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On truthful constrained heterogeneous facility location with max-variant cost

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ABSTRACT

We consider a problem where agents have private positions on a line, and public approval preferences over two facilities, and their cost is the maximum distance from their approved facilities. The goal is to decide the facility locations to minimize the total and the max cost, while incentivizing the agents to be truthful. We design a strategyproof mechanism that is simultaneously 11- and 5-approximate for these two objective functions, thus improving the previously best-known bounds of $2n + 1$ and 9.

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1. Introduction

We consider the following truthful heterogeneous facility location problem with max-variant cost: There is a set N of n agents with private positions on the line of real numbers (x_i for agent i), and public approval preferences $p_i \in \{0, 1\}^2$ over two facilities F_1 and F_2 with $p_{i1} + p_{i2} \geq 1$ (i.e., each agent approves at least one facility). Let N_j be the set of agents that approve facility $j \in [2]$. Clearly, N_1 and N_2 need not be disjoint since there might be agents that approve both facilities; we denote by $N_1 \setminus N_2$ the set of agents that approve only F_1 , $N_2 \setminus N_1$ the set of agents that approve only F_2 , and $N_1 \cap N_2$ the set of agents that approve both facilities.

There is also a finite set C of given candidate locations, where the facilities can be placed. For any agent i and point x of the line, let $d(i, x) = |x_i - x|$ denote the distance between the position of i and x . Given a feasible solution $\mathbf{y} = (y_1, y_2)$ consisting of the location y_1 , where F_1 is placed, and the location $y_2 \neq y_1$, where F_2 is placed, the individual cost of an agent i is her distance to the farthest facility among the ones she approves (hence, the term max-variant cost in the title of the problem), that is,

$$\text{cost}_i(\mathbf{y}) = \max_{j: p_{ij}=1} d(i, y_j).$$

The social cost of a solution $\mathbf{y} = (y_1, y_2)$ is the total cost of all agents:

$$\text{SC}(\mathbf{y}) = \sum_{i \in N} \text{cost}_i(\mathbf{y}).$$

The max cost is the maximum cost over all agents:

$$\text{MC}(\mathbf{y}) = \max_{i \in N} \text{cost}_i(\mathbf{y}).$$

A mechanism takes as input the positions of the agents and, together with the public information about their preferences, decides a feasible solution, which we will typically denote by $\mathbf{w} = (w_1, w_2)$ in our analysis. For any $f \in \{\text{SC}, \text{MC}\}$, the approximation ratio of a mechanism in terms of f is the worst-case (over all possible instances) of the ratio $\frac{f(\mathbf{w})}{\min_{\mathbf{y}} f(\mathbf{y})}$. Our goal is to design mechanisms that achieve a low approximation ratio in terms of both the social cost and the max cost simultaneously, while at the same time being strategyproof, that is, do not provide incentive to the agents to misreport their private positions and decrease their individual cost.

Zhao et al. [16] were the first to consider this constrained heterogeneous facility location problem with max-variant cost; a version of the problem where the facilities can be placed at the same locations was previously studied by Chen et al. [2] under different informational assumptions (known positions and private preferences). Zhao et al. [16] showed that there are strategyproof mechanisms which achieve approximation ratios of $2n + 1$ and 9 for the social cost and max cost objectives, respectively. For both objectives, the mechanisms of Zhao et al. [16] switch between cases depending on whether there is at least one agent that approves both facilities or not. When there is such an agent, the

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facilities are placed at the candidate locations that are closest to a designated agent (the median for the social cost and the leftmost for the max cost). When no agent approves both facilities, each facility is separately placed to the available candidate location that is closest to a designated agent among the ones that approve it (again, median for the social cost or leftmost for the max cost); for the social cost, the facility that is placed first is the one that the majority of agents approve.

Our contribution

In this work, we design a different strategyproof mechanism and show that it simultaneously achieves an approximation ratio of at most 11 for the social cost and at most 5 for the maximum cost, thus improving both bounds at the same time; for the social cost, the improvement is significant as the approximation decreases from linear to constant. Our mechanism works as follows. For simplicity, suppose that F_1 is the facility that most agents approve. The mechanism switches between two cases depending on whether the number of agents that approve F_1 is at least the number of agents that approve both facilities. If this is indeed the case, F_1 is placed at the candidate location closest to the median among the set of agents that approve only it, while F_2 is placed at the available candidate location closest to the median agent among the ones that approve it (including the agents that approve both facilities). Otherwise, the two facilities are placed at the two candidate locations that are closest to the median agent among those that approve both facilities. Our analysis uses a combination of simple tools (such as the triangle inequality and the fact that the median of a set of points minimizes their total distance to any point) and worst-case characterization which reveals the properties that worst-case instances (where the approximation ratio is maximized) have. These results are presented in Section 2.

