Risk sharing with Lambda value at risk

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Abstract

In this paper, we study the risk sharing problem among multiple agents using Lambda value at risk (ΛVaR) as their preferences via the tool of inf-convolution, where ΛVaR is an extension of Value-at-Risk (VaR). We obtain explicit formulas of the inf-convolution of multiple ΛVaR with monotone Λ, and explicit forms of the corresponding optimal allocations, extending the results of the inf-convolution of VaR. It turns out that the inf-convolution of several ΛVaR is still a ΛVaR under some mild condition. Moreover, we investigate the inf-convolution of one ΛVaR and a general monotone risk measure without cash-additivity, including ΛVaR, expected utility and rank-dependent expected utility as special cases. The expression of the inf-convolution and the explicit forms of the optimal allocation are derived, leading to some partial solution of the risk sharing problem with multiple ΛVaR for general Λ functions. Finally, we discuss the risk sharing problem with $\Lambda\mathrm{VaR}^+$, another definition of Lambda value at risk. We focus on the inf-convolution of $\Lambda\mathrm{VaR}^+$ and a risk measure that is consistent with the second-order stochastic dominance, deriving very different expression of the inf-convolution and the forms of the optimal allocations.

Key-words: Lambda value at Risk; Value-at-Risk; Risk sharing; Inf-convolution; Expected utility; Rank-dependent expected utility; Distortion risk measure; Expected shortfall

1 Introduction

A risk sharing problem in risk management and game theory concerns the redistribution of the aggregate risk among multiple agents. The preference of the agents can be characterized by e.g., expected utility or risk measures. In the past two decades, the Pareto-optimal risk sharing problem has been extensively studied with the preferences of agents represented by some risk measures. These risk measures are chosen to be either *coherent* or *convex* introduced by Artzner et al. (1999), Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2005) or non-convex

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such as quantile-based risk measures in Embrechts et al. (2018) or VaR-type risk measures in Weber (2018).

A mathematical tool to study the risk sharing problem in the cooperative environment is inf-convolution. The inf-convolution of convex-type risk measures has been studied in e.g., Barrieu and El Karoui (2005) and Jouini et al. (2008) and Filipović and Svindland (2008), showing the existence of the optimal allocation and it can be chosen to be *comonotonic*[1]. In recent years, some works investigate the inf-convolution of non-convex risk measures such as Value-at-Risk (VaR), Range-Value-at-Risk (RVaR) and tail risk measures. The analytical expressions of the inf-convolution and the explicit forms of the optimal allocation for VaR and RVaR are given in Embrechts et al. (2018). These results are extended to the mixture of left and right VaR, and tail-risk measures in Liu et al. (2022) and VaR-type risk measures in Weber (2018). Moreover, Embrechts et al. (2019) study the risk sharing problem with VaR and expected shortfall (ES) as risk measures under heterogeneous beliefs. We refer to Cai and Chi (2020) for a review of the risk sharing problem in the context of the design of insurance and reinsurance contracts using risk measures.

In this paper, we focus on the risk sharing problem with multiple agents, where the preferences of agents are characterized by Lambda value at risk (ΛVaR), which is introduced by Frittelli et al. (2014) as an extension of VaR by changing the fixed probability level to a function of loss $1-\Lambda$. The idea is to incorporate the dependence of the probability level and the amount of loss in the definition of VaR. The choice of Λ function is very flexible and may be problem-driven; see Hitaj et al. (2018) for the choice of Λ function. As discussed in Frittelli et al. (2014), the Λ function is usually assumed to be monotone, representing risk manager's individual risk appetite. For decreasing Λ, Lambda value at risk is able to capture the tail risk by controlling the probability of exceeding the loss, which is in a different way from Expected Shortfall (ES). For increasing Λ, ΛVaR may incorporate some additional requirement such as risk manager's judgement in the process of risk management; see Bellini and Peri (2022). Moreover, ΛVaR satisfies quasi-convexity with respect to distributions, the same as VaR; see Frittelli et al. (2014). Compared with VaR and ES, one potential application of ΛVaR is to measure the catastrophic risk such as the losses caused by tsunamis, hurricanes and earthquakes because these losses are typically modelled by infinite-mean distributions; see Bignozzi et al. (2020). The study of other properties of ΛVaR such as robustness, elicitability and consistency can be bound in Burzoni et al. (2017). The backtesting, estimation and risk contribution of ΛVaR are studied in Corbetta and Peri (2018), Hitaj et al. (2018) and Ince et al. (2022), respectively. Most recently, Bellini and Peri (2022) offer an axiomatization of ΛVaR for increasing Λ functions and study some further properties of ΛVaR, and Han et al. (2021) provide a

[1]We say $X_1,\dots,X_n$ are comonotonic if there exist non-decreasing functions $f_1,\dots,f_n$ satisfying $f_1(x)+\cdots+f_n(x)=x,\ x\in\mathbb{R}$ such that $X_1=f_1(X_1+\cdots+X_n),\dots,X_n=f_n(X_1+\cdots+X_n)$.

representation of ΛVaR in terms of VaR for increasing Λ functions. Some economic interpretation of ΛVaR is available in Frittelli et al. (2014) and Bellini and Peri (2022). An extension of Lambda value at risk, called lambda-fixed point risk measure, is introduced and studied in Balbás et al. (2023), where its applications to reinsurance contracts and premium calculation are also discussed.

In Section 3, we study the inf-convolution of multiple ΛVaR with monotone Λ. This is the most applicable case and the monotonicity of Λ in ΛVaR reflects the risk appetites of agents as discussed in Frittelli et al. (2014). For increasing Λ, we allow a higher exceeding probability for a higher level of loss; For decreasing Λ, we only accept a higher level of loss with a lower exceeding probability. We give a thorough discussion for the cases that all Λ functions are increasing or decreasing by deriving the expression of the inf-convolution and finding explicit forms of the optimal allocations. Surprisingly, the inf-convolution of ΛVaR is still a ΛVaR with a slight requirement on the Λ functions. Our findings extend the results of the inf-convolution of multiple VaR in Embrechts et al. (2018).

Section 4 is devoted to the study of the inf-convolution of one ΛVaR and a general monotone risk measure without cash-additivity. This is motivated by the fact that some important risk functionals do not satisfy cash-additivity such as expected utility (EU) with convex utility function, rank-dependent expected utility (RDEU) and ΛVaR for general Λ functions; see Han et al. (2021) for more examples on the risk measures without cash-additivity. Here we only require the right continuity of Λ. The risk measure considered is either law-invariant monotone risk measure or a monotone ε-tail risk measure depending on the domain of the risk measures. We obtain an expression of the inf-convolution and explicit forms of the optimal allocation. Based on this result, we further obtain the result for the inf-convolution of multiple ΛVaR with general Λ functions. However, it is not solved thoroughly due to the heterogeneity and complexity of Λ functions. The case with the mixture of increasing and decreasing Λ functions is a special case of our result, indicating different risk appetites of agents. Finally, we compute some examples with the risk measures being EU and RDEU. Our results in this section generate the corresponding results in Liu et al. (2022), where the inf-convolution of VaR and a monetary ε-tail risk measure is considered.

In Section 5, we consider another definition of Lambda value at risk, $\Lambda\mathrm{VaR}^+$ (see the definition in (5) in Section 2), which can capture the tail risk by controlling the probability of exceeding the loss. The inf-convolution of two $\Lambda\mathrm{VaR}^+$ is a very difficult problem and is beyond the scope of this paper as it involves the robust risk aggregation of two $\Lambda\mathrm{VaR}^+$ with fixed marginal distributions and unknown dependence structure. Instead, we consider the inf-convolution of $\Lambda\mathrm{VaR}^+$ with general Λ and a risk measure that is consistent with the second-order stochastic dominance such as convex or coherent risk measures. The expression of the inf-convolution and the explicit forms of the optimal allocations are derived, which are very different from the ones in previous sections because the function Λ is showing up in these expressions. To illustrate our main result, we consider the inf-convolution of $\Lambda\mathrm{VaR}^+$ and one expected utility/distortion risk measure as examples.

The notation and definitions are displayed in Section 2 and all the proofs are postponed to Appendices A-E.

2 Notation and Definitions

For a given atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $L^p, p \in (0, \infty]$ denote the collection of all random variables with finite L^p norm and L^0 be the set of all finite random variables. Moreover, let $\mathcal{X}$ be a set of random variables containing L^∞. We say $\mathcal{X}$ is unbounded if $\mathcal{X} \supsetneq L^\infty$. If we do not specify $\mathcal{X}$, we tacitly suppose $\mathcal{X} \supseteq L^\infty$ is a set of random variables with good enough properties to conduct our study, such as $\mathcal{X} = L^p$ for some $p \in [1, \infty]$. For any $X \in \mathcal{X}$, the positive value of X represents the financial loss and its distribution function is denoted as F_X. A *risk measure* is a mapping from $\mathcal{X}$ to $(-\infty, \infty)$. A risk measure ρ is *law-invariant* if for all $X, Y \in \mathcal{X}$,

$$X \stackrel{\mathrm{d}}{=} Y \quad \Rightarrow \quad \rho(X) = \rho(Y), \tag{1}$$

where $\stackrel{\mathrm{d}}{=}$ stands for equality in distribution; ρ is *monotone* if $X \leqslant Y$ implies $\rho(X) \leqslant \rho(Y)$; and ρ is *cash-additive* if $\rho(X + c) = \rho(X) + c$ for $X \in \mathcal{X}$ and $c \in \mathbb{R}$. We say ρ is a *monetary* risk measure if ρ is monotone and cash-additive. We refer to Föllmer and Schied (2016) for more details on risk measures.

For $X \in \mathcal{X}$, F_X^{-1} represents its left-quantile, which is defined by

$$F_X^{-1}(p) = \inf\{x : F_X(x) \geqslant p\}, \;\; p \in (0, 1]$$

with the convention that $\inf \emptyset = \infty$. For any $X \in \mathcal{X}$, we denote by U_X a uniform random variable on $[0, 1]$ such that $X = F_X^{-1}(U_X)$ a.s.. The existence of such U_X for any random variable X is guaranteed by e.g., Lemma A.32 of Föllmer and Schied (2016). We next define tail risk measure, which is important in our results later. For a random variable $X \in \mathcal{X}$ and $p \in (0, 1]$, we call

$$X_p = F_X^{-1}(1 - p + pU_X)$$

the tail risk of X beyond its $(1 - p)$-quantile. The distribution of X_p is given by

$$\mathbb{P}(X_p \leqslant x) = \frac{(F_X(x) - (1 - p))_+}{p}, \;\; x \in \mathbb{R},$$

where $x_+ = \max(x, 0)$.

We say ρ is a p-tail risk measure for some $p \in (0,1)$ if $X_p \stackrel{\mathrm{d}}{=} Y_p$ implies $\rho(X) = \rho(Y)$ for all $X, Y \in \mathcal{X}$. One can refer to Liu and Wang (2021) and Liu et al. (2022) for more details on the definition, properties and applications of tail risk measures.

Next, we define inf-convolution. For a random variable $X \in \mathcal{X}$, define the set of *allocations* of X as

$$\mathbb{A}_n(X) = \left\{ (X_1, \dots, X_n) \in \mathcal{X}^n : \sum_{i=1}^n X_i = X \right\}. \tag{2}$$

The *inf-convolution* of risk measures $\rho_1, \dots, \rho_n$ is the mapping $\square_{i=1}^n \rho_i : \mathcal{X} \to [-\infty, \infty)$, defined as

$$\mathop{\square}_{i=1}^{n} \rho_i(X) = \inf \left\{ \sum_{i=1}^n \rho_i(X_i) : (X_1, \dots, X_n) \in \mathbb{A}_n(X) \right\}. \tag{3}$$

An n-tuple $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ is called *an optimal allocation* of X for $(\rho_1, \dots, \rho_n)$ if $\sum_{i=1}^n \rho_i(X_i) = \square_{i=1}^n \rho_i(X)$. If the risk measures are interpreted as the capital charge for a financial institution to take risky positions, then $\square_{i=1}^n \rho_i(X)$ represents the smallest possible aggregate capital for the total risk X in financial system. We refer to Delbaen (2012), Rüschendorf (2013) and Embrechts et al. (2018) for more economic interpretations on the inf-convolution.

Finally, we define Lambda value at risk. For $\Lambda : \mathbb{R} \to [0,1]$, the Lambda value at risk are given by

$$\Lambda\mathrm{VaR}(X) = \inf\{x \in \mathbb{R} : F_X(x) \geqslant 1 - \Lambda(x)\}, \tag{4}$$

and

$$\Lambda\mathrm{VaR}^+(X) = \sup\{x \in \mathbb{R} : F_X(x) < 1 - \Lambda(x)\}, \tag{5}$$

where $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. Note that our definition is a bit different from the one in Bellini and Peri (2022). This modification is to be consistent with the definition of VaR as in Embrechts et al. (2018) and Liu et al. (2022) to focus more on the tail probability. Here, Λ function is used to control the tail probability. Note that $\Lambda\mathrm{VaR}(X) = \Lambda\mathrm{VaR}^+(X)$ if Λ is increasing; Otherwise, it may not be true; see Proposition 6 of Bellini and Peri (2022). In fact, there are two other definitions of ΛVaR given by Bellini and Peri (2022). They are not discussed in this paper as they perform similarly as ΛVaR and $\Lambda\mathrm{VaR}^+$ in the risk sharing problem. In this paper, we are particularly interested in ΛVaR with Λ being monotone functions. If Λ is a constant, then ΛVaR boils down to VaR, i.e., VaR at level $p \in [0,1)$ is

$$\mathrm{VaR}_p(X) = \Lambda\mathrm{VaR}(X) = F_X^{-1}(1-p), \qquad X \in \mathcal{X},$$

for $\Lambda = p$. Although VaR has been widely applied in practice for risk measurement due to its simplicity and possession of some nice properties, VaR is always criticised that it cannot capture the tail risk. See McNeil et al. (2015) and the references therein for more detailed discussion on VaR. Compared to VaR, $\Lambda\mathrm{VaR}^+$ is able to capture the tail risk; see e.g., Frittelli et al. (2014) and Hitaj et al. (2018).

The following result shows the essential difference between VaR and ΛVaR.

Proposition 1. *Let $\rho = \Lambda\mathrm{VaR}$ or $\Lambda\mathrm{VaR}^+$ with $\Lambda : \mathbb{R} \to (0, 1)$ being a monotone function. Then ρ is a* VaR *if and only if it satisfies cash-additivity.*

As shown in Bellini and Peri (2022), ΛVaR and VaR share many properties. The above result shows that the only difference between VaR and ΛVaR is whether it satisfies cash-additivity. This somehow explains the fact that the inf-convolution of ΛVaR and that of VaR share the similar forms of the optimal allocations in Section 3.

Let $\mathcal{H}_I$ be the collection of all increasing and right-continuous function $\Lambda : \mathbb{R} \to [0, 1]$, where Λ is not a constant 1, and $\mathcal{H}_D$ be the collection of all decreasing and left-continuous function $\Lambda : \mathbb{R} \to [0, 1]$. Here increasing (decreasing) means non-decreasing (non-increasing). We denote $\mathcal{H}$ as the collection of all $\Lambda : \mathbb{R} \to [0, 1]$, where Λ is right-continuous and not a constant 1. Hereafter, for any Λ, we denote $\lambda^- = \inf_{x\in\mathbb{R}} \Lambda(x)$ and $\lambda^+ = \sup_{x\in\mathbb{R}} \Lambda(x)$. We say that a constant λ is *attainable* for Λ if there exists $x \in \mathbb{R}$ such that $\Lambda(x) = \lambda$.

3 Inf-convolution of multiple ΛVaR with monotone Λ

In this section, we consider the most attractive scenarios: all Λ functions are monotone in the same direction. The cases with Λ being other type of functions will be studied in Section 4. The set $\mathcal{X}$ is general and the aggregate risk is shared among n agents using ΛVaR as their preferences. Hence we consider risk sharing problem with Λ_iVaR, $i = 1, \ldots, n$. Recall that $\lambda_i^- = \lim_{x\to\infty} \min(\Lambda_i(x), \Lambda_i(-x))$ and $\lambda_i^+ = \lim_{x\to\infty} \max(\Lambda_i(x), \Lambda_i(-x))$.

We denote $\Lambda^{\mathbf{x}_{n-1}}(x) = \left(\Lambda_n(x - x_{n-1}) + \sum_{i=1}^{n-1} \Lambda_i(x_i - x_{i-1})\right) \wedge 1$ for $n \geqslant 2$, where $\mathbf{x}_{n-1} = (x_1, \ldots, x_{n-1})$, $x \wedge y = \min(x, y)$ and $x_0 = 0$. For convenience, we let $\Lambda^{\mathbf{x}_0} = \Lambda_1$, $\bigvee_{i=1}^n x_i = \max_{i=1}^n x_i$ and $\bigwedge_{i=1}^n x_i = \min_{i=1}^n x_i$. Moreover, we denote

$$\Lambda^*(x) = \sup_{\mathbf{x}_{n-1}\in\mathbb{R}^{n-1}} \Lambda^{\mathbf{x}_{n-1}}(x) = \left(\sup_{\sum_{i=1}^n x_i = x} \sum_{i=1}^n \Lambda_i(x_i)\right) \bigwedge 1, \ x \in \mathbb{R}. \tag{6}$$

Note that $0 \leqslant \Lambda^* \leqslant 1$ and Λ^* is increasing (decreasing) if all Λ_i are increasing (decreasing).

3.1 Inf-convolution of multiple $\Lambda\mathrm{VaR}$ with decreasing Λ

In this subsection, we consider the inf-convolution of several $\Lambda\mathrm{VaR}$ with decreasing and left-continuous Λ. As discussed in Frittelli et al. (2014), in this setup, higher losses should be tolerated with lower probability for all agents. We give a thorough discussion for this case by obtaining an expression of the inf-convolution and explicit forms of optimal allocations.

