this Mandelbrot proposes a way out by starting with an unobservable process for the unarbitraged price changes (the so called "function of causes " with the exception of arbitraging itself) which is non-Gaussian with finite variance. The appropriate perfect arbitrage which does not entail linear least squares prediction will bring about a Pareto-Levy distribution for the arbitraged price increments. Methodologically, as there is only one price process, Mandelbrot's set up of a relationship between the 'price before arbitrage' and 'the price after arbitrage', as if there is any objective status to the former will be in stark contrast with the latest attempts to justify the Levy Stable distribution for asset returns.

homogeneity of beliefs or forecast rules is clearly inconsistent with the notion of a **REE**. The pressure to be in a minority make agents to be contrarian of any global attractor that is building up and the price regimes will irregularly cycle between periods of bullishness, bearishness and quiesence.

Chen and Yeh (2001) make an explicit case for **EEMH** (Emergent Efficient Market Hypothesis). Homogeneous micro rational behaviour based on random walk believers is no longer considered to be a necessary condition of macro-level unpredictability of prices in terms of serially uncorrelated returns. Hence, a combination of the latter with a critical level of heterogeneous agents with diverse forecast rules is considered necessary to **EEMH**[22]. The important point in Chen and Yeh (2000) is that agents who are non-random walk believers are not deluded; some significant proportion of them at any one time make profitable trades. In other words, till recently the **EMH** was at odds with technical trading which is seen as a sign of irrationality that will in time be driven out of the market. However, the view here is that the generic algorithmic unsolvability of learning **REE** or the absence of a unique decision procedure for this will produce, to use the language of SFI, a coevolving ecology of traders with heterogeneous belief models.

Unfortunately, given the diverse micro architecture of extant artificial stock markets to date, it is not easy to draw definitive conclusions regarding what if any generic conditions for self-organizing apply to these simulated markets. We will briefly consider two classes of artificial stock market models in this section and leave a further third class to the next section. There are those based on the SFI micro architecture for a single risky asset which generally has a fundamental value determined by an AR(1) process for the dividends. In contrast, there are minority game stock market models which is not unlike the structure of Arthur's (1994) El Farol game. In these models (see, Section 2.1), the asset price does not have an explicit fundamental value determining process. The SFI stock market models build on the standard portfolio choice of a single risky asset that uses the Sharpe ratio (see, Arthur *et. al.*,1997). Significantly, in all variants of the SFI stock market model (eg. Chen and Yeh, 2000, Brock and Holmes, 1998, Le Baron, 2001, etc) as well as the mi-

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[22] It is interesting to note from Chen and Yeh (2001) that heterogeniety of trader beliefs and strategies is not sufficient to generate unpredictable asset market prices. For a certain multi-agent micro architecture of learning, the market asset price dynamics was simply an ARMA process.

nority stock market games, agents are identical except in how they make inductive inference. There is a finite set of forecast rules at the start. Agents are either randomly allotted a subset of these or have access to all of them with selection of and innovation to forecast rules proceeding according to evolutionary computing methods given in column III of ***Table 1***.

A number of common themes can be gleaned from the simulation results of the first two classes of models. As the market environment is maintained relatively stationary with a low intensity of search by **GA**s or **GP**s, the unique homogenous rational expectations **HREE** result followed in Arthur *et. al.* (1997). In contrast when the rate of **GA** exploration for 'better' predictors was speeded up, the stock market prices began to show historically observed properties of volatility clustering. Further, convergence to at least two belief classes of agents followed. In contrast, in Brock and Holmes (1998) the intensity of search parameter is endogenous. In cases where agents' memory was lengthened as in le Baron (2001) and Brock and Holmes (1997) (and costs of rational expectations are zero) non-random walk believers will be driven out and the market converges to a **HREE**. In other words, factors that lead to inertia in inductive experimentation results in a more stationary trading environment, making it possible to incrementally learn the fixed point of the market price function.

In the standard minority game where all N traders buy, N$_b$, or sell, N$_s$ (viz. there is no option of not to trade as specified in 3.c) the 'social' gains from trade are optimized when after some large number of trading periods the following variance function

$$\sigma^2 = \frac{1}{T}\sum_{t=1}^{T}(N_{bt} - \frac{N}{2})^2 \tag{7}$$

is at a minimum. What is interesting is that the mixed strategy Nash equilibrium where agents randomly buy or sell with probability half is not in fact Pareto optimal or the emergent outcome. There is a generic degree of heterogeneity of strategies[23] denoted by m* for any N and the total number of strategies is m, $m > m*$, from which selection is being made. Note that N is assumed to be odd and $N \geq m$. At such a m*, the σ^2 function in (7) is minimized at well below what is obtained by

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[23] Note, Cavagna (1999) has proved that what is important here is not the length of past historical data points that agents were allowed access , viz. memory, but rather the size of the strategy set.

the Nash equilibrium random mixed strategies. Savit *et. al.* (1999) use mean-field-like arguments to show that σ^2/N is a function of z= 2^m/N , where 2^m is the dimension of the strategy set. Further, it was discovered that for numerous runs of the minority game with different N and m, the plots for σ^2/N with respect to $z = 2^m/N$ fall on the same curve. The minimum of this generic curve is near $z = 2^m/N =$ 0.5. Thus, the microsimulation of agents shows remarkable emergent coordination where by the agents adaptively adopt the precise m* which brings about the optimal level of social coordination, viz. the σ^2 in (7) is well below that for the Nash equilibrium random mixed strategies. This is remarkable because only the experimenter and not the agents knows the σ^2 function and all local Hayek information constraints exist. However, once no trade strategy as in (3c) is allowed and gains from being in the minority are not unitary, it is clear that the criterion as in (7) is no longer capable of assessing what constitutes optimal social coordination.