In Section 3, we present an interesting connection between the max-variant truthful heterogeneous two-facility location problem (which is the main focus of this paper) and the sum-variant considered by Kanellopoulos et al. [10], in which the individual cost of an agent is the total distance from the facilities she approves. We show that any mechanism that is strategyproof in both variants and achieves an approximation ratio of at most ρ in terms of the social cost or the maximum cost in one of the variants, also achieves an approximation ratio of at most 2ρ in the other variant. This allows us to show that a constant approximation ratio (at most 22) can be achieved for the social cost in the sum-variant, which is the first constant bound for this version of the problem as well; Kanellopoulos et al. [10] previously claimed an upper bound of 3 but with a mechanism that turned out to not be strategyproof.

Other related work

The seminal paper of Procaccia and Tennenholtz [13] initiated the study of facility location problems under the prism of approximate mechanism design without money. Since their work, a plethora of different variants have been studied in the literature under various assumptions, always aiming to design strategyproof mechanisms with an as low approximation ratio as possible in terms of some social objective. Indicatively, different variants have been considered depending on the number of facilities to locate [7,12], whether the facilities are desirable or obnoxious [3], whether the preferences of the agents are homogeneous or heterogeneous [2,4,5,9,11,14], and whether there are constraints about where the facilities can be placed [6,10,15] in combination with min or sum individual costs. For a detailed exposition of the truthful facility location literature, we refer the interested reader to the survey of Chan et al. [1].

Some useful notation and observations

We will extensively use the *triangle inequality* stating that $d(x, y) \leq d(x, z) + d(z, y)$ for any three points x, y, z of the line. We will sometimes use directly the following version of the triangle inequality (which can be shown by applying the classic one twice): $d(i, x) \leq d(i, y) + d(j, y) + d(j, x)$ for any two agents i, j and locations x, y .

When the objective (either the social cost or the maximum cost) is clear from context, we will denote by $\mathbf{o} = (o_1, o_2)$ an optimal solution, that is, \mathbf{o} is a feasible solution that minimizes the objective. In addition, for any agent i , we will denote by $o(i)$ the location of the facility that determines the cost of i in $\mathbf{o} = (o_1, o_2)$. Observe that, for any $j \in [2]$, if $i \in N_j$, then $d(i, o_j) \leq d(i, o(i)) = \text{cost}_i(\mathbf{o})$; this follows by the fact that either $o_j = o(i)$ or o_j is closer to i than $o(i) = o_{3-j}$. Finally, we will denote by $t(i)$ and $s(i)$ the candidate locations that are closest and second-closest to the position of agent i . Observe that, by definition, $d(i, t(i)) \leq d(i, o(i))$.

2. Results

We present and analyze the mechanism for the case where $|N_1| \geq |N_2|$; for the case $|N_2| > |N_1|$, it suffices to swap 1 and 2 in the description below.

- **(Case 1)** If $|N_1 \setminus N_2| \geq |N_1 \cap N_2|$, then place F_1 at the location $t(m_1)$ that is closest to the median agent $m_1 \in N_1 \setminus N_2$, and place F_2 at the location that is closest to the median agent $m_2 \in N_2$ from the set of available locations; so, either $t(m_2) \neq t(m_1)$, or $s(m_2)$ if $t(m_2) = t(m_1)$.
- **(Case 2)** Otherwise, place F_1 at $t(m_{12})$ and F_2 at $s(m_{12})$, where m_{12} is the median agent of $N_1 \cap N_2$.

We refer to this mechanism as **CONDITIONAL-MEDIAN**.

Theorem 2.1. **CONDITIONAL-MEDIAN** is strategyproof.

Proof. First observe that the two cases considered by the mechanism only depend on the preferences of the agents, which are publicly known. Hence, no agent can force the mechanism to go from (Case 1) to (Case 2) or vice versa by misreporting. We now discuss each case separately.