Theorem 1. *For $\Lambda_i \in \mathcal{H}_D$ with $0 < \lambda_i^- \leqslant \lambda_i^+ < 1$, we have the following conclusion.*

(i) If $\sum_{i=1}^n \lambda_i^+ < 1$, then

$$\mathop{\square}_{i=1}^{n} \Lambda_i \mathrm{VaR}(X) = \inf_{\mathbf{x}_{n-1}\in\mathbb{R}^{n-1}} \Lambda^{\mathbf{x}_{n-1}}\mathrm{VaR}(X); \tag{7}$$

If in addition, all $\lambda_i^\pm$ are attainable, then

$$\mathop{\square}_{i=1}^{n} \Lambda_i \mathrm{VaR}(X) = \Lambda^* \mathrm{VaR}(X); \tag{8}$$

(ii) If $\sum_{i=1}^n \lambda_i^+ > 1$, then $\square_{i=1}^n \Lambda_i \mathrm{VaR} = -\infty$;

(iii) Moreover, for $\sum_{i=1}^n \lambda_i^+ < 1$, the existence of the optimal allocation of the inf-convolution is equivalent to the existence of the minimizer of (7). If $\mathbf{x}_{n-1} \in \arg\min_{\mathbf{x}_{n-1}\in\mathbb{R}^{n-1}} \Lambda^{\mathbf{x}_{n-1}}\mathrm{VaR}(X)$, then one optimal allocation is recursively given by

$$\begin{aligned}
X_n &= (X - x_{n-1})\mathbb{1}_{\{U_X \leqslant 1-\Lambda^{\mathbf{x}_{n-2}}(x_{n-1})\}} - m_{n-1}\mathbb{1}_{\{U_X > 1-\Lambda^{\mathbf{x}_{n-2}}(x_{n-1})\}},\\
X_k &= (x_k - x_{k-1})\mathbb{1}_{\{U_X \leqslant 1-\Lambda^{\mathbf{x}_{k-1}}(x_k)\}} + \left(X + \sum_{i=k}^{n-1} m_i - x_{k-1}\right)\mathbb{1}_{\{1-\Lambda^{\mathbf{x}_{k-1}}(x_k) < U_X \leqslant 1-\Lambda^{\mathbf{x}_{k-2}}(x_{k-1})\}}\\
&\quad - m_{k-1}\mathbb{1}_{\{U_X > 1-\Lambda^{\mathbf{x}_{k-2}}(x_{k-1})\}},\ k = n-1, \dots, 2,\\
X_1 &= x_1\mathbb{1}_{\{U_X \leqslant 1-\Lambda_1(x_1)\}} + \left(X + \sum_{i=1}^{n-1} m_i\right)\mathbb{1}_{\{U_X > 1-\Lambda_1(x_1)\}},
\end{aligned}$$

where $m_k,\ k = n-1, \dots, 1$ satisfy $m_k > x_k - F_X^{-1}(1 - \sum_{i=1}^n \lambda_i^+)$ and $\sum_{i=n+1}^n m_i = 0$.

Remark 1. If all Λ_i are decreasing and right-continuous, our findings in Theorem 1 still hold except (8). To guarantee the validity of (8) in this setup, we only need to impose an additional assumption on Λ_i: all Λ_i are continuous.

Remark 2. Note that the optimal allocation given in Theorem 1 is not unique. At least, the choice of $m_k,\ k = 1, \dots, n-1$ is not unique. The result given in Theorem 1 indicates that the optimal risk allocation can be obtained by allocating the total risk onto a partition of the whole probability space, sharing the similar structure as the one for the inf-convolution of multiple VaR or Range-Value-at-Risk in Embrechts et al. (2018). Moreover, if $\Lambda_i,\ i=1,\dots,n$ are all constants, Theorem 1 boils down to Corollary 2 of Embrechts et al. (2018).

Note that we do not discuss the boundary scenario $\sum_{i=1}^n \lambda_i^+ = 1$ in Theorem 1 because it is tricky and it involves the assumptions on whether the maximum of Λ_i can be attained. We next discuss the boundary case.

Proposition 2. *Suppose all $\Lambda_i \in \mathcal{H}_D$ with $0<\lambda_i^- \leqslant \lambda_i^+ < 1$ and $\sum_{i=1}^n \lambda_i^+ = 1$.*

(i) If all λ_i^+ are attainable, then $\square_{i=1}^n \Lambda_i \mathrm{VaR}(X) = -\infty$;

(ii) If one of $\lambda_i^+,\ i=1,\dots,n-1$ is not attainable,

$$\mathop{\square}_{i=1}^{n} \Lambda_i \mathrm{VaR}(X) = \inf_{\mathbf{x}_{n-1}\in\mathbb{R}^{n-1}} \Lambda^{\mathbf{x}_{n-1}} \mathrm{VaR}(X).$$

We notice that in Proposition 2, we do not discuss the scenario that only λ_n^+ is not attainable. This can actually be transformed to case (ii) of Proposition 2 by changing the order of the inf-convolution with the aid of Lemma 2 in Liu et al. (2020).

To illustrate our result in Theorem 1, we next consider a special case where Λ_i are step functions with two values.

Example 1. Let $\Lambda_i(x) = \lambda_i^+ \mathbb{1}_{\{x\leqslant b_i\}} + \lambda_i^- \mathbb{1}_{\{x>b_i\}}$, where $0<\lambda_i^- \leqslant \lambda_i^+ < 1$ and $\sum_{i=1}^n \lambda_i^+ < 1$. Then we have

$$\mathop{\square}_{i=1}^{n} \Lambda_i \mathrm{VaR}(X) = (\lambda^* \vee \Lambda^\diamond)\mathrm{VaR}(X),$$

where $\lambda^* = \bigvee_{i=1}^{n-1}(\lambda_i^- + \sum_{j\neq i}\lambda_j^+)$ and $\Lambda^\diamond(x) = \sum_{i=1}^{n-1}\lambda_i^+ + \Lambda_n(x - \sum_{i=1}^{n-1} b_i)$.

3.2 Inf-convolution of multiple $\Lambda\mathrm{VaR}$ with increasing Λ

In this subsection, we will focus on the inf-convolution of several ΛVaR with increasing and right-continuous Λ. This indicates that all agents are accepting higher losses with higher probability of exceeding the loss. As discussed in Example 7 of Bellini and Peri (2022), this type of ΛVaR can be used to describe the contrasting objectives of the risk manager in setting the capital reserve: to be conservative but not too much. We obtain an expression for the inf-convolution and find the forms of optimal allocations.

Theorem 2. *For $\Lambda_i \in \mathcal{H}_I$ with $0<\lambda_i^- \leqslant \lambda_i^+ \leqslant 1$, we have the following conclusion.*

(i) If $\bigvee_{i=1}^n(\lambda_i^- + \sum_{j\neq i}\lambda_j^+) < 1$, then

$$\mathop{\square}_{i=1}^{n} \Lambda_i \mathrm{VaR}(X) = \inf_{\mathbf{x}_{n-1}\in\mathbb{R}^{n-1}} \Lambda^{\mathbf{x}_{n-1}} \mathrm{VaR}(X); \tag{9}$$

If in addition, all $\lambda_i^{\pm}$ are attainable, then

$$\mathop{\square}_{i=1}^{n} \Lambda_i \mathrm{VaR}(X) = \Lambda^* \mathrm{VaR}(X); \tag{10}$$

(ii) If $\bigvee_{i=1}^n (\lambda_i^- + \sum_{j\neq i} \lambda_j^+) > 1$, then $\square_{i=1}^n \Lambda_i \mathrm{VaR} = -\infty$;

(iii) Moreover, for $\bigvee_{i=1}^n (\lambda_i^- + \sum_{j\neq i} \lambda_j^+) < 1$, the existence of the optimal allocation is equivalent to the existence of the minimizer of (9). If $\mathbf{x}_{n-1} \in \arg\min_{\mathbf{x}_{n-1}\in\mathbb{R}^{n-1}} \Lambda^{\mathbf{x}_{n-1}} \mathrm{VaR}(X)$, then the optimal allocation is recursively given by

$$\begin{aligned}
X_n &= (X - x_{n-1}) \mathbb{1}_{\{U_X \leqslant 1-\Lambda^{\mathbf{x}_{n-2}}(x_{n-1})\}} - m_{n-1} \mathbb{1}_{\{U_X > 1-\Lambda^{\mathbf{x}_{n-2}}(x_{n-1})\}}, \\
X_k &= (x_k - x_{k-1}) \mathbb{1}_{\{U_X \leqslant 1-\Lambda^{\mathbf{x}_{k-1}}(x_k)\}} + \left(X + \sum_{i=k}^{n-1} m_i - x_{k-1} \right) \mathbb{1}_{\{1-\Lambda^{\mathbf{x}_{k-1}}(x_k) < U_X \leqslant 1-\Lambda^{\mathbf{x}_{k-2}}(x_{k-1})\}} \\
&\quad - m_{k-1} \mathbb{1}_{\{U_X > 1-\Lambda^{\mathbf{x}_{k-2}}(x_{k-1})\}}, \ k = n-1, \dots, 2, \\
X_1 &= x_1 \mathbb{1}_{\{U_X \leqslant 1-\Lambda_1(x_1)\}} + \left(X + \sum_{i=1}^{n-1} m_i \right) \mathbb{1}_{\{U_X > 1-\Lambda_1(x_1)\}},
\end{aligned}$$

where m_k, $k = n-1, \dots, 1$ satisfy $m_k > x_k - F^{-1}_{X-\sum_{i=k+2}^n X_i}(1-\Lambda^{\mathbf{x}_{k-1}}(x_k))$ and $\Lambda_{k+1}(-m_k) + F_{X-\sum_{i=k+2}^n X_i}(-m_k + x_k) < 1 - \Lambda_k(x_k)$ with $\sum_{i=n+1}^n m_i = 0$.

For increasing and decreasing Λ, ΛVaR possesses very different properties (see Bellini and Peri (2022)). However, their applications in the risk sharing problem result in similar expressions of the inf-convolution and similar forms of the optimal risk allocations as shown in Theorems 1-2. Surprisingly, the inf-convolution of ΛVaR is still a ΛVaR for monotone Λ functions if the supremum and the infimum are attainable. This conclusion will be extended to the case with general Λ functions in Theorem 4 in next section under some more strict conditions on Λ functions.

Note that by Proposition 6 of Bellini and Peri (2022), we have $\Lambda_i' \mathrm{VaR}(X) = \Lambda_i \mathrm{VaR}(X)$, where $\Lambda_i' : \mathbb{R} \to [0,1]$ is increasing and Λ_i represents its right-continuous version. This allows us to extend Theorem 2 to the cases with increasing Λ functions. More specifically, $\square_{i=1}^n \Lambda_i' \mathrm{VaR}(X) = \square_{i=1}^n \Lambda_i \mathrm{VaR}(X) = \Lambda^* \mathrm{VaR}(X)$.

In light of Theorem 3.1 in Han et al. (2021), we obtain some new expressions on the inf-convolution in Theorem 2.

Proposition 3. *For $\Lambda_i \in \mathcal{H}_I$ with $0 < \lambda_i^- \leqslant \lambda_i^+ \leqslant 1$, if $\bigvee_{i=1}^n (\lambda_i^- + \sum_{j\neq i} \lambda_j^+) < 1$, then*

$$\mathop{\square}_{i=1}^{n} \Lambda_i \mathrm{VaR}(X) = \inf_{y_1,\dots,y_n\in\mathbb{R}} \left(\mathrm{VaR}_{\sum_{i=1}^n \Lambda_i(y_i)}(X) \bigvee (\sum_{i=1}^n y_i) \right).$$

If in addition, all $\lambda_i^{\pm}$ are attainable, then

$$\mathop{\square}_{i=1}^{n} \Lambda_i \mathrm{VaR}(X) = \inf_{x\in\mathbb{R}} \left(\mathrm{VaR}_{\Lambda^*(x)}(X) \vee x\right).$$

Remark 3. We can show that if $a := \bigvee_{i=1}^{n}(\lambda_i^- + \sum_{j\neq i}\lambda_j^+) < 1$, then $\square_{i=1}^{n} \Lambda_i \mathrm{VaR}(X) > -\infty$. Observe that

$$\lim_{y_1+\cdots+y_n\to-\infty} \mathrm{VaR}_{\sum_{i=1}^{n}\Lambda_i(y_i)}(X) \bigvee (\sum_{i=1}^{n} y_i) \geqslant \mathrm{VaR}_{a+\varepsilon}(X) > -\infty,$$

where $0 < \varepsilon < 1 - a$. Hence

$$\mathop{\square}_{i=1}^{n} \Lambda_i \mathrm{VaR}(X) = \inf_{y_1,\ldots,y_n\in\mathbb{R}} \left(\mathrm{VaR}_{\sum_{i=1}^{n}\Lambda_i(y_i)}(X) \bigvee (\sum_{i=1}^{n} y_i)\right) > -\infty.$$

Note that we do not discuss the boundary scenario $\bigvee_{i=1}^{n}(\lambda_i^- + \sum_{j\neq i}\lambda_j^+) = 1$ in Theorem 2 because it is tricky and it involves the assumptions on whether the maximum of Λ_i can be attained. We next discuss the boundary case.

Proposition 4. *Suppose all $\Lambda_i \in \mathcal{H}_I$, $0 < \lambda_i^- < 1$ and $\bigvee_{i=1}^{n}(\lambda_i^- + \sum_{j\neq i}\lambda_j^+) = 1$.*

(i) If $\lambda_i^- + \sum_{j\neq i}\lambda_j^+ = 1$ for some $i = 1,\ldots,n$ and λ_j^+ are attainable for all $j \neq i$, then $\square_{i=1}^{n} \Lambda_i \mathrm{VaR}(X) = -\infty$;

(ii) Otherwise,

$$\mathop{\square}_{i=1}^{n} \Lambda_i \mathrm{VaR}(X) = \inf_{\mathbf{x}_{n-1}\in\mathbb{R}^{n-1}} \Lambda^{\mathbf{x}_{n-1}} \mathrm{VaR}(X).$$

To illustrate our Theorem 2, we next consider a special case where Λ_i are step functions with two steps.

Example 2. Let $\Lambda_i(x) = \lambda_i^- \mathbb{1}_{\{x<b_i\}} + \lambda_i^+ \mathbb{1}_{\{x\geqslant b_i\}}$, where $0 < \lambda_i^- \leqslant \lambda_i^+ < 1$ and $\bigvee_{i=1}^{n}(\lambda_i^- + \sum_{j\neq i}\lambda_j^+) < 1$. Then we have

$$\mathop{\square}_{i=1}^{n} \Lambda_i \mathrm{VaR}(X) = (\lambda^* \vee \Lambda^\diamond)\mathrm{VaR}(X),$$

where $\lambda^* = \bigvee_{i=1}^{n-1}(\lambda_i^- + \sum_{j\neq i}\lambda_j^+)$ and $\Lambda^\diamond(x) = \left(\sum_{i=1}^{n-1}\lambda_i^+ + \Lambda_n(x - \sum_{i=1}^{n-1} b_i)\right) \wedge 1$.

Remark 4. Note that in Bellini and Peri (2022), Lambda value at risk is defined differently from (4). As in Bellini and Peri (2022), for $\overline{\Lambda} : \mathbb{R} \to [0,1]$, the Lambda value at risk is given by

$$\overline{\Lambda}\mathrm{VaR}(X) = \inf\{x \in \mathbb{R} : F_X(x) \geqslant \overline{\Lambda}(x)\}, \tag{11}$$

where $\inf \emptyset = \infty$. The corresponding $\Lambda^{\mathbf{x}_{n-1}}(x)$ and $\Lambda^*(x)$ become a bit more complicated, i.e.,

$$
\begin{aligned}
\overline{\Lambda}^{\mathbf{x}_{n-1}}(x) &= 1 - \left(1 - \overline{\Lambda}_n(x - x_{n-1}) + \sum_{i=1}^{n-1} \left(1 - \overline{\Lambda}_i(x_i - x_{i-1})\right)\right) \wedge 1 \\
&= \left(\overline{\Lambda}_n(x - x_{n-1}) + \sum_{i=1}^{n-1} \overline{\Lambda}_i(x_i - x_{i-1}) - n + 1\right)_+
\end{aligned}
$$

and

$$
\overline{\Lambda}^*(x) = \inf_{\mathbf{x}_{n-1} \in \mathbb{R}^{n-1}} \overline{\Lambda}^{\mathbf{x}_{n-1}}(x) = \inf_{\sum_{i=1}^n x_i = x} \left(\sum_{i=1}^n \overline{\Lambda}_i(x_i) - n + 1\right)_+,
$$

where $x_+ = \max(x, 0)$. We will give the corresponding results of Theorems 1 and 2 in terms of definition (11). Let $\overline{\lambda}_i^- = \lim_{x\to\infty} \min(\overline{\Lambda}_i(x), \overline{\Lambda}_i(-x))$ and $\overline{\lambda}_i^+ = \lim_{x\to\infty} \max(\overline{\Lambda}_i(x), \overline{\Lambda}_i(-x))$. By replacing Λ_i by $1 - \overline{\Lambda}_i$ in Theorems 1 and 2, we arrive at the following results.

- For $\overline{\Lambda}_i : \mathbb{R} \to [0, 1]$ being increasing and left-continuous with $0 < \overline{\lambda}_i^- \leqslant \overline{\lambda}_i^+ < 1$, we have the following conclusion.

  (i) If $\sum_{i=1}^n \overline{\lambda}_i^- > n - 1$, then

  $$
  \mathop{\square}_{i=1}^{n} \overline{\Lambda}_i \mathrm{VaR}(X) = \inf_{\mathbf{x}_{n-1} \in \mathbb{R}^{n-1}} \overline{\Lambda}^{\mathbf{x}_{n-1}} \mathrm{VaR}(X);
  $$

  If in addition, all $\overline{\lambda}_i^{\pm}$ are attainable, then

  $$
  \mathop{\square}_{i=1}^{n} \overline{\Lambda}_i \mathrm{VaR}(X) = \overline{\Lambda}^* \mathrm{VaR}(X);
  $$

  (ii) If $\sum_{i=1}^n \overline{\lambda}_i^- < n - 1$, then $\square_{i=1}^n \overline{\Lambda}_i \mathrm{VaR} = -\infty$;

  (iii) If $\sum_{i=1}^n \overline{\lambda}_i^- > n - 1$ and $\mathbf{x}_{n-1} \in \arg\min_{\mathbf{x}_{n-1} \in \mathbb{R}^{n-1}} \overline{\Lambda}^{\mathbf{x}_{n-1}} \mathrm{VaR}(X)$, the optimal allocation is given in (iii) of Theorem 1 by replacing $1 - \Lambda^{\mathbf{x}_{k-1}}(x_k)$ by $\overline{\Lambda}^{\mathbf{x}_{k-1}}(x_k)$.