The observed features of historical price dynamics in terms of their power laws clearly requires non-convergence to a **HREE**. Indeed, the main finding of the micro-simulations literature is that the **EMH** on the absence of arbitrage opportunities is that a critical degree of non-homogeneity of trader forecast rules or strategies must exist. This must also sustain the observed volatility clustering, non-Gaussian fat tails in asset returns and other features that are associated with power laws produced by emergent coordination that is endogenous to asset market trading.

2.4. Power Laws In Investor Wealth Distribution and EEMH

Certain macro dynamic variables arising from large numbers of micro interacting agents in both natural and social settings are known to display the power law distribution. Till recently few have ventured an endogenous explanation for power laws. In the case of stock market returns, Solomon (1998, 2000) and Levy and Solomon (1996) have made one such explanation using a micro architecture similar to the SFI model. As will be briefly outlined below, in these papers the efficient market outcome emerges from the coevolution of agents' forecast rules with efficiency and the no arbitrage result are clearly an unintended consequence of traders' objectives. In contrast, Kirman and Teyserrie (1991), Lux and Marchesi (1999) use a micro-architecture in which there are two fixed classes of traders: the fundamentalists who expect the price to follow the discounted value of expected future earnings and the noise trader who uses trading strategies based on price trends and by imitating other traders. Based on certain specification of trader reaction to economic signals, there are endogenous probabilities of switching from one or the other group with waves of optimism and pessimism in buying. The stock prices are endogenously adjusted according to excess demand (supply). The power law is observed in the asset returns as well as long memory in the volatility of returns[24]. High volatility is seen to coincide with a larger proportion of noise traders.

In the Levy-Solomon thesis on endogenous emergence of power law in stock returns, the power law is defined in terms of the distribution of investor wealth in large microagent system of N agents. The probability distribution (or the proportion of individuals in a population with wealth of size w) is given as

$$P(w) \sim w^{-1-\alpha}. \tag{8}$$

Here, w (integer valued)[25] is a certain value of wealth w in the population. The total wealth is generated from the N sub/micro agent systems is

$$W(t) = w_{1(t)} + w_{2(t)} + ... + w_N(t). \tag{9}$$

Solomon and Levy discovered that dynamics characterized by generalized Lotka Voltera equations for each micro system can under certain conditions bring about the power law distribution in (8). The general form of this is

$$w_i(t+1) = \lambda_i(t)w_i(t) - cW(t) + vW(t), i = 1, 2, ..., N. \tag{10}$$

Here, $\lambda_i(t)$ is the random multiplicative wealth generating factor which arises due to the performance of each agent's forecasting model and strategy to buy, sell or not to trade in relation to the market's generation of the spot price which is common to all traders. Factor c in (10) relates to

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[24] Kirman and Teyssiere (2000) is not aimed at an endogenous explanation of power law in asset returns nor on the emergence of efficiency in markets.

[25] Note that P(w) = 0 for w $\leq$ 0 and lim P(w) = 0 as $w \rightarrow \infty$. That is, the distribution is zero if w is negative or tends to infinity.

the interaction (such as correlation) between individual wealth held in asset i and total market capitalization W(t). The factor v in (10) relates to the amount received from external sources.

I will summarize below the main results from Levy and Solomon (1996) and Solomon (2000). **Levy-Solomon Result 1**: The power law in (8) follows if and only if the multiplicative coefficient $\lambda_i(t)$ in (10) on agent's wealth, at a point of emergence, becomes independent of agent i factors and all agents' payoffs from strategies must be drawn from a uniform probability distribution. The emergence of the efficient market hypothesis and the absence of arbitrage opportunities follow when the population of strategies/ forecast rules that evolve are precisely ones that have the same probability of obtaining a payoff in a given finite range. That is, on average no strategy has an undue advantage over another strategy in obtaining higher than average payoff at each t. The important point here is that this is an emergent phenomena amongst traders who are each trying to find rules to 'beat ' the market and assiduously select 'good' forecast rules by generic evolutionary fitness criteria of rewarding those rules that increase investor wealth shares, w_i(t)/ W(t). Thus, **EEMH** is sustained in microsimulation models under circumstances very different from what is traditionally associated with trader rationality. That is, under **EEMH** at each t a range of strategies/forecast rules are used so that investor wealth distribution far from being egalitarian or equal on average over time (as might be the case if agents were making random choices), satisfies the power law distribution in (8).

