ACCEPTANCE SAMPLING: ROBUST ALTERNATIVES FOR SAMPLING BY VARIABLES

by

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(VOLUME I)

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Dedication

To my spiritual leader and "Father" Ustaz Mahmoud Mohammed Taha with my love. To him I pay my sincerest tribute and utmost respect.

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GLOSSARY OF TERMS

1. Property "Q": A sampling plan has property "Q" if, and only if, there is no pair of sample configurations such that the first sample leads to rejection and the second to acceptance of a batch, but the first sample is preferred to the second. One sample is preferred to another of the same sample size, n, if the units can be arranged into n pairs, one from each sample, so that in every pair the unit from the preferred sample is at least as good as the unit from the other sample and is strictly better in at least one pair.

2. Lower Specification Limit, A: This is the value of the tested characteristic such that an inspected item in a sample is classified as defective if its characteristic has a value less than A.

3. Upper Classification Limit, B: This is the value of the tested characteristic such that an inspected item from a batch is classified as marginal if its characteristic has a value between A and B, and an item is classified as effective if its characteristic has a value greater than B.

4. Attribute Sampling Schemes and Sampling By-Variables Schemes: In the decision making process the observation or the unit in the sample can either be on a quantitative basis or a qualitative basis, e.g. defective, effective and marginal. An acceptance sampling scheme which uses the qualitative characteristic is called an attribute sampling scheme. An acceptance sampling scheme which bases the decision on the values taken by a quantitative characteristic is called a by-variables sampling scheme. Typically the attribute

sampling schemes will require larger sample sizes than by-variables schemes which have broadly similar power.

5. BS6001 and BS6002: These are the British Standards relating to acceptance sampling procedures for by-attributes and by-variables schemes respectively. They comprise parameters, decision rules and algorithms with examples of applications of the schemes and tables and operating characteristics curves.

MIL-STD-105D: This is the American Military Standard for acceptance sampling by-attribute. It is broadly equivalent to BS6001.

MIL-STD-414: is the by-variables counterpart and is broadly equivalent to BS6002.

DEF-131: The British Ministry of Defence Standard for acceptance sampling schemes replaced by BS6001.

6. Acceptable Quality Level (AQL): The proportion of defective items in a batch which is regarded as acceptable by the consumer of the batch. A batch with quality at AQL has a high probability of acceptance. The exact probability depends on the sampling plan used, but for sampling schemes such as BS6001 and BS6002 the values of AQL are used to index the plans within the scheme so that the probability of acceptance is greater than 0.88.

7. Quality Scoring Functions, $Q(x)$: These are functions for scoring the quality of an item in a batch depending on the value of the characteristic X. Denoted by $Q(x)$ these quality functions are

simple monotonic functions non-decreasing with X.

8. Robustness: Many test procedures involving probability levels depend for their correctness on assumptions concerning the data generating mechanism (e.g. that the parent distribution of the sample is Normal). If the inferences are but little affected by departures from those assumptions, e.g. if the significance points of a test vary little if the population departs quite substantially from the normality the test is said to be robust. Distinction has been made between criterion robustness which refers to the probability distribution of the various decisions being little affected by the assumption and inferential robustness which refers to the inference on a particular occasion depending little on the assumed form of the distribution. The thesis is concerned with criterion robustness.

9. Robustness Measure: Robustness is the degree of insensitivity of a procedure to changes in the assumption not under test with the condition that the procedure is powerful for the parameters under test. It relates to the changes in power of the procedure when the assumptions on which it is theoretically based are violated. One measure of this robustness is given by the size of the deviation of the operating characteristic curve under specific background assumptions different from those assumed when deriving the test. The deviations are standardised to measure the bias in terms of the standard error of the single test decision which sets a random variable D to 0 or 1 as follows:

accept = 1, reject = 0.

10. Splines: Splines are piece-wise polynomials to fit a continuous function to a set of grid-points representing a functional relation f(x). They are used for interpolation purposes. To interpolate function values f(x) at the points $x = t_1, t_2, \ldots, t_m$ a spline S(x) of order n with prescribed interior knots (or nodes) $x_1, x_2, \ldots, x_{N-1}$ could be found, such that it satisfies the condition that

$$S(x_j) = f(x_j), \quad j = 1, 2, \ldots, N-1$$

and certain smoothness constraints on the derivatives of S(x) are also satisfied. As, e.g., in cubic splines this may be that the first derivative of S exists.

ABSTRACT

Existing acceptance sampling schemes have diverse advantages and disadvantages with the two main considerations of economy by sample savings and robustness to prior assumptions, conflicting. The thesis provides for a new class of schemes which preserves most of the advantages of the current schemes and sheds some of the disadvantages.

The thesis sets out the properties to be maintained in the proposed new schemes, in the order of priority: robustness, sample savings, Property "Q" and ease of practice and application. The research deals extensively with robustness and sample savings of the existing "orthodox" schemes from simple two-class attributes schemes (which are "ideally" robust but with large sample sizes) to sampling-by-variables schemes (which are not robust but do save on sample sizes). The designs are formulated with Property "Q" as well as these two properties in mind. This shaped the use of scoring functions such as the Ramp, Logistic and Cumulative Normal quality scoring functions where we define the notion of "quality function" (denoted by Q(X)) as a monotonic function of the process quality r.v., X.

One characteristic of the two-class attribute plans is the leap from defective to effective quality. The "by-variables" schemes, on the other hand, have a gradually changing "score" with the measured variable. This makes the by-attribute plans susceptible to sometimes costly errors as a consequence of misclassifications based on sharp division into "0"'s and "1"'s. This disadvantage does not arise in the by-variables schemes. The use of quality functions, $Q(X_i)$, provides

the sampling plans with the Property "Q" if monotonicity in X_i is maintained as is the case in our "new" schemes.

The new schemes in this thesis set out to fulfil the following criteria:

(1) Fill in the gap between the two existing extreme schemes by enumerating alternatives, and hence satisfy the demands of industry which is currently stranded on the two extremes.

(2) Incorporate revised forms of the orthodox schemes in the new design hence preserving almost all of the properties and levels of discrimination of quality that are already experienced in the practice.

(3) Allow more flexibility and versatility by giving some wide choice of trade-offs between robustness and sample savings. This was achieved by using a design parameter B with which one can change from a "by-variables" to a "by-attributes" or vice versa.

(4) Theoretically, give an assessment of the robustness and equivalence properties of the old schemes compared with the new ones.

For the new designs, computational procedures are devised to derive the distributions of the test statistic (average Q(X)). Equivalences are then established between the "new" and "orthodox" schemes in terms of O.C. curves for some typical plans. Sample sizes were compared for possible savings.

On the basis of these equivalences, tests for robustness are made

using a representative range of non-normal assumptions. Conclusions about the performance of different, equivalent, schemes are linked with the sample savings results.

Robustness studies are made using analytical procedures, whenever possible, otherwise simulation is adopted. Thus, analytical methods are used in the two cases of non-normal process model of the uniform distribution, and the two-point distribution. Simulation is used for the Contaminated Normal and the Lognormal process models.

Some developments and extensions of the "new" schemes into double sampling are considered and compared with attri-var plans by simulation of an example.

Some concluding remarks are made with recommendations which are highlighted in terms of B (to reflect on robustness) and of n (to reflect on sample saving).

Volume II of this thesis is an extension of the appendices. Some technical articles and notes which have some bearing on the aspects of the thesis are put there. Also, in Volume II are the important computer programs and subroutines with detailed comments, especially in the simulation computer programs and as much as possible of the main tables and figures.

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CHAPTER 1

INTRODUCTION

1.1 The General Background

Acceptance sampling schemes deal with assessing the quality of batches or lots of products by inspection of a sample of the items of these batches or lots as they are supplied in the market. Normally the outcome of the assessment is acceptance or rejection of the batch. Sometimes the fate of the batch is negotiated between the producer and consumer as far as their interests (or risks) can tolerate, or alternatively the batch can possibly undergo a remedial process.

The science of statistics constantly provides refined techniques of acceptance sampling schemes that would better protect the consumer from accepting "rejectable" (or "bad") products and the producer from offering them (*). These two view-points are translated in the statistical concepts of producer's and consumer's risks. Simply, these are the minimum degree of protection against rejection of given good quality that is tolerated by the producer and against acceptance of given bad quality tolerated by the consumer.

(*) It is worth mentioning here that, sometimes, the distinction between acceptance sampling and quality control is not properly made. The quality control is an internal "industrial" or production concern which is to do with the technical aspect of the production process of the item while acceptance sampling has a wider scope to be an "external" concern to do with the marketability (or utility) of the overall batch of items (which may originate from same or different processes). Deming (1982) and Shewhart (1931) and others would like more emphasis on control and support the idea that the goal is to "make the product right in the first place" and that without any exception the rule holds: "It is never cost-effective to inspect quality into a product"!

Under these basic concepts many sampling schemes were set up and used extensively. In fact most of today's users of such schemes give almost all of their attention to these degrees of protection. We claim that, although important, this is not enough to obtain effective results in statistical acceptance schemes that penetrate to the core of the problem. Other aspects and properties may be and indeed are as important as the degree of protection and may even determine it. The appeal of this point of view will become evident as we discuss the new properties which are endorsed in this research, especially robustness (*) and "Property Q" (Farlie 1981). The basic idea of Property "Q" is that it requires that the decision on batch quality should mirror the decision on sample quality. One of our main intentions in this work is to try to weed out any chances of decision-making in acceptance sampling that are made under the illusions of false assumptions. The assumptions are usually (if not always) about the background distributional forms that model the process or batch configuration.

With existing acceptance schemes each possessing diverse advantages and disadvantages, and with the two main attractive properties of sample savings and robustness - unfortunately pulling in opposite directions - our work set out in the quest for a new design of schemes that could preserve most of the advantages of current schemes with few (if not none) of the disadvantages. This is the objective we aspired to achieve with the very satisfying and still promising results to be detailed later.

(*) A provisonal definition of robustness, in this context, could be: "the degree of insensitivity of the decision rules to the 'assumed' background form of distribution of the process under inspection". A more rigorous definition of this essential concept is given in Chapter 5 with some details and discussions.

1.1.1 Some General Important Properties of Schemes

We set the favourable properties in any acceptance sampling schemes, beside efficiency of maintaining the assumed degree of protection, to be:

a. Robustness.

b. Sample savings.

c. Property "Q".

d. Ease of operation and application.

in the natural order of priority and importance.

It was felt that the properties (c) and (d) could be acquired partly by design as in the "new" schemes proposed below. A comment on how the orthodox schemes fare in these two cases (of (c) and (d)) will be given later.

As there was little work (Anscombe (1960)) in assessing (a) and (b) jointly, we need to study them rather rigorously, their paramount importance in the field of acceptance sampling made it necessary to give them a lot of emphasis and research. Anscombe (1960) argued that robust methods could be treated as paying a premium for protection. Essentially this argument was a perturbation type of argument for small departures from assumptions. Some work has been done in each of these two areas separately. Pearson et al (1977) and Owen (1964, 1969) worked on non-normality alone. We are not aware of any serious efforts directed mainly at sample savings per se except perhaps in the general context that relies on the inherent savings of the by-variables scheme.

1.2 Specification of the Problem and the Contribution of this Thesis

The research basically deals with the robustness of the existing orthodox schemes ranging from the simple two-class attribute plans which enjoy insensitivity to the exact distributional forms of the sampled process, to the "by-variables" plans that are heavily dependent on the distributional form (and more specially the "known σ" variables plans whence exact knowledge of σ is crucial). Coupled with this polarisation in the robustness area we find that the sample sizes are very different varying between the low levels of the "by-variables" and the high levels of their equivalent "by attributes" schemes.

We have a limited scope for manoeuvre in setting up any new alternatives in acceptance sampling: the need for such new alternative schemes is argued further on. We cannot have a scheme more robust than an attributes scheme, and if "σ" is known and the distributional form is normal the minimum possible sample size will be those of the "known σ" by-variables equivalent scheme.

The thesis suggests various alternative schemes and assesses the performance of the existing (orthodox) as well as the new ones in terms of the four properties ((a), (b), (c) and (d), listed above). The schemes are formulated with these four properties in mind but particularly on the basis of "Property Q" and the ease of operation.

One characteristic feature noticed about the existing acceptance schemes is that there is a "leap" from "defective" to "effective" quality of an item in the case of "two-class" attributes schemes. On the other hand there is a gradual change in score of quality along the quality scale

in the "by-variables" schemes. This is a favourable characteristic of the variables plans which we sought to reproduce in our designs and which we were able to maintain. This characteristic is never maintained in the attributes plans and makes them susceptible to sometimes costly errors as a consequence of misclassifying defectives as effectives or vice versa. "Marginal" cases are sharply classified as effectives or defectives. This point attracted some workers in acceptance sampling to suggest ways of dealing with marginal cases in their suggested designs though for very different reasons and aims from ours. For example, Bray, Lyon and Burr (1973) in their "three-class attribute" scheme were motivated by the need for the recognition of "marginal" items. Another attempt though not using the same words was made by what is known in the literature as the "mixed or attri-var dependent plans" (Schilling and Dodge (1969)). They were implicitly talking about marginal batches awaiting more information before sentence, this information is obtained both by effectively increasing the data and the robustness, in other words by taking more items but reducing the assumptions.