- For $\overline{\Lambda}_i : \mathbb{R} \to [0, 1]$ being decreasing and right-continuous with $0 < \overline{\lambda}_i^- \leqslant \overline{\lambda}_i^+ < 1$, we have the following conclusion.

  (i) If $\bigvee_{i=1}^n (n - \overline{\lambda}_i^+ - \sum_{j \neq i} \overline{\lambda}_j^-) < 1$, then

  $$
  \mathop{\square}_{i=1}^{n} \overline{\Lambda}_i \mathrm{VaR}(X) = \inf_{\mathbf{x}_{n-1} \in \mathbb{R}^{n-1}} \overline{\Lambda}^{\mathbf{x}_{n-1}} \mathrm{VaR}(X);
  $$

If in addition, all $\overline{\lambda}_i^{\pm}$ are attainable, then

$$\mathop{\square}_{i=1}^{n} \overline{\Lambda}_i \mathrm{VaR}(X) = \overline{\Lambda}^* \mathrm{VaR}(X);$$

(ii) If $\bigvee_{i=1}^n (n - \overline{\lambda}_i^+ - \sum_{j\neq i} \overline{\lambda}_j^-) > 1$, then $\square_{i=1}^n \overline{\Lambda}_i \mathrm{VaR} = -\infty$;

(iii) If $\bigvee_{i=1}^n (n - \overline{\lambda}_i^+ - \sum_{j\neq i} \overline{\lambda}_j^-) < 1$ and $\mathbf{x}_{n-1} \in \arg\min_{\mathbf{x}_{n-1}\in\mathbb{R}^{n-1}} \overline{\Lambda}^{\mathbf{x}_{n-1}} \mathrm{VaR}(X)$, the optimal allocation is given in (iii) of Theorem 2 by replacing $1 - \Lambda^{\mathbf{x}_{k-1}}(x_k)$ by $\overline{\Lambda}^{\mathbf{x}_{k-1}}(x_k)$.

4 Inf-convolution of ΛVaR and a risk measure without cash-additivity

In this section, we consider the inf-convolution of one ΛVaR and a general monotone risk measure without cash-additivity. Here we do not impose any monotonicity or measurability assumptions on Λ. We only assume that Λ is right-continuous. Note that many well-known risk functionals do not satisfy cash-additivity such as expected utility (EU) with convex utility function and rank-dependent expected utility (RDEU). As our main concern, ΛVaR is another example of risk measure without cash-additivity.

Theorem 3. *Suppose $\Lambda \in \mathcal{H}$ with $0 < \lambda^- \leqslant \lambda^+ < 1$, and ρ is monotone. If either (i) $\mathcal{X} = L^\infty$ and ρ is law-invariant, or (ii) $\mathcal{X}$ is unbounded and ρ is an ε-tail risk measure for some $\varepsilon \in (0,1)$, we have*

$$\Lambda\mathrm{VaR} \,\square\, \rho(X) = \inf_{x\in\mathbb{R}} \inf_{y\in\mathbb{R}} \{x + \rho(X_{x,y})\}, \tag{12}$$

where $X_{x,y} = (X - x)\mathbb{1}_{\{U_X \leqslant 1-\Lambda(x)\}} + y\mathbb{1}_{\{U_X > 1-\Lambda(x)\}}$.

Moreover, the optimal allocation exists if and only if the minimizer of (12) exists. If (x^*, y^*) is the minimizer of (12), then one optimal allocation is given by

$$X_1^* = X - X_{x^*,y^*}, \; X_2^* = X_{x^*,y^*}. \tag{13}$$

Remark 5. The conclusion in Theorem 3 also holds for all $\Lambda \in \mathcal{H}_D$ with $0 < \lambda^- \leqslant \lambda^+ < 1$. The right-continuity of Λ is only used to derive $F_{X_1}(x_1) \geqslant 1 - \Lambda(x_1)$ if $\Lambda\mathrm{VaR}(X_1) = x_1$. Clearly, this inequality still holds for $\Lambda \in \mathcal{H}_D$.

Note that our result in Theorem 3 converts the risk sharing problem to an optimization problem with two real parameters. Our findings generalize Theorem 2 of Liu et al. (2022), where the inf-convolution of VaR and a monetary ε-tail risk measure is considered.

Applying Theorem 3 recursively, we can arrive at the following conclusion on the inf-convolution of multiple ΛVaR with general Λ functions.

Theorem 4. *For $\Lambda_i \in \mathcal{H}$ with $0 < \lambda_i^- \leqslant \lambda_i^+ < 1$, we have the following conclusion.*

(i) If $\sum_{i=1}^n \lambda_i^+ < 1$, then

$$\mathop{\square}_{i=1}^{n} \Lambda_i \mathrm{VaR}(X) = \inf_{\mathbf{x}_{n-1} \in \mathbb{R}^{n-1}} \Lambda^{\mathbf{x}_{n-1}} \mathrm{VaR}(X); \tag{14}$$

If in addition, all Λ_i are continuous and $\lambda_i^{\pm}$ are attainable, then

$$\mathop{\square}_{i=1}^{n} \Lambda_i \mathrm{VaR}(X) = \Lambda^* \mathrm{VaR}(X); \tag{15}$$

(ii) If $\bigvee_{i=1}^n (\lambda_i^+ + \sum_{j \neq i} \lambda_j^-) > 1$, then $\square_{i=1}^n \Lambda_i \mathrm{VaR} = -\infty$;

(iii) If $\sum_{i=1}^n \lambda_i^+ < 1$, then the existence of the optimal allocation is equivalent to the existence of the minimizer of (14). If $\mathbf{x}_{n-1} \in \arg\min_{\mathbf{x}_{n-1} \in \mathbb{R}^{n-1}} \Lambda^{\mathbf{x}_{n-1}} \mathrm{VaR}(X)$, then one optimal allocation is given by (iii) of Theorem 1.

We notice that there is a gap between cases (i) and (ii) in Theorem 4. However, due to the heterogeneity of the Λ functions and the pathological issue caused by the general Λ functions, we currently cannot fill in this gap. It is worth pointing out that the case with the mixture of increasing and decreasing Λ functions is covered by Theorem 4, which represents different type of risk appetites of the agents in the risk sharing problem.

We next consider a special case of Theorem 3: $\rho = \mathrm{EU}$, which is defined as

$$\rho(X) = \mathbb{E}(u(X)), \tag{16}$$

where $u : \mathbb{R} \to \mathbb{R}$ is an increasing function. Note that EU can also be used to quantify the risk if u is a convex function. Note that EU is monotone and law-invariant but it does not satisfy cash-additivity generally. To avoid the integrability issue, we set $\mathcal{X} = L^\infty$ in the following proposition.

Proposition 5. *Suppose $\Lambda \in \mathcal{H}$ with $0 < \lambda^- \leqslant \lambda^+ < 1$ and ρ is an* EU *defined in (16). If $a := \lim_{x \to -\infty} u(x) > -\infty$, then*

$$\Lambda \mathrm{VaR} \,\square\, \rho(X) = \inf_{x \in \mathbb{R}} \left\{ x + a\Lambda(x) + \int_0^{1-\Lambda(x)} u(F_X^{-1}(t) - x) \mathrm{d}t \right\}; \tag{17}$$

Otherwise, $\Lambda \mathrm{VaR} \,\square\, \rho(X) = -\infty$.

Moreover, if a is attainable ($u(x_0) = a$ for some $x_0 \in \mathbb{R}$) and x^* is the minimizer of (17), then one optimal risk allocation is given by

$$X_1 = X - X_{x^*,y},\ \ X_2 = X_{x^*,y}$$

with $y \leqslant x_0$.

Note that the above proposition is also valid on unbounded $\mathcal{X}$ if a is attainable.

Another example is the rank-dependent expected utility (RDEU) (see, e.g., Quiggin (1982) and Quiggin (1993)). We set $\mathcal{X} = L^\infty$ and for $X \in \mathcal{X}$, define

$$\rho(X) = \int_0^\infty g(\mathbb{P}(u(X) > x))\mathrm{d}x + \int_{-\infty}^0 (g(\mathbb{P}(u(X) > x)) - 1)\mathrm{d}x, \tag{18}$$

where u is an increasing and left-continuous function and $g : [0, 1] \to [0, 1]$ is an increasing and left-continuous function satisfying $g(0) = 0$ and $g(1) = 1$. Under the above assumption, RDEU can be rewritten in a Lebesgue–Stieltjes integral form: $\rho(X) = \int_0^1 u(\mathrm{VaR}_t(X))\mathrm{d}g(t)$. Note that REDU is monotone and law-invariant but it is not cash-additive. If u is the identity function, then (18) is reduced to a distortion risk measure denoted by ρ_g, where $\rho_g(X) = \int_0^1 \mathrm{VaR}_t(X)\mathrm{d}g(t)$; see e.g., Yaari (1987) and Föllmer and Schied (2016) for more discussions on the distortion risk measures.

Proposition 6. *Let $\Lambda \in \mathcal{H}$ with $0 < \lambda^- \leqslant \lambda^+ < 1$ and ρ be a* RDEU *defined in (18). Moreover, suppose λ^+ is attainable. If $g(1 - \lambda^+) = 1$, then*

$$\Lambda\mathrm{VaR} \,\square\, \rho(X) = \inf_{x\in\mathbb{R}} \left\{ x + \int_0^{1-\Lambda(x)} u(F_X^{-1}(1 - t - \Lambda(x)) - x)\mathrm{d}g(t)) \right\}; \tag{19}$$

If $g(1 - \lambda^+) < 1$ and $a := \lim_{x\to-\infty} u(x) > -\infty$, then

$$\Lambda\mathrm{VaR} \,\square\, \rho(X) = \inf_{x\in\mathbb{R}} \left\{ x + a(1 - g(1 - \Lambda(x)) + \int_0^{1-\Lambda(x)} u(F_X^{-1}(1 - t - \Lambda(x)) - x)\mathrm{d}g(t)) \right\}; \tag{20}$$

Otherwise, $\Lambda\mathrm{VaR} \,\square\, \rho(X) = -\infty$.

For $g(1 - \lambda^+) = 1$, if x^* is the minimizer of (19), then the optimal risk allocation is given by

$$X_1 = X - X_{x^*,y},\ \ X_2 = X_{x^*,y},\ \ y \in \mathbb{R}; \tag{21}$$

For $g(1 - \lambda^+) < 1$, if a is attainable ($u(x_0) = a$ for some $x_0 \in \mathbb{R}$) and x^* is the minimizer of (20), then the optimal allocation has the form of (21) with $y < x_0 \wedge (\text{ess-inf}X - x^*)$.

It is worth pointing out that Proposition 6 can also be extended to unbounded $\mathcal{X}$. The conclusion in Theorem 3 holds on unbounded $\mathcal{X}$ requiring that ρ is an ε-tail risk measure for some $\varepsilon \in (0,1)$. This is equivalent to the condition that $g(1-\varepsilon)=1$ with some $0<\varepsilon<1$ for RDEU.

5 Inf-convolution of $\Lambda\mathrm{VaR}^+$ and a SSD-consistent risk measure

In this section, we set $L^\infty \subseteq \mathcal{X} \subseteq L^1$. As discussed in Frittelli et al. (2014) and Hitaj et al. (2018), $\Lambda\mathrm{VaR}^+$ is able to capture the tail risk for decreasing Λ. Note that if Λ is increasing, then $\Lambda\mathrm{VaR}^+ = \Lambda\mathrm{VaR}$ in light of Proposition 6 of Bellini and Peri (2022). However, this is not valid for general Λ. The study the inf-convolution of two $\Lambda\mathrm{VaR}^+$ with decreasing Λ is beyond the scope of this paper. We will give a comment on it later. Instead, in this section, we investigate the inf-convolution of $\Lambda\mathrm{VaR}^+$ and another risk measure that is consistent with second-order stochastic dominance(SSD). For two random variables $X, Y \in \mathcal{X}$, we denote $X \leqslant_{\mathrm{icx}} Y$ if $\mathbb{E}(f(X)) \leqslant \mathbb{E}(f(Y))$ for all increasing and convex function f. A mapping $\rho : \mathcal{X} \to \mathbb{R}$ is said to be SSD-consistent if $X \leqslant_{\mathrm{icx}} Y$ implies $\rho(X) \leqslant \rho(Y)$. For any $\Lambda \in \mathcal{H}$, we denote $\Lambda_x(z) = \inf_{x \leqslant t \leqslant z} \Lambda(t),\ z \geqslant x$. Then $\Lambda_x(z)$ is decreasing and right-continuous for $z \geqslant x$. If Λ is decreasing and right-continuous, then $\Lambda_x(z) = \Lambda(z),\ \ z \geqslant x$; and if $\Lambda \in \mathcal{H}_I$, then $\Lambda_x(z) = \Lambda(x),\ \ z \geqslant x$. The following result is valid for general $\Lambda \in \mathcal{H}$.

Theorem 5. *Suppose $\Lambda \in \mathcal{H}$ with $0 < \lambda^- \leqslant \lambda^+ < 1$, and ρ is SSD-consistent. If either (i) $\mathcal{X} = L^\infty$, or (ii) $\mathcal{X}$ is unbounded and ρ is additionally an ε-tail risk measure with $0<\varepsilon<1$, then*

$$\Lambda\mathrm{VaR}^+ \,\square\, \rho(X) = \inf_{x\in\mathbb{R}} \inf_{y \geqslant x \vee x_0} \left\{ x + \rho\left(X - F^{-1}_{x,y}(U_X)\right)\right\}, \tag{22}$$

where $x_0 = 2F_X^{-1}(1-\lambda^-)$ and

$$F_{x,y}(z) = \begin{cases} 0, & z < x \\ 1 - \Lambda_x(z), & x \leqslant z < y \\ F_X(z - y/2), & z \geqslant y \end{cases}.$$

Moreover, the optimal allocation of the inf-convolution exists if and only if the minimizer of (22) exists. If (x^*, y^*) is the minimizer, then one optimal allocation is given by

$$X_1^* = F^{-1}_{x^*,y^*}(U_X),\ \ X_2^* = X - F^{-1}_{x^*,y^*}(U_X). \tag{23}$$

Remark 6. We notice that $F_{x,y}$ is decreasing with respect to y for $y \geqslant x \vee x_0$. Hence $\rho\left(X - F^{-1}_{x,y}(U_X)\right)$

is also decreasing with respect to y. This means

$$\inf_{y\geqslant x\vee x_0}\left\{x+\rho\left(X-F_{x,y}^{-1}(U_X)\right)\right\}=x+\lim_{y\to\infty}\rho\left(X-F_{x,y}^{-1}(U_X)\right). \tag{24}$$

This fact is very helpful to find more explicit expressions of the inf-convolution for some specific ρ.

Remark 7. It is worth mentioning that the inf-convolution in Theorem 5 involves the robust risk aggregation of ρ for two random variables: $\inf_{X\sim F,Y\sim G}\rho(X+Y)$ with two distributions F and G. This is a very difficult problem and we only know some limited results except that ρ is SSD-consistent. For for $\rho=\mathrm{VaR}$, we refer to Wang and Wang (2016), Jakobsons et al. (2016) and Blanchet et al. (2023). Note that the case $\rho=\mathrm{VaR}$ is covered by Theorem 3.

Remark 8. The optimal allocation given in Theorem 5 is very different from the ones in previous sections and in the literature for the inf-convolution of convex risk measures or quantile-based risk measures; see, e.g. Jouini et al. (2008), Filipović and Svindland (2008), Embrechts et al. (2018) and Liu et al. (2022). Using Theorem 5, we have

$$\begin{aligned}
X_1^*&=x^*\mathbb{1}_{\{U_X\leqslant 1-\Lambda(x^*)\}}+(1-\Lambda_{x^*})^{-1}(U_X)\mathbb{1}_{\{1-\Lambda(x^*)<U_X\leqslant 1-\Lambda_{x^*}(y)\}}\\
&\quad+y\mathbb{1}_{\{1-\Lambda_{x^*}(y)<U_X\leqslant F_X(y/2)\}}+(X-y/2)\mathbb{1}_{\{U_X>F_X(y/2)\}},\\
X_2^*&=(X-x^*)\mathbb{1}_{\{U_X\leqslant 1-\Lambda(x^*)\}}+(X-(1-\Lambda_{x^*})^{-1}(U_X))\mathbb{1}_{\{1-\Lambda(x^*)<U_X\leqslant 1-\Lambda_{x^*}(y)\}}\\
&\quad+(X-y)\mathbb{1}_{\{1-\Lambda_{x^*}(y)<U_X\leqslant F_X(y/2)\}}+\frac{y}{2}\mathbb{1}_{\{U_X>F_X(y/2)\}},
\end{aligned}$$

where $y\geqslant y^*$ and $(1-\Lambda_x)^{-1}(t)=\inf\{z\geqslant x:1-\Lambda_x(z)\geqslant t\}$ for $1-\Lambda(x^*)<t<1-\Lambda_{x^*}(\infty)$ with $\Lambda_{x^*}(\infty)=\lim_{y\to\infty}\Lambda_{x^*}(y)$. Over $\{1-\Lambda(x^*)<U_X\leqslant 1-\Lambda_{x^*}(y)\}$ with $y>y^*$, the relation of $(1-\Lambda_{x^*})^{-1}(U_X)$ and $X-(1-\Lambda_{x^*})^{-1}(U_X)$ depends on the relation of F_X and Λ, which is somehow arbitrary. Hence in general X_1^* and X_2^* are neither comonotonic nor possessing the similar structure as in Theorems 2-5 (splitting the probability space).