The steps in the proof of the result requires that the α parameter in (8) has to be positive. For this Solomon *et. al.* specify that there has to be a lower bound, $w_{min(t)}$, which dictates that the central limit theorem no longer applies at large t and log normal distributions do not follow for w_i(t). The lower bound $w_{min(t)}$ is specified as

$$w_{min}(t) = q\overline{w}(t), \tag{11}$$

where $\overline{w}$(t) = W(t)/N, ie. $\overline{w}(t)$ is the mean income at time t.

Consider the wealth dynamics for an agent in a single asset market with the second term in (9) is zero and there is no new injection of resources at each date (viz. v in (9) is zero). On placing the lower bound constraint on minimum wealth given above, the income dynamics for each agent is defined as

$$w_i(t+1) = \lambda_i(t) w_i(t) \tag{12}$$

with the lower cut-off,

$$w_i(t+1) \geq q\overline{w}(t). \tag{13}$$

Levy-Solomon Result 2: When the power law in (8) holds for the system dynamics in (12,13), for given N and q , q in range of $1 > q > 1/\ln$ N, the exponent α in (8) is given by

$$\alpha = 1/(1-q). \tag{14}$$

Typically financial stock market data gives $\alpha \sim 3/2$ and q $\sim$ 1/3 with q being interpreted as the 1/r where r is the long term market impact factor.

Levy-Solomon Result 3: Short term returns distribution on stocks satisfies the truncated Levy stable distribution (see, footnote, 21). This follows because the returns appropriately defined is the sum of the N(t) agents' trades at time t where each w_i(t) satisfies the power law.

It may be said that λ_i(t) in (12) satisfies i-independence as a consequence of the ergodic properties of logarithmic scales. Further, Solomon (2000) claims that " almost every realistic microscopic market model we have studied in the past shares this characteristic of w-independent $\Pi(\lambda)$ distribution". However, as the i-dependence of λ_i(t) is due to the stochastic performance of the inductive choice by agent i of its forecast rule $\hat{f_i}$, what has not be sufficiently investigated is the emergent nature of i-independent strategies by traders who are attempting to achieve the opposite. Secondly, it can be conjectured that the w-independence of the strategies is the consequence of the feature called the Red Queen Effect , Ray (1992) when it gets harder and harder to maintain the superiority of a strategy in a population of coevolving strategies. Finally, the contrast between the Levy-Solomon endogenous theory of the power law in asset prices and that of the herding models of Lux-Marchesi cannot be greater. In the former, market efficiency and no arbitrage with power law in asset prices emerges only when there is no statistical distinction whatsoever in the payoffs to any strategy in terms of categories such as smart money/fundamentalists and chartists/noise traders. In the herding models these distinctions are retained. Thus, while great advance has been made for a better understanding of the endogenous reasons for market efficiency, more work needs to be done to be able to discriminate more finely as to what micro architectures in artificial stock markets are necessary and sufficient for the emergent properties of **EMH**.

3. The Ubiquitous Structure of Opposition, Emergence of Innovation and Irregular Structure Changing Dynamics

Despite many an astute observer's views of capitalist growth that it is inevitably accompanied by dislocating changes, creative destruction and the like, innovative growth has remained a veritable loose cannon on the deck of neoclassical economic theory. The evolution of cooperation, Axelrod (1984) has received a lot of attention. But, the ubiquitous structure of opposition that necessitates secrecy and emergence of innovation though intuitively familiar has not received formal attention in economic models. Recent work in this direction by Ray's Tierra (1992) and Hillis (1992) show how in complex system simulations both cooperative and competitive structures develop where competitors soon learn the advantages of adopting secrecy and surprise or innovative strategies that their rivals cannot predict when there is competition for scare resources. This can move the system in unpredictable structure changing directions which are highly disippative of the old order.

In extant game theory[26] whether eductive or evolutionary there is no notion of innovation being a Nash equilibrium strategy let alone one that is necessitated as a best response by a structure of opposition. Innovation is either brought about by random mutation or is an ad hoc addition in the form of trend growth.

I will briefly make two points here to show how the confluence of the mathematics of incompleteness and non-recursiveness with the tools of emergent phenomena (Columns II and III in ***Table 1***) is essential to handle the theoretical and analytical demands of innovation and irregu-

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[26] There is, however, a long tradition in the macro policy literature design seminally put forward by Lucas (1972). Lucas postulated the necessity of secrecy, ambiguity and surprise strategies in policy against the possibility of a private sector that can contravene policy and render it ineffective if policy outcomes can be rationally expected. The Lucas Critique in Lucas (1976) indicates that there is a problem of predictive failure of policy outcomes by meta (econometric) models. Though not fully recognized yet (see, Markose, 2001b) this is the exact same logic from formalist settings that demonstrate incompleteness and undecidable dynamics.

lar structure changing dynamics in complex economic systems[27]. First, as perceptively observed by Witt (1993, p.92) in extant economic theory "the methodological implications of the nonanticipability condition " of innovations are violated due to the absence of an appropriate mathematical paradigm that can model the structures of surprise. Witt claims that a number of papers that deal with innovation use the optimization algorithm and hence "presuppose systematic knowledge on the part of the decision makers concerning the innovations - knowledge which simply does not exist in the pre-revelation context" (Ibid., p.92)[28]. I intend to show why both in the Ray's Tierra simulation of complexity with computational agents and in the Wolfram-Chomsky **Type 4** dynamics with continuing innovation, agents with powers of making self referencial mappings are necessary. Secondly, the non-recursive implementation or the emergent nature of self-organizing change in an evolutionary system has some profound consequences notably on the endemic role of error that economists confined to problem solving on the classical domain (Column I, ***Table 1***) have had no need to address.