We felt correctly that this graduation effect would reduce some of the costs of wrong decisions possible under the strict classification of "0" and "1". On the other hand we saw, and will discuss below, how the graduation effect is responsible for the "paradox" (discussed by Farlie (1981)) which led to the advocation of "Property Q". This immediately suggested a mix of the two systems, the gradual variable middle (or marginal) zone and the "0" and "1". Thus, censoring the values below the lower classification limit and above the upper classification limit, (where classification limit here can be either a specification limit or a conventional limit to mark the bounds of marginal item quality). As will be discussed in the next paragraph, this has the advantage of bringing

our schemes in line with "Property Q". The point made in the work of Farlie (1981) is that "Property Q" failed in some of the orthodox schemes (namely, the "unknown σ" variables scheme and all the "attri-var" schemes that use them). The others fared well.

1.2.1 What this Thesis is Offering

The dilemma, faced by the users of acceptance sampling, namely the "robustness versus minimum sampling" puts users in the awkward position of choosing to foresake one or the other of these two desirable properties, in addition they may have to forego "Property Q". In this context our schemes set out to satisfy the following:

(i) Fill in the gap between the two extreme schemes by a flexible series of schemes. This flexibility of choice is enhanced by the ease of use of the new schemes.

(ii) Incorporate revised forms of orthodox schemes into the new system, so preserving almost all of the properties and levels of discrimination between qualities that are already experienced in the practice. This is done by varying the design parameter B (defined in section (2.2.3.1)) as shown later.

(iii) Give more flexibility and versatility by allowing a wider choice of trade-offs between robustness and sample savings; for this purpose the "equivalence" notion allows us to match our new plans to existing ones in such a way that the approximate overall O.C. behaviour is maintained.

(iv) Give, in the theory of acceptance sampling, a rigorous assessment of the robustness performance of the old schemes beside the new ones. Practically, this will give a fair assessment of the confidence to be attached to the existing schemes.

1.3 The Concept of "Scoring Functions"

We present a very useful concept called scoring functions which are meant to translate the quality of the inspected item to a scale ranging from 0 to 1. Denoted by Q(X) they are simple monotonic functions non-decreasing with the item quality , x. Defined as such they help in unifying all the schemes at least in the notation. In the next few sections we will discuss some basic aspects of the new as well as the orthodox procedures and relate them to this concept of scoring functions.

1.3.1.

Generally, the existing designs of inspection schemes are predominantly polarised into by-variables or by-attributes. The research will be dealing with the by-variables though we will consider the by-attributes and the mixed attri-var plans for comparative purposes and in helping to show how our new designs would be a suitable alternative to each of these. The powers and properties of our schemes will be revealed by comparison with orthodox schemes. The existing procedures for inspection of quality have their properties tested against several characteristics. These are the powers of discrimination, the sample size, property "Q", and the indifference of the test rules to the distribution form and other important assumptions.

1.3.2 Description of the Scoring Functions

There are seven single schemes that could be considered, namely the by variables with its two subdivisions into known and unknown-σ schemes, the two-class attribute schemes, the three-class attribute schemes, the "Ramp", the "Logistic" and the "Normal Cumulative" schemes.

For this description and thereafter (except of course when we come to the robustness studies) we will assume that the quality of the process under inspection is given by the random variables $X_i (i = 1,..,n)$ which are identically and independently distributed. For the sake of explaining the scoring schemes and their performance we use the convention that this random variable, X_i, is normally distributed with mean μ and standard deviation parameter σ. We assume quality is an increasing function of X_i and without any loss of generality for variables having a normal distribution the Lower Specification Limit, A, can be set to be 0. This convention is adopted for most of the following discussion. A new parameter B defined as an "Upper Limit" of marginal quality is used but some further connotations will be attached to it and discussed later. With these considerations the scoring functions for each of these schemes are shown below:

(i) The Simple By-Variables Scheme:

$$Q(x) = c.x + d, \qquad \text{for all } x$$

Notice that this does not restrict Q(X) to the interval (0,1), however a suitable truncation of the Normal distribution with a practically trivial degree of departure from the normality does

enable the variables plan to be added to the list as an extreme case of the Ramp scheme (see below).

(ii) The Two-Class Attribute Scheme:

$$Q(x) = 0 \quad , \quad x \leqslant A$$
$$= 1 \quad , \quad \text{otherwise.}$$

(iii) The Three-Class Attribute:

$$Q(x) = 0 \quad , \quad x \leqslant A$$
$$= w \quad , \quad A < x < B$$
$$= 1 \quad \text{elsewhere}$$

where w is a constant lying in $(0,1)$.

(iv) The Ramp Scheme:

$$Q(x) = 0 \quad , \quad x \leqslant A$$
$$= (x-A)/(B-A) \quad , \quad A < x < B$$
$$= 1 \quad , \quad x \geqslant B.$$

(v) The Normality Curve:

$$Q(x) = \Phi(z) \quad , \quad \text{for all } x$$

where $m = (B+A)/2$, $z = 6.18(x-m)/(B-A)$ and Φ is the cumulative normal distribution function. This will set

$$Q(x) \leqslant 0.001 \text{ at } x \leqslant A$$

and $$Q(x) \geqslant 0.999 \text{ at } x \geqslant B.$$

(vi) The Logistic:

$$Q(x) = \exp(r.Z)/(1+\exp(r.Z))$$

where r is a suitable scaling factor such as to let Q(x) approach 0 as $x \to A$, and approach 1 as $x \to B$, and Z is as in (v).

1.3.2.1. Note that there are some basic similarities in the general tendency of the graphs of these schemes. Practical considerations have enforced a redefinition of (v) and (vi) as hinted above (see the numerical computations chapter and the Figure (2.2) in Chapter 2). The graphs of (v) and (vi) have similar behavioural patterns and consequently led to the preference of one over the other on the basis of convenience.

1.3.3 The Implications on and Descriptions of the Decision Rules

Based on these scoring functions we have some decision rules guaranteed to satisfy Property "Q". We have looked at the average score as the most suitable and handy statistic for indicating level of quality. As the purpose is to establish robustness we build all the schemes on the basis of a unifying background distribution of X_i (the assumed normal distribution), and we note the random sample by: $X_1, X_2, \ldots, X_n$, where the required sample size n assumes different values across the schemes. The decision rules for the various schemes are as follows:

1.3.3.1. For scheme (i):

Accept if $\bar{X}_n - A \geq K.s$

Reject otherwise.

where $\bar{x}_n$ is the mean of the n randomly sampled items.

A is the user's defined Single Lower Specification Limit below which the quality of a single item is deemed defective. (For many purposes we will choose the origin to be at A so that A becomes effectively zero)

s is σ if the variance of the process is known, and in the case of the unknown variance it is the sample standard deviation. (In the building process of the design and without any loss of generality the known-σ variance is set to equal 1).

K as the criterion test constant, is the percentage point corresponding to a given probability α of rejecting the quality p_0 derived from a process mean μ. It is the solution of the following system if σ is known (and an analogous one in terms of the Non-central t-distribution and the sample variance if σ is unknown):

$$p_{0_i} = \Phi(A-\mu_i)$$

$$\alpha_i = \Phi((K-\mu_i)\sqrt{n}),$$

where i = 1 (2) indicates producer (consumer) risk point

The dependence of these plans on the normality assumption and the correct knowledge of the variance of the process (in case of known-σ plan) is vital to the efficient performance of the procedure for discrimination of the quality of the batches and, consequently, the fate of the batches.

The significant shift of the Operating Characteristic curves on the O.C. surface (*) due to the change in the variance indicates how non-robust the σ-known plans are to Variance-effect. Results of these are shown in Appendix (D.2.) for some representative code letters.

The producer could suffer from poor quality of production if the process is not as assumed. Reasons could come from one of two sources:

I. The true distributional model is skewed and/or more or less heavy-tailed than his assumed model or historical evidence would indicate. Technical failures or inefficiencies are some of the causes, for example a production line of properly controlled machines contaminated by one or more "out-of-control" machines could give rise to a skewed distribution.

II. The variance of the process has changed (as is normal in any process e.g. because of deterioration in the manufacturing equipment) but the time of a drastic critical change passed unnoticed and the variance assumption is no longer realistic. Here, too, the example given above could apply if the "out-of-control" machines were displaced in both directions from the correct setting.

(*) The O.C. surface could be looked on from one of two perspectives, either as:

(i) $Pa(p_1, p_2)$ i.e. as a function of the proportions marginal and effective,

or (ii) $Pa(\mu, \sigma)$ i.e. a function of the mean and variance of the background (Normal) process.

These two sources relate to the measures of central tendency and spread, these are what σ-plans (and to a lesser degree the s-plans) significantly relate to, even though the assumption of normality allows this relation to extend to proportions outside limits.

The decision rules for the s-plans and σ-plans are alike for the most part but with some differences, the most obvious one being that σ is replaced by S_n, the sample standard deviation (with the advantages of adjusting for the realised variability instantaneously which translates itself into more robustness in the variance-effect but with the inevitable disadvantage of increasing the sample size). Another difference of a theoretical nature is the shift from the standard normal to the derived Non-central t distribution which leads to the decision rules for s-plans to be:

$$\text{Accept if } \bar{X}_n \geq K_s \cdot S_n + A$$

$$\text{Reject otherwise}$$

where $n = n_s$ is the plan sample size for the "unknown-σ" by-variables, and $\bar{X}_n$ and S_n are the sample mean and the unbiased sample standard deviation

$$(S_n = \sqrt{([\Sigma (x_i - \bar{X}_n)^2)/(n - 1)}).$$

And the value of K_s is provided as the relevant theoretical percentage point of the Non-central t distribution with (n-1) degrees of freedom.

1.3.3.2. For scheme (ii):

The decision rules (under the assumption of a fixed n and independent r.v. X_i) relate to the Binomial distribution with parameters (n,p_0) as follows:

Accept if $r_0 \leqslant c$

Reject otherwise

where

r_0 is the sample number of defectives.

c is the maximum allowable number of defectives (determined by the producer's and consumer's points on the O.C.

There is much to be said about the good and bad properties of this scheme. We have its 'perfect' robustness, it is not influenced by the distributional form. However, it has an unavoidably large size of sample. This must count as a major disadvantage, it can be very expensive or sometimes impractical to use, e.g., in destructive tests required by acceptance sampling.

1.3.3.3. For scheme (iii):

This scheme is not greatly used yet. The evaluation of power for its decision rules depends on the Trinomial distribution. The main work in this area is due to Bray, Lyon and Burr (1973), who dealt with setting up the three-class scheme and showed some of its properties most extensively in the limited cases of accept 0 defectives, with the consequence that, for our purposes, their results are not satisfactory. However, in accordance with our convention of using the scoring functions, Q(X), for all schemes, the Three-class scoring scheme defined above depends on the

"marginal" weighting parameter w, (0 < w < 1). Due to the difficulty of applying the Three-class plans and their insignificance in current practice we will not concern ourselves with them to any great extent.

[[Some research has been done with an interesting result about the behaviour of w. As m (the maximum allowable number of marginals) increases then under the assumption of no defectives allowed, both bounds of the interval containing w will increase but get closer to each other. In compact form w lies in the interval:

$$(a(m-1)/m, \quad a.m/(m+1))$$

where the fraction a is determined by the choice of n and m.]]

1.3.3.4. For schemes (iv), (v) and (vi):

The decision rules for these three schemes are virtually the same in form:

Accept if $\Sigma\, Q(x)/n > t(B,AQL,n)$

Reject otherwise.

As will be seen later B could act as a "robustness-versus-sample-saving" adjustment or parameter. The schemes (iv), (v) and (vi) are a modified form of "by-variables" and we need to have the distributions of the mean Q(x) to determine the values of t(B,AQL,n), the relevant percentage point in the convolution distribution of the sum of the n random variables $Q(X_1)$, $Q(X_2)$, ..., $Q(X_n)$. These percentage points together with their corresponding n and AQL's will form the basis for

the above decision rules as will be shown later on. In our research these t values are defined by the 95% and 50% points on the relevant O.C. curve.

More details on the derivation processes and the determination of the parameters of the decision rules are discussed in the next two chapters.

1.4 Property "Q" in the Schemes under Study

The idea of property "Q" originated in the work of Farlie (1981). Its basis lies in the paradoxical preference shown by certain decision rules for inferior samples rather than for better ones. The following two definitions make the concept clearer and more precise:

Definition (1): For the same sample size, n, a sample "X" is preferred to another sample "Y" if and only if, $x_{(i)} \geq y_{(i)}$, for $i = 1,2,\ldots,n$ and $x_{(i)} > y_{(i)}$ for some i, where $\{x_i\}$, $\{y_i\}$ are the respective samples and $x_{(i)}$, $y_{(i)}$ are the order statistics of these samples respectively.

Definition (2): Property "Q":

Farlie (1981) defined this notion as: "A sampling plan has Property "Q" if, and only if, there is no sample configuration leading to the rejection of the batch which is preferred to one or more of the sample configurations leading to the acceptance of the batch".

In other words some plans may enter into a paradoxical situation whereby in two samples the better quality sample leads to rejection of the batch and the poorer sample leads to its acceptance. An

example of such a plan is the"unknown-σ" by-variables plan. Suppose we have two samples each of size 3: Sample (I) is [0.2, 1.0, 1.8] and Sample (II) is [0.2, 1.0, 2.8]. For "unknown-σ" plans of sample size n=3 the test procedure is to accept only if the mean, $m \geq 1.12\ S_n$, where S_n is the sample standard deviation, here A = 0.