Remark 9. For the case of $\mathcal{X}=L^\infty$ in Theorem 5, the conclusion still holds if

$$F_{x,y}(z)=\begin{cases}0, & z<x\\ 1-\Lambda_x(z), & x\leqslant z<y\\ 1, & z\geqslant y\end{cases}.$$

We next illustrate our main result using two specific example of ρ: EU and distortion risk measures. Here for simplicity, we suppose $\Lambda\in\mathcal{H}$ is decreasing. Hence $\Lambda_x(z)=\Lambda(z),\ z\geqslant x$. By the application of Theorem 5 and Remark 9, we arrive at the following results.

Proposition 7. *Suppose $\mathcal{X}=L^\infty$, $\Lambda\in\mathcal{H}$ is decreasing with $0<\lambda^-\leqslant\lambda^+<1$ and $\rho(X)=$*

$\mathbb{E}(u(X))$, *where* u *is an increasing and convex function with* $a = \lim_{x\to-\infty} u(x)$*. If* $a > -\infty$*, then*

$$\Lambda\mathrm{VaR}^+ \,\square\, \rho(X) = \inf_{x\in\mathbb{R}} \left\{ x + \int_0^{1-\Lambda(x)} u(F_X^{-1}(t) - x)\mathrm{d}t \right. \\ \left. + \int_{1-\Lambda(x)}^{1-\lambda^-} u(F_X^{-1}(t) - (1-\Lambda)^{-1}(t))\mathrm{d}t \right\} + \inf_{y\in\mathbb{R}} \int_{1-\Lambda(y)}^{1} u(F_X^{-1}(t) - y)\mathrm{d}t;$$

If $a = -\infty$*, then* $\Lambda\mathrm{VaR}^+ \,\square\, \rho(X) = -\infty$*.*

Moreover, if (x^*, y^*) *is the minimizer of above infimum, then the optimal allocation has the form of* (23)*.*

Proposition 8. *Suppose* $\mathcal{X} = L^\infty$*,* $\Lambda \in \mathcal{H}$ *is decreasing with* $0 < \lambda^- \leqslant \lambda^+ < 1$ *and* $\rho(X) = \rho_g(X)$*, where* g *is a concave distortion function. Moreover, we assume* λ^- *is attainable* ($\Lambda(x_1) = \lambda^-$)*. If* $g(1-\lambda^-) = 1$*, then*

$$\Lambda\mathrm{VaR}^+ \,\square\, \rho(X) = \inf_{x\in\mathbb{R}} \left\{ x + \int_0^{1-\lambda^-} K_x^{-1}(1-t)\mathrm{d}g \right\},$$

where $K_x(z) = \lambda^- + F_X(x+z) \wedge (1-\Lambda(x)) + \mathbb{P}(X - (1-\Lambda)^{-1}(U_X) \leqslant z, 1-\Lambda(x) < U_X \leqslant 1-\lambda^-)$*,* $z \in \mathbb{R}$*; Otherwise,* $\Lambda\mathrm{VaR}^+ \,\square\, \rho(X) = -\infty$*. If* x^* *is the minimizer of the above infimum, then the optimal allocation has the form of* (23) *with* (x^*, y^*) *for* $y^* > (\text{ess-sup}X - \text{ess-inf}X + x^* \vee x_1) \vee x_0$*.*

6 Conclusion

In this paper, we consider the inf-convolution of multiple ΛVaR for three different scenarios: all Λ_i are increasing, decreasing, or right-continuous. Moreover, we also consider the inf-convolution of two risk measures: i) ΛVaR and one law-invariant monotone risk measure without cash-additivity; ii) $\Lambda\mathrm{VaR}^+$ and one SSD-consistent risk measure. For all these cases, we obtain the expressions of the inf-convolution and the forms of the optimal allocations.

There are still some unsolved problems such as the inf-convolution of multiple $\Lambda\mathrm{VaR}^+$, and the inf-convolution of multiple ΛVaR under heterogenous beliefs. They deserve future investigation.

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References

Albrecher, H., Beirlant, J. and Teugels, J.L. (2017). Reinsurance: Actuarial and Statistical Aspects. John & Wiley Sons.

Artzner, P., Delbaen, F., Eber, J.-M. and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, **9**(3), 203–228.

Balbás, A., Balbás, B. and Balbás, R. (2023). Lambda-quantiles as fixed points. *SSRN: https://ssrn.com/abstract=4583950*.

Barrieu, P. and El Karoui, N. (2005). Inf-convolution of risk measures and optimal risk transfer. *Finance and Stochastics*, **9**, 269—298.

Bellini, F. and Peri, I. (2022). An axiomatization of Λ-quantiles. *SIAM Journal on Financial Mathematics*, **13**(1), 26–38.

Bignozzi, V., Burzoni, M. and Munari, C. (2020). Risk measures based on benchmark loss distributions. *The Journal of Risk and Insurance*, **87**(2), 437–475.

Blanchet, J., Lam, H., Liu, Y. and Wang, R. (2023). Convolution bounds on quantile aggregation. *arXiv:* 2007.09320.

Burzoni, M., Peri, I. and Ruffo, C.M. (2017). On the properties of the Lambda value at risk: robustness, elicitability and consistency. *Quantitative Finance*, **17**, 1735–1743.

Cai, J. and Chi, Y. (2020). Optimal reinsurance designs based on risk measures: a review. *Statistical Theory and Related Fields*, **4**(1), 1–13.

Corbetta, J. and Peri, I. (2018). Backtesting Lambda value at risk. *The European Journal of Finance*, **24**, 1075–1087.

Delbaen, F. (2012). *Monetary Utility Functions.* Osaka University Press, Osaka.

Embrechts, P. (2017). A Darwinian view on internal models. *Journal of Risk*, **20**(1), 1–21.

Embrechts, P., Liu, H. and Wang, R. (2018). Quantile-based risk sharing. *Operations Research*, **66**(4), 936–949.

Embrechts, P., Liu, H., Mao, T. and Wang, R. (2020). Quantile-based risk sharing with heterogeneous beliefs. *Mathematical Programming Series B*, **181**(2), 319-347.

Filipović, D. and Svindland, G. (2008). Optimal capital and risk allocations for law- and cash-invariant convex functions. *Finance and Stochastics*, **12**, 423–439.

Föllmer, H. and Schied, A. (2002). *Convex measures of risk and trading constraints. Finance and Stochastics*, **6**(4) 429—447.

Föllmer, H. and Schied, A. (2016). *Stochastic Finance. An Introduction in Discrete Time.* Walter de Gruyter, Berlin, Fourth Edition.

Frittelli, M. and Rosazza Gianin, E. (2005). Law-invariant convex risk measures. *Advances in*

Mathematical Economics, **7**, 33—46.

Frittelli, M., Maggis, M., and Peri, I. (2014). Risk Measures on $\mathcal{P}(\mathbb{R})$ and Value at Risk with Probability/Loss function. *Mathematical Finance*, **24**(3), 442–463.

Han, X., Wang, Q., Wang, R. and Xia, J. (2021). Cash-subadditive risk measures without quasi-convexity. *arXiv: 2110.12198.*

Hitaj, A., Mateus, C. and Peri, I. (2018). Lambda value at risk and regulatory capital: a dynamic approach to tail risk. *Risks*, **6**, 17.

Jakobsons, E., Han, X. and Wang, R. (2016). General convex order on risk aggregation. *Scandinavian Actuarial Journal*, **2016**(8), 713–740.

Jouini, E., Schachermayer, W. and Touzi, N. (2008). Optimal risk sharing for law invariant monetary utility functions. *Mathematical Finance*, **18**(2), 269–292.

Kusuoka, S. (2001). On law invariant coherent risk measures. *Advances in Mathematical Economics*, **3**, 83–95.

Liu, F. and Wang, R. (2021). A theory for measures of tail risk. *Mathematics of Operations Research*, **46**(3), 1109–1128.

Liu, F., Mao, T., Wang, R. and Wei, L. (2022). Inf-convolution, optimal allocations, and model uncertainty for tail risk measures. *Mathematics of Operations Research*, **47**(3), 2494–2519.

Liu, P., Wang, R. and Wei, L. (2020). Is the inf-convolution of law-invariant preferences law-invariant? *Insurance: Mathematics and Economics*, **91**, 144–154.

McNeil, A. J., Frey, R. and Embrechts, P. (2015). *Quantitative Risk Management: Concepts, Techniques and Tools.* Revised Edition. Princeton, NJ: Princeton University Press.

Ince, A., Peri, I. and Pesenti, S. (2022). Risk contributions of lambda quantiles. *Quantitative Finance*, **22**(10), 1871–1891.

Quiggin, J. (1982). A theory of anticipated utility. *Journal of Economic Behavior & Organization*, **3**(4), 323–343.

Quiggin, J. (1993). *Generalized Expected Utility Theory: The Rank-Dependent Model.* Kluwer, the Netherlands.

Rüschendorf, L. (2013). *Mathematical Risk Analysis. Dependence, Risk Bounds, Optimal Allocations and Portfolios.* Springer, Heidelberg.

Wang, B. and Wang, R. (2016). Joint mixability. *Mathematics of Operations Research*, **41**(3), 808–826.

Wang, R. and Wei, Y. (2020). Characterizing optimal allocations in quantile-based risk sharing. *Insurance: Mathmatics and Economics*, **93**, 288–300.

Weber, S. (2018). Solvency II, or how to sweep the downside risk under the carpet. *Insurance:*

Mathematics and Economics, **82**, 191–200.

Yaari, M. E. (1987). The dual theory of choice under risk. *Econometrica*, **55**(1), 95–115.

A Proof of Proposition 1

Proof of Proposition 1. The "only if" part follows directly from the property of VaR. We next show the "if" part. We first focus on the case $\rho = \Lambda\mathrm{VaR}$ for some function $\Lambda : \mathbb{R} \to (0,1)$. We will consider two scenarios separately. First, suppose Λ is an increasing function. We assume by contradiction that there exist $x_1 < x_2$ such that $\Lambda(x_1) < \Lambda(x_2)$. Let $X \in \mathcal{X}$ with the following distribution

$$F_X(x) = \frac{2 - \Lambda(x_1) - \Lambda(x_2)}{2} \mathbb{1}_{\{x_2 \leqslant x < x_2 + a\}} + \mathbb{1}_{\{x \geqslant x_2 + a\}}, \ x \in \mathbb{R}$$

for some $a > 0$. It follows that $\rho(X) = x_2$. Note that

$$F_{X-(x_2-x_1+a)}(x) = \frac{2 - \Lambda(x_1) - \Lambda(x_2)}{2} \mathbb{1}_{\{x_1 - a \leqslant x < x_1\}} + \mathbb{1}_{\{x \geqslant x_1\}}, \ x \in \mathbb{R}.$$

Hence, by (4), we have $\rho(X - (x_2 - x_1 + a)) = x_1$. However, cash-additivity implies $\rho(X - (x_2 - x_1 + a)) = \rho(X) - (x_2 - x_1 + a) = x_1 - a$, leading to a contradiction. Hence, Λ is a constant on $\mathbb{R}$.

If Λ is a decreasing function, suppose by contradiction that there exist $x_1 < x_2$ such that $\Lambda(x_1) > \Lambda(x_2)$. Let $X \in \mathcal{X}$ with the following distribution

$$F_X(x) = (1 - \Lambda(x_1)) \mathbb{1}_{\{x_1 - 1 \leqslant x < x_1\}} + \mathbb{1}_{\{x \geqslant x_1\}}, \ x \in \mathbb{R}.$$

Direct calculation gives $\rho(X) = x_1 - 1$. Moreover, by definition, we have $\rho(X + x_2 - x_1 + 1) = x_2 + 1$. Using cash-additivity, it follows that $\rho(X + x_2 - x_1 + 1) = \rho(X) + x_2 - x_1 + 1 = x_2$, which yields a contradiction. Hence Λ is a constant.

For $\rho = \Lambda\mathrm{VaR}^+$, we could similarly show that cash-additivity implies that Λ is a constant. This completes the proof. □

B Proof of Section 3.1

In this section, we display all the proofs of the results from Section 3.1.

Proof of Theorem 1. The proof of (8) is a direct application of Proposition 9, which will be given later. Now, we consider all the other conclusions. We first consider cases (i) and (ii). Let us start with the case $n = 2$. For any $X_1 \in \mathcal{X}$, we denote $x_1 = \Lambda_1\mathrm{VaR}(X_1)$. For some $m \in \mathbb{R}$, let

$$X_2 = x_1 \mathbb{1}_{\{U_{X_1} \leqslant 1-\Lambda_1(x_1)\}} + (x_1 \vee X_1 \vee (X+m))\, \mathbb{1}_{\{U_{X_1} > 1-\Lambda_1(x_1)\}}.$$

By the definition of $\Lambda\mathrm{VaR}$, we have $F_{X_1}(x_1) \geqslant 1 - \Lambda_1(x_1)$, which implies $x_1 \geqslant F_{X_1}^{-1}(1 - \Lambda_1(x_1))$. Hence we have $\Lambda_1\mathrm{VaR}(X_2) = x_1$ and $X_1 \leqslant X_2$. By monotonicity of $\Lambda_2\mathrm{VaR}$, we have

$$\Lambda_1\mathrm{VaR}(X_1) + \Lambda_2\mathrm{VaR}(X - X_1) \geqslant \Lambda_1\mathrm{VaR}(X_2) + \Lambda_2\mathrm{VaR}(X - X_2).$$

Observe that

$$X - X_2 = (X - x_1)\mathbb{1}_{\{U_{X_1} \leqslant 1-\Lambda_1(x_1)\}} + ((X - x_1) \wedge (X - X_1) \wedge (-m))\, \mathbb{1}_{\{U_{X_1} > 1-\Lambda_1(x_1)\}}.$$

Direct computation shows $\mathbb{P}(X - X_2 \leqslant -m) \geqslant \Lambda_1(x_1)$. If $\lambda_1^+ + \lambda_2^+ > 1$, there exists $x_1 \in \mathbb{R}$ such that $\Lambda_1(x_1) > 1 - \lambda_2^+$. This implies that $\mathbb{P}(X - X_2 \leqslant -m) \geqslant 1 - \Lambda_2(-m)$ if $m > m_0$ for some $m_0 \in \mathbb{R}$. Hence $\Lambda_2\mathrm{VaR}(X - X_2) \leqslant -m$. Consequently, $\lim_{m\to\infty}(\Lambda_1\mathrm{VaR}(X_2) + \Lambda_2\mathrm{VaR}(X - X_2)) \leqslant \lim_{m\to\infty}(x_1 - m) = -\infty$. This implies $\Lambda_1\mathrm{VaR} \,\square\, \Lambda_2\mathrm{VaR}(X) = -\infty$.

We next consider the case $\lambda_1^+ + \lambda_2^+ < 1$. Let

$$X_3 = x_1 \mathbb{1}_{\{U_X \leqslant 1-\Lambda_1(x_1)\}} + (X+m)\mathbb{1}_{\{U_X > 1-\Lambda_1(x_1)\}}, \tag{25}$$

where $m > x_1 - F_X^{-1}(1 - \Lambda_1(x_1))$. Using the fact that $\mathbb{P}(X \leqslant x + x_1, U_{X_1} \leqslant 1 - \Lambda_1(x_1)) \leqslant \mathbb{P}(X \leqslant x + x_1, U_X \leqslant 1 - \Lambda_1(x_1))$, we have for $x \geqslant -m$,

$$\begin{aligned}\mathbb{P}(X - X_2 \leqslant x) &= \Lambda_1(x_1) + \mathbb{P}(X \leqslant x + x_1, U_{X_1} \leqslant 1 - \Lambda_1(x_1)) \\ &\leqslant \Lambda_1(x_1) + \mathbb{P}(X \leqslant x + x_1, U_X \leqslant 1 - \Lambda_1(x_1)) = \mathbb{P}(X - X_3 \leqslant x).\end{aligned}$$

We set

$$m > x_1 - F_X^{-1}(1 - \lambda_1^+ - \lambda_2^+). \tag{26}$$

For $x < -m$, it follows that

$$\begin{aligned}&\mathbb{P}(X - X_2 \leqslant x) \leqslant F_X(x + x_1) \wedge (1 - \Lambda_1(x_1)) + \Lambda_1(x_1) < 1 - \Lambda_2(x), \\ &\mathbb{P}(X - X_3 \leqslant x) = F_X(x + x_1) \wedge (1 - \Lambda_1(x_1)) < 1 - \Lambda_2(x).\end{aligned}$$

This indicates that $\Lambda_2\mathrm{VaR}(X - X_2) \geqslant -m$ and $\Lambda_2\mathrm{VaR}(X - X_3) \geqslant -m$. Hence we have $\Lambda_2\mathrm{VaR}(X -$

$X_3) \leqslant \Lambda_2\mathrm{VaR}(X - X_2)$. Direct computation yields $\mathbb{P}(X - X_3 \leqslant x) = (\Lambda_1(x_1) + F_X(x + x_1)) \wedge 1$ for $x \geqslant -m$. Note also that $\Lambda_1(x_1) + F_X(x + x_1) < 1 - \Lambda_2(x)$ for $x < -m$, and $\Lambda_2\mathrm{VaR}(X_3) = \Lambda_2\mathrm{VaR}(X_2) = x_1$. Consequently, we have

$$\begin{aligned}
\Lambda_1\mathrm{VaR}(X_3) + \Lambda_2\mathrm{VaR}(X - X_3) &= x_1 + \inf\{x : \mathbb{P}(X - X_3 \leqslant x) \geqslant 1 - \Lambda_2(x)\} \\
&= x_1 + \inf\{x : \Lambda_1(x_1) + F_X(x + x_1) \geqslant 1 - \Lambda_2(x)\} \\
&= x_1 + \inf\{y - x_1 : F_X(y) \geqslant 1 - \Lambda_2(y - x_1) - \Lambda_1(x_1)\} \\
&= \inf\{y : F_X(y) \geqslant 1 - \Lambda_2(y - x_1) - \Lambda_1(x_1)\} = \Lambda^{x_1}\mathrm{VaR}(X),
\end{aligned}$$

where the last three equalities are actually the so-called Λ cash-additivity in Frittelli et al. (2014). Therefore, for any $X_1 \in \mathcal{X}$ with $x_1 = \Lambda_1\mathrm{VaR}(X_1)$, we have

$$\Lambda_1\mathrm{VaR}(X_1) + \Lambda_2\mathrm{VaR}(X - X_1) \geqslant \Lambda^{x_1}\mathrm{VaR}(X) \geqslant \inf_{x_1\in\mathbb{R}} \Lambda^{x_1}\mathrm{VaR}(X).$$

This implies $\Lambda_1\mathrm{VaR} \,\square\, \Lambda_2\mathrm{VaR}(X) \geqslant \inf_{x_1\in\mathbb{R}} \Lambda^{x_1}\mathrm{VaR}(X)$. Moreover, for any $x_1 \in \mathbb{R}$, we could construct X_3 as in (25) such that $\Lambda_1\mathrm{VaR}(X_3) + \Lambda_2\mathrm{VaR}(X - X_3) = \Lambda^{x_1}\mathrm{VaR}(X)$. Hence

$$\Lambda_1\mathrm{VaR} \,\square\, \Lambda_2\mathrm{VaR}(X) \leqslant \inf_{x_1\in\mathbb{R}} \Lambda^{x_1}\mathrm{VaR}(X).$$

We establish the claim for $n = 2$.