3.1. A Computational Theory of Actor Innovation

Goldberg (1995) claims that the mystery shrouding innovation can be dispelled .. "by a heavy dose of *mechanism*. Many of the difficulties in the social sciences comes from a lack of a *computational theory* of actor innovation . . population oriented systems are dominated by what economists call the law of unintended consequences (which is itself largely the result of the innovative capability of the actors) and interacting with **GA**s provides hands-on experience in understanding what for most people is counterintuitive behaviour", (*Ibid.* p.28).

Despite Binmore's (1987) seminal work that introduced to game theory the requisite dose of mechanism with players with powers of Turing Machines, and along with it 'the spectre of Gödel', the computational theory of actor innovation did not follow. Binmore's critique of traditional game theory is that it cannot accommodate a generic model

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[27] Kaufmann (1993, pp 369-404) suggests the use of random grammars to model the evolution of complexity and novelty which necessarily incorporate computationally incomplete and undecidable problems.

[28] See also, Martens (2000, p.16):"Neoclassical paradigm of perfect competition based on perfect information and exogenously fixed consumer preferences and production technology, is an unsuitable starting point from which to introduce innovation into economic models. Attempts to do so by so called endogenous growth theory and by neo-Schumpetarian School has ended up in models that are basically inconsistent with the neoclassical paradigm".

of a rule breaker. The centre piece of Gödel (1931, p.19) is a formal analogue of the Liar[29] which proved the limits of calculation in self-referential structures. The Liar strategy in a two person game is defined as being capable of systematically contradicting the mutually predicted outcomes of the other person's strategy[30]. This structure of opposition is universal and *ipso facto* renders the transparent or predictable rule inoptimal. In Ray's Tierra (1992) when there is recognition on part of some agents that others are parasitic on them if their whereabouts is public knowledge, secrecy and unpredictable strategies are adopted. We will proceed to show that when there is mutual knowledge of the Liar qua rule breaker, viz. at the fixed point with the Liar, we are at Gödel's famous uncomputable fixed point. Technically, from the latter the only total computable best response function is one that maps into a domain of the player's strategy set that cannot be algorithmically enumerated. Further, we can show that this surprise strategy function can implement a new action/institution outside extant action sets. Formally, as the surprise or innovative strategy involves a total computable response function that corresponds to the productive function, the encoding of which provides an ever extendible set of explicit 'witnesses' for incompleteness in set theoretic proofs of the 1931 Gödel result developed by Post (1943) (see, also Cutland,1980). The analogue of this in the formalisation of a game with computational agents is that when they have mutually identified the Liar or the structure of opposition, then not only does secrecy become paramount for their objectives, but it also follows that the only Nash equilibrium strategies thereof are surprises implemented by the productive function.

With computational agents, all decision procedures for the determination of Nash equilibrium strategies involve computable functions and so do their best response functions. It is the latter in that they correspond to finitely encodable procedures fully defined in all states, viz. a total computable function, that will permit a mathematical characterization of an institutional innovation that differs from existing actions in an algorithmically non-anticipating way. As we will see, the innovation itself is emergent, but only agents with full powers of Turing Machines capable

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[29] Gödel's analogue of the Liar proposition is the undecidable proposition, say A, which has the following structure : $A \leftrightarrow \sim P(A)$. That is, A says of itself that it is not provable ($\sim P$). However, there is no paradox here as it is indeed true that this is so. Any attempt to prove the proposition A results in a contradiction with both A and $\sim A$, its negation, being provable in the system.

[30] Self-subversion is possible. In this case, the player breaks his own rule when its outcomes are desired by the other.

of self-referential mappings can deduce the necessity to innovate. To do so is also the Nash equilibrium of the game. These main points can be sketched for a two person game.

A major implication of computational agents is that all meta-information with regard to the outcomes of the game for any given set of state variables, s $\in$ S, can be effectively organized by the so called prediction function $\phi_{\sigma(x,y)}$(s) in an infinite chequer board like matrix Ξ of the enumeration of all partial computable functions, given in Figure 1, Cutland (1980). The tuple (x,y) identifies the row and column of this matrix Ξ whose rows are denoted as Ξ_i, i= 0,1,2,...... .