Sample (I) has m=1.0 and S_n=0.8 [hence accept since

$$m=1.0 > 1.12\ S_n=0.896].$$

Sample (II) has m=1.33 and S_n=1.78 [hence reject since

$$m=1.33 < 1.12\ S_n=1.9936]$$

Sample (II) which is better leads to rejection while its inferior, Sample (I), leads to acceptance. This illustration reveals how important it is that we concern ourselves with such a crucial property. It is believed that Property "Q" could have a significant role in shaping the "new" scoring functions by trying to use the results of the research done on property "Q". Actually, these new schemes were chosen with specific reference to censorship and monotonocity as a way of getting away from the paradox. In the work on property "Q" it has been shown that the orthodox plans which have property "Q" include the "σ-known" variables plan and the two-class attributes plan. The above example gives direct evidence that the s-plans lack this important property. The striking feature about the configuration in the example above is the effect of extreme values on the variance. In our new designs of the scoring schemes we have suppressed lower extreme values of quality to "0" quality, and the upper ones to the quality value "1". This stems from the censoring and monotonocity principles advocated by the work of Farlie (1981). In his work Farlie dealt with quality in terms of X_i but could be extended to our $Q(X_i)$ very easily. Then $X_{(i)} < X_{(j)}$,

implies $Q(X_{(i)}) \leqslant Q(X_{(j)})$ if Q is monotonic.

The design of the new scoring functions so as to have a censoring effect on both tails of the sample together with the monotonicity of Q(X) builds strong foundations for the Property "Q" in our new systems. This is practically important where the "outlying" tendency of the observations can never be statistically recognized as such especially for small sample sizes (e.g. n=3, above). Incidently, the variables plans are mostly desired when minimum sample sizes are sought as first priority.

To show an example of the impact of using the schemes, suppose in the data of the example we chose a value B=1.8, say, (which is in line with our recommendations for plans that mimic variables plans), then the value of 2.8 in Sample (II) will be treated no differently from its counterpart 1.8 (in Sample (I)), and consequently the two samples decisions will not be paradoxical. It is needless to emphasise the dramatic consequences upon all the arguments made above if the sample size n is equal to the batch size N.

1.5 Comments and Criticisms of Existing "Orthodox" Schemes and their Rationale

1.5.1 In the beginning acceptance sampling was influenced by the simple use of go/no-go gauges. These resulted in "attribute" data which led naturally to a predominance of by-attributes plans as a basis for all the major developments in the subsequent statistical literature in the field. One benefit of this "0 and 1" classification of quality has been that the assumption of the distribution of the actual product quality is not required.

On the other side of the ledger, a heavy price has been imposed by the fact that attributes destroy the information in the sample that is utilised under a by-variables scheme. Another costly consequence is that the percent defective is impossible to control at very low levels of AQL of, e.g., 0.001 unless the attribute sample size is increased as a way of reducing the discontinuities resulting from discreteness of the numbers n and c (the sample size and the acceptance number). That leads to increased costs in sampling. This last point is not merely artificially imposed, since in today's market such AQL's are being considered for many production processes. Marquardt (1984) argued for the need to move to by-variables schemes by remarking:

> "My experience is primarily in process industry products where it is not routinely practical to work with attributes data. Today, however, even in mechanical industries, automated meterology systems make possible the more desirable variables data. To meet this challenge of the more stringent market requirements, I believe that much greater emphasis must be given to developing automated measurement procedures capable of measuring variables so that processes can be controlled on the by-variables basis. Statisticians should take the lead in showing the economic incentive for this new direction ."

We believe that the by-variables scheme could be newly treated and presented away from its disabilities of non-robustness and as close to its glamours of sample saving and informativeness as possible. Variables plans have been available since at least 1955 but missed their rightful popularity. Owen (1969) supposed the lack of enthusiasm for such variables plans to be due to the difficulties in application by users especially in translating standardised deviates into proportions defective, beside the uncertainty

of the normality assumption. He also raised questions about the appropriateness of by-variables plans and suggested ways of meeting the requirements of their assumptions. The revival of by-variables plans was one of the motives for Owen's work when he gave some suggestions (which we will take in the robustness section later). He also reviewed the past developments in connection with showing the great academic interest in by-variables. For its conciseness and relevence we quote his review:

> "Kao (1966) considers mixed attributes variables sampling plans as have Gregory and Resnikov (1955). Continuous sampling plans are prepared by White (1966) and Hillier (1964). Dodge and Stephens (1965) study chain sampling inspection. Hald (1968) designs attribute sampling plans for continuous prior distributions. Zeigler and Tietjen (1968) examine double sampling plans based on the variance. Stange (1960) and (1966), Freeman and Weiss (1964), Flehinger and Miller (1964), Campling (1968) and Lieberman (1965) consider other aspects of acceptance sampling inspection.
>
> Folks, Pierce and Stewart (1965), among other things, give an estimator of the proportion, p..... Wheeler (1968) shows that this estimator is equivalent to the one given by Bowker and Goode (1952) and by Lieberman and Resnikoff (1955), and gives an additional form of the estimator and discusses its variance. Ellison (1964) gives another derivation of Bowker and Goode (1952) estimator.
>
> Theodorescu and Vaduva (1967) give a procedure for the control of several variables simultaneously based on the generalized range.
>
> In summary, there is activity on theoretical problems of acceptance sampling, but it is not as great as one might expect when one

considers the potential usefulness of the techniques which can be developed."

We strongly feel that the interests of theoreticians may repay well if more of it is addressed to resolving the problems that dissuade the practitioners from using the by-variables more extensively.

1.5.2. In looking at the variables schemes as e.g. in BS6002 and working tables our work had arrived at the same conclusive evidence as the results postulated by Bravo and Wetherill (1980). They were surprised to find that matching (as is recognised in many published tables of the current orthodox variables schemes) is certainly very poor. Comparing our results in this respect with the published equivalences we feel that the present published tables are theoretically unfounded or too approximate to rely on in establishing the equivalences needed for the comparative study and the performances of the schemes in terms of the properties under study (mentioned before).

1.5.3. In the "by-variables" plans sampled items are presented in batches for inspection and on the basis of the sample evidence they must be sentenced as "accept" or "reject". The convention is that the variable measurement of an item, x_i may be distributed normally with expectation μ and variance σ^2. Of course, one of the limitations of "known-σ by-variables" schemes as proposed, e.g., in the BS6002 lies more especially with small samples. Wetherill and Kollerstrom (1979) mentioned that the estimation of p_0, the proportion defective, in the batch assuming σ known, is non-robust and unreliable unless double checked as they put it. And if n is small this is impossible.

1.5.4. A good survey of the different schemes and their rationale is given by Hill (1962). The monograph by Wetherill (1969) provides a valuable sketch of mathematical and practical bases of acceptance sampling.

1.5.5. In sampling "by attributes" the sample size, n, and the maximum permissible number of defective items, c, are determined on statistical grounds by considering how the probability distribution of the number of defectives, d, in the sample of n varies with the true proportion defective, π, in the batch of N items (usually, $N >> n$). In by-variables a similar technique is used but rather than the random variable being the number of defectives the mean value of the n random variables is used as a criterion.

Early methods specified two points on the O.C. curve so that the probabilities of offering a batch of a specific poor quality of π_0 or of rejecting a batch of a better quality π_1 (less then π_0) will be kept at levels that could be tolerated by both supplier and customer.

In the past one would not speak of optimum π_0, π_1 and their corresponding risk levels, instead these were subjectively formulated. They are still so chosen because so much changing human behaviour influences their setting. Although tradition has it that arbitrarily chosen values could be used for the levels of AQL (Acceptable Quality Level) the recent conventions in the literature and to an extent the practice have considered the rationalised series [0.001, 0.0015, 0.0025, 0.004, 0.005, 0.0065, 0.010, 0.015, 0.025, 0.04, 0.05, 0.065, 0.1] which provides sufficient variety for practical purposes. The values of π_0 and π_1 were meant to accommodate two different considerations: the cost implications of not offering "reasonable quality" batches or of offering substandard batches,

and the prevailing distribution of the batches being produced. Theoretically, this second consideration refers to the so-called "process curve". The concept of AQL was advocated in an attempt to accommodate these two factors mentioned. The AQL characterised the design and application of different acceptance sampling schemes such as the ASF and SRG tables of plans or the military standards MIL-STD-105 and DEF-131, ... etc. The point of importance in the construction of these is that they all agree to fixing and referring to one single point on the O.C. curve namely the AQL. The sample size is related to the batch size in a specified (if somewhat) arbitrary manner. The historical record of the batch-to-batch quality variation (theoretically reflected by the process curve though not necessarily applied there) is expected to dictate whether to operate "tightened" inspection (if quality drops), "reduced" inspection (if acceptance persists) or "normal" (otherwise). Barnett (1974) criticised the 'traditional methods'. He among other things was referring to the single point on the O.C. curve which is implicitly acknowledged in the idea developed by Bravo and Wetherill (1980) who agreed with the principle of matching at two points due to Hamaker and Von Strik (1955). The difference between Hamaker and Bravo and Wetherill is that Hámaker is looking at the mechanics of the O.C. curve at the two points while the latter just fixed these two points. To us there is no basic difference because of the nature and predictable behaviour of the O.C. curve in general. We picked the Bravo and Wetherill idea because it is more convenient and a short cut over Hamaker's which is more useful when the mathematical form of the O.C. is readily available, which is not our case.

1.5.6. We still believe that the producer's interests and the AQL as the mechanism for protection acting as summary of them could resolve the problems of incorporating the cost of supplying (witholding) poor (good)

batches. This is under the assumption that the motive of the producer is profit maximisation via greater economic efficiency. As for the costs of operating the inspection system, since different circumstances of inspection apply, we feel that this could be indirectly incorporated in the schemes by the reduction of the sample size, n. In our new schemes in theory we can on the one hand reduce n for a given discrimination efficiency as much as we are willing to forfeit robustness on the other hand. Our plans have generally reduced the sample size substantially from the by-attribute peaks without sacrificing too much robustness. As for the variation found from batch-to-batch we feel that the tradition of the process curve is worth preserving but any technique of incorporating the detection of the variation in the day to day operation of the systems of inspection will jeopardise the much sought after simplicity of operation.

CHAPTER 2

IMPLICATIONS OF THE SCORING SYSTEMS

2.1 General

In all the discussions of the "new" single scoring functions we had emphasised the monotonic property of Q(x). This had the favourable effect of guaranteeing the "new" schemes, using these Q(x), would possess property "Q". An earlier discussion of this point was done in Chapter 1. It is worth mentioning that Huber (1964) has advocated robust estimates which have similar properties to those of our Q(x) functions in the precise sense of effectively excluding the extreme values of the sample from influencing the estimates. The other characteristic of these new schemes is that they are more adaptable and flexible and so can encompass all the existing orthodox schemes and permit more scheme options.

There is no practical difference between the Logistic and the Cumulative Normal scoring system as could be anticipated and consequently the one with the greater difficulty in application can be dropped. In practice the Logistic is easier for users to evaluate. The Cumulative Normal system showed some irregularities in the convolution process when deriving the distributional form while the Ramp case was well-behaved which gave another of its credits, i.e. ease and manageability.

The idea of scoring quality against a price scale would give the acceptance sampling a new dimension in the re-evaluation of inspected products or commodities.

2.2 Rationale of These Implications

The point related to property "Q" is dealt with in Chapter 1. As for the adaptability and flexibility characteristic we will deal with in a later separate section.

2.2.1. The graphs of the new scoring schemes are given in one typical case (as in the inset of Figure 2.2 below).

No basic difference is noticed between the average Logistic and the Cumulative Normal curves. One feature of either of these two systems is that their single scoring functions tend to stay close to the tails (i.e. 0 and 1) for a longer time and then rapidly rise in the neighbourhood of their points of inflexion. This feature has an unfavourable implication on the behaviour of the derived distributions: they meander and become curly (as shown by the typical graphs in Figure 2.3), and this behaviour is of course compounded by the other jumps discussed in Chapter 3 which are due to the convolution of the discrete probabilities initiated by the spike probabilities p_0 and p_2 (the proportions defective and effective in the batch) of the single scores at 0 and 1 respectively. In the diagram the distribution of the Cumulative Normal scoring function (and that of the Logistic too) show curls which are not shown in the Ramp case. This is even more obvious when we come to the process of removing the discreteness later on. [More details on this are given in the numerical computations section of Chapter 3 together with the technical aspects of the derivations of the distributions].

The curls (and indeed the discrete jumps too) give the added complexity of making the determination of the distribution percentage

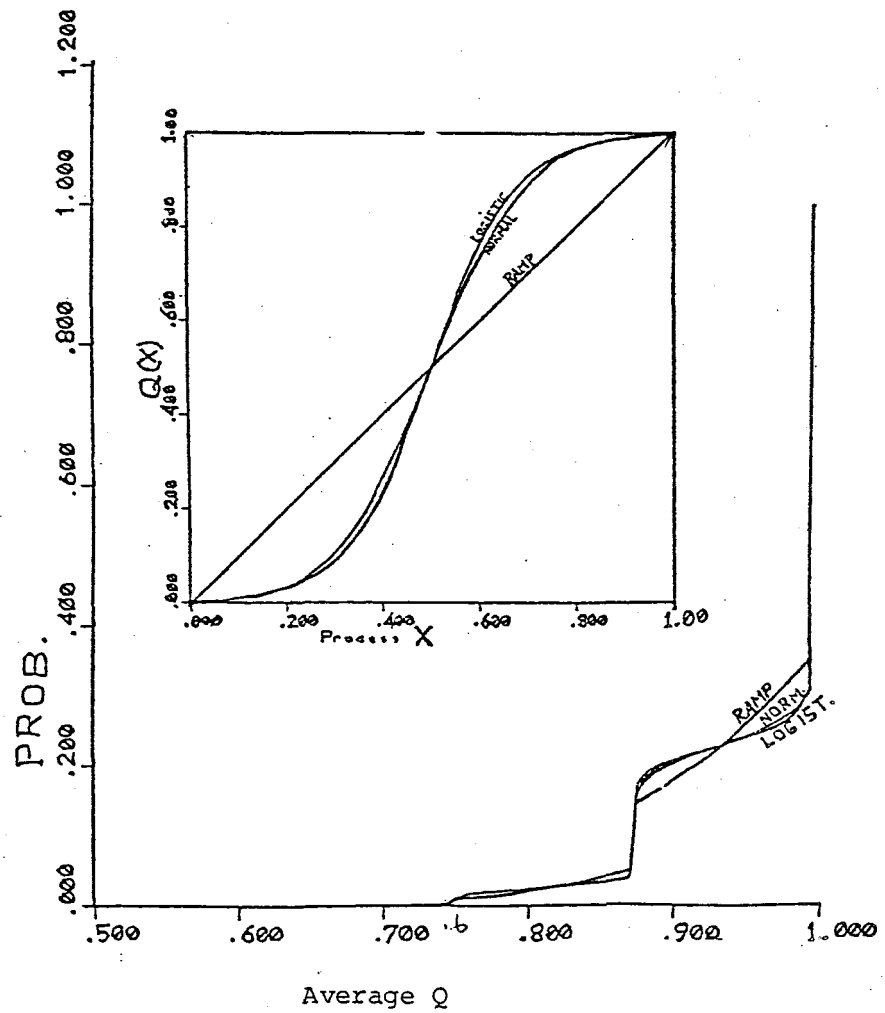


Figure (2.2): CDF of $\bar{Q}$ for B = 0.50, μ = 2.1, n = 8 for the average quality of NORMAL, LOGISTIC and RAMP (inset: single Q(x) functions for B = 1.0).