- **(Case 1)** Since F_1 is placed at $t(m_1)$, the cost of m_1 is minimized. To change the location of F_1 , an agent $i \in (N_1 \setminus N_2) \setminus \{m_1\}$ would have to misreport a position such that the median of $N_1 \setminus N_2$ changes, which would either not change the location of F_1 or move it farther away. Clearly, no agent of N_2 can affect the location of F_1 , and, for the same reason, has no incentive to misreport. In particular, given the location of F_1 , the cost of m_2 is minimized, and any other agent $i \in (N_2 \setminus N_1) \setminus \{m_2\}$ would have to change the median of N_2 , which would either not change the location of F_2 or move it farther away.
- **(Case 2)** No agent of $(N_1 \setminus N_2) \cup (N_2 \setminus N_1)$ can affect the outcome of the mechanism in this case. The cost of m_{12} is clearly minimized, and no other agent $i \in (N_1 \cap N_2) \setminus \{m_{12}\}$ can misreport; similarly to (Case 1), i would have to change the median of $N_1 \cap N_2$, which would either not change the location of the facilities or move them farther away.

Consequently, in any of the two cases, no agent has incentive to misreport, and thus the mechanism is strategyproof. \square

Theorem 2.2. For the social cost, the approximation ratio of **CONDITIONAL-MEDIAN** is at most 11.

Proof. Let $\mathbf{w} = (w_1, w_2)$ be the solution computed by the mechanism, and $\mathbf{o} = (o_1, o_2)$ be an optimal solution. We consider each of the two cases of the mechanism separately. In each case, we combine known properties related to the median of a set of points on the line, the triangle inequality, and some with properties of worst-case instances (which achieve the largest possible approximation ratio).

(Case 1). We consider two subcases depending on whether m_2 is closer to w_2 than to o_2 . Observe that this is definitely true when $t(m_1) \neq t(m_2)$ and might be true when $t(m_1) = t(m_2)$. If it is not true when $t(m_1) = t(m_2)$, then, since $w_2 = s(m_2)$, we have more information about the structure of the instance, in particular, we have that $w_1 = o_2$. For any $j \in [2]$, let S_j be the subset of $N_1 \cap N_2$ that includes agents whose cost in \mathbf{w} is determined by w_j .

(Case 1.1): $d(m_2, w_2) \leq d(m_2, o_2)$. The social cost of \mathbf{w} can be written as follows:

$$\begin{aligned} \text{SC}(\mathbf{w}) &= \sum_{i \in N_1 \setminus N_2} d(i, w_1) + \sum_{i \in S_1} d(i, w_1) + \sum_{i \in (N_2 \setminus N_1) \cup S_2} d(i, w_2) \\ &\leq \sum_{i \in N_1 \setminus N_2} d(i, w_1) + \sum_{i \in S_1} d(i, w_1) + \sum_{i \in N_2} d(i, w_2) \end{aligned}$$

For the agents in $N_1 \setminus N_2$, using the triangle inequality, the fact that $d(m_1, w_1) \leq d(m_1, o_1)$, $o_1 = o(i)$ for every $i \in N_1 \setminus N_2$, and the property of m_1 which minimizes the total distance of all agents in $N_1 \setminus N_2$ from any point on the line, including o_1 , we obtain

$$\begin{aligned} \sum_{i \in N_1 \setminus N_2} d(i, w_1) &\leq \sum_{i \in N_1 \setminus N_2} \left(d(i, m_1) + d(m_1, w_1) \right) \\ &\leq \sum_{i \in N_1 \setminus N_2} \left(d(i, m_1) + d(m_1, o(i)) \right) \\ &\leq \sum_{i \in N_1 \setminus N_2} \left(2 \cdot d(i, m_1) + d(i, o(i)) \right) \\ &\leq 3 \cdot \sum_{i \in N_1 \setminus N_2} d(i, o(i)). \end{aligned} \quad (1)$$

For the agents in S_1 , we have that $|S_1| \leq |N_1 \cap N_2| \leq |N_1 \setminus N_2|$. In addition, $d(i, o_1) \leq d(i, o(i))$ for every agent $i \in S_1$, $d(m_1, w_1) \leq d(m_1, o_1)$, and $o_1 = o(i)$ for any agent $i \in N_1 \setminus N_2$. So, using the triangle inequality and the property of m_1 which minimizes the total distance of all agents in $N_1 \setminus N_2$ from any point on the line, including o_1 , we obtain

$$\begin{aligned} \sum_{i \in S_1} d(i, w_1) &\leq \sum_{i \in S_1} \left(d(i, o_1) + d(m_1, o_1) + d(m_1, w_1) \right) \\ &\leq \sum_{i \in S_1} d(i, o(i)) + \sum_{i \in N_1 \setminus N_2} 2 \cdot d(m_1, o_1) \\ &\leq \sum_{i \in S_1} d(i, o(i)) \\ &\quad + \sum_{i \in N_1 \setminus N_2} \left(2 \cdot d(i, m_1) + 2 \cdot d(i, o(i)) \right) \\ &\leq \sum_{i \in S_1} d(i, o(i)) + 4 \cdot \sum_{i \in N_1 \setminus N_2} d(i, o(i)). \end{aligned} \quad (2)$$