We next prove the result for $n \geqslant 3$ by induction. Suppose the conclusion holds true for $n \leqslant k$, where $k \geqslant 2$. We next show that the conclusion is also correct for $n = k + 1$. For $n = k + 1$, we first consider the case $\sum_{i=1}^{k+1} \lambda_i^+ < 1$. Using Lemma 2 in Liu et al. (2020) and the conclusion for $n \leqslant k$, we have

$$\begin{aligned}
\mathop{\square}_{i=1}^{k+1} \Lambda_i\mathrm{VaR}(X) &= \left(\mathop{\square}_{i=1}^{k} \Lambda_i\mathrm{VaR}\right) \square\, \Lambda_{k+1}\mathrm{VaR}(X) \\
&= \left(\inf_{\mathbf{x}_{k-1}\in\mathbb{R}^{k-1}} \Lambda^{\mathbf{x}_{k-1}}\mathrm{VaR}\right) \square\, \Lambda_{k+1}\mathrm{VaR}(X) \\
&= \inf\left\{Y \in \mathcal{X} : \inf_{\mathbf{x}_{k-1}\in\mathbb{R}^{k-1}} \Lambda^{\mathbf{x}_{k-1}}\mathrm{VaR}(Y) + \Lambda_{k+1}\mathrm{VaR}(X - Y)\right\} \\
&= \inf_{\mathbf{x}_{k-1}\in\mathbb{R}^{k-1}} \inf\left\{Y \in \mathcal{X} : \Lambda^{\mathbf{x}_{k-1}}\mathrm{VaR}(Y) + \Lambda_{k+1}\mathrm{VaR}(X - Y)\right\} \\
&= \inf_{\mathbf{x}_k\in\mathbb{R}^k} \Lambda^{\mathbf{x}_k}\mathrm{VaR}(X).
\end{aligned}$$

Next, we consider the case $\sum_{i=1}^{k+1} \lambda_i^+ > 1$. If $\sum_{i=1}^{k} \lambda_i^+ > 1$, then by the assumption, we have $\square_{i=1}^{k} \Lambda_i\mathrm{VaR}(X) = -\infty$. It follows from Lemma 2 in Liu et al. (2020) that $\square_{i=1}^{k+1} \Lambda_i\mathrm{VaR}(X) = -\infty$. If $\sum_{i=1}^{k} \lambda_i^+ \leqslant 1$, for $i = 1, \dots, k$, let $\tilde{\Lambda}_i = \Lambda_i - \varepsilon$ with $\varepsilon < \frac{\min(\wedge_{j=1}^{k+1}\lambda_j^-, \sum_{j=1}^{k+1}\lambda_j^+ - 1)}{k+1}$. Using monotonicity of $\Lambda\mathrm{VaR}$ with respect to Λ, we have

$$\mathop{\square}_{i=1}^{k+1} \Lambda_i \mathrm{VaR}(X) \leqslant \left(\mathop{\square}_{i=1}^{k} \tilde{\Lambda}_i \mathrm{VaR}\right) \square\, \Lambda_{k+1}\mathrm{VaR}(X).$$

It follows from the fact that $\sum_{i=1}^k \lambda_i^+ - k\varepsilon < 1$ and the conclusion for $n = k$ that $\square_{i=1}^k \tilde{\Lambda}_i \mathrm{VaR}(X) = \inf_{\mathbf{x}_{k-1}\in\mathbb{R}^{k-1}} \tilde{\Lambda}^{\mathbf{x}_{k-1}}\mathrm{VaR}(X)$, where $\tilde{\Lambda}^{\mathbf{x}_{k-1}}(x) = \left(\tilde{\Lambda}_k(x - x_{k-1}) + \sum_{i=1}^{k-1} \tilde{\Lambda}_i(x_i - x_{i-1})\right)$. Hence, for any $\mathbf{x}_{k-1} \in \mathbb{R}^{k-1}$,

$$\mathop{\square}_{i=1}^{k+1} \Lambda_i \mathrm{VaR}(X) \leqslant \tilde{\Lambda}^{\mathbf{x}_{k-1}}\mathrm{VaR}\,\square\, \Lambda_{k+1}\mathrm{VaR}(X).$$

We could choose $\mathbf{x}_{k-1} \in \mathbb{R}^{k-1}$ such that $\sup_{x\in\mathbb{R}} \tilde{\Lambda}^{\mathbf{x}_{k-1}}(x) \geqslant \sum_{i=1}^k \lambda_i^+ - (k+1)\varepsilon$, which indicates $\lambda_{k+1}^+ + \sup_{x\in\mathbb{R}} \tilde{\Lambda}^{\mathbf{x}_{k-1}}(x) \geqslant \sum_{i=1}^{k+1} \lambda_i^+ - (k+1)\varepsilon > 1$. The conclusion for $n = 2$ above implies $\tilde{\Lambda}^{\mathbf{x}_{k-1}}\mathrm{VaR}\,\square\,\Lambda_{k+1}\mathrm{VaR}(X) = -\infty$, which further indicates $\square_{i=1}^{k+1} \Lambda_i\mathrm{VaR}(X) = -\infty$. We establish the claim for $n \geqslant 3$.

Finally, we focus on (iii). Suppose the optimal allocation of the inf-convolution is $(X_1, \dots, X_n)$, i.e., $\sum_{i=1}^n X_i = X$ and $\square_{i=1}^n \Lambda_i\mathrm{VaR}(X) = \sum_{i=1}^n \Lambda_i\mathrm{VaR}(X_i)$. Then we have $\square_{i=1}^k \Lambda_i\mathrm{VaR}(\sum_{i=1}^k X_i) = \sum_{i=1}^k \Lambda_i\mathrm{VaR}(X_i)$ for all $k = 2, \dots, n$. Using the above argument for $n = 2$, we could find $x_1 \in \mathbb{R}$ such that $\Lambda_1\mathrm{VaR}(X_1) + \Lambda_2\mathrm{VaR}(X_2) \geqslant \Lambda^{\mathbf{x}_1}\mathrm{VaR}(X_1 + X_2)$. Using the conclusion in (i), $\Lambda_1\mathrm{VaR}\,\square\,\Lambda_2\mathrm{VaR}(X_1 + X_2) = \inf_{\mathbf{y}_1\in\mathbb{R}} \Lambda^{\mathbf{y}_1}\mathrm{VaR}(X_1 + X_2) \leqslant \Lambda^{\mathbf{x}_1}\mathrm{VaR}(X_1 + X_2)$. Consequently, $\Lambda_1\mathrm{VaR}\,\square\,\Lambda_2\mathrm{VaR}(X_1+X_2) = \Lambda^{\mathbf{x}_1}\mathrm{VaR}(X_1+X_2)$. Using the above argument for $n = 2$, we could find $x_2 \in \mathbb{R}$ such that $\Lambda^{\mathbf{x}_1}\mathrm{VaR}(X_1+X_2)+\Lambda_3\mathrm{VaR}(X_3) \geqslant \Lambda^{\mathbf{x}_2}\mathrm{VaR}(X_1+X_2+X_3)$. Using the conclusion in (i), we have $\square_{i=1}^3 \Lambda_i\mathrm{VaR}(X_1+X_2+X_3) = \inf_{\mathbf{y}_2\in\mathbb{R}^2} \Lambda^{\mathbf{y}_2}\mathrm{VaR}(X_1+X_2+X_3) \leqslant \Lambda^{\mathbf{x}_2}\mathrm{VaR}(X_1+X_2+X_3)$. Hence we have $\square_{i=1}^3 \Lambda_i\mathrm{VaR}(X_1 + X_2 + X_3) = \Lambda^{\mathbf{x}_2}\mathrm{VaR}(X_1 + X_2 + X_3)$. We continue this process and finally we could find $\mathbf{x}_{n-1} \in \mathbb{R}^{n-1}$ such that $\square_{i=1}^n \Lambda_i\mathrm{VaR}(X) = \Lambda^{\mathbf{x}_{n-1}}\mathrm{VaR}(X)$. This implies that $\mathbf{x}_{n-1}$ is the minimizer of (7).

Suppose $\mathbf{x}_{n-1}$ is the minimizer of (7), i.e., $\Lambda^{\mathbf{x}_{n-1}}\mathrm{VaR}(X) = \square_{i=1}^n \Lambda_i\mathrm{VaR}(X)$. Let $Y_n = X$ and for $k = n-1, \dots, 1$,

$$Y_k = x_k \mathbb{1}_{\{U_X \leqslant 1-\Lambda^{\mathbf{x}_{k-1}}(x_k)\}} + (Y_{k+1} + m_k)\mathbb{1}_{\{U_X > 1-\Lambda^{\mathbf{x}_{k-1}}(x_k)\}},$$

with m_k satisfying $m_k > x_k - F_X^{-1}(1 - \sum_{i=1}^n \lambda_i^+)$. Note that U_X and Y_k are comonotonic for $k = n, \dots, 1$. Hence, direct calculation yields, for $k = 2, \dots, n$,

$$\Lambda^{\mathbf{x}_{n-k}}\mathrm{VaR}(Y_{n-k+1}) + \Lambda_{n-k+2}\mathrm{VaR}(Y_{n-k+2} - Y_{n-k+1}) = \Lambda^{\mathbf{x}_{n-k+1}}\mathrm{VaR}(Y_{n-k+2}).$$

Adding up both sides of the above equalities, we have

$$\Lambda_1(Y_1) + \sum_{i=2}^{n} \Lambda_i \mathrm{VaR}(Y_i - Y_{i-1}) = \Lambda^{\mathbf{x}_{n-1}} \mathrm{VaR}(X) = \mathop{\square}_{i=1}^{n} \Lambda_i \mathrm{VaR}(X).$$

This implies the optimal allocation is

$$X_n = Y_n - Y_{n-1}, X_{n-1} = Y_{n-1} - Y_{n-2}, \ldots, X_1 = Y_1.$$

We can obtain the expression of the optimal allocation by noting that for $k = n-1, \ldots, 1$,

$$Y_k = x_k \mathbb{1}_{\{U_X \leqslant 1 - \Lambda^{\mathbf{x}_{k-1}}(x_k)\}} + \left(X + \sum_{i=k}^{n-1} m_i \right) \mathbb{1}_{\{U_X > 1 - \Lambda^{\mathbf{x}_{k-1}}(x_k)\}}.$$

This completes the proof. $\square$

The next proposition shows the validity of (8) in Theorem 1.

Proposition 9. *For $\Lambda_i \in \mathcal{H}_D$ with $0 < \lambda_i^- \leqslant \lambda_i^+ < 1$, we have*

$$\inf_{\mathbf{x}_{n-1} \in \mathbb{R}^{n-1}} \Lambda^{\mathbf{x}_{n-1}} \mathrm{VaR}(X) \geqslant \Lambda^* \mathrm{VaR}(X).$$

Moreover, the above inequality holds as an equality if all $\lambda_i^\pm$ are attainable.

Proof. First, note that Λ^* is a decreasing function. Using the monotonicity of $\Lambda\mathrm{VaR}$ with respect to Λ, we have $\Lambda^{\mathbf{x}_{n-1}} \mathrm{VaR}(X) \geqslant \Lambda^* \mathrm{VaR}(X)$, which implies

$$\inf_{\mathbf{x}_{n-1} \in \mathbb{R}^{n-1}} \Lambda^{\mathbf{x}_{n-1}} \mathrm{VaR}(X) \geqslant \Lambda^* \mathrm{VaR}(X).$$

Next, we show the equality holds under the assumption that all $\lambda_i^\pm$ are attainable. By this assumption, there exist x_i^- and x_i^+ such that $\Lambda_i(x_i^-) = \lambda_i^-$ and $\Lambda_i(x_i^+) = \lambda_i^+$. Note that $\Lambda^*(x) = \sup_{\mathbf{x}_{n-1} \in \mathbb{R}^{n-1}} \Lambda^{\mathbf{x}_{n-1}}(x) = \sup_{y_1 + \cdots + y_n = x} (\sum_{i=1}^n \Lambda_i(y_i)) \wedge 1$. We next prove the existence of the maximizer $\mathbf{x}_{n-1}$ or equivalently $(y_1, \ldots, y_n)$. By contradiction, we suppose the maximizer does not exist. Then there exists a sequence $(y_1^{(k)}, \ldots, y_n^{(k)}),\ k \geqslant 1$ such that $y_1^{(k)} + \cdots + y_n^{(k)} = x$ and $\sum_{i=1}^n \Lambda_i(y_i^{(k)}) \uparrow \Lambda^*(x)$. By choosing a subsequence, we could force $y_i^{(k)}$ to be monotone with respect to k. Let $A_1 = \{i \in \{1, \ldots, n\} : \lim_{k \to \infty} y_i^{(k)} = \infty\}$, $A_2 = \{i \in \{1, \ldots, n\} : \lim_{k \to \infty} y_i^{(k)} = -\infty\}$ and $A_3 = \{i \in \{1, \ldots, n\} : \lim_{k \to \infty} y_i^{(k)} \in (-\infty, \infty)\}$. Note that $A_1 = \emptyset$ is equivalent to $A_2 = \emptyset$. Now, we suppose $A_1 \neq \emptyset$. Then $A_2 \neq \emptyset$ and we have $\sum_{i \in A_1} y_i^{(k)} + \sum_{i \in A_2} y_i^{(k)} = x - \sum_{i \in A_3} y_i^{(k)}$. Let $\tilde{y}_i^{(k)} = x_i^+ + M_{k,1}$ for $i \in A_1$, $\tilde{y}_i^{(k)} = x_i^- - M_{k,2}$ for $i \in A_2$, and $\tilde{y}_i^{(k)} = y_i^{(k)}$ for $i \in A_3$. We can choose $M_{k,1} > 0$ and $M_{k,2} > 0$ such that

$$\sum_{i\in A_1\cup A_2} \tilde{y}_i^{(k)} = \sum_{i\in A_1} x_i^+ + \sum_{i\in A_2} x_i^- + n_1 M_{k,1} - n_2 M_{k,2} = x - \sum_{i\in A_3} y_i^{(k)},$$

where $n_1 = Card(A_1)$ and $n_2 = Card(A_2)$. Note that $M_{k,1}$ and $M_{k,2}$ can be chosen to be bounded, indicating that $(\tilde{y}_1^{(k)}, \dots, \tilde{y}_n^{(k)})$ is a bounded sequence. Moreover, $\sum_{i=1}^n \Lambda_i(\tilde{y}_i^{(k)}) \uparrow \Lambda^*(x)$, and we force $\tilde{y}_i^{(k)}$ to be monotone by picking a subsequence. We denote $y_i^* = \lim_{k\to\infty} \tilde{y}_i^{(k)}$. If $\tilde{y}_i^{(k)} \leqslant y_i^*$ for all $k \geqslant 1$, by the left-continuity of Λ_i, we have $\lim_{k\to\infty} \Lambda_i(\tilde{y}_i^{(k)}) = \Lambda_i(y_i^*)$; If $\tilde{y}_i^{(k)} \geqslant y_i^*$ for all $k \geqslant 1$, using the left-continuity and monotonicity of Λ_i, we have $\lim_{k\to\infty} \Lambda_i(\tilde{y}_i^{(k)}) \leqslant \Lambda_i(y_i^*)$.

Then we have $\Lambda^*(x) \leqslant (\sum_{i=1}^n \Lambda_i(y_i^*)) \wedge 1$, implying $\Lambda^*(x) = (\sum_{i=1}^n \Lambda_i(y_i^*)) \wedge 1$. Denoting $x_i = \sum_{j=1}^i y_j^*$, we have $\Lambda^{\mathbf{x}_{n-1}}(x) = \Lambda^*(x)$. This shows the existence of the maximizer.

For simplicity, we denote $x^* = \Lambda^*\mathrm{VaR}(X)$. If $x^* = -\infty$, then there exists a sequence $y_k \to -\infty$ as $k \to \infty$ such that $F_X(y_k) \geqslant 1 - \Lambda^*(y_k)$. Using the previous conclusion, there exists a sequence $\mathbf{x}_{n-1}^{(k)}$ such that $\Lambda^{\mathbf{x}_{n-1}^{(k)}}(y_k) = \Lambda^*(y_k)$. This implies that $F_X(y_k) \geqslant 1 - \Lambda^{\mathbf{x}_{n-1}^{(k)}}(y_k)$. Hence we have $\inf_{\mathbf{x}_{n-1}\in\mathbb{R}^{n-1}} \Lambda^{\mathbf{x}_{n-1}}\mathrm{VaR}(X) \leqslant \Lambda^{\mathbf{x}_{n-1}^{(k)}}\mathrm{VaR}(X) \leqslant y_k \to \infty$ as $k \to \infty$.