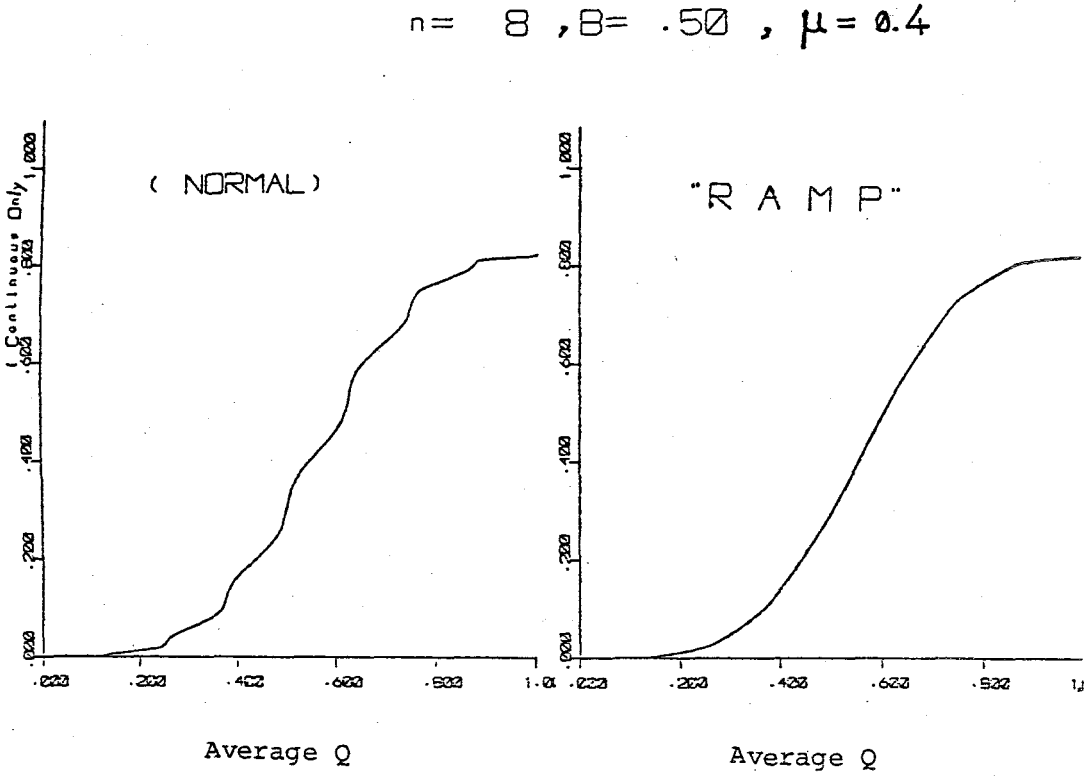


Figure (2.3): The continuous part of distribution function of average Q(x) (i.e. jumps removed), revealing the "curls" in (NORMAL) plan.

points (as needed in the establishing and determination of the plans parameters) a numerically ill-conditioned problem and hence a computationally demanding one.

The Ramp case has the discrete jumps but not the curls, so it is relatively well behaved in this respect. This coupled with the other favourable qualities (like ease of handling in practice and the fact that its n-convolution distribution acts as a good approximant of the Normal Cumulative we have every good reason to put emphasis on the Ramp system as our favoured one. Rather than dealing in depth, we will bring to light a very interesting implication of the new scoring systems that could widen and enrich the uses of the acceptance sampling. This relates to the graduating nature of the new scoring systems and how they could be used to re-evaluate the market value of the products or commodities after the inspection. Under such basis an acceptance scheme will not only answer the question of 'Accepted or rejected'?, but also new questions such as 'What is the market value of the batch'?. This takes advantage of the presence of the dealers in the transactions when the inspection takes place and/or when the decision is made.

2.2.2. The new quality functions being essentially by-variables help to evaluate the quality of the item (and hence the whole batch) in "real" money terms. The need here is to find a workable real price scale that is related to the quality level of each unit. In fact the characteristic measured in the process could be the market-worth of the item and one method to standardise this to the "0"-and-"1" scale could possibly be by assuming a minimum and maximum price quality by using ratios or by dividing by the range, ... etc.

2.2.3 Flexibility and Adaptability of the New Schemes

2.2.3.1. The parameter B, in terms of which all the scoring functions are defined, is the key behind the power, flexibility and adaptability of the new schemes. It plays two major roles:

(a) The parameter B could be scaled using the variance σ^2 of the process or batch into a new variable

$$B_\sigma = B.\sigma$$

which could act as a scale equivalent form of B, whenever we need to accommodate any changes in the variance σ^2. This relates to the study of robustness against the variance-effect which will be dealt with in the robustness section (Chapter 4), and where this role will be seen in operation and allowed a longer argument for its appreciation.

Moreover, if, as we did, we let $B_\sigma = B$ for $\sigma=1$ then B_σ/σ is the B for σ not equal to the assumed value of 1. That is to say, an undetected change in σ corresponds to an offsetting change in B when using our tabulated results. It also relates to the process curve as this deals with the variability between batches. This role can be better appreciated if we consider the following:

If A_σ and B_σ are the lower and upper (though not in the conventional "upper" specification sense) limits, then

$$A_\sigma = \mu + K_0.\sigma$$

$$B_\sigma = \mu + K_1.\sigma$$

where μ is the parental Normal distribution mean (of the process by assumption) and K_0 and K_1 are $\Phi^{-1}(p_0)$ and $\Phi^{-1}(1-p_2)$ respectively. (Here, Φ is the conventional standard Normal distribution function). The displacement of A_σ from B_σ is: $B_\sigma - A_\sigma = B_\sigma$ since A_σ is chosen to be zero,

$$= (K_1 - K_0).\sigma$$

which is non-standardised. The standardised B-value is, then,

$$B = B_\sigma/\sigma.$$

(b) B would also work as a standard "selector" (or "adaptor") of plans and schemes so that if reasonably inflated, we can approach the "by-variables" plans; and when compressed towards "0" we will effectively have the "by-attributes" plans. Moreover, between these two extremes a wide range of plans are available using the trade-off technique described below.

These two roles of B are basic to the rest of the thesis. We leave the implications of role (a), i.e. the scale-variant, to the robustness chapter and give in next section some rigorous discussion to role (b) but leave until Chapter 6 concerning the extensions of our schemes to reveal the attraction of that role of B as a "selector" of schemes particularly in our new "attri-var" dependent mixed plans. We are content at this stage to close this chapter with the following section balancing robustness demands against sample savings leaving the discussion of further points on the implications of the scoring functions until they arise naturally in the remainder of the chapters.

2.2.3.2. The Trade-offs Between Robustness and Sample Savings

With the already established fact that sample size savings and robustness are conflicting requirements, it would suggest a decision path of possible trade-offs between robustness and sample savings. That is, for a given situation in terms of the preferred property of these two (or a proportionate combination of them) we could present a clearly established plan where these requirements are met by choosing B, as postulated above. With prior knowledge of the premium which we are willing to pay for foresaking a degree of robustness for sample savings or vice versa we can immediately fix a plan to match any desired protection levels (or O.C. curve).

For the major properties in this thesis (taken to be the robustness and the reduction in sample size), the available decision levels to give alternative plans for these two properties could be made distinctly clear by the following diagram, where:

d_1 : decision to maximally save in sampling at the expense of robustness,

d_2 : decision to guarantee maximum robustness for whatever level of sample size (a large n).

Actually, the decisions available will range from d_1 to d_2. From one angle the level of decision is a function of n, the sample size; from the other angle it is a function of the degree of robustness. The parameter B could be thought of as a device controlled by these two properties and is responsible for determining the "best" decision level. To put it differently; for each given n there is a B implying a certain

risk of non-robustness while n and B together correspond to a given protection level via the "equivalence" process defined later.

The diagram is helpful in summarising these trends in the above discussion and revealing its own potentialities. In the diagram it is important to note that the two curves shown refer to different vertical domains, with different measures (and possibly different scale). But the two curves agree in their horizontal, B-axis.

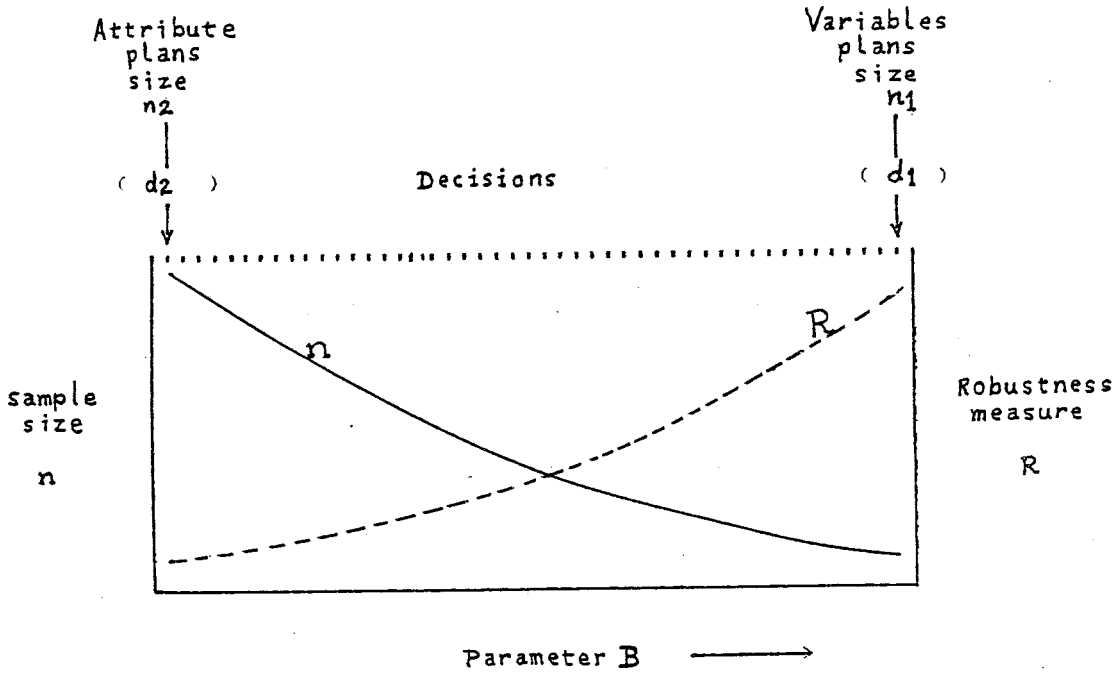


Figure 2.4: Decision Levels and Trade-Offs between Robustness and Sample Size Savings.

The diagram ties in neatly with the decision-making and also with Development III of our schemes (see Chapter 6 on Extensions and Developments).

The diagram could be looked on as (and in fact is) a nomogram; and consequently no one should be confused by the intersection point of the two curves, it is not significant and does not have any practical implications. The point of intersection can be arbitrarily altered by merely rescaling either the sample size or robustness scales.

2.3 The "Marginality" Concepts and the Scoring Systems

A subsidiary role of B, which acts as its original definition, is that it is the demarcation point between the "marginal" and the "effective" quality in the single scores. Any single item quality between A and B (the classification limits) is labelled "marginal". By this single score concept of marginality our schemes reproduce the three-classes suggested by Bray, Lyonn and Burr (1973).

In another context marginality has a somewhat different sense as suggested (though not quite explicitly stated) by the work of "attri-var" (or "mixed") plans whereby a whole batch rather than a single item is classified as marginal in the sense of awaiting a more thorough inspection (i.e. double-sampled). No confusion should arise between the two senses of this concept since use of either sense in this thesis will be clear in context. The two notions have been used intensively in the next chapters especially Chapter 5 and have been used in different distinct contexts. The first sense is directly responsible for the graduation of the quality functions and the powerfulness of B as a selector of schemes.

The second sense gives more rigorous mathematical content to replace the vague and subjectively defined notion of the "not-very-clear case" quality cited in the work of Schilling and Dodge (1969). The marginality of batches under inspection would be predetermined by the choice of B specified prior to the inspection.