For the agents in N_2 , using the triangle inequality, the fact that $d(m_2, w_2) \leq d(m_2, o(i))$ for every $i \in N_2$, and the property of m_2 which minimizes the total distance of all agents in N_2 from any point on the line, including o_2 , we obtain

$$\begin{aligned} \sum_{i \in N_2} d(i, w_2) &\leq \sum_{i \in N_2} \left(d(i, m_2) + d(m_2, w_2) \right) \\ &\leq \sum_{i \in N_2} \left(d(i, m_2) + d(m_2, o(i)) \right) \\ &\leq \sum_{i \in N_2} \left(2 \cdot d(i, m_2) + d(i, o(i)) \right) \\ &\leq 3 \cdot \sum_{i \in N_2} d(i, o(i)) \end{aligned} \quad (3)$$

Putting everything together, and using the facts that $N = (N_1 \setminus N_2) \cup N_2$ and $(N_1 \setminus N_2) \cup S_1 \subseteq N_1 \subseteq N$, we have

$$\begin{aligned} \text{SC}(\mathbf{w}) &\leq 7 \cdot \sum_{i \in N_1 \setminus N_2} d(i, o(i)) + \sum_{i \in S_1} d(i, o(i)) + 3 \cdot \sum_{i \in N_2} d(i, o(i)) \\ &\leq 7 \cdot \text{SC}(\mathbf{o}), \end{aligned}$$

that is, the approximation ratio is at most 7.

(Case 1.2): $d(m_2, w_2) > d(m_2, o_2)$. First observe that if $w_1 = t(m_1)$ and $w_2 = t(m_2)$, then it would have to be the case that $d(m_2, w_2) \leq d(m_2, o_2)$, which is already captured by (Case 1.1). So, it must be the case that $w_1 = t(m_1) = t(m_2)$ and $w_2 = s(m_2)$, which, in combination with the fact that $d(m_2, w_2) > d(m_2, o_2)$, implies that $w_1 = o_2$.

Since m_1 is closer to $w_1 = o_2$ than to o_1 , the same must be true for at least half of the agents in $N_1 \setminus N_2$, which implies the following lower bound on the optimal social cost:

$$\text{SC}(\mathbf{o}) \geq \frac{|N_1 \setminus N_2|}{2} \cdot \frac{d(o_1, o_2)}{2} = \frac{|N_1 \setminus N_2|}{4} \cdot d(o_1, o_2).$$

Now, Inequalities (1) and (2) are still true since $w_1 = t(m_1)$. We will use (1) in this case as well to bound the contribution of the agents in $N_1 \setminus N_2$ to the social cost of \mathbf{w} , but for the agents in S_1 we will use a different bound obtained by applying the triangle inequality once, as follows:

$$\begin{aligned} \sum_{i \in S_1} d(i, w_1) &\leq \sum_{i \in S_1} \left(d(i, o_1) + d(o_1, w_1) \right) \\ &= \sum_{i \in S_1} d(i, o_1) + |S_1| \cdot d(o_1, o_2). \end{aligned} \quad (4)$$

We will now bound the contribution of the agents in $(N_2 \setminus N_1) \cup S_2$ by using the triangle inequality, the fact that $d(m_2, w_2) \leq d(m_2, o_1)$ (which is true since $t(m_2) = w_1 = o_2$ and $s(m_2) = w_2$), the fact that $(N_2 \setminus N_1) \cup S_2 \subseteq N_2$, and the fact that m_2 minimizes the total distance of all agents in N_2 from any point of the line, including o_2 . We have