$$\begin{array}{lllllll}
\Xi_0 & \phi\sigma\,(0,0) & \phi\sigma\,(0,1) & \phi\sigma\,(0,2) & \phi\sigma\,(0,3) & \cdots\, \phi\sigma\,(0,y) & \cdots \\
\Xi_1 & \phi\sigma\,(1,0) & \phi\sigma\,(1,1) & \phi\sigma\,(1,2) & \phi\sigma\,(1,3) & \cdots\, \phi\sigma\,(1,y) & \cdots \\
\Xi_2 & \phi\sigma\,(2,0) & \phi\sigma\,(2,1) & \phi\sigma\,(2,2) & \phi\sigma\,(1,3) & \cdots\, \phi\sigma\,(2,y) & \cdots \\
. & & & & & & \\
\Xi_x & \phi\sigma\,(x,0) & \phi\sigma\,(x,1) & \phi\sigma\,(x,2) & \phi\sigma\,(x,3) & \cdots\, \phi\sigma\,(x,x) & \cdots \\
. & & & & & & \\
. & & & & & &
\end{array}$$

Figure 1.1.

The function $\phi_{\sigma(x,y)}$(s) if defined at a given state s and σ(x,y) yields

$$\phi_{\sigma(x,y)}(s) = q. \tag{15}$$

Here in (15), q in some code, is the vector of state variables determining the outcome of the game. Note, σ(x,y) is the index of the program for this function ϕ that produces the output of the game when the first player plays strategy x and the second player plays a strategy *that is consistent with his belief that the first player has used strategy y.*

We will now adopt this generic framework for the analysis of a two person game with computational agents. The game can best be interpreted as one of regulatory arbitrage where an oppositional structure can arise between the players.

The policy game is played by the authorities (g) and the private sector (p) where with no loss of generality the codes (g,p) denote their respective objective functions. Each player optimizes using strategies denoted by (β_p^*, β_g^*) which involve computable functions that specify the optimal response functions (f_p , f_g) which incorporate elements from the respective action sets A= (A_p ,A_g) *and* given mutual beliefs of one another's optimal strategy. Here, $b_p^{\hat{}}$ denotes (codes of) p's meta -representations of g's optimal strategy and $b_g^{\hat{}}$ denotes (codes of) g's meta-representations of p's optimal strategy. Further, (g,p), (A_p ,A_g) and all archival information of past and current state variables (where s is a given vector of state variables) are assumed to be in the public domain. All meta calculations on the strategies played and capable of being played in the game are based on this information and recorded in the matrix Ξ.

The determination of Nash equilibrium strategies[31] involve the use of total computable best response functions (f_p , f_g) which operate directly on points such as $\sigma(x,x)$ to effect computable transformations of the system from one row to another of matrix Ξ with special reference to its diagonal array, see, Figure 1. Thus,

$$\phi_{f_i \sigma(x,x)}, \qquad i \in (p,g). \tag{16}$$

Again, proceeding very informally, all fixed points and Nash equilibria have to be elements along the diagonal array of this matrix. A typical Nash equilibrium are at points defined by $\sigma(x,x)$, viz. player p plays x and g correctly identifies this. Off diagonal elements along any row defined by strategy, say y, employed by the private sector, cannot be Nash equilibria, as these off diagonal terms imply that authorities are choosing their strategy assuming the wrong meta representation of p's play. Consistent alignment of beliefs by which we have $\sigma(x,x)$ is a necessary condition of a Nash equilibrium as is the condition that there is rational expectations and both agents choose their optimal Nash equilibrium

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[31] Thus, the codes for the optimal strategy functions (β_p^*, β_g^*) are obtained recursively from the respective codes used in the calculation f_p , f_g and the codes of the players' meta representations of one another's play. In the calculation of Nash equilibrium strategies, codes denoted by (b_p^E ,b_g^E), in general, we need to specify two iterations. Note, the objective functions of players are computable functions Π_i , i $\in$ (p,g) defined over the partial recursive payoff/outcome functions specified in state variables in (15). Thus,

$$ArgMax_{b_i \in B_i} \Pi_i(\phi_{\sigma(b_i, b_j^{\hat{}})}(s)), i, j \in (p,g).$$

strategies such that each identifies the same function as producing the outcome of the game.

The best response functions f_i, i ∈ (p,g) that are total computable functions can belong to one of the following classes -

$$f_i = \begin{cases} \text{1 (Identity Function)} & \text{- Rule Abiding} \\ \mathrm{f}_i^{+} & \text{- Rule Bending} \\ \mathrm{f}_i^{\neg} & \text{- Rule Breaking (Liar Strategy)} \\ \mathrm{f}_i! & \text{- Surprise} \end{cases} \tag{17}$$

such that the codes of f_i are contained in set $\Re$,

$$\Re = \{m | f_i = \phi_m, \phi_m \text{ is total computable}\}. \tag{18}$$

The set $\Re$ which is the set of all total computable functions is not recursively enumerable. The proof of this is standard, Cutland (1980).