CHAPTER 3

DERIVATIONS OF THE SAMPLING DISTRIBUTIONS

3.1 Introductory Note

We use the search procedure of first trying to explore solutions to the problem of derivation of the distributions (and indeed any other problem) by analytical methods if possible and next numerical methods are explored. If both of these ventures fail then simulation is the last resort due to its expensiveness. In this respect we could say in summary that:

(i) Analytical methods in the derivation of the distribution worked for the simple by-variables, the two-class and the three-class attribute schemes, but for the other ("new") schemes analytical handling was not amenable and so we resorted to:

(ii) Numerical methods: In this category we have tried forming the distributions of the Ramp, Normal Cumulative and the Logistic systems using:

(a) Johnson Systems for approximating the distributions, which makes use of the fact that unimodal distributions could have their shapes determined by using their first four moments (if available). The required distributions could be approximated by equivalent standard (Johnson) systems with the same first four moments. These Systems have four parameters such that their standard random variables are transformable into (standard) normal

variables. The four moments are used to solve for these transformation parameters. The idea is to use the first two moments (the mean and variance) to indicate location and scale and leave the major role of dictating the general shape to the standard third and fourth moments ($\sqrt{\beta_1}$ and β_2). Our feeling is that it could represent the general tendency of the shape of the distribution but with the inevitable loss of the details. And as will be clear, the details (i.e. the discontinuities) are very essential to our distributions. Hence the Johnson systems were to prove not very useful in fitting our distributions. A more elaborate note on these discontinuities and irregular features is given in section 2.2.1. of the previous chapter.

(b) Special Numerical convolution, was the most suitable method for generating the derived distributions in the cases of the Ramp, Normal Cumulative and Logistic scoring systems. It is an exact deterministic method with the usual but controlled numerical errors arising from;

- accuracy considerations
- tolerable truncation of very small probabilities which are collapsed to facilitate the convolution process, (discussed later and implemented in the Computer Fortran Program CONVOL listed in the Appendix (B.1.)).

The superiority of the Numerical Convolution method over the Johnson approximating system (when they both work) is the preservation of the details; and the build-up of the mixed convolution. The jumps due to the discrete contribution in the convolution are not tractable and traceable in the Johnson system but are in the numerical method.

(iii) Simulation had not been used in any of the derivations of the distributions of the basic scoring functions since the methods of (i) and (ii) above have solved the problem of derivations. It was only used in the robustness studies where we needed to treat each and every scoring system with a background process (or batch) of a non-normal distribution, (see robustness studies of Chapter 5).

3.1.1 Derivations of the "New" Distributions by Numerical Convolution

The derived distribution of the random variable statistic $\bar{Q}_n(x)$ is a convolution of n copies of Q(x) derived from the basic distribution of this single random variable. Generally, the single r.v. Q(x) (other than in the orthodox "pure" uncontaminated cases) have a discrete part at the ends together with a continuous part at the middle "marginal" zone [see the discription of these random variables in section (1.3.2) above]. Our choice of the Logistic or the Cumulative Normal as score functions was meant for the preservation of continuity of the random variable Q(x). Yet not only did these two functions have the discontinuity (due to truncation) but also they gave rise to another problem. That was the curls and meanders (as discussed earlier in section (2.2.1)) on top of the usual problem of the jumps. So, besides the relative ease of calculating Q(x), we benefit by adopting the Ramp case by the removal of the irregular form of the cumulative distribution function.

The special implications of this discrete-continuous mixing on the derived distributions (read convolutions) of the "new" scoring functions are discussed for each separate case. These "new" distributions have been derived using a specially designed algorithm, called CONVOL, and

is described later on and listed in Appendix (B.1.).

The distributions of these "new" schemes had to be derived numerically rather than analytically (which would have been best) mainly because of the nature of the mixture (being a contamination of a continuous random variable by a discrete contribution). The discrete probabilities in the single random variable are p_0 and p_2 (the proportions of defective and effective quality respectively). These show graphically as spikes at Q(x) equal to 0 and 1 (in that order) when n=1 and they are directly responsible for the jumps in the distribution function for n > 1, (see pictures of graphs, for example in Figure 2.2, and for more examples see Appendix (D.1.)).

The convolution runs for those values of n as in the terms of the geometric progression $\{ 2^k : k \in \mathbb{N} \}$. The factor 2, here, is due to the convolution being of the r.v. on itself (i.e. the sum of two i.i.d. random variables) at each k-th convolution. In other words, we have basically a r.v. X which is normally distributed and since this r.v. X is linearly transformed (by the relevant scoring function) into a random variable Q(x), we take the sum of copies of the previous sum each time for (k-1) times. If we denote the j-th sum by S_j then we have the following system:

$$S_0 = Q(X_1)$$

$$(*)\quad S_j(X_1,\ldots,X_m) = S_{j-1}(X_1,\ldots,X_{m/2}) + S_{j-1}(X_{m/2+1}, \ldots, X_m) \quad \text{for } m=2^j$$

In other words for sample size n we have n=m.

$$(*) \qquad S_n = \sum_{i=1}^{n} S_0(X_i), \quad n=1,2,4,8,\ldots$$

The distribution of this is the derived distribution using the numerical convolution details of which are cited in algorithm CONVOL (appended). Here we will give the theoretical framework behind the algorithm. For each k ($k \in N$) we have $n = 2^k$ and an n-fold recursive convolution of a probability distribution G with itself, where G in the initial stage is the "single score" probability distribution. This convolution is defined as:

$$G(t,k) = G(t) \qquad k = 0$$

$$= \int_0^t g(t-v,k-1)\, dG(v,k-1) \qquad k > 0$$

Though this is the rigorous mathematical background there were some numerical considerations. Firstly, a change from the strictly continuous concept of the integration sign to a summation sign because of the implications of the continuous-discrete mixture of the r.v. Q(x). The concept of the histogram was borrowed as a representation of G in terms of many small cells. Here we are assuming that the continuous portion of probability per cell is uniformly distributed within each cell. For this the whole range of Q(x) has been split into M+2 cells. The discrete part is split into its natural components p_0 and p_2 with their contents put in the first cell and the last cell, respectively. The rest of the initial distribution (i.e. the continuous part) was subdivided in the obvious manner whereby each cell (i.e. sub-interval) of the remaining equally spaced M cells of Q(x) contains its corresponding probability.

We can get more insight into this sub-division process pictorialy in Figure 3.1.1 at the end of this section. (In the program M = 100 cells).

As mentioned above we divide the probability distribution of the average of the n Q(X)'s into a large number of separate cells. Let p(i,n) represent the probability contents of the ith cell (i.e. of Q(X) = i/M), for sample size n. Initially, at n=1, we have the basic probability of the single score Q(X) sub-divided in M+2 cells as mentioned. This p(i,1) is the first unit for building the convolution which is evaluated recursively as follows:

$$d(s,2) = \sum_{i+j=s} \{C_{ij}(n) \cdot p(i,1) \cdot p(j,1)\}$$

where $C_{ij}(n)$ is the combinatorial number of occurrences as in the usual Binomial coefficient.

This gives a distribution for double the range of Q(X). The next stage is to shift from the sum of Q(X)'s to the average, then the resulting shrinking in the scale implies amalgamating the probabilities of groups of 2 neighbouring cells in one. This is done by the following simple operation:

$$p(i,2) = d(2i-1,2) + d(2i,2) \qquad \text{for } i=1,2,\ldots,M$$

which is then used in the next stage as in:

$$d(s,4) = \sum_{i+j=s} \{C_{ij}(n) \cdot p(i,2) \cdot p(j,2)\}$$

Again and in a similar manner, we have to shrink this doubled range by considering:

$$p(i,4) = d(2i-1,4) + d(2i,4)$$

$$\vdots$$

and so on.

By this method the range doubles each time and so the distribution would have to be collapsed each time to represent the distribution of the average. Each convolution would require $(M+2)^2$ additions and multiplications where M+2 is the number of partitioned cells. The Central Limit Theorem would also reduce the effective range of the average $\bar{Q}(X)$ and hence the simple compression back to M+2 cells will not be needed all the time. This necessitated a careful programming in keeping track of the shrinkage in the probability distribution. The cells number 0 and M+2 contain the spike probabilities p_0 and p_2, respectively, and of course these two end cells contain purely discrete probabilities. In the convolution these are responsible for the discrete-contamination of the whole distribution. When the range doubles we know that the compression may necessitate averaging two neighbouring cells. This averaging of the cells leads the discrete contamination to affect more cells than is intuitively expected. Consequently some care is required in the tracking of these contaminations as this is necessary for their subsequent removal to give a smooth and pure representation of the continuous part. (As is clear in the comments and details of the Fortran routine CONVOL this process of tracking is accomplished by physically separable arrays used to store the contaminant (discrete) and the continuous parts).

To complete the presentation of the convolution process there are few additional points to raise:

(i) The truncation (or folding) of the "negligible" probability contents at the tails is done within tolerable limits. Whenever more than M/2 cells at the tails contain a sum of probability less than ε (a carefully determined level of tolerable probability error), each of the relevant Q-cells are collapsed (truncated) and the corresponding part of the sum is added to the next end cells, as appropriate and relevant.

(ii) The value of μ, of the normal distribution of the process X_i, is directly related to p_0, the proportion defective. The value of A determines value of p_0 in relation to the normal distribution function. So as to hold the general conventions of values of AQL (or p_0), we found that the equi-spaced values of μ with intervals of 0.1 would represent such convention very well indeed. The range of μ used is (0.4 to 3.4) since they cover the range of values of p_0 that are of interest in the theory and application in acceptance sampling.

We have taken the case of the Ramp as the representative case of our "new" schemes. The Ramp as far as the distributional computations has the same typical numerical features as the "Logistic" and the "Normality Cumulative" schemes. It will therefore suffice to only consider the Ramp case in this work.

For the Ramp $Q(X_1)$, with values in [0,1], is a piece-wise linear transformation, (i.e. composed of a constant, a linear trend and a constant;) of X_1, the normally distributed process with mean μ and standard deviation 1. But in the open interval (0,1), Q(X) is a linear transformation of

X_1, and so the probability contents of the ith cell - [i.e. p(i,1) of the initial distribution of the single $Q(X_1)$] - are relatively easy to deduce by using the fact that X is the inverse function of Q(X). To calculate p(i,1), the probability content of the cell demarked from below and above by $q_1 = i/M$ and $q_2 = (i+1)/M$ respectively we used the linear relation of Q and X. By inverting Q(X) this p(i,1) corresponds to the probability defined as:

$$p(i,1) = \Phi(A-\mu) \qquad \text{for } Q(X_1) = 0$$

$$= 1 - \Phi(B-\mu) \qquad \text{for } Q(X_1) = 1$$

$$= \Pr\{Q^{-1}(q_1) < X < Q^{-1}(q_2)\} \qquad \text{elsewhere}$$

where the two bounds of X (within the curled brackets above) are simply the inverses of the quality function Q(X). And since our X is normally distributed with mean μ and standard deviation 1, this probability is known and is given by the standard normal theory. The following figure puts the above argument diagrammatically, and shows clearly how i-th cell is assigned its probability content p(i,1) before the convolution is operated, as in Figure (3.1.1.) below.

As is obvious (earlier in (*)), the values of the sample size, n, visited by the numerical convolution in CONVOL have increasingly yawning gaps for larger values of n. This is no major problem since:

- we could interpolate successfully for lower values of n, since the spacing between the values of n is small then.

- For larger values of n we resort to the Central Limit Theorem and the normal approximation to take over early enough.

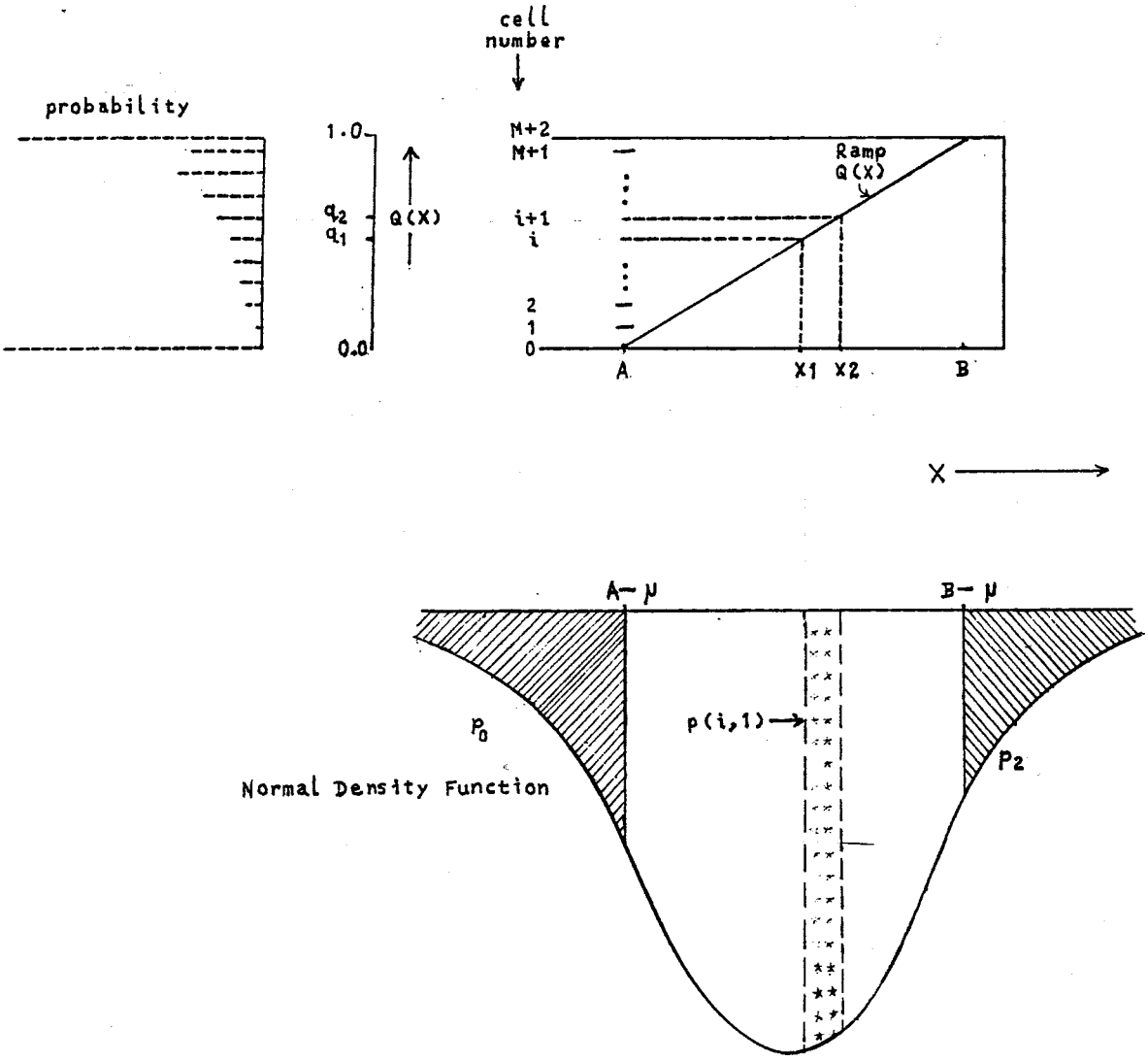


Figure (3.1.1): The relation of X to the partitioned Q-cells used in evaluating p(i,1) (the Probability Content of the i-th cell), and how the cumulative probability for Ramp Q(X) is formed.