$$\begin{aligned} \sum_{i \in (N_2 \setminus N_1) \cup S_2} d(i, w_2) &\leq \sum_{i \in (N_2 \setminus N_1) \cup S_2} \left(d(i, m_2) + d(m_2, w_2) \right) \\ &\leq \sum_{i \in (N_2 \setminus N_1) \cup S_2} \left(d(i, m_2) + d(m_2, o_1) \right) \\ &\leq \sum_{i \in (N_2 \setminus N_1) \cup S_2} \left(d(i, m_2) + d(m_2, o_2) + d(o_1, o_2) \right) \\ &\leq 2 \cdot \sum_{i \in N_2} d(i, m_2) \end{aligned}$$

$$\begin{aligned}
& + \sum_{i \in (N_2 \setminus N_1) \cup S_2} d(i, o_2) + \left(|N_2 \setminus N_1| + |S_2| \right) \cdot d(o_1, o_2) \\
\leq & 2 \cdot \sum_{i \in N_2} d(i, o(i)) \\
& + \sum_{i \in (N_2 \setminus N_1) \cup S_2} d(i, o_2) + \left(|N_2 \setminus N_1| + |S_2| \right) \cdot d(o_1, o_2). \quad (5)
\end{aligned}$$

Combining Inequalities (1), (4) and (5), the fact that $(N_2 \setminus N_1) \cup S_2 \cup S_1 = N_2$, the fact that $|N_2| \leq 2 \cdot |N_1 \setminus N_2|$ (which is true since $|N_1 \setminus N_2| \geq |N_1 \cap N_2|$ and $|N_1 \setminus N_2| \geq |N_2 \setminus N_1|$), and the lower bound on $SC(\mathbf{o})$, we finally have:

$$\begin{aligned}
SC(\mathbf{w}) & \leq 3 \cdot SC(\mathbf{o}) + |N_2| \cdot d(o_1, o_2) \\
& \leq 3 \cdot SC(\mathbf{o}) + 2 \cdot |N_1 \setminus N_2| \cdot d(o_1, o_2) \\
& \leq 11 \cdot SC(\mathbf{o}),
\end{aligned}$$

that is, the approximation ratio is at most 11.

(Case 2) Here, we have that $w_1 = t(m_{12})$ and $w_2 = s(m_{12})$. Since the decision does not depend on the positions of the agents in $(N_1 \setminus N_2) \cup (N_2 \setminus N_1)$, these agents can be moved anywhere without affecting the outcome of the mechanism. Hence, in a worst-case instance, to maximize the approximation ratio, all these agent can be positioned at exactly the optimal locations of the two facilities, namely, all agents of $N_1 \setminus N_2$ are positioned exactly at o_1 and all agents of $N_2 \setminus N_1$ are positioned exactly at o_2 .

Since $|N_1 \setminus N_2| \leq |N_1 \cap N_2|$, in a worst-case instance, the contribution of the agents of $N_1 \setminus N_2$ to the social cost of the solution \mathbf{w} computed by the mechanism is

$$\sum_{i \in N_1 \setminus N_2} d(i, w_1) = \sum_{i \in N_1 \setminus N_2} d(o_1, w_1) \leq \sum_{i \in N_1 \cap N_2} d(o_1, w_1)$$

By the triangle inequality, we further have that

$$\begin{aligned}
\sum_{i \in N_1 \setminus N_2} d(i, w_1) & \leq \sum_{i \in N_1 \cap N_2} \left(d(i, o_1) + d(i, w_1) \right) \\
& \leq \sum_{i \in N_1 \cap N_2} \left(d(i, o(i)) + d(i, w(i)) \right), \quad (6)
\end{aligned}$$

where $w(i)$ is the location of the facility that determines the cost of agent i in \mathbf{w} .

Since $|N_2| \leq |N_1|$, we also have that $|N_2 \setminus N_1| \leq |N_1 \setminus N_2| \leq |N_1 \cap N_2|$. Consequently, similarly to above, in a worst-case instance, the contribution of the agents of $N_2 \setminus N_1$ to the social cost of \mathbf{w} can be upper-bounded as follows:

$$\sum_{i \in N_2 \setminus N_1} d(i, w_2) \leq \sum_{i \in N_1 \cap N_2} \left(d(i, o(i)) + d(i, w(i)) \right) \quad (7)$$

The social cost of \mathbf{w} in a worst-case instance can now be written as follows:

$$\begin{aligned}
SC(\mathbf{w}) & = \sum_{i \in N_1 \setminus N_2} d(i, w_1) + \sum_{i \in N_2 \setminus N_1} d(i, w_2) \\
& + \sum_{i \in N_1 \cap N_2} d(i, w(i)) \\
& \leq 2 \cdot \sum_{i \in N_1 \cap N_2} d(i, o(i)) + 3 \cdot \sum_{i \in N_1 \cap N_2} d(i, w(i)). \quad (8)
\end{aligned}$$