As will be clear, (18) draws attention to issues on how innovative actions/institutions can be constructed from existing action sets. The remarkable nature of the set $\Re$ is that potentially there is an uncountable infinite number of ways in which 'new' institutions can be constructed from extant action sets A. In standard rational choice models of game theory, the optimization calculus in the choice of best response requires choice to be restricted to given actions sets. Hence, strategy functions map from a relevant tuple that encodes meta information of the game into given action sets

$$\beta_i(f_i\sigma(x,x)), s, A) \rightarrow A \text{ and } f_i = \phi_m, m \in A, i \in (p,g). \tag{19}$$

Unless this is the case, as the set $\Re$ is not recursively enumerable there is in general no computable decision procedure that enables a players to determine the other player's response functions. However, in principle, a strategic decision procedure $(\beta_g, \beta_p\ )$ for choice of best response, $f_i = \phi_m, m \in \Re, i \in (p,g)$, can map into $\Re - A$, implying that an innovative action not previously in given action sets is used. We define the surprise/innovation producing strategy as follows:

$$\beta_i(f_i\sigma(x,x)), s, A) \rightarrow \Re - A \text{ and } f_i = f_i^{!} = \phi_m, m \in \Re - A, i \in (p,g). \tag{20}$$

It has indeed been noted in passing by Anderlini and Sabourian (1995, p.1351), based on the work of Holland (1975), that heterogeneity in forms do not arise primarily by random mutation but by algorithmic recombinations that operate on existing patterns. However, a number of preconceptions from traditional game theory such as the 'givenness' of actions sets prevent Anderlini and Sabourian(1995) from positing that players who as in (20), equipped with the wherewithal for algorithmic recombinations of existing actions, do indeed innovate from strategic necessity rather than by random mutation. The innovation *per se* is emergent phenomena, but the strategic necessity for it is fully deducible. Hence, it is adduced that in the Wolfram-Chomsky schema on dynamical systems with computationally intelligent agents, agents with full deductive powers of Turing Machines capable of self-referential calculations are necessary to bring about innovation based structure changing dynamics. It only remains to show the specific structure of opposition that logically and strategically necessitates surprise strategies in the Nash equilibrium of the game.

Consider the state of affairs given by

$$\phi_{\sigma(b_a,b_b)}(s) = q, \tag{21}$$

where outcomes of a policy rule a is predictable and q is the desired outcome that g wants in state variables when applying this policy rule a. Here, in our two place notation $\sigma(b_a, b_b)$, the first b_a is the code of the program, as adopted by p to simulate the impact of the policy rule a that p believes that g will follow and the second place b_a denotes that g believes and acts on the basis that the private sector has simulated policy rule a . It is convenient to assume that policy rule a is optimal for g if the private sector is rule abiding. By rule abiding is meant that p will leave the system unchanged in terms of the row b_a of matrix Ξ.

However, for player p, for the given (a,s) it is optimal for p to apply the Liar strategy, $f_p^{\neg}\ \sigma(b_a, b_b)$, the code of which is, say, $b_a^{\neg}$. Formally, the Liar strategy has the following generic structure. For any state s when the rule a applies,

$$\phi_{f_p^{\neg}\sigma(b_a,b_b)}(s) = q^{\sim}, q^{\sim} \notin E_{\sigma_{b_a}} \leftrightarrow \phi_{\sigma_{(b_a,b_b)}}(s) = q, q \in E_{\sigma_{b_a}}. \tag{22}$$

For all s when policy rule a does not apply,

$$f_p^{\neg} = 0, \text{viz. do nothing.} \tag{23}$$

The Liar can successfully subvert with certainty in (22) if and only if ($\leftrightarrow$) the policy rule is transparent with predictable outcomes and $f_p^{\neg}$ itself is total computable. Also, $f_p^{\neg} = \phi_m$, m $\in A_p$, must include a codified description of an action rule if undertaken by the Liar can subvert the predictable outcomes of the policy rule a. Formally, if q is predicted then the application of $f_p^{\neg}$ to $\sigma(b_a, b_a)$ will bring about an outcome $q^{\sim}$ $\notin$ $E_{\sigma_{b_a}}$ which belongs to a set disjoint from the set that contains the desired output of rule a for all s for which rule a applies, viz. $E_{\sigma_{b_a}}$ $\bigcap$ $E_{\sigma_{b_a}\neg} = \emptyset$. The outcomes ($q^{\sim}$, q) can be zero sum but in general we refer to property $q^{\sim}$ $\notin$ $E_{\sigma_{b_a}}$ in (22) as being oppositional or subversive. This underpins the intuition behind Ray's Tierra simulation where agents recognize the necessity for secrecy. This is also well known from the Lucas (1972) postulate on policy ineffectiveness in the case of fully anticipated policy and the wisdom behind the panacea that to forestall subversion, the policy rule must be undefined and fraught with ambiguity.

Thus, we come to the point as why agents who precipitate the Wolfram-Chomsky **Type 4** dynamics with innovation have to have powers of self-referential calculation. Firstly, g acknowledges the identity of the Liar in (22) and understands that transparent rule a cannot be implemented rationally as the outcome defined[32] by $\phi_{\sigma(b_a^{\neg}, b_a)} = q^{\sim}$. The latter is out of equilibrium. Player g, updates beliefs so that formally we obtain the fixed point involving the Liar which is $\sigma(b_a^{\neg}, b_a^{\neg})$ where $b_a^{\neg}$[33] is the code for the Liar strategy in (22). Now, the Liar, p, knows that g knows that p is the Liar. The prediction function indexed by the fixed point of the Liar/rule breaker best response function $f_p^{\neg}$ in (24) is not computable and corresponds to the famous Gödel uncomputable fixed point.

$$\phi_{f_p^{\neg}\sigma(b_a^{\neg}, b_a^{\neg})}(s) = \phi_{\sigma(b_a^{\neg}, b_a^{\neg})}(s) \tag{24}$$

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[32] In out two place notation, the first $b_a\neg$ is code for p's Liar strategy and ba is code for g's mistaken belief of p's strategy.