Asymptotically the discontinuities will tend to be smaller and vanish.

- We are not interested in a comprehensive tabulation of the "new" schemes, we only want to emphasise their potentialities and explore them as alternatives, so a selective representation of such schemes would serve our objectives.

Existing schemes have the ratio of successive sample size of 1.6:1, hence widening gaps are already a common place in the literature.

3.1.2 The Tables of the Numerical Distribution

So, as a result of the numerical derivations and its procedures advocated as above we have the tabulations of the distribution for equi-spaced average Q(X) for each of the n values that lie within the path of the specific convolution process defined above. The tables are voluminous and are stored in the computer disc storage, but some representative tabular forms are displayed in the Appendix (C.1.). An example of the tabular distribution is given by table (3.1.2(a)) below. For completeness some percentiles for each case were contained in the tables (see Appendix (C.2.). A display of some of their contents is given in table (3.1.2(b)) below.

Though the main distribution tables could be seen as self-explanatory a description of their presented form may be helpful in revealing their contents. For a given sample size n we have a table for each of the values of μ. This table has serially numbered rows representing probability distribution for equi-spaced Q-values, and has columns standing for

Table (3.1.2.(a)): One example of the tabular distribution for the Ramp average Q(x).

Sample size = 8 B = 0.5(0.2)1.9 AMU = 1.3 (P0 = 0.0968)

P1 0.1151 0.1775 0.2478 0.3239 0.4032 0.4825 0.5586 0.6289

B	**(0.5)	**(0.7)	**(0.9)	**(1.1)	**(1.3)	**(1.5)	**(1.7)	**(1.9)
100 QMIN	0.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000
100 SCAL	1.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000
QMAX	1.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000
[0]	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
[1]	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
[2]	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
[3]	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
[4]	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
[5]	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
[6]	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
[7]	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
[8]	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
[9]	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
[10]	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
[11]	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
[12]	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
[13]	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
[14]	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001
[15]	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001
[16]	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001
[17]	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001	0.0002
[18]	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0003
[19]	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001	0.0002	0.0004
[20]	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0003	0.0005
[21]	0.0000	0.0000	0.0000	0.0001	0.0001	0.0002	0.0004	0.0007
[22]	0.0000	0.0000	0.0000	0.0001	0.0002	0.0003	0.0005	0.0009
[23]	0.0000	0.0000	0.0001	0.0001	0.0002	0.0004	0.0007	0.0012
[24]	0.0000	0.0000	0.0001	0.0001	0.0003	0.0005	0.0009	0.0016
[25]	0.0000	0.0000	0.0001	0.0002	0.0003	0.0006	0.0012	0.0021
[26]	0.0000	0.0001	0.0001	0.0002	0.0004	0.0008	0.0015	0.0027
[27]	0.0000	0.0001	0.0001	0.0003	0.0006	0.0010	0.0019	0.0034
[28]	0.0001	0.0001	0.0002	0.0004	0.0007	0.0013	0.0024	0.0043
[29]	0.0001	0.0001	0.0003	0.0005	0.0009	0.0017	0.0031	0.0054
[30]	0.0001	0.0002	0.0003	0.0006	0.0011	0.0021	0.0038	0.0068
[31]	0.0001	0.0002	0.0004	0.0008	0.0014	0.0027	0.0048	0.0084
[32]	0.0001	0.0003	0.0005	0.0010	0.0018	0.0033	0.0059	0.0104
[33]	0.0002	0.0003	0.0006	0.0012	0.0022	0.0041	0.0073	0.0127
[34]	0.0002	0.0004	0.0008	0.0015	0.0028	0.0050	0.0090	0.0155
[35]	0.0002	0.0005	0.0010	0.0018	0.0034	0.0062	0.0109	0.0187
[36]	0.0003	0.0006	0.0012	0.0022	0.0042	0.0075	0.0132	0.0226
[37]	0.0003	0.0007	0.0014	0.0027	0.0051	0.0091	0.0159	0.0270
[38]	0.0004	0.0009	0.0018	0.0033	0.0061	0.0110	0.0191	0.0321
[39]	0.0006	0.0011	0.0022	0.0041	0.0074	0.0132	0.0228	0.0380
[40]	0.0007	0.0014	0.0027	0.0050	0.0090	0.0158	0.0270	0.0447
[41]	0.0009	0.0018	0.0033	0.0060	0.0108	0.0188	0.0319	0.0524
[42]	0.0012	0.0022	0.0040	0.0073	0.0129	0.0223	0.0375	0.0610
[43]	0.0014	0.0027	0.0049	0.0087	0.0153	0.0263	0.0439	0.0707
[44]	0.0017	0.0032	0.0059	0.0104	0.0182	0.0309	0.0511	0.0815
[45]	0.0020	0.0039	0.0070	0.0124	0.0214	0.0361	0.0592	0.0935
[46]	0.0024	0.0046	0.0083	0.0146	0.0251	0.0421	0.0683	0.1068
[47]	0.0028	0.0054	0.0098	0.0172	0.0294	0.0488	0.0784	0.1214
[48]	0.0032	0.0063	0.0115	0.0202	0.0342	0.0563	0.0896	0.1373
[49]	0.0037	0.0074	0.0135	0.0235	0.0396	0.0647	0.1020	0.1546
[51]	0.0055	0.0105	0.0187	0.0319	0.0528	0.0845	0.1304	0.1934
[52]	0.0068	0.0126	0.0220	0.0371	0.0606	0.0960	0.1466	0.2149
[53]	0.0083	0.0150	0.0258	0.0429	0.0694	0.1087	0.1641	0.2377
[54]	0.0099	0.0177	0.0301	0.0495	0.0791	0.1225	0.1829	0.2619
[55]	0.0117	0.0207	0.0349	0.0568	0.0898	0.1375	0.2030	0.2873
[56]	0.0136	0.0241	0.0402	0.0649	0.1015	0.1539	0.2245	0.3139
[57]	0.0158	0.0279	0.0462	0.0738	0.1144	0.1714	0.2473	0.3415
[58]	0.0181	0.0320	0.0528	0.0837	0.1284	0.1903	0.2713	0.3701
[59]	0.0207	0.0366	0.0601	0.0945	0.1436	0.2105	0.2965	0.3995
[60]	0.0235	0.0416	0.0681	0.1063	0.1600	0.2319	0.3227	0.4296
[61]	0.0266	0.0471	0.0769	0.1192	0.1776	0.2546	0.3501	0.4602
[62]	0.0299	0.0532	0.0865	0.1331	0.1964	0.2784	0.3783	0.4911
[63]	0.0345	0.0606	0.0975	0.1485	0.2166	0.3035	0.4073	0.5222
[64]	0.0407	0.0696	0.1101	0.1653	0.2382	0.3296	0.4369	0.5534
[65]	0.0473	0.0794	0.1237	0.1833	0.2610	0.3567	0.4671	0.5843
[66]	0.0544	0.0900	0.1383	0.2026	0.2849	0.3847	0.4976	0.6149
[67]	0.0620	0.1014	0.1541	0.2230	0.3100	0.4135	0.5283	0.6449
[68]	0.0702	0.1137	0.1709	0.2447	0.3361	0.4430	0.5590	0.6742
[69]	0.0788	0.1269	0.1889	0.2675	0.3632	0.4729	0.5895	0.7027
[70]	0.0881	0.1409	0.2080	0.2915	0.3912	0.5032	0.6197	0.7301
[71]	0.0979	0.1559	0.2282	0.3166	0.4200	0.5337	0.6494	0.7564
[72]	0.1083	0.1718	0.2496	0.3427	0.4494	0.5643	0.6784	0.7814
[73]	0.1192	0.1886	0.2720	0.3697	0.4793	0.5946	0.7066	0.8051
[74]	0.1309	0.2064	0.2955	0.3976	0.5096	0.6247	0.7338	0.8272
[75]	0.1432	0.2250	0.3199	0.4262	0.5399	0.6542	0.7598	0.8479
[76]	0.1593	0.2464	0.3458	0.4552	0.5700	0.6829	0.7845	0.8670
[77]	0.1762	0.2687	0.3723	0.4845	0.5999	0.7107	0.8079	0.8845
[78]	0.1938	0.2917	0.3996	0.5142	0.6296	0.7376	0.8299	0.9005
[79]	0.2120	0.3156	0.4276	0.5441	0.6588	0.7635	0.8505	0.9150
[80]	0.2309	0.3402	0.4561	0.5741	0.6874	0.7882	0.8695	0.9280
[81]	0.2504	0.3655	0.4851	0.6040	0.7153	0.8116	0.8870	0.9395
[82]	0.2705	0.3915	0.5145	0.6337	0.7424	0.8336	0.9030	0.9497
[83]	0.2912	0.4182	0.5441	0.6630	0.7683	0.8542	0.9174	0.9585
[84]	0.3125	0.4453	0.5738	0.6918	0.7931	0.8733	0.9303	0.9662
[85]	0.3344	0.4730	0.6035	0.7198	0.8167	0.8908	0.9418	0.9728
[86]	0.3568	0.5010	0.6330	0.7470	0.8388	0.9067	0.9519	0.9783
[87]	0.3797	0.5293	0.6623	0.7732	0.8594	0.9211	0.9607	0.9829
[88]	0.4007	0.5545	0.6880	0.7961	0.8774	0.9335	0.9681	0.9867
[89]	0.4196	0.5763	0.7100	0.8158	0.8929	0.9441	0.9743	0.9898
[90]	0.4385	0.5979	0.7315	0.8345	0.9072	0.9535	0.9795	0.9923
[91]	0.4574	0.6194	0.7523	0.8522	0.9202	0.9617	0.9839	0.9942
[92]	0.4764	0.6405	0.7724	0.8687	0.9319	0.9688	0.9876	0.9958
[93]	0.4953	0.6612	0.7917	0.8840	0.9424	0.9748	0.9905	0.9970
[94]	0.5142	0.6816	0.8100	0.8981	0.9516	0.9799	0.9928	0.9978
[95]	0.5329	0.7014	0.8274	0.9110	0.9596	0.9841	0.9947	0.9985
[96]	0.5515	0.7206	0.8437	0.9226	0.9665	0.9875	0.9960	0.9989
[97]	0.5700	0.7392	0.8589	0.9329	0.9723	0.9902	0.9971	0.9992
[98]	0.5882	0.7571	0.8730	0.9419	0.9771	0.9923	0.9978	0.9995
[99]	0.6062	0.7742	0.8858	0.9498	0.9809	0.9938	0.9983	0.9996
[100]	0.6239	0.7905	0.8975	0.9564	0.9839	0.9949	0.9986	0.9996
[101]	0.6239	0.7905	0.8975	0.9564	0.9839	0.9949	0.9986	0.9996

Table (3.1.2.(b)): An example of some percentage points of the distribution of average Q(x). (These relate directly to table (a) above).

* S.SIZE= 8 " Ramp case" AMU=1.30(i.e. PO= 0.0968) *							
* QMIN x100: 0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000 *
* SCALEx100: 1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000 *
* P1 x 100: 11.51	17.75	24.78	32.39	40.32	48.25	55.86	62.89 *
P[Q < T]	.....TEST CRITERION ,T						
0.01	0.5241	-0.1000	0.4702	0.4371	0.4055	0.4000	0.3456	0.3183
0.02	0.5769	0.5413	0.5101	0.4791	0.4456	0.4135	0.3826	0.3535
0.03	0.6126	0.5706	0.5369	0.5042	0.4712	0.4331	0.4062	0.3761
0.04	-0.1000	0.5931	0.5573	0.5237	0.4905	0.4566	0.4240	0.3931
0.05	0.7500	0.6117	0.5740	0.5396	0.5055	0.4717	0.4386	0.4070
0.06	0.6419	-0.1000	0.5882	0.5531	0.5187	0.4845	0.4509	0.4189
0.49	-0.1000	0.8379	0.8020	0.7671	0.7329	0.6953	0.6574	0.6196
0.50	-0.1000	0.8415	0.8054	0.7705	0.7362	0.6986	0.6607	0.6228
0.51	-0.1000	0.8451	0.8088	0.7739	0.7395	0.7019	0.6639	0.6260
0.89	-0.1000	0.9802	0.9465	0.9141	0.8802	0.8481	0.8113	0.7731
0.90	-0.1000	0.9860	0.9525	0.9202	0.8866	0.8542	0.8175	0.7795
0.91	-0.1000	0.9919	0.9586	0.9266	0.8933	0.8606	0.8242	0.7862
0.94	-0.1000	-0.1000	0.9789	0.9484	0.9164	0.8820	0.8476	0.8102
0.95	-0.1000	-0.1000	0.9865	0.9568	0.9255	0.8917	0.8571	0.8201
0.96	-0.1000	-0.1000	0.9947	0.9661	0.9357	0.9027	0.8681	0.8315

N.B.