For any agent $i \in N_1 \cap N_2$, by the triangle inequality, the fact that $d(m_{12}, w_1) \leq d(m_{12}, w_2) \leq d(m_{12}, o(m_{12}))$, and the fact that $d(i, o(m_{12})) \leq d(i, o(i))$, we have

$$\begin{aligned}
d(i, w(i)) & \leq d(i, m_{12}) + d(m_{12}, w(i)) \\
& \leq d(i, m_{12}) + d(m_{12}, o(m_{12})) \\
& \leq 2d(i, m_{12}) + d(i, o(m_{12})) \\
& \leq 2d(i, m_{12}) + d(i, o(i))
\end{aligned}$$

Hence,

$$\sum_{i \in N_1 \cap N_2} d(i, w(i)) \leq 2 \cdot \sum_{i \in N_1 \cap N_2} d(i, m_{12}) + \sum_{i \in N_1 \cap N_2} d(i, o(i)).$$

Since m_{12} minimizes the total distance of the agents in $N_1 \cap N_2$ from any point of the line, including o_1 , and the fact that $d(i, o_1) \leq d(i, o(i))$ for any agent $i \in N_1 \cap N_2$, we have

$$\begin{aligned}
\sum_{i \in N_1 \cap N_2} d(i, w(i)) & \leq 2 \cdot \sum_{i \in N_1 \cap N_2} d(i, o_1) + \sum_{i \in N_1 \cap N_2} d(i, o(i)) \\
& \leq 3 \cdot \sum_{i \in N_1 \cap N_2} d(i, o(i)). \quad (9)
\end{aligned}$$

Replacing (9) to (8) and using the obvious fact that $N_1 \cap N_2 \subseteq N$, we finally obtain that

$$SC(\mathbf{w}) \leq 11 \cdot \sum_{i \in N_1 \cap N_2} d(i, o(i)) \leq 11 \cdot SC(\mathbf{o}),$$

that is, the approximation ratio is at most 11 in this case, completing the proof. \square

We also show that the bound of 11 on the approximation ratio of the mechanism in terms of the social cost is tight with the following example.

Example 2.3. Let $\varepsilon > 0$ be an infinitesimal and consider an instance with the following four candidate locations: $\{0, \varepsilon, 1, 1 + \varepsilon\}$. There are $n/3$ agents that approve F_1 positioned at 0, $n/3$ agents that approve F_2 positioned at 0, $n/6$ agents that approve both facilities positioned at 0, and another $n/6 + 1$ agents that approve both facilities all positioned $1/2 + 2\varepsilon$. Clearly, this instance is captured by (Case 2), and thus the mechanism places the two facilities at the candidate locations closest to the median of $N_1 \cap N_2$ who is positioned at 1. Hence, $w_1 = 1$ and $w_2 = 1 + \varepsilon$, for a social cost of approximately $2n/3 + n/6 + n/12 = 11n/12$. However, the optimal solution is to place the two facilities at 0 and ε for a social cost of approximately $n/12$, leading to an approximation ratio of 11. \square

We now turn our attention to the max cost objective.

Theorem 2.4. For the maximum cost, the approximation ratio of CONDITIONAL-MEDIAN is at most 5.

Proof. Let i be the agent with the maximum individual cost over all agents for the solution $\mathbf{w} = (w_1, w_2)$ computed by the mechanism. Also, let $\mathbf{o} = (o_1, o_2)$ be an optimal solution. We consider the following two cases depending on which facility determines the individual cost of i .

Case (a): The individual cost of i in \mathbf{w} is determined by F_1 .

Clearly, since the cost of i is determined by F_1 , $i \in N_1$, and thus $d(i, o_1) \leq d(i, o(i))$. Let m be the agent that determines the location of F_1 ; so, m is either m_1 in (Case 1) or m_{12} in (Case 2). Observe that w_1 is the candidate location closest to m , and thus $d(m, w_1) \leq d(m, o(m))$. Also, since $m \in N_1$, $d(m, o_1) \leq d(m, o(m))$. Hence, by the triangle inequality, we have that

$$\begin{aligned} MC(\mathbf{w}) &= d(i, w_1) \leq d(i, o_1) + d(m, o_1) + d(m, w_1) \\ &\leq d(i, o(i)) + 2 \cdot d(m, o(m)) \\ &\leq 3 \cdot MC(\mathbf{o}). \end{aligned}$$

Case (b): The individual cost of i in \mathbf{w} is determined by F_2 .