Next, we consider the case $x^* > -\infty$. By definition, $F_X(x^*) \geqslant 1 - \Lambda^*(x^*)$ or there exist $y_k \downarrow x^*$ such that $F_X(y_k) \geqslant 1 - \Lambda^*(y_k)$. If $F_X(x^*) \geqslant 1 - \Lambda^*(x^*)$, using the previous conclusion, there exists $\mathbf{x}_{n-1} \in \mathbb{R}^{n-1}$ such that $\Lambda^{\mathbf{x}_{n-1}}(x^*) = \Lambda^*(x^*)$. Hence, we have $F_X(x^*) \geqslant 1 - \Lambda^{\mathbf{x}_{n-1}}(x^*)$. This implies $\Lambda^{\mathbf{x}_{n-1}}\mathrm{VaR}(X) \leqslant x^*$. Hence, $\inf_{\mathbf{x}_{n-1}\in\mathbb{R}^{n-1}} \Lambda^{\mathbf{x}_{n-1}}\mathrm{VaR}(X) \leqslant x^*$.

If there exist $y_k \downarrow x^*$ such that $F_X(y_k) \geqslant 1 - \Lambda^*(y_k)$, using the previous conclusion, there exists a sequence of $\mathbf{x}_{n-1}^{(k)} \in \mathbb{R}^{n-1}$ such that $\Lambda^{\mathbf{x}_{n-1}^{(k)}}(y_k) = \Lambda^*(y_k)$. Hence, we have $F_X(y_k) \geqslant 1 - \Lambda^{\mathbf{x}_{n-1}^{(k)}}(y_k)$. This implies $\Lambda^{\mathbf{x}_{n-1}^{(k)}}\mathrm{VaR}(X) \leqslant y_k$. Hence $\inf_{\mathbf{x}_{n-1}\in\mathbb{R}^{n-1}} \Lambda^{\mathbf{x}_{n-1}}\mathrm{VaR}(X) \leqslant y_k \to x^*$ as $k \to \infty$. We establish the claim. $\square$

Proof of Proposition 2. Using $\sum_{i=1}^{n-1} \lambda_i^+ < 1$, Lemma 2 in Liu et al. (2020) and the expression in Theorem 1, we have

$$\left(\mathop{\square}_{i=1}^{n-1} \Lambda_i\mathrm{VaR}\right) \square\, \Lambda_n\mathrm{VaR}(X) = \inf_{\mathbf{x}_{n-2}\in\mathbb{R}^{n-2}} \left(\Lambda^{\mathbf{x}_{n-2}}\mathrm{VaR} \,\square\, \Lambda_n\mathrm{VaR}(X)\right).$$

We first show case (i). Note that $\Lambda^{\mathbf{x}_{n-2}}(x) = \Lambda_1(x_1) + \Lambda_2(x_2 - x_1) + \cdots + \Lambda_{n-1}(x - x_{n-2})$. We can choose $\mathbf{x}_{n-2} \in \mathbb{R}^{n-2}$ such that $\Lambda^{\mathbf{x}_{n-2}}(x) = \sum_{i=1}^{n-2} \lambda_i^+ + \Lambda_{n-1}(x - x_{n-2})$. We fix $\mathbf{x}_{n-2}$. Then there exists $x_{n-1} \in \mathbb{R}$ such that $\Lambda^{\mathbf{x}_{n-2}}(x_{n-1}) = \sum_{i=1}^{n-1} \lambda_i^+$. Moreover, let $x_n \in \mathbb{R}$ such that $\Lambda_n(x_n) = \lambda_n^+$. We define X_1 as

$$X_1 = x_{n-1}\mathbb{1}_{\{U_X \leqslant 1-\Lambda^{\mathbf{x}_{n-2}}(x_{n-1})\}} + (X+m)\mathbb{1}_{\{U_X > 1-\Lambda^{\mathbf{x}_{n-2}}(x_{n-1})\}}.$$

It follows that

$$\begin{aligned}\mathbb{P}(X - X_1 \leqslant -m) &= \Lambda^{\mathbf{x}_{n-2}}(x_{n-1}) + \mathbb{P}(X \leqslant -m + x_{n-1}, U_X \leqslant 1 - \Lambda^{\mathbf{x}_{n-2}}(x_{n-1})) \\ &\geqslant \sum_{i=1}^{n-1} \lambda_i^+ = 1 - \Lambda_n(-m),\end{aligned}$$

implying $\Lambda_n \mathrm{VaR}(X - X_1) \leqslant -m$. Hence we have

$$\begin{aligned}\Lambda^{\mathbf{x}_{n-2}}\mathrm{VaR} \,\square\, \Lambda_n \mathrm{VaR}(X) &\leqslant \Lambda^{\mathbf{x}_{n-2}}\mathrm{VaR}(X_1) + \Lambda_n \mathrm{VaR}(X - X_1) \\ &\leqslant x_{n-1} + \Lambda_n \mathrm{VaR}(X - X_1) \leqslant x_{n-1} - m \to -\infty\end{aligned}$$

as $m \to \infty$, implying $\left(\square_{i=1}^{n-1} \Lambda_i \mathrm{VaR}\right) \square\, \Lambda_n \mathrm{VaR}(X) = -\infty$.

We next consider case (ii). We denote $\lambda' = \sup_{x\in\mathbb{R}} \Lambda^{\mathbf{x}_{n-2}}(x)$. Our assumption implies $\lambda' + \lambda_n^+ \leqslant 1$. For any $X, X_1 \in \mathcal{X}$, we denote $y_1 = \Lambda^{\mathbf{x}_{n-2}}\mathrm{VaR}(X_1)$ and let

$$X_2 = y_1 \mathbb{1}_{\{U_{X_1} \leqslant 1 - \Lambda^{\mathbf{x}_{n-2}}(y_1)\}} + y_1 \vee X_1 \vee (X + m) \mathbb{1}_{\{U_{X_1} > 1 - \Lambda^{\mathbf{x}_{n-2}}(y_1)\}}.$$

Then we have $\Lambda^{\mathbf{x}_{n-2}}\mathrm{VaR}(X_2) = y_1$ and $X_1 \leqslant X_2$. This implies $\Lambda^{\mathbf{x}_{n-2}}\mathrm{VaR}(X_1) + \Lambda_n \mathrm{VaR}(X - X_1) \geqslant \Lambda^{\mathbf{x}_{n-2}}\mathrm{VaR}(X_2) + \Lambda_n \mathrm{VaR}(X - X_2)$. Let

$$X_3 = y_1 \mathbb{1}_{\{U_X \leqslant 1 - \Lambda^{\mathbf{x}_{n-2}}(y_1)\}} + (X + m) \mathbb{1}_{\{U_X > 1 - \Lambda^{\mathbf{x}_{n-2}}(y_1)\}},$$

where $m > y_1 - F_X^{-1}(1 - \Lambda^{\mathbf{x}_{n-2}}(y_1))$. Then we have $\Lambda^{\mathbf{x}_{n-2}}\mathrm{VaR}(X_3) = y_1$. For $x \geqslant -m$, we have $\mathbb{P}(X - X_2 \leqslant x) \leqslant \mathbb{P}(X - X_3 \leqslant x)$. Note that our assumption implies $\lambda_n^+ + \Lambda^{\mathbf{x}_{n-2}}(y_1) < 1$. By setting $m > y_1 - F_X^{-1}(1 - \lambda_n^+ - \Lambda^{\mathbf{x}_{n-2}}(y_1))$, it follows that for $x < -m$,

$$\begin{aligned}&\mathbb{P}(X - X_2 \leqslant x) \leqslant \Lambda^{\mathbf{x}_{n-2}}(y_1) + \mathbb{P}(X \leqslant x + y_1, U_{X_1} \leqslant 1 - \Lambda^{\mathbf{x}_{n-2}}(y_1)) < 1 - \Lambda_n(x), \\ &\mathbb{P}(X - X_3 \leqslant x) = \mathbb{P}(X \leqslant x + y_1) \wedge (1 - \Lambda^{\mathbf{x}_{n-2}}(y_1)) < 1 - \Lambda_n(x).\end{aligned}$$

This implies $\Lambda_n \mathrm{VaR}(X - X_2) \geqslant -m$ and $\Lambda_n \mathrm{VaR}(X - X_3) \geqslant -m$. Hence, we have $\Lambda_n \mathrm{VaR}(X - X_3) \leqslant \Lambda_n \mathrm{VaR}(X - X_2)$. Following the same argument as in the proof of Theorem 1 for $n = 2$, we can show $\Lambda^{\mathbf{x}_{n-2}}\mathrm{VaR} \,\square\, \Lambda_n \mathrm{VaR}(X) = \inf_{x_{n-1}\in\mathbb{R}} \Lambda^{\mathbf{x}_{n-1}}\mathrm{VaR}(X)$. This completes the proof. □

Proof of Example 1. Applying Theorem 1, we have

$$\mathop{\square}_{i=1}^{n} \Lambda_i \mathrm{VaR}(X) = \Lambda^* \mathrm{VaR}(X).$$

In order to compute Λ^*, we split $\mathbb{R}^{n-1}$ into 2^{n-1} disjoint subsets. Let $\mathcal{N}_i \subset \{1, \ldots, n-1\}, i =$

$1,\dots,2^{n-1}$, be distinct sets. We denote $B_i = \{\mathbf{x}_{n-1} \in \mathbb{R}^{n-1} : x_j - x_{j-1} \leqslant b_j$ only for $j \in \mathcal{N}_i\}$. Then we have B_i are disjoint and $\cup_{i=1}^{2^{n-1}} B_i = \mathbb{R}^{n-1}$. Without loss of generality, we suppose $\mathcal{N}_1 = \emptyset$ and $\mathcal{N}_2 = \{1,\dots,n-1\}$. Over B_i, we have

$$\Lambda^{\mathbf{x}_{n-1}}(x) = \Lambda_n(x - x_{n-1}) + \sum_{j\in\mathcal{N}_i} \lambda_j^+ + \sum_{j\in\mathcal{N}_i^\complement} \lambda_j^-,$$

where $N_i^\complement = \{1,\dots,n-1\} \setminus \mathcal{N}_i$. Note that for $i \geqslant 3$, under the constraint $\mathbf{x}_{n-1} \in B_i$, x_{n-1} can take any value in $\mathbb{R}$. Moreover, over B_1, the range of x_{n-1} is $(\sum_{j=1}^{n-1} b_j, \infty)$, and over B_2, the range of x_{n-1} is $(-\infty, \sum_{j=1}^{n-1} b_j]$. Direct computation shows for $i = 1$,

$$\sup_{\mathbf{x}_{n-1}\in B_i} \Lambda^{\mathbf{x}_{n-1}}(x) = \lambda_n^+ + \sum_{i=1}^{n-1} \lambda_i^-;$$

for $i = 2$,

$$\sup_{\mathbf{x}_{n-1}\in B_i} \Lambda^{\mathbf{x}_{n-1}}(x) = \sum_{i=1}^{n-1} \lambda_i^+ + \Lambda_n(x - \sum_{i=1}^{n-1} b_i) = \Lambda^\diamond(x);$$

and for $i \geqslant 3$,

$$\sup_{\mathbf{x}_{n-1}\in B_i} \Lambda^{\mathbf{x}_{n-1}}(x) = \lambda_n^+ + \sum_{j\in\mathcal{N}_i} \lambda_j^+ + \sum_{j\in\mathcal{N}_i^\complement} \lambda_j^-.$$

These equations imply $\Lambda^*(x) = \lambda^* \vee \Lambda^\diamond(x)$. Hence, by Theorem 1, we have

$$\mathop{\square}_{i=1}^{n} \Lambda_i\mathrm{VaR}(X) = (\lambda^* \vee \Lambda^\diamond)\mathrm{VaR}(X).$$

□

C Proof of Section 3.2

This section is devoted to the proofs of all the results in Section 3.2.

Proof of Theorem 2. The proof of (10) is given by Proposition 10. Now we only focus on the proof of other conclusions. The proof is similar to that of Theorem 1. We first consider cases (i) and (ii). Let us start with the case $n = 2$. For any $X_1 \in \mathcal{X}$, we denote $x_1 = \Lambda_1\mathrm{VaR}(X_1)$. For some $m \in \mathbb{R}$, let

$$X_2 = x_1 \mathbb{1}_{\{U_{X_1} \leqslant 1-\Lambda_1(x_1)\}} + (x_1 \vee X_1 \vee (X + m))\, \mathbb{1}_{\{U_{X_1} > 1-\Lambda_1(x_1)\}}.$$

Note that $\Lambda_1\mathrm{VaR}(X_2) = x_1$ and $X_1 \leqslant X_2$. Hence, by monotonicity of $\Lambda_2\mathrm{VaR}$, we have

$$\Lambda_1\mathrm{VaR}(X_1) + \Lambda_2\mathrm{VaR}(X - X_1) \geqslant \Lambda_1\mathrm{VaR}(X_2) + \Lambda_2\mathrm{VaR}(X - X_2).$$

Observe that

$$X - X_2 = (X - x_1)\mathbb{1}_{\{U_{X_1} \leqslant 1-\Lambda_1(x_1)\}} + ((X - x_1) \wedge (X - X_1) \wedge (-m))\,\mathbb{1}_{\{U_{X_1} > 1-\Lambda_1(x_1)\}}.$$

Direct computation shows $\mathbb{P}(X - X_2 \leqslant -m) \geqslant \Lambda_1(x_1)$. If $\lambda_1^+ + \lambda_2^- > 1$, there exists $x_1 \in \mathbb{R}$ such that $\Lambda_1(x_1) > 1 - \lambda_2^-$. This implies that $\mathbb{P}(X - X_2 \leqslant -m) > 1 - \lambda_2^- \geqslant 1 - \Lambda_2(-m)$. Hence $\Lambda_2\mathrm{VaR}(X - X_2) \leqslant -m$. Consequently, $\lim_{m\to\infty}(\Lambda_1\mathrm{VaR}(X_2) + \Lambda_2\mathrm{VaR}(X - X_2)) \leqslant \lim_{m\to\infty}(x_1 - m) = -\infty$. This implies $\Lambda_1\mathrm{VaR} \,\square\, \Lambda_2\mathrm{VaR}(X) = -\infty$. We could analogously obtain the same conclusion if $\lambda_1^- + \lambda_2^+ > 1$.

We next consider the case $(\lambda_1^- + \lambda_2^+) \vee (\lambda_1^+ + \lambda_2^-) < 1$. Let

$$X_3 = x_1\mathbb{1}_{\{U_X \leqslant 1-\Lambda_1(x_1)\}} + (X + m)\mathbb{1}_{\{U_X > 1-\Lambda_1(x_1)\}}, \tag{27}$$

where

$$m > x_1 - F_X^{-1}(1 - \Lambda_1(x_1)). \tag{28}$$

It follows that for $x \geqslant -m$,

$$\begin{aligned}\mathbb{P}(X - X_2 \leqslant x) &= \Lambda_1(x_1) + \mathbb{P}(X \leqslant x + x_1, U_{X_1} \leqslant 1 - \Lambda_1(x_1)) \\ &\leqslant \Lambda_1(x_1) + \mathbb{P}(X \leqslant x + x_1, U_X \leqslant 1 - \Lambda_1(x_1)) = \mathbb{P}(X - X_3 \leqslant x).\end{aligned}$$

Note that we could find $m \in \mathbb{R}$ such that

$$\Lambda_1(x_1) + F_X(-m + x_1) < 1 - \Lambda_2(-m). \tag{29}$$

For $x < -m$, it follows that

$$\begin{aligned}&\mathbb{P}(X - X_2 \leqslant x) \leqslant \mathbb{P}(X \leqslant x + x_1, U_{X_1} \leqslant 1 - \Lambda_1(x_1)) + \Lambda_1(x_1) < 1 - \Lambda_2(x), \\ &\mathbb{P}(X - X_3 \leqslant x) = F_X(x + x_1) \wedge (1 - \Lambda_1(x_1)) < 1 - \Lambda_2(x).\end{aligned}$$

This indicates that $\Lambda_2\mathrm{VaR}(X - X_2) \geqslant -m$ and $\Lambda_2\mathrm{VaR}(X - X_3) \geqslant -m$. Hence we have $\Lambda_2\mathrm{VaR}(X - X_3) \leqslant \Lambda_2\mathrm{VaR}(X - X_2)$. Direct computation yields $\mathbb{P}(X - X_3 \leqslant x) = (\Lambda_1(x_1) + F_X(x + x_1)) \wedge 1$

for $x \geqslant -m$. Note also that $\Lambda_1\mathrm{VaR}(X_3) = \Lambda_1\mathrm{VaR}(X_2) = x_1$. Consequently, we have

$$\begin{aligned}\Lambda_1\mathrm{VaR}(X_3) + \Lambda_2\mathrm{VaR}(X - X_3) &= x_1 + \inf\{x : \mathbb{P}(X - X_3 \leqslant x) \geqslant 1 - \Lambda_2(x)\}\\ &= x_1 + \inf\{x : \Lambda_1(x_1) + F_X(x + x_1) \geqslant 1 - \Lambda_2(x)\}\\ &= \inf\{y : F_X(y) \geqslant 1 - \Lambda_2(y - x_1) - \Lambda_1(x_1)\} = \Lambda^{\mathbf{x}_1}\mathrm{VaR}(X).\end{aligned}$$

Therefore, for any $X_1 \in \mathcal{X}$ with $x_1 = \Lambda_1\mathrm{VaR}(X_1)$, we have

$$\Lambda_1\mathrm{VaR}(X_1) + \Lambda_2\mathrm{VaR}(X - X_1) \geqslant \Lambda^{\mathbf{x}_1}\mathrm{VaR}(X) \geqslant \inf_{x_1\in\mathbb{R}} \Lambda^{\mathbf{x}_1}\mathrm{VaR}(X).$$

This implies $\Lambda_1\mathrm{VaR} \,\square\, \Lambda_2\mathrm{VaR}(X) \geqslant \inf_{x_1\in\mathbb{R}} \Lambda^{\mathbf{x}_1}\mathrm{VaR}(X)$. Moreover, for any $x_1 \in \mathbb{R}$, we could construct X_3 as in (27) such that $\Lambda_1\mathrm{VaR}(X_3) + \Lambda_2\mathrm{VaR}(X - X_3) = \Lambda^{\mathbf{x}_1}\mathrm{VaR}(X)$. Hence

$$\Lambda_1\mathrm{VaR} \,\square\, \Lambda_2\mathrm{VaR}(X) \leqslant \inf_{x_1\in\mathbb{R}} \Lambda^{\mathbf{x}_1}\mathrm{VaR}(X).$$

We establish the claim for $n = 2$. We could prove the result for $n \geqslant 3$ by induction analogously as in the proof of Theorem 1. Following the same argument as in the proof of (iii) of Theorem 1, we can show (iii) by noting (27), (28) and (29). The detailed proof is omitted. □

The conclusion in next proposition is sufficient to show (10) in Theorem 2.