Each of the negative entries is a convenient way of indicating that a percentage point is "non-existent".

different levels of B (B ranges in steps of 0.2 from 0.5 to 1.9 across the table). At the head of these tables are four rows labelled QMIN, SCALE, QMAX and P_1. Respectively, these stand for the minimum value of the convoluted (and hence, possibly, compressed) Q-value, the scale which changes due to shrinkage compression process, the maximum value of the convoluted average Q and the proportion marginal determined by the value of B.

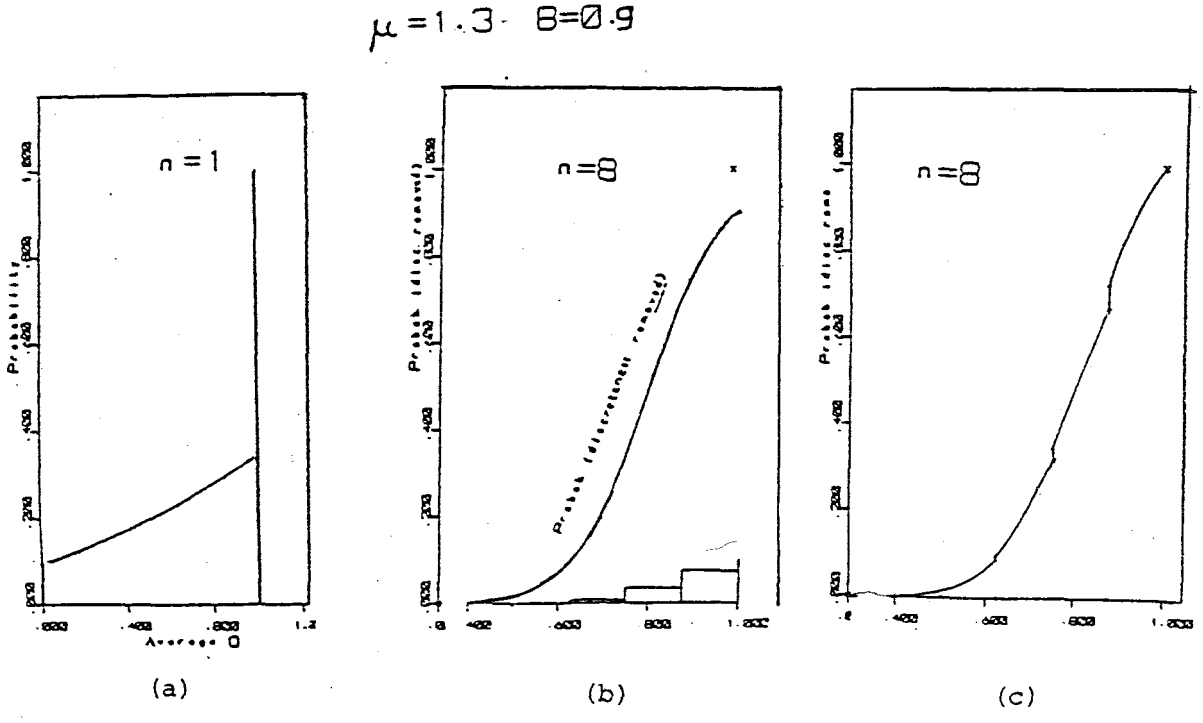
In these tables the average Q values are indicated by the serial numbers (in the square brackets). The addition of the product of these serial numbers and the scale to the minimum Q, QMIN, gives the Q values. That is to say:

$$Q_i = \text{QMIN} + i \,.\, \text{SCALE}$$

To relate this to the previous section the probability content corresponding to Q_i is p(i,n) as defined above.

3.1.3. Graphical representations of the distributions are given in the Appendix (D.1.). Here below, is one distribution graph to give a typical pictorial representation of the tabulated distributions. As an example here and for explanatory purposes in this chapter we took the sample size n=8, B=0.9 and μ=1.3 (i.e. part of the same example used in tables (3.1.2) above). Figure (3.1.3(c)) below gives the final shape of the distribution of the case of μ=1.3 for B=0.9 (which refers to the third column of each of the tables above). This case when considered from the initial single Q(X) until the convolved final form is enough to reflect on how the convolution evolves in a manner typical of all other cases. Of course, different values of p_2 are determined amongst

Figure (3.1.3): Some illustrative graphical forms of the distributions (a typical example Ramp case of n = 8, μ = 1.3, B = 0.9).



(a) (b) (c)

other factors by the value of B such that if other things remain the same a low B value dictates a high p_2, and vice versa. This does change the graphs as is clear on the appended diagrams, a higher B value permits more continuity than a lower one.

The initial single r.v. Q(X) from which the graph for $\mu=1.3$ emerged as a convolution is given in Figure (3.1.3(a)) so as to give an idea of the transition from the initial to the final convolved form shown in Figure (c) . These two figures can be used to illustrate the formation of the jumps in the final form (as in (c)) from the two spike probabilities of the initial single stage (as is seen at the tails in Figure (a)).

Figure (3.1.3(b)) shows the splitting of the tabulated distributions into the continuous and discrete part. This is needed (especially in section (3.3) below) so as to smoothly connect the grid points of the purely continuous part separately by the cubic-spline representation. Once this spline representation is made and stored the purely discrete part could always be precisely brought in.

3.2 A Note on our Experience with Johnson Distribution-Fitting Systems And on the Work on Convolutions

It is note-worthy that the area of deriving such distributions (with a discrete-continuous mixture) did not receive much (if any) attention in the literature. Consequently, and to our belief, it is still fertile for different new techniques and methods. In the rest of this section we give as a basis for future developments in this area a note of our results when attempting different methods before our numerical

convolution. Besides we cite some of the relevant work in the area of convolutions which were only in the context of a purely continuous or purely discrete nature but no mixtures.

3.2.1. As mentioned earlier we have attempted analytical handling in deriving the "new" distributions but it was not possible to arrive at a compact form for these distributions. / The only analytically viable tool was the derivation of the moment generating function. The moments have always been a focus of attention in the early Pearsonian Distribution Systems (see Pearson (1963)), and in their counterparts in what is known as the Johnson Systems (Johnson (1949)). These lean, in essence, on the fact that any two unimodal distributions with the same first four moments have the same general shape and behaviour. And out of the Johnson analysis came the recent enhancement by Hill et al (1976) to put these findings of the Johnson Systems into algorithmic forms.

Though in principle we could evaluate the moment generating function, and hence the first four moments, the Johnson Systems (and by analogy the Pearsonian ones) would not reproduce the "new" distributions. The reasons are four-fold:

(a) The most specific reason is that our "new" distributions are characterised by the discontinuities (or jumps) which are most significant for the smaller n values.

(b) Our moments lie close to, but not on, the boundaries of the different systems and according to Hill et al (1976) such cases are not easy to deal with; such cases were excluded by them from their algorithm.

(c) The empirical fact that the fourth moment is numerically unstable.

(d) "Moments fits" are known (Pearson (1963)) to fail at the steep tail (see also Pearson and Stephens (1962)). According to Pearson long tailed distributions stand a better chance in moment-fits than short tailed ones.

In Appendix (E.1.) we have the graph for the (β_1, β_2)-plane (the basic tool of the Johnson (and Pearsonian) Systems. Shown, also there, are some points of our distributions on that plane. Some remarks on these points are also included there followed by the derivation of the moment generating function.

The theory of the Johnson Systems as cited by Johnson (1949) tells us that even at their best the moments are expected only to provide approximations to our distributions at the general trend level. That is to say, the jumps (discontinuities) resulting from contribution to the convolution by the discrete random variable and originating from the initial spike probabilities p_0 and p_2 are ignored. Of course, these details of the jumps are essential, especially when we deal with deciding on the percentage points for the purposes of formulating the decision rules of the "new" schemes. A look at the graph of the numerically convolved distributions (shown in the representative example above) may clarify this point more. We should not expect the Johnson System to reveal these discontinuities. Its only effectiveness goes as far as matching the four moments and so may be suitable for mimicking our distributions for large values of n (i.e. large enough for the discontinuities to vanish or become "tolerably" negligible).

3.2.2. The work done in the development of techniques to evaluate distributions that arise in a convolution context include Baxter (1981), Cleroux and McConalogue (1976), McConalogue (1978 and 1981). Though their ideas apply effectively to purely continuous distributions or purely discrete ones we still feel that the n-fold convolutions of mixtures of discrete-continuous random variables did not receive a direct attention. The reason for lack of developments in the area of mixtures may possibly be that there are few problems in applied statistics which give rise to such "mixed" models.

3.3 Evaluation of the Percentage Points from the Tabulated Convolutions

As developed earlier we have a certain grid of the distribution of the statistic parameter mean $\bar{Q}$ in a tabular form. Generally, and for our work, we would like to be able to find the values of this statistic corresponding to any set of prespecified values of probability. We are interested in determining the percentage point, say t, such that:

$$\Pr\{\bar{Q} \geq t\} = \alpha,$$

where α is given.

3.3.1. The major difficulty is that due to the distribution being a mixture of convoluted discrete and continuous probabilities there arises certain jumps in the distribution function at the specific points where $t = r/n$, for $r=0,1,\ldots,n$ (where n is the sample size). This difficulty becomes more serious if we know that these jumps may or may not be precisely reflected by the tabulations and we need to locate them beside measuring them. This could only be done by:

(i) splitting the distribution into its continuous part and discrete one,

(ii) removing the contamination from all the grid points, and

(iii) specially treating the points of the continuous part with a cubic spline representation to link the otherwise tabular points of the distribution (for which we used the FORTRAN library NAG routines : E01BAF and E02BBF of spline fitting).

In other words, the discontinuities at t = r/n, for r=0,1,...,n, may coincide with the required probabilities. If the required probability lies at the top or the bottom of the jump no new problem arises. However, if it lies within the jump the required probability is not directly accessible except by the device of an additional random experiment to smooth all the discontinuities. We do not believe this to be good sampling practice and we restrict the cases to those in which the required p is directly accessible. The spline will provide us with a 'continuous' continuous part of the distribution, and since we know the precise position and size of the jumps these could be added to the cubic spline representation to form the whole of the distribution with the position and the extent of the discontinuities precisely known.

The size of the jumps is given by the distribution of the discrete probability originating from the spikes p_0 and p_2 at the ends of the range of the mean of Q(X)'s and will show as a trinomial distribution in the n-fold convolution. Formally, we have this discrete distribution as follows:

$$\Pr\{i \text{ effectives}, 0 \text{ marginals}, (n-i) \text{ defectives: given } n\}$$

$$= C(n,0,i) \quad p_2^i \, p_1^0 \, p_0^{(n-i)} \qquad \text{for } i=0,1,2,\ldots,n$$

$$= (p_0 + p_2)^n \left[C(n,0,i) \cdot (p_2/(p_0+p_2))^i \cdot (p_0/(p_o+p_2))^{(n-i)}\right]$$

where,

$$C(n,0,i) = n!/(i!\ 0!\ (n-i)!),$$

$$p_0 = \Phi(A-\mu),$$

$$p_2 = 1 - \Phi(B-\mu),$$

$$p_0 + p_1 + p_2 = 1,$$

and A, B are the specifications limits as before.

The rectangular brackets contain the usual term of standard (or proper) Binomial distribution model. The distribution jump represented by this trinomial form would be called the Trinomial jump at the ith position. It can be added at the ith position to its continuous counterpart produced by the cubic spline. These specific positions may need to be interpolated and that is where the spline-fit has its vital role.

3.3.2 Problem of "Non-Existence"

All the above made it generally possible to find the percentage points as given in the tables (showing some of the important percentage points in the Appendix (C.2.)) with one major exception. This is the non-

existence of percentage points for certain points of the distribution (in the tables in the Appendix these show, for convenience, as negative values). This apparent side issue, gave rise to some complications of non-existence of some schemes. The feature is likened to the problem of percentage points in the discrete random variables such as in the Binomial model. In fact the same basis for the argument of non-existence holds in our case giving rise to non-existent inverses of our composite probability function. This problem will reappear in the context of establishing the parameters of our new schemes as will be discussed in Chapter 4.

3.3.3 Mathematical Vision of the "New" Distributions as Related to Trinomial Jumps

As another way of perceiving the above discussion, and though the distribution produced by removing the contaminating discrete part is stored in a tabular grid of M+2 points, it can be theoretically conceived of as a function F(y: n,A,B) defined as:

$$F(y) = \lim_{d \to 0} \{F(y-d) + F(y+d)\}/2$$

The form of the distribution is useful in finding the probability for any cut-off percentage point of the mean quality for given sample size, $n \in \{2^k : k=0,1,2,3,...\}$, of X_i (distributed as $N(\mu,1)$, and given the specification limit A=0 and a level of B. Thus:

$$Pr[\bar{Q}_n \leq t \quad \text{given } n, A, B]$$

$$= F(t : n, A, B) + \sum_{i=0}^{j(t)} H(i : n, A, B)$$

where;

$j(t)$ = integral part of $(n.t)$

and $H(r : n,A,B)$ is a Trinomial probability term of 0 marginals, r (total number of defectives plus effectives), with our usual parameters n, p_0, p_1, p_2.

What we have done is to replace the "continuous" part (given by the grid points) by a smoothly connected spline representation so as to enable interpolation. Then the Trinomial jumps are allocated precisely and integrated within this new representation.

CHAPTER 4

ON THE DETERMINATION OF TEST CRITERIA FOR NEW SCHEMES AND EQUIVALENCE

4.1 General

In the previous Chapter 3, the percentage points for the selected percentages of the cumulative probability of the "new" average quality statistic, $\bar{Q}_n(X)$, were evaluated using the continuous spline-fit and the discrete Trinomial jumps. These results were used here below to set up and establish the decision rules for our "new" schemes. Once these were determined the "equivalences" (or matchings) with the orthodox schemes were established next. In all these matchings interpolation was used extensively. The problem [discussed earlier in section 3.3.2] of "non-existence" of the percentage points for some probabilities of the statistic $\bar{Q}_n(X)$ arises more significantly for lower n values and tends to diminish as n gets larger.

The determination of the decision rules and their parameters are based on the idea of the O.C. curve. Two points on the O.C. curve are felt to be enough for fixing the plan parameters. If one or both of these two O.C. points correspond to percentage points that are "non-existent" then a plan with such O.C. specifications will not be available.

Two points on the O.C. curve are also used in the matching (or equivalence) procedure of our schemes with other schemes. In the next sections we give a review of the matching procedures in the literature together with our chosen method. The sections following it (sections

(4.5.1 - 4.5.4)) will consider the equivalence (or matching) of our schemes with the orthodox ones, and the techniques used in making them. These are followed by sections with notes and comments on equivalences and the performance of sample savings of our "new" schemes in relation to the orthodox ones.

4.2 Equivalence and Matching Procedures

4.2.1. The concept of "equivalence" in the literature: In theory, two plans with different parameters and decision procedures are said to be "equivalent" if their O.C. curves coincide. In practice this is not needed exactly as it is a stringent requirement to meet. All the work done in this respect is based on the fact that near the top (or bottom) of the O.C. curves the would-be-equivalent curves should be very close and so close elsewhere that the protection levels are practically coincident.

In the literature, there are basically two definitions to this effect; one is attributable to Hamaker and Van Strik (1955), and the other is due to Bravo and Wetherill (1980). Hamaker and Van Strik (1955) defined O.C. curves to be "equivalent" when they share the point of control p50 (i.e. fix the indifference point of 50% acceptance) and have the same relative slope h at that point of p50; where

$$h = - (d \ln P_a / d \ln p)_{/p50} = - 2(p.d P_a/dp)_{/p50}.$$

Bravo and Wetherill (1980) suggested matching at the two points corresponding to the AQL and the point of indifference quality stipulated to be where the probability of acceptance is $P_a = 50\%$. (There is a

variant of this that uses 10% rather than the 50% but we have numerical evidence that this would not matter in principle).

4.2.2. We find that the one point control on the O.C. curve as in the Hamaker and Van Strik methods with its mathematical elegance necessarily brings about unnecessary complications in our case especially when we know that our probability distribution of the test statistic $\bar{Q}_n(X)$ is a numerical tabulation rather than a compact closed form. Moreover, as can be seen, either definition reveals the same basic concept and that the procedure due to Bravo and Wetherill (1980) is more appealing and easier to implement especially for our tabulated distributions, and hence it is adopted in furthering this research.

In line with this discussion our parameterisation and determination of the decision criteria would be made on the same basis, i.e. by solving for two points of probability of acceptance as in the following section (section (4.3)), and then uses these two points to find "equivalents". Once "equivalences" are established tests on the properties under study ((as cited in section (1.1.1) of Chapter 1)) will be viable and possible to carry out.

4.3. Determination of Decision Parameters and Criteria for the "New" Schemes

As mentioned earlier the determination of the decision rules for a plan in our "new" schemes and its parameters will be based on the idea of the O.C. curve. Two points on the O.C. curve fix the plan as follows:

(a) Given n (to be chosen from the set of $\{2^k$, for k = 1,2,...$\}$,

and an AQL (appropriately chosen from the p_0 values in the tables), then for a fixed value of B we have a unique percentage point, t, that gives a cumulative probability level α (corresponding to an acceptance probability of $P_a = 1-\alpha$).

(b) Having determined the t value and the α-risk point on the O.C. curve, we then search for the value of p_0 that corresponds to the acceptance level of β with the same percentage point t.

In other words the cut-off point, t, of the distribution of the average quality is determined such that a batch with $p_0(1)$ proportion defective [read AQL] will have a probability of acceptance $P_a = 1-\alpha$ (for prespecified α) while a batch with $p_0(2)$ defective proportion, $(p_0(1) < p_0(2))$ is accepted with probability $P_a = \beta$, $(\beta < 1-\alpha)$. The O.C. curve will then pass through the points (AQL, $1-\alpha$) and $(p_0(2), \beta)$ which, as mentioned before will act as a frame of reference for finding equivalents later on and it is the base of that "equivalence".

The result of this process of determining these two points arrives at a unique point of (n, AQL, $p_0(2)$, t) on the parametric space of our scheme, and hence determine the plan with the same protection levels as the O.C. curve passing through the above-mentioned two O.C. points. This set of parameters could also be translated into (n, $\mu(1-\alpha)$, $\mu(\beta)$, B) or other obvious forms depending on the circumstances and convenience.

There are two problems facing the evaluation process of these two points on the O.C. curve. The first is that interpolation in the tables is needed to find the $p_0(2)$ (or alternatively its corresponding $\mu(\beta)$ for

fixing the second O.C. point. The other problem is that of "non-existence" of the particular percentage point in the distribution representation.

a. Interpolation: We need to interpolate for the $p_0(2)$ corresponding to our t value because the distribution percentage points are, again, on a grid of equi-spaced μ's. This equi-spacing was originally intended to facilitate interpolation. Linear interpolation between the μ's is made possible and reliable by the smallness of the spacing between the μ's. The only complication for interpolation procedure is the possible occurrence of the problem of "non-existence" of one or both of the bounding t-values needed as interpolation parameters. There is a solution to such a problem as is explained in details in the next procedure section (section (4.3.1)).

b. "Non-existence": Non-existence in this context could only prevail as a hindrance to the interpolation process above. The problem that the interpolated value of $p_0(2)$ corresponds to a non-existent t-value does not arise at all, obviously.

4.3.1 Procedure for Determining the Decision Parameters (given n,B,α and β)

In our work we chose $\alpha = 0.05$ and $\beta = 0.50$ (i.e. a 95% producer's risk and 50% consumer's risk (or indifference point)). We have a prespecified set of AQL's for 95% level (namely, the set of p_0 values of {0.0107, 0.0139, 0.0179, 0.0228, 0.0287, 0.0359, 0.0446, 0.0548, 0.0668, 0.0808, 0.0968} corresponding to μ values of {2.3(-0.1)1.3}, respectively.

The procedure for determining the decision parameters is as in the

following major steps:

(i) Choose a specific AQL (for 95% level) and so determine a value for t, the test parameter for the test statistic.

(ii) Interpolate for the p_0 (or μ) value in the percentage point table at the 50% level such that its percentage point is equal to t.

If no problem of "non-existence" appears this procedure will determine $p_0(2)$ the proportion defective for the other O.C. point, hence a plan is determined.

The interpolation procedure changes its grounds from being in terms of the p_0 values to be in terms of their corresponding μ values because these are equally spaced in the tables. In between two successive μ values the interpolation used is linear since the interval spacing (of 0.1) of the μ values of the percentage points is well behaved except, of course, in cases next to the jumps. This is when we encounter a problem of non-existence of a percentage point. In other words if no discontinuity is caused by the value i/n for t then linear interpolation should suffice; most of the cases are solved by this simple interpolation. (Note here i = 0,1,2,...,n designates the points of discontinuities).

If there is a discontinuity problem then we need either to extrapolate from above or below depending on the location of the discontinuity. All the above search procedure is best summarised in the following steps, which apply to the tables of the percentage points shown in Appendix (C.2).

Firstly, let μ_+ denote the smallest μ for which the tabulated percentage point, T, is such that $T > t$, where t is the test criterion determined by fixing the first O.C. point of (AQL, $1-\alpha$). Similarly, let μ_- be the largest μ for which $T < t$. In both cases ignore those μ's for which T is negative. (Remember that in the tables of percentage points a negative T value is a convenient way of denoting a "non-existence" of a percentage point for the specific given level of probability).

If μ_+ and μ_- are spaced 0.1 apart, and the interval $(T(\mu_+), T(\mu_-))$ does not include a critical jump value then the mentioned interval is a normal interval and linear interpolation in μ's is carried out.

If not this case we then consider two situations:

(1) For t less than the critical position of i/n we extrapolate from μ_- upwards.

(2) For $t > i/n$ we extrapolate from μ_+ downwards.

And since under such circumstances we cannot interpolate between μ_+ and μ_- we will need four μ's to be able to fix our extrapolated μ then four values will be returned from these steps to be used in the extrapolation. These four values are $T(\mu_+)$, $T((\mu_+) + 0.1)$, $T(\mu_-)$ and $T((\mu_-) - 0.1)$. The steps of all this exercise are included in the computer program EQUIV.FOR in Appendix (B.4.).

4.4 The Basic Ramp Decision Criteria: As a result of all the above discussion we can get a $p_0(2)$ for the 100 β% (second) point on the O.C. curve. The protection level by this O.C. curve which passes through

(AQL, 1-α) and ($p_0(2)$, β) is determined by the solution values of n, AQL, t and B fixed in the above sections.

These decision parameters are used as in the following typical and basic decision rules:

* Decide the values of n, B.

* Determine t value by choice of protection level.

* Take a sample of n r.v.'s $x_1, x_2, \ldots, x_n$ drawn independently from a batch, believed to be normally distributed, evaluate $Q(x_i)$ for each and every x.

* Evaluate the average of these $Q(x_i)$'s, denote it by $\bar{Q}_n(x)$.

* If $\bar{Q}_n(x) > t$ accept, otherwise reject the batch from which the sample was drawn.

4.5 Determination of the Orthodox "Equivalents" of our Plans

For the established plans of the "new" schemes as given by the O.C. curve fixed by the two points as shown above we can use such points in moving the information to determine the "equivalents" to each and every "new" plan in the orthodox field of schemes. The basis for this equivalence is the same O.C. curve on which the "new" plan was determined.

In the following sections we will discuss methods of finding the "equivalents" to our determined plans in the cases of two-class attributes and σ-known and unknown variables plans. In each case we pick the resultant O.C. points of (AQL, 95%) and ($p_0(2)$, 50%) which were arrived

at from establishing the "new" plans above, and use them to find a "matching" (or "eqivalent") plan of the well established orthodox schemes. Results for all the matchings made are cited in the tables in Appendix (C.3.).

4.5.1 The Two-Class Attribute "Equivalence"

The parameters of importance in the two-class attribute schemes are n, the attribute sample size, and c, the acceptance number of defectives. In the equivalence process in principle the two given probability of acceptance points of (AQL, $(1-\alpha)$) and ($p_0(2)$, β) could be solved for n and c to determine the attribute plan which is equivalent to the given "Ramp" plan.

Because the Binomial model (behind the acceptance probability of the attribute case) is discrete it may not be a convenient method since it will be difficult and sometimes impossible to satisfy the two O.C. points exactly. Suggested legitimate flexibilities in the way of approaching the problem are introduced as follows.

Let $P_a(p)$ denote acceptance probability for p proportion defective then a way of finding a convenient form to work is to use the approximating system of the following inequalities:

$$P_a(AQL) \geq 1-\alpha$$

$$P_a(p_o(2)) \leq \beta$$

where,

$$P_a(p) = \sum_{i=0}^{c} \binom{n}{i} p^i (1-p)^{n-i}$$

Also, we make use of the fact that

$$B(c;\, n,\, p) \doteq P(c,\, np) \doteq \psi(2np,\, 2(c+1))$$

where,

$B(c;n,p)$ is the cumulative Binomial up to c (with parameters n and p),

$P(c, np)$ is the corresponding cumulative Poisson with mean np,

and $\psi(2np, 2(c+1))$ is the χ^2-distribution with "(c+1) d.f. and a level corresponding to a percentage point $\chi^2 > 2np$".

These are well accepted approximations of one another (see Hald (1977) and Wetherill (1969)). These would lead us to an easier system to solve in terms of the χ^2-distribution rather than the more complicated Binomial system of inequalities. So we have:

$$\Pr\{ \chi^2 > 2np : \text{ given } 2(c+1) \text{ d.f.}\} \geqslant 1-\alpha \qquad (5.1)$$

$$\Pr\{ \chi^2 > 2np : \text{ given } 2(c+1) \text{ d.f.}\} \leqslant \beta \qquad (5.2)$$

These inequalities should be constrained such as to provide a little tolerance and no major discrepancies. They should be as close as possible to the values on the right-hand-side of the inequalities. A simple method of minimising c will have such an effect and is described below.

A method due to Hald (1976-77) proved to be invaluable in solving for n and c, was also used in Bravo and Wetherill (1980), and is followed in the solution of (5.1) and (5.2) simultaneously, next.

Given AQL and $p_0(2)$ corresponding to $\alpha = 0.05$ and $\beta = 0.50$ respectively, then we require a solution for (n, c) such that

$$P_a(AQL) \geq 1-\alpha$$

$$P_a(p_0(2)) \leq \beta$$

and c is as small as possible, where AQL, $p_0(2)$, α and β are given values such that:

$$0 < AQL < p