[33] Formally, $b_a^{\neg}$ may be viewed as the g code of a refutable proposition in a formal system. A refutable proposition is one whose negation (b_a here) is provable in the system. As theoremhood is a computable relationship, the g code of the refutable proposition cannot belong to the domain of any computable function. However, as b_a is provable, the set of all such refutable functions is a recursively enumerable subset of the domain of calculations such as $\phi_x(x)$, for all x, do not terminate.

The proof is standard[34] .

There is no paradox in stating that as both players can prove the non-computability of (24) they will have mutual knowledge that the only Nash equilibrium strategies for both players that is consistent with meta information in the fixed point in (24), is one that involves strategies that elude prediction from within the system. On substituting the fixed point $\sigma(b_a^{\neg}, b_a^{\neg})$ in (24) for $\sigma(x, x)$ in (20), g's Nash equilibrium strategy β_g^E with code b_g^E implemented by an appropriate total computable function must be such that

$$\beta_g^E(f_g \sigma(b_a^{\neg}, b_a^{\neg}), s, A) \rightarrow \Re - A \text{ and } f_g = f_g! = \phi_m, m \in \Re - A. \quad (25)$$

That is, $f_g!$ implements an innovation and $b_g^E!$ is the code of the surprise strategy function in (25) and hence is the fixed point of $f_g!$.

Likewise for player p, $f_p!$ implements an innovation in (26) and $b_p^E!$ is the code of the surprise strategy function viz. the fixed point of $f_g!$. Thus,

$$\beta_p^E(f_p \sigma(b_a^{\neg}, b_a^{\neg}), s, A) \rightarrow \Re - A \text{ and } f_p = f_p! = \phi_m, m \in \Re - A. \quad (26)$$

The intuition here is that from the uncomputable fixed point with the Liar, the total computable best response function implementing the Nash equilibrium strategies can only map as above into domains of the action and strategy sets of the players that cannot be algorithmically enumerated in advance[35].

As in the Tierra and Hilles simulation models, once computational agents have enough capabilities to detect rivalrous behaviour that is inimical to them, they learn to use secrecy and surprises. In a two person game with computational agents, this can be fully formalized using the Gödelian result on incompleteness. To show how with parallel computing agents, we have cooperation and competition not simply as in Prisoners Dilemma, but with the use of periodic adoption of new institutions outside of extant action sets, we need the new technology of virtual models of emergent phenomena.

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[34] Assume (24) is computable and the **RHS** of (24) produces the output $q^{\sim}$ and the **LHS** by the definition of the Liar strategy produces output q. However, if (24) is computable then we have $q = q^{\sim}$ which is a contradiction.

[35] A rigorous proof of this is omitted here and can be found in Markose (2001b).

The logician Koons (1992) has noted that the ubiquitous structure of the Liar "will play a crucial role in effecting the transition from rational agent model to understanding society in terms of institutions, rules and practices (*Ibid*, ix) ...the cognitive blind spots have a wider set of implications concerning the relationship between institutionalist social theory and the rational agent model, and, in ethics, the relations between deontic rule based ethical theories and consequentialism"(*Ibid*, p.151). The latter is intended to mean that the vulnerability of formalist control of society to the Liar and innovative behaviour makes unwritten constitutions, deontic virtue, truth, meaning and such tacit values the predominant guides to spontaneous order. In the new regulatory arbitrage theory of institutional innovation (see, Miller,1986, Schanze, 1995, etc.) the latter is brought about by rule breaking regulatees who in their attempt to contravene regulation with predictable outcomes do so by 'exit' and innovation. Schanze (1995) calls such innovation regulatory bifurcation for which the constitutional structure of rules may have to be modified to accommodate innovation.

3.2. The Endemic Role of Error in Non-Recursive Systems

On the nature of the error driven path of the capitalist mode of production, the work of Karl Marx remains one of the best accounts of how in fact selection takes place among the rivalrous entrepreneurial goals and objectives in the system. Marx's so called "impersonal" market forces determine the economic fate of individuals that produce and trade in the system. The nature of the periodic crisis of overproduction and subsequent slumps are also described in great detail. "The capitalist mode of production, while on the one hand, enforcing economy in each individual business, on the other hand, begets by its anarchical system of competition, the most outrageous squandering of labour power and of means of employments, at present dispensable, but in themselves superfluous", Marx (circa 1869). As a way of dealing with the wasteful nature of the selective process of emerging patterns in capitalist growth, the Marxist agenda of Scientific Socialism sought radical institutional reform favouring the communal ownership of capital to avoid the possibility of a multiplicity of conflicting and competing investment decisions. Neoclassical economics sought solace in viewing market outcomes solely from equilibrium outcomes where inconsistent objectives of agents are assumed to be resolved and no agent needs to be thwarted in his expectations or actions. The excessive focus on equilibrium positions has come at a great price of our continuing ignorance of the nature of error in a system where the global ordering principle has no recursive implementation.