Proposition 10. *For $\Lambda_i \in \mathcal{H}_I$ with $0 < \lambda_i^- \leqslant \lambda_i^+ < 1$, we have*

$$\inf_{\mathbf{x}_{n-1}\in\mathbb{R}^{n-1}} \Lambda^{\mathbf{x}_{n-1}}\mathrm{VaR}(X) \geqslant \Lambda^*\mathrm{VaR}(X).$$

Moreover, the above inequality holds as an equality if all $\lambda_i^\pm$ are attainable.

Proof. First, note that $\sum_{i=1}^n \lambda_i^- \leqslant \Lambda^*(x) \leqslant 1$. Moreover, one can easily check that Λ^* is an increasing function. Using the monotonicity of $\Lambda\mathrm{VaR}$ with respect to Λ, we have $\Lambda^{\mathbf{x}_{n-1}}\mathrm{VaR}(X) \geqslant \Lambda^*\mathrm{VaR}(X)$, which implies $\inf_{\mathbf{x}_{n-1}\in\mathbb{R}^{n-1}} \Lambda^{\mathbf{x}_{n-1}}\mathrm{VaR}(X) \geqslant \Lambda^*\mathrm{VaR}(X)$.

Next, we show the equality holds under the assumption. By the assumption, there exist x_i^- and x_i^+ such that $\Lambda_i(x_i^-) = \lambda_i^-$ and $\Lambda_i(x_i^+) = \lambda_i^+$. Note that $\Lambda^*(x) = \sup_{y_1+\cdots+y_n=x}(\sum_{i=1}^n \Lambda_i(y_i)) \wedge 1$. In the proof of Proposition 9, we have shown that there exists a bounded sequence $(y_1^{(k)}, \ldots, y_n^{(k)})$ such that $y_1^{(k)} + \cdots + y_n^{(k)} = x$ and $\sum_{i=1}^n \Lambda_i(y_i^{(k)}) \uparrow \Lambda^*(x)$ as $k \to \infty$. Moreover, we can force $y_i^{(k)}$ to be monotone by picking a subsequence. We denote $y_i^* = \lim_{k\to\infty} y_i^{(k)}$. If $y_i^{(k)} \geqslant y_i^*$, by the right continuity of Λ_i, we have $\lim_{k\to\infty} \Lambda_i(y_i^{(k)}) = \Lambda_i(y_i^*)$. If $y_i^{(k)} \leqslant y_i^*$, by the monotonicity of Λ_i, we have $\lim_{k\to\infty} \Lambda_i(y_i^{(k)}) \leqslant \Lambda_i(y_i^*)$. Consequently, $\Lambda^*(x) \leqslant (\sum_{i=1}^n \Lambda_i(y_i^*)) \wedge 1$, which implies $\Lambda^*(x) = (\sum_{i=1}^n \Lambda_i(y_i^*)) \wedge 1$. Denoting $x_i = \sum_{j=1}^i y_j^*$, we have $\Lambda^{\mathbf{x}_{n-1}}(x) = \Lambda^*(x)$. This shows the existence of the maximizer. Following exactly the same argument as in the proof of Proposition 9, we obtain $\inf_{\mathbf{x}_{n-1}\in\mathbb{R}^{n-1}} \Lambda^{\mathbf{x}_{n-1}} \leqslant \Lambda^*\mathrm{VaR}(X)$. This completes the proof. $\square$

Proof of Proposition 3. Note that $\Lambda^{\mathbf{x}_{n-1}}$ and Λ^* are increasing functions. Hence, in light of Theorem 3.1 in Han et al. (2021) and Theorem 2, we have if $\bigvee_{i=1}^n(\lambda_i^- + \sum_{j\neq i}\lambda_j^+) < 1$,

$$\begin{aligned}\mathop{\square}_{i=1}^{n} \Lambda_i\mathrm{VaR}(X) &= \inf_{\mathbf{x}_{n-1}\in\mathbb{R}^{n-1}} \Lambda^{\mathbf{x}_{n-1}}\mathrm{VaR}(X)\\ &= \inf_{\mathbf{x}_{n-1}\in\mathbb{R}^{n-1}} \inf_{x_n\in\mathbb{R}} \mathrm{VaR}_{\Lambda^{\mathbf{x}_{n-1}}(x_n)}(X) \vee x_n\\ &= \inf_{y_1,\ldots,y_n\in\mathbb{R}} \left(\mathrm{VaR}_{\sum_{i=1}^n \Lambda_i(y_i)}(X) \bigvee (\sum_{i=1}^n y_i)\right).\end{aligned}$$

If in addition, all $\lambda_i^\pm$ are attainable, then $\square_{i=1}^n \Lambda_i\mathrm{VaR}(X) = \Lambda^*\mathrm{VaR}(X) = \inf_{x\in\mathbb{R}}\left(\mathrm{VaR}_{\Lambda^*(x)}(X)\vee x\right)$. This completes the proof. $\square$

Proof of Proposition 4. The proof is the same as that of Proposition 2. Hence it is omitted. $\square$

Proof of Example 2. Applying Proposition 10, we only need to compute $\sup_{\mathbf{x}_{n-1}\in\mathbb{R}^{n-1}} \Lambda^{\mathbf{x}_{n-1}}(x)$. To compute this value, we split $\mathbb{R}^{n-1}$ into 2^{n-1} disjoint subsets. Let $\mathcal{N}_i \subset \{1,\ldots,n-1\}, i = 1,\ldots,2^{n-1}$, be distinct sets. We denote $B_i = \{\mathbf{x}_{n-1}\in\mathbb{R}^{n-1} : x_j - x_{j-1} < b_j \text{ only for } j\in\mathcal{N}_i\}$. Then we have B_i are disjoint and $\cup_{i=1}^{2^{n-1}} B_i = \mathbb{R}^{n-1}$. Without loss of generality, we suppose $B_1 = \emptyset$ and $B_2 = \{1,\ldots,n-1\}$. Over B_i, we have

$$\Lambda^{\mathbf{x}_{n-1}}(x) = \left(\Lambda_n(x - x_{n-1}) + \sum_{j\in\mathcal{N}_i}\lambda_j^- + \sum_{j\in\mathcal{N}_i^\complement}\lambda_j^+\right)\wedge 1,$$

where $N_i^\complement = \{1,\ldots,n-1\}\setminus\mathcal{N}_i$. Note that for $i \geqslant 3$, under the constraint $\mathbf{x}_{n-1}\in B_i$, x_{n-1} can take any value in $\mathbb{R}$. Moreover, over B_1, the range of x_{n-1} is $[\sum_{j=1}^{n-1} b_j, \infty)$, and over B_2, the range of x_{n-1} is $(-\infty, \sum_{j=1}^{n-1} b_j)$. It follows that, for $i = 1$,

$$\sup_{\mathbf{x}_{n-1}\in B_i} \Lambda^{\mathbf{x}_{n-1}}(x) = \Lambda^\diamond(x);$$

for $i = 2$,

$$\sup_{\mathbf{x}_{n-1}\in B_i} \Lambda^{\mathbf{x}_{n-1}}(x) = \lambda_n^+ + \sum_{i=1}^{n-1}\lambda_i^-;$$

and for $i \geqslant 3$,

$$\sup_{\mathbf{x}_{n-1}\in B_i} \Lambda^{\mathbf{x}_{n-1}}(x) = \lambda_n^+ + \sum_{j\in\mathcal{N}_i}\lambda_j^- + \sum_{j\in\mathcal{N}_i^\complement}\lambda_j^+.$$

Combing the above equalities, we have $\sup_{\mathbf{x}_{n-1}\in\mathbb{R}^{n-1}} \Lambda^{\mathbf{x}_{n-1}}(x) = \lambda^*\vee\Lambda^\diamond(x)$. In light of Theorem 2, we establish the claim. $\square$

D Proof of Section 4

In this section, we give all the proofs of the results in Section 4.

Proof of Theorem 3. For any $x, y \in \mathbb{R}$, define $X_1 = x\mathbb{1}_{\{U_X \leqslant 1-\Lambda(x)\}} + (X-y)\mathbb{1}_{\{U_X > 1-\Lambda(x)\}}$. Note that $\mathbb{P}(X_1 \leqslant x) \geqslant 1-\Lambda(x)$, implying $\Lambda\mathrm{VaR}(X_1) \leqslant x$. Hence we have

$$\Lambda\mathrm{VaR}(X_1) + \rho(X - X_1) \leqslant x + \rho\left(X_{x,y}\right).$$

This implies $\Lambda\mathrm{VaR} \,\Box\, \rho(X) \leqslant x + \rho\left(X_{x,y}\right)$. As x, y are chosen arbitrarily, we have

$$\Lambda\mathrm{VaR} \,\Box\, \rho(X) \leqslant \inf_{x\in\mathbb{R}} \inf_{y\in\mathbb{R}} \left\{x + \rho\left(X_{x,y}\right)\right\}.$$

We next show the inverse inequality. For any $X_1 \in \mathcal{X}$, note that $\Lambda\mathrm{VaR}(X_1) \in (-\infty, \infty)$. Let $x_1 = \Lambda\mathrm{VaR}(X_1)$. Define $X_2 = x_1\mathbb{1}_{\{U_{X_1} \leqslant 1-\Lambda(x_1)\}} + (x_1 \vee X_1 \vee (X-m))\,\mathbb{1}_{\{U_{X_1} > 1-\Lambda(x_1)\}}$. Clearly, $F_{X_2}(x_1) \geqslant 1-\Lambda(x_1)$. This implies that $\Lambda\mathrm{VaR}(X_2) = x_1$ and $X_1 \leqslant X_2$ a.s.. By monotonicity of ρ, we have $\Lambda\mathrm{VaR}(X_1) + \rho(X - X_1) \geqslant x_1 + \rho(X - X_2)$. Define $X_3 = x_1\mathbb{1}_{\{U_X \leqslant 1-\Lambda(x_1)\}} + (X - m)\mathbb{1}_{\{U_X > 1-\Lambda(x_1)\}}$ with $m < F_X^{-1}(1-\Lambda(x_1)) - x_1$ and recall $X_{x_1,m} = (X - x_1)\mathbb{1}_{\{U_X \leqslant 1-\Lambda(x_1)\}} + m\mathbb{1}_{\{U_X > 1-\Lambda(x_1)\}}$. For $x \geqslant m$, it follows that

$$\begin{aligned}
\mathbb{P}(X - X_2 \leqslant x) &= \Lambda(x_1) + \mathbb{P}(X - x_1 \leqslant x, U_{X_1} \leqslant 1-\Lambda(x_1)) \\
&\leqslant \Lambda(x_1) + \mathbb{P}(X - x_1 \leqslant x, U_X \leqslant 1-\Lambda(x_1)) \\
&= \mathbb{P}(X - X_3 \leqslant x) = \mathbb{P}(X_{x_1,m} \leqslant x).
\end{aligned}$$

If $\mathcal{X} = L^\infty$, by choosing $m < (\text{ess-inf}(X - X_1)) \wedge \text{ess-inf}(X - x_1)$, it follows that for $x < m$,

$$\begin{aligned}
\mathbb{P}(X - X_2 \leqslant x) &= \mathbb{P}(X - x_1 \leqslant x, U_{X_1} \leqslant 1-\Lambda(x_1)) \\
&\leqslant \mathbb{P}(X - x_1 \leqslant x, U_X \leqslant 1-\Lambda(x_1)) \\
&= \mathbb{P}(X - X_3 \leqslant x) = \mathbb{P}(X_{x_1,m} \leqslant x) = 0.
\end{aligned}$$

By monotonicity and law-invariance of ρ, we have $\rho(X - X_2) \geqslant \rho(X - X_3) = \rho(X_{x_1,m})$. If ρ is an ε-tail risk measure, we choose m small enough such that $\mathbb{P}(X - X_2 < m) \vee \mathbb{P}(X - X_3 < m) < 1-\varepsilon$. Then we have, for $x \in \mathbb{R}$,

$$(\mathbb{P}(X - X_2 \leqslant x) - 1 + \varepsilon)_+ \leqslant (\mathbb{P}(X - X_3 \leqslant x) - 1 + \varepsilon)_+ = (\mathbb{P}(X - X_{x_1,m} \leqslant x) - 1 + \varepsilon)_+.$$

It follows from the monotonicity of ρ and the fact that ρ is an ε-tail risk measure that $\rho(X - X_2) \geqslant \rho(X - X_3) = \rho(X_{x_1,m})$. Note that $\Lambda\mathrm{VaR}(X_2) = \Lambda\mathrm{VaR}(X_3) = x_1$. Therefore, with m small enough, we have

$$\begin{aligned}\Lambda\mathrm{VaR}(X_1) + \rho(X - X_1) &\geqslant x_1 + \rho(X - X_2) \geqslant x_1 + \rho(X - X_3) \\ &= x_1 + \rho(X_{x_1,m}) \geqslant \inf_{x\in\mathbb{R}} \inf_{y\in\mathbb{R}} \{x + \rho(X_{x,y})\}.\end{aligned}$$

This implies the inverse inequality.

We next show that the existence of the optimal allocation implies the existence of the minimizer of (12). We assume that there exists $X_1 \in \mathcal{X}$ such that $\Lambda\mathrm{VaR}(X_1) + \rho(X - X_1) = \Lambda\mathrm{VaR} \,\square\, \rho(X)$. Following the same argument as above to show the inverse inequality, there exists $y_1 \in \mathbb{R}$ such that $\Lambda\mathrm{VaR}(X_1) + \rho(X - X_1) \geqslant x_1 + \rho(X_{x_1,y_1})$ with $x_1 = \Lambda\mathrm{VaR}(X_1)$. Hence $\Lambda\mathrm{VaR} \,\square\, \rho(X) \geqslant x_1 + \rho(X_{x_1,y_1}) \geqslant \inf_{x\in\mathbb{R}} \inf_{y\in\mathbb{R}} \{x + \rho(X_{x,y})\} = \Lambda\mathrm{VaR} \,\square\, \rho(X)$. This implies that (x_1, y_1) is the minimizer of (12), leading to a contradiction.

If (x^*, y^*) is the minimizer of (12), then (X_1^*, X_2^*) is well defined. We have $\mathbb{P}(X_1^* \leqslant x^*) \geqslant 1 - \Lambda(x^*)$. By the definition, we have $\Lambda\mathrm{VaR}(X_1^*) \leqslant x^*$. Hence we have

$$\Lambda\mathrm{VaR}(X_1^*) + \rho(X - X_1^*) \leqslant x^* + \rho(X_{x^*,y^*}) = \inf_{x\in\mathbb{R}} \inf_{y\in\mathbb{R}} \{x + \rho(X_{x,y})\} = \Lambda\mathrm{VaR} \,\square\, \rho(X).$$

Therefore, the optimal allocation exists and one of them is $(X_1^*, X - X_1^*)$. $\square$

Proof of Theorem 4. We first focus on the case $n = 2$. In light of Theorem 3, we have

$$\Lambda_1\mathrm{VaR} \,\square\, \Lambda_2\mathrm{VaR}(X) = \inf_{x\in\mathbb{R}} \inf_{y\in\mathbb{R}} \{x + \Lambda_2\mathrm{VaR}(X_{x,y})\},$$

where $X_{x,y} = (X - x)\mathbb{1}_{\{U_X \leqslant 1-\Lambda_1(x)\}} + y\mathbb{1}_{\{U_X > 1-\Lambda_1(x)\}}$. Direct computation gives $F_{X_{x,y}}(z) = (F_X(z + x) + \Lambda_1(x)) \wedge 1,\ z \geqslant y$ and $F_{X_{x,y}}(z) = F_X(z + x) \wedge (1 - \Lambda_1(x)),\ z < y$. We first fix x and let $y < F_X^{-1}(1 - \lambda_1^+ - \lambda_2^+) - x$. Then for $z < y$,

$$F_{X_{x,y}}(z) \leqslant F_X(z + x) < 1 - \lambda_1^+ - \lambda_2^+ \leqslant 1 - \Lambda_2(z).$$

This implies that $\Lambda_2\mathrm{VaR}(X_{x,y}) \geqslant y$. Moreover, for $z \geqslant y$,

$$F_X(z + x) + \Lambda_1(x) < 1 - \lambda_1^+ - \lambda_2^+ + \Lambda_1(x) \leqslant 1 - \Lambda_2(z).$$

Hence for $y < F_X^{-1}(1-\lambda_1^+ - \lambda_2^+) - x$,

$$\begin{aligned}
x + \Lambda_2 \mathrm{VaR}(X_{x,y}) &= x + \inf\{z \in \mathbb{R} : F_{X_{x,y}}(z) \geqslant 1 - \Lambda_2(z)\} \\
&= x + \inf\{z \geqslant y : (F_X(z+x) + \Lambda_1(x)) \wedge 1 \geqslant 1 - \Lambda_2(z)\} \\
&= x + \inf\{z \in \mathbb{R} : F_X(z+x) + \Lambda_1(x) \geqslant 1 - \Lambda_2(z)\} \\
&= \inf\{z \in \mathbb{R} : F_X(z) \geqslant 1 - \Lambda_2(z-x) - \Lambda_1(x)\} \\
&= \Lambda^x \mathrm{VaR}(X),
\end{aligned}$$

where $\Lambda^x(z) = (\Lambda_1(x) + \Lambda_2(z-x)) \wedge 1$. Using the monotonicity of $\Lambda_2\mathrm{VaR}(X_{x,y})$ with respect to y, we have $\inf_{y\in\mathbb{R}} \{x + \Lambda_2 \mathrm{VaR}(X_{x,y})\} = \Lambda^x \mathrm{VaR}(X)$. Therefore, we conclude that if $\lambda_1^+ + \lambda_2^+ < 1$,

$$\Lambda_1 \mathrm{VaR} \,\square\, \Lambda_2 \mathrm{VaR}(X) = \inf_{\mathbf{x}_1 \in \mathbb{R}} \Lambda^{\mathbf{x}_1} \mathrm{VaR}(X).$$