If we are in (Case 1) and $w_2 = t(m_2)$, then we can get an upper bound of 3 similarly to Case (a). We can also derive an upper bound of 3 if we are in (Case 2), where $w_2 = s(m_{12})$, since this implies that for some of $o \in \{o_1, o_2\}$, $d(m_{12}, w_2) \leq d(m, o) \leq d(m, o(m))$; using this and the fact that $d(m, o_2) \leq d(m, o(m))$, we can apply the triangle inequality as in Case (a) to get the bound of 3.

So, let us now assume that we are in (Case 1) and $w_2 = s(m_2)$, which implies that $t(m_2) = w_1 = t(m_1)$. Clearly, if m_2 is (weakly) closer to w_2 than to o_2 , then the same argument as in Case (a) can again lead to an upper bound of 3. So, we can further assume that $t(m_2) = w_1 = t(m_1) = o_2$. Due to this, m_2 is (weakly) closer to $w_2 = s(m_2)$ than to o_1 . Therefore, using repeatedly the triangle inequality and this relation between these points, we obtain

$$\begin{aligned} MC(\mathbf{w}) &= d(i, w_2) \\ &\leq d(i, o_2) + d(m_2, o_2) + d(m_2, w_2) \\ &\leq d(i, o(i)) + d(m_2, o(m_2)) + d(m_2, o_1) \\ &\leq 2 \cdot MC(\mathbf{o}) + d(m_2, o_1) \\ &\leq 2 \cdot MC(\mathbf{o}) + d(m_2, o_2) + d(m_1, o_2) + d(m_1, o_1) \\ &\leq 2 \cdot MC(\mathbf{o}) + d(m_2, o(m_2)) + d(m_1, t(m_1)) + d(m_1, o(m_1)) \\ &\leq 5 \cdot MC(\mathbf{o}), \end{aligned}$$

and the proof is complete. \square

We also show that the bound of 5 shown in the above theorem is tight.

Example 2.5. Let $\varepsilon > 0$ be an infinitesimal and consider an instance with the following three candidate locations: $\{0, 2, 6\}$. There are three agents that approve F_1 all positioned at $1 + \varepsilon$, one agent that approves F_2 positioned at 1, and another two agents that approve F_2 positioned at $3 + \varepsilon$. This instance is captured by (Case 1), and thus the mechanism places F_1 at $w_1 = t(m_1) = 2$, and since $t(m_2) = 2$ as well, F_2 is placed at $w_2 = s(m_2) = 6$. The max cost of this solution is 5, realized by the agent that approves F_2 positioned at 1. However, the optimal solution is to place F_1 at 0 and F_2 at 2 for a max cost of approximately 1, leading to an approximation ratio of 5. \square

3. A connection between the max- and the sum-variant

In this short section, we discuss an interesting connection between the max-variant heterogeneous two-facility location problem that is the focus of this paper and the *sum-variant* that has been studied by Kanellopoulos et al. [10]. In the sum-variant, the individual cost of an agent i for a feasible solution \mathbf{y} is her distance to all the facilities she approves, in particular,

$$\text{cost}_i^{\text{sum}}(\mathbf{y}) = \sum_{j \in [2]} p_{ij} \cdot d(i, y_j).$$

In contrast, recall that, in the max-variant, the individual cost of an agent i is her distance to the farthest facility among the ones she approves, i.e.,

$$\text{cost}_i^{\text{max}}(\mathbf{y}) = \max_{j: p_{ij}=1} d(i, y_j).$$

It is not hard to observe that $\frac{1}{2} \cdot \text{cost}_i^{\text{sum}}(\mathbf{y}) \leq \text{cost}_i^{\text{max}}(\mathbf{y}) \leq \text{cost}_i^{\text{sum}}(\mathbf{y}) \leq 2 \cdot \text{cost}_i^{\text{max}}(\mathbf{y})$ for any agent i , which implies the following statement.