I will now suggest a new way of understanding what many have observed to be typical of the growth in capitalist systems by drawing an analogy between this and the Penrose theory of the non-recursive patterns pertaining to quasi-crystalline growth structures. The latter display a simultaneous emergence of not one but many alternative arrangements of attaching atoms and these must coexist till an particular arrangement or a reduced superposition of the same is singled out by some non-recursive process that is known to govern general tiling problems (Penrose, 1989, p.436-438). What is significant here is that in the non-recursive nature of emerging patterns in which tiles abut one another in specific ways, it becomes necessary that some sections of the preexisting structures of tiles have to be discarded to make way for the new order in such a fashion that at no time can the entire order be brought about by an act of construction that can avoid the profligate abortions of preexisting structures even if the latter may satisfy lowest energy configurations locally. Why the system can not move directly to a particular pattern without the intervening stages of several alternative arrangements, many of which are then aborted, is the precise consequence of the pattern having no recursive implementation. On a non-recursive domain, as there is no effective procedure that can determine in an *a priori* way on the effectiveness or the so called halting behaviour of any given set of alternative arrangements within the overall pattern and hence there is no mechanism that can circumvent the abortion of the many accretions that do not fit the whole. Many would identify such profligate wastefulness of resources in all evolutionary and emergent processes.

Path breaking work by Anderson, Bak, Priyogne, Penrose and others, have enhanced our understanding as to why nature resorts to such dissipative forces with the seemingly wasteful use of resources in the implementation of non-recursive patterns. In parallel, there is Gödel's discovery that a global criterion such as the internal consistency of formal systems, the hall mark of a rational order is an undecidable proposition within it has no recursive implementation. The latter, we saw is on account of computationally intelligent agents who can by *contra* positions, qua the Liar, exit and innovate from any given listable formalistic structure with predictable outcomes. Thus, interestingly, we may currently have an easier task of explaining the pervasive existence of spontaneous or emergent order with non-recursive selective processes in the complex institutions of society with computationally creative agents rather than in purely physical systems![36]

4. Concluding Remarks

The sciences of complex adaptive systems have been a truly interdisciplinary venture being spearheaded by physicists, biologists, computer scientists, mathematicians, economists and others. In the case of the evolution of complexity in intelligent social systems, it is not far off the mark to say that the modern evolutionary agenda on spontaneous or emergent order was clearly identified at the provenance of Economics in the 18^{th} century. A survey of classical political economy has been given here to redress the balance in the growing contributions by physicists and mathematicians on the evolution of complexity. The *sin qua non* of a system capable of complexity is manifested in the disjunction between system wide outcomes and the micro level computational capabilities and also in the non-anticipating or surprise producing features of the system. The normative necessity of a structure of constitutional rules that permits this was clearly articulated in classical liberal tradition even if its implications were and are not yet fully understood. Over two centuries have had to elapse till we have the methodological tools necessary to counter, on grounds of computational impossibility, the ultra rationalist position that sees no distinction between what can be constructed by calculation and what can only emerge. Scientifically, the latter category of events has had a tenuous status till recently. The advances in the mathematics of non-linear dynamics, emergence and evolutionary computation has made it possible to pin down the elusive properties of non-computable dynamics produced by computationally intelligent agents.

I started with the premise that when the domain of a decision problem is non-recursive, the loss of a categoric decision procedure prompts a multiplicity of models for inductive inference. Indeed, we saw a critical degree of competing heterogeneous 'world views' are needed to achieve the emergent coordination and enhanced social welfare even in the simple minority game. The explanation of power laws in asset prices as a manifestation of ever persisting endogenous heterogeneity of agent behaviour has progressed greatly. Section 3 sketched a computational theory of actor innovation based on the mathematics of Gödel incompleteness which permits computational agents to exit and innovate when caught up in

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[36] See, footnote 9 and the Langton (1992) thesis on this.

an oppositional structure. This ubiquitous dynamic behind innovation has been observed in **Type 4** dynamics of Wolfram-Chomsky cellular automata and more recently been seen in the Tierra and Hillis (1992) Artificial Life simulations. Despite efforts in this direction as in Easley and Rustichini (1999), it is clear that economists have a long way to go to understand complexity in terms of the Langton-Kaufmann novelty producing world with attendant problems of incompleteness and undecidability. Section 3.2 was included to highlight the less palatable side to the evolution of complexity and the inevitable consequence of decision procedures being on non-recursive domains. Error is endemic in such systems and they have real consequences. To conclude, therefore, the formalist limits on algorithms and adaptive methodology of evolution and emergence (viz. columns II and III of ***Table 1***)are two sides of the same coin that should be brought to bear on discussions of complex adaptive systems.

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