Now we consider the case (ii). Without loss of generality, we suppose $\lambda_1^+ + \lambda_2^- > 1$. For any $0 < \varepsilon < \lambda_1^+ + \lambda_2^- - 1$, we could find $x_0 \in \mathbb{R}$ such that $\Lambda_1(x_0) \geqslant \lambda_1^+ - \varepsilon$. Note that

$$\begin{aligned}
F_{X_{x_0,y}}(y) &= (F_X(y + x_0) + \Lambda_1(x_0)) \wedge 1 \\
&\geqslant \Lambda_1(x_0) \geqslant \lambda_1^+ - \varepsilon > 1 - \lambda_2^- \geqslant 1 - \Lambda_2(y).
\end{aligned}$$

This implies $\Lambda_2\mathrm{VaR}(X_{x_0,y}) \leqslant y$. Letting $y \to \infty$, we have

$$\inf_{x\in\mathbb{R}} \inf_{y\in\mathbb{R}} \{x + \Lambda_2 \mathrm{VaR}(X_{x,y})\} \leqslant x_0 + \Lambda_2 \mathrm{VaR}(X_{x_0,y}) \leqslant x_0 + y \to -\infty.$$

We have shown (i) and (ii) for case $n = 2$. Using the same argument as in the proof of Theorem 1, we can extend (i) to the case $n \geqslant 3$ by induction. For (ii), without loss of generality, suppose $\lambda_n + \sum_{i=1}^{n-1} \lambda_i^- > 1$. By the monotonicity of $\Lambda\mathrm{VaR}$, we have $\Lambda_i\mathrm{VaR}(X) \leqslant \mathrm{VaR}_{\lambda_i^-}(X)$. In light of Lemma 2 in Liu et al. (2020), we have

$$\mathop{\square}_{i=1}^{n} \Lambda_i \mathrm{VaR}(X) = (\mathop{\square}_{i=1}^{n-1} \Lambda_i \mathrm{VaR}) \,\square\, \Lambda_n \mathrm{VaR}(X) \leqslant (\mathop{\square}_{i=1}^{n-1} \mathrm{VaR}_{\lambda_i^-}) \,\square\, \Lambda_n \mathrm{VaR}(X).$$

It follows from Theorem 1 and Proposition 2 that $\square_{i=1}^{n-1} \mathrm{VaR}_{\lambda_i^-} = -\infty$ if $\sum_{i=1}^{n-1} \lambda_i^- \geqslant 1$, which indicates that $\square_{i=1}^{n} \Lambda_i\mathrm{VaR}(X) = -\infty$. Moreover, if $\sum_{i=1}^{n-1} \lambda_i^- < 1$, by Lemma 2 in Liu et al. (2020) or Theorem 1, we have $\square_{i=1}^{n-1} \mathrm{VaR}_{\lambda_i^-} = \mathrm{VaR}_{\sum_{i=1}^{n-1} \lambda_i^-}$. Hence, $\square_{i=1}^{n} \Lambda_i\mathrm{VaR}(X) \leqslant \mathrm{VaR}_{\sum_{i=1}^{n-1} \lambda_i^-} \square\, \Lambda_n\mathrm{VaR}(X)$. It follows from the fact that $\lambda_n + \sum_{i=1}^{n-1} \lambda_i^- > 1$ and the conclusion from the case $n = 2$ that $\square_{i=1}^{n} \Lambda_i\mathrm{VaR}(X) \leqslant \mathrm{VaR}_{\sum_{i=1}^{n-1} \lambda_i^-} \square\, \Lambda_n\mathrm{VaR}(X) = -\infty$. We establish the claim for $n \geqslant 3$.

We can establish (15) using exactly the same argument as in the proof of Proposition 9.

Moreover, we can also prove (iii) using the same argument as in the proof of (iii) of Theorem 1. Hence the details are omitted. □

Proof of Proposition 5. Applying Theorem 3, we have

$$\begin{aligned}\Lambda\mathrm{VaR}\,\Box\,\rho(X) &= \inf_{x\in\mathbb{R}}\inf_{y\in\mathbb{R}}\left\{x+\mathbb{E}\left(u\left((X-x)\mathbb{1}_{\{U_X\leqslant 1-\Lambda(x)\}}+y\mathbb{1}_{\{U_X>1-\Lambda(x)\}}\right)\right)\right\}\\ &= \inf_{x\in\mathbb{R}}\inf_{y\in\mathbb{R}}\left\{x+\mathbb{E}\left(u\left((X-x)\mathbb{1}_{\{U_X\leqslant 1-\Lambda(x)\}}\right)+u(y)\Lambda(x)\right)\right\}\\ &= \inf_{x\in\mathbb{R}}\left\{x+a\Lambda(x)+\mathbb{E}\left(u\left((X-x)\mathbb{1}_{\{U_X\leqslant 1-\Lambda(x)\}}\right)\right)\right\}.\end{aligned}$$

Note that $\Lambda\mathrm{VaR}\,\Box\,\rho(X)=-\infty$ if $a=-\infty$. This completes the proof. □

Proof of Proposition 6. First, applying Theorem 3, we have

$$\Lambda\mathrm{VaR}\,\Box\,\rho(X)=\inf_{x\in\mathbb{R}}\inf_{y\in\mathbb{R}}\left\{x+\int_0^1 u(\mathrm{VaR}_t(X_{x,y}))\mathrm{d}g\right\},$$

where $X_{x,y}=(X-x)\mathbb{1}_{\{U_X\leqslant 1-\Lambda(x)\}}+y\mathbb{1}_{\{U_X>1-\Lambda(x)\}}$. We now let $y<\text{ess-inf}X-x$. Direct calculation gives

$$\mathrm{VaR}_t(X_{x,y})=\begin{cases}F_X^{-1}(1-t-\Lambda(x))-x, & t<1-\Lambda(x)\\ y, & t\geqslant 1-\Lambda(x)\end{cases}.$$

Hence

$$\int_0^1 u(\mathrm{VaR}_t(X_{x,y}))\mathrm{d}g=\int_0^{1-\Lambda(x)} u(F_X^{-1}(1-t-\Lambda(x))-x)\mathrm{d}g+u(y)(1-g(1-\Lambda(x))).$$

Note that if $g(1-\lambda^+)<1$, there exists $x_0\in\mathbb{R}$ such that $g(1-\Lambda(x_0))<1$. If in addition, $a=-\infty$, we have

$$\inf_{y\in\mathbb{R}}\int_0^1 u(\mathrm{VaR}_t(X_{x_0,y}))\mathrm{d}g(t)=-\infty.$$

For the cases of $g(1-\lambda^+)=1$, or $g(1-\lambda^+)<1$ and $a>-\infty$, the conclusion can be obtained by a direct computation. □

E Proof of Section 5

This section is dedicated to the proof of all the results appeared in Section 5.

Proof of Theorem 5. First, note that SSD-consistency implies that ρ is monotone and law-invariant. For any $X_1\in\mathcal{X}$, let $x=\Lambda\mathrm{VaR}^+(X_1)$. Using the fact that $X-F_{X_1}^{-1}(U_X)\leqslant_{\mathrm{icx}} X-X_1$

and ρ is SSD-consistent, we obtain $\rho(X - X_1) \geqslant \rho(X - F_{X_1}^{-1}(U_X))$. Hence we have

$$\Lambda\mathrm{VaR}^+(X_1) + \rho(X - X_1) \geqslant x + \rho(X - F_{X_1}^{-1}(U_X)).$$

For simplicity, we denote $X_2 = F_{X_1}^{-1}(U_X)$. Next, we will show $\rho(X - X_2) \geqslant \inf_{y \geqslant x \vee x_0} \rho(X - F_{x,y}^{-1}(U_X))$.

If $\mathcal{X} = L^\infty$, let $y > 2(\text{ess-sup}|X| \vee \text{ess-sup}|X_1| \vee |x|)$. Then we have $F_{x,y}(z) = F_{X_1}(z) = 1$ if $z \geqslant y$. By the definition of $\Lambda\mathrm{VaR}^+$ and the right-continuous of Λ, we have $F_{X_1}(z) \geqslant 1 - \Lambda(z)$ for $z \geqslant x$, which together with the monotonicity of F_{X_1} implies $F_{X_1}(z) \geqslant 1 - \Lambda_x(z)$ for $z \geqslant x$. By noting that $F_{x,y}(z) = F_{X_1}(z) = 1$ for $z \geqslant y$, we have $F_{x,y}(z) \leqslant F_{X_1}(z)$ for all $z \in \mathbb{R}$, which implies $X_2 \leqslant F_{x,y}^{-1}(U_X)$. Consequently, $\rho(X - X_2) \geqslant \inf_{y \geqslant x \vee x_0} \rho(X - F_{x,y}^{-1}(U_X))$ by noting $F_{x,y}$ is decreasing in y.

Next, we consider the unbounded $\mathcal{X}$. We set $y \geqslant x \vee x_0$. Then we can write

$$\begin{aligned} X - F_{x,y}^{-1}(U_X) &= (X - F_{x,y}^{-1}(U_X))\mathbb{1}_{\{U_X \leqslant F_{X_1}(y) \wedge F_X(y/2)\}} \\ &\quad + (X - F_{x,y}^{-1}(U_X))\mathbb{1}_{\{F_{X_1}(y) \wedge F_X(y/2) < U_X \leqslant F_X(y/2)\}} + (X - F_{x,y}^{-1}(U_X))\mathbb{1}_{\{U_X > F_X(y/2)\}} \end{aligned}$$

and $X - X_2 = (X - X_2)\mathbb{1}_{\{U_X \leqslant F_{X_1}(y) \wedge F_X(y/2)\}} + (X - X_2)\mathbb{1}_{\{U_X > F_{X_1}(y) \wedge F_X(y/2)\}}$. It follows from the definition of $\Lambda\mathrm{VaR}$ that $F_{X_1}(z) \geqslant 1 - \Lambda(z)$ for all $z \geqslant x$. This implies $F_{X_1}(z) \geqslant F_{x,y}(z)$ for all $z < y$. Consequently, on $\{U_X \leqslant F_{X_1}(y) \wedge F_X(y/2)\}$, we have $X_2 \leqslant F_{x,y}^{-1}(U_X)$, implying $X - X_2 \geqslant X - F_{x,y}^{-1}(U_X)$. Let $y > 2|F_{X-X_2}^{-1}(1-\varepsilon)|$. Then on $\{F_{X_1}(y) \wedge F_X(y/2) < U_X \leqslant F_X(y/2)\}$, we have $X - F_{x,y}^{-1}(U_X) = X - y \leqslant y/2 - y = -y/2 < F_{X-X_2}^{-1}(1-\varepsilon)$. Finally, over $U_X > F_X(y/2)$, $X - F_{x,y}^{-1}(U_X) = -y/2 < F_{X-X_2}^{-1}(1-\varepsilon)$.

Combing the above information, we have for $z \geqslant F_{X-X_2}^{-1}(1-\varepsilon)$,

$$\begin{aligned} &\mathbb{P}(X - F_{x,y}^{-1}(U_X) \leqslant z) \\ &= 1 - F_{X_1}(y) \wedge F_X(y/2) + \mathbb{P}(X - F_{x,y}^{-1}(U_X) \leqslant z, U_X \leqslant F_{X_1}(y) \wedge F_X(y/2)) \\ &\geqslant 1 - F_{X_1}(y) \wedge F_X(y/2) + \mathbb{P}(X - X_2 \leqslant z, U_X \leqslant F_{X_1}(y) \wedge F_X(y/2)) \geqslant \mathbb{P}(X - X_2 \leqslant z). \end{aligned}$$

This implies for $z \geqslant F_{X-X_2}^{-1}(1-\varepsilon)$, $(\mathbb{P}(X - X_2 \leqslant z) - 1 + \varepsilon)_+ \leqslant (\mathbb{P}(X - F_{x,y}^{-1}(U_X) \leqslant z) - 1 + \varepsilon)_+$. The above inequality also holds for $z < F_{X-X_2}^{-1}(1-\varepsilon)$ as $(F_{X-X_2}(z) - 1 + \varepsilon)_+ = 0$. Using the fact that ρ is a monotone and ε-tail risk measure, we have $\rho(X - X_2) \geqslant \rho(X - F_{x,y}^{-1}(U_X))$ for some $y \geqslant x \vee x_0$. Hence, $\rho(X - X_2) \geqslant \inf_{y \geqslant x \vee x_0} \rho(X - F_{x,y}^{-1}(U_X))$ also holds.

This further implies

$$\Lambda\mathrm{VaR}(X_1) + \rho(X - X_1) \geqslant x + \rho(X - X_2) \geqslant x + \inf_{y \geqslant x \vee x_0} \rho(X - F_{x,y}^{-1}(U_X))$$
$$\geqslant \inf_{x \in \mathbb{R}} \inf_{y \geqslant x \vee x_0} \left\{ x + \rho\left(X - F_{x,y}^{-1}(U_X)\right) \right\}.$$

As X_1 is chosen arbitrarily, we can conclude that

$$\Lambda\mathrm{VaR} \,\square\, \rho(X) \geqslant \inf_{x \in \mathbb{R}} \inf_{y \geqslant x \vee x_0} \left\{ x + \rho\left(X - F_{x,y}^{-1}(U_X)\right) \right\}.$$

We next show the inverse inequality. For $x \in \mathbb{R}$ and $y \geqslant x \vee x_0$, we have $\Lambda\mathrm{VaR}(F_{x,y}^{-1}(U_X)) = x$. Hence we have

$$x + \rho\left(X - F_{x,y}^{-1}(U_X)\right) = \Lambda\mathrm{VaR}(F_{x,y}^{-1}(U_X)) + \rho\left(X - F_{x,y}^{-1}(U_X)\right) \geqslant \Lambda\mathrm{VaR} \,\square\, \rho(X),$$

implying the desired inverse inequality.

The same argument as in the proof of Theorem 3 can show the existence of the optimal allocation of the inf-convolution is equivalent to the existence of the minimizer of (22). Hence we omit the details. Finally, one can easily check that the allocation (X_1^*, X_2^*) is optimal. □

Proof of Proposition 7. In light of Theorem 5 and Remark 9, we have

$$\Lambda\mathrm{VaR}^+ \,\square\, \rho(X) = \inf_{x \in \mathbb{R}} \inf_{y \geqslant x \vee x_0} \left\{ x + \mathbb{E}\left(u(X - F_{x,y}^{-1}(U_X))\right) \right\},$$

where $x_0 = 2F_X^{-1}(1 - \lambda^-)$. Using (24), we only need to compute $\lim_{y \to \infty} \mathbb{E}\left(u(X - F_{x,y}^{-1}(U_X))\right)$. This can be done by noting

$$X - F_{x,y}^{-1}(U_X) = (X - x)\mathbb{1}_{\{U_X \leqslant 1 - \Lambda(x)\}} + (X - (1 - \Lambda)^{-1}(U_X))\mathbb{1}_{\{1 - \Lambda(x) < U_X \leqslant 1 - \Lambda(y)\}}$$
$$+ (X - y)\mathbb{1}_{\{U_X > 1 - \Lambda(y)\}}.$$

The optimal allocation is easy to check using the result in Theorem 5.

Proof of Proposition 8. By Theorem 5 and Remark 9, we have

$$\Lambda\mathrm{VaR}^+ \,\square\, \rho(X) = \inf_{x \in \mathbb{R}} \inf_{y \geqslant x \vee x_0} \left\{ x + \rho_g\left(X - F_{x,y}^{-1}(U_X)\right) \right\},$$

where $x_0 = 2F_X^{-1}(1 - \lambda^-)$. Using (24), we only need to compute $\lim_{y \to \infty} \rho_g\left(X - F_{x,y}^{-1}(U_X)\right)$.

Recall that $\Lambda(x_1) = \lambda^-$. For $y > x_1$, it follows that

$$X - F_{x,y}^{-1}(U_X) = (X - x)\mathbb{1}_{\{U_X \leqslant 1-\Lambda(x)\}} + (X - (1-\Lambda)^{-1}(U_X))\mathbb{1}_{\{1-\Lambda(x)<U_X\leqslant 1-\lambda^-\}}$$
$$+ (X - y)\mathbb{1}_{\{U_X > 1-\lambda^-\}}.$$

If $y > \text{ess-sup}X - \text{ess-inf}X + x \vee x_1$, we have $\mathbb{P}(X - F_{x,y}^{-1}(U_X) \leqslant z) = \mathbb{P}(X - y \leqslant z, U_X > 1-\lambda^-) < \lambda^-$ for $z < \text{ess-sup}X - y$; $\mathbb{P}(X - F_{x,y}^{-1}(U_X) \leqslant z) = \lambda^-$ for $z = \text{ess-sup}X - y$; and $\mathbb{P}(X - F_{x,y}^{-1}(U_X) \leqslant z) = K_x(z)$ for $z > \text{ess-sup}X - y$. Hence, we have $\mathrm{VaR}_t(X - F_{x,y}^{-1}(U_X)) = F_X^{-1}(1-\lambda^- + 1 - t) - y$ for $t \in [1-\lambda^-, 1]$ and $\mathrm{VaR}_t(X - F_{x,y}^{-1}(U_X)) = K_x^{-1}(1-t)$ for $t \in (0, 1-\lambda^-)$. Hence, for $y > \text{ess-sup}X - \text{ess-inf}X + x \vee x_1$, we have

$$\rho_g\left(X - F_{x,y}^{-1}(U_X)\right) = \int_0^{1-\lambda^-} K_x^{-1}(1-t)\mathrm{d}g + \int_{1-\lambda^-}^1 (F_X^{-1}(1-\lambda^- + 1 - t) - y)\mathrm{d}g.$$

Clearly, $g(1-\lambda^-) = 1$ implies $\rho_g\left(X - F_{x,y}^{-1}(U_X)\right) = \int_0^{1-\lambda^-} K_x^{-1}(1-t)\mathrm{d}g$ and $g(1-\lambda^-) < 1$ implies $\lim_{y\to\infty} \rho_g\left(X - F_{x,y}^{-1}(U_X)\right) = -\infty$. This completes the proof. $\square$