Theorem 3.1. *Let M be any mechanism that is strategyproof for the max-variant and the sum-variant. If the approximation ratio of M in terms of the social cost or the maximum cost is at most $\rho \geq 1$ in one variant, then it is at most $2 \cdot \rho$ in the other variant.*

Proof. Let \mathbf{w} be the solution computed by M . Also, let \mathbf{o}^{max} and \mathbf{o}^{sum} be the optimal solutions in terms of the objective $f \in \{\text{SC}, \text{MC}\}$ for the max- and the sum-variant, respectively. For convenience, we write $f^{\text{max}}(\mathbf{y})$ and $f^{\text{sum}}(\mathbf{y})$ for the objective value of solution \mathbf{y} in terms of f in the max- and the sum-variant, respectively. Due to the relation between the individual costs of the agents in the two variants mentioned above and the fact that f is either the sum of the costs or the cost of some agent, we have that $f^{\text{sum}}(\mathbf{w}) \leq 2 \cdot f^{\text{max}}(\mathbf{w})$ and $f^{\text{sum}}(\mathbf{o}^{\text{sum}}) \geq f^{\text{max}}(\mathbf{o}^{\text{sum}})$. Combined with the fact that $f^{\text{max}}(\mathbf{o}^{\text{sum}}) \geq f^{\text{max}}(\mathbf{o}^{\text{max}})$, we obtain

$$\frac{f^{\text{sum}}(\mathbf{w})}{f^{\text{sum}}(\mathbf{o}^{\text{sum}})} \leq \frac{2 \cdot f^{\text{max}}(\mathbf{w})}{f^{\text{max}}(\mathbf{o}^{\text{sum}})} \leq 2 \cdot \frac{f^{\text{max}}(\mathbf{w})}{f^{\text{max}}(\mathbf{o}^{\text{max}})} \leq 2 \cdot \rho.$$

Similarly, we have that $f^{\text{max}}(\mathbf{w}) \leq f^{\text{sum}}(\mathbf{w})$ and $f^{\text{max}}(\mathbf{o}^{\text{max}}) \geq \frac{1}{2} \cdot f^{\text{sum}}(\mathbf{o}^{\text{max}})$. Combined with the fact that $f^{\text{sum}}(\mathbf{o}^{\text{max}}) \geq f^{\text{sum}}(\mathbf{o}^{\text{sum}})$, we obtain

$$\frac{f^{\text{max}}(\mathbf{w})}{f^{\text{max}}(\mathbf{o}^{\text{max}})} \leq \frac{f^{\text{sum}}(\mathbf{w})}{\frac{1}{2} \cdot f^{\text{sum}}(\mathbf{o}^{\text{max}})} \leq 2 \cdot \frac{f^{\text{sum}}(\mathbf{w})}{f^{\text{sum}}(\mathbf{o}^{\text{sum}})} \leq 2 \cdot \rho,$$

as desired. \square

It is not hard to observe that CONDITIONAL-MEDIAN is strategyproof not only in the max-variant but also in the sum-variant (in particular, the arguments in the proof of Theorem 2.1 follow through for the sum-variant). Consequently, by Theorems 2.2, 2.4 and 3.1, CONDITIONAL-MEDIAN achieves constant approximation ratio in terms of the social cost and the maximum cost in the sum-variant. As already mentioned, this fixes a result of Kanellopoulos et al. [10] who claimed an upper bound of 3 for the social cost with a mechanism that turned out to not be strategyproof.

Corollary 3.2. *In the sum-variant, the approximation ratio of CONDITIONAL-MEDIAN is at most 22 in terms of the social cost and at most 10 in terms of the maximum cost.*

4. Open problems

In this paper, we focused on the truthful constrained heterogeneous facility location problem with max-variant cost. We showed that constant bounds of 11 and 5 on the approximation ratio in terms of the social cost and maximum cost, respectively, can be achieved *simultaneously* by a rather simple and natural strategyproof mechanism, thus improving upon the previous best-known bounds. The most direct open question is whether improved approximations can be simultaneously achieved in terms of these two objectives. Building on this idea (which has recently been considered by Han et al. [8] for non-strategic voting and single-facility location), one could also consider other objectives that interpolate between the social cost and the max cost, such as the sum of the $k \in [n]$ max individual agent costs; observe that this objective coincides with the social cost for $k = n$ and with the max cost for $k = 1$. Another direction would be to consider different assumptions, such as when both the positions and the preferences of the agents are private.

CRediT authorship contribution statement

Mohammad Lotfi: Methodology, Formal analysis. **Alexandros A. Voudouris:** Writing – review & editing, Writing – original draft, Methodology, Formal analysis, Conceptualization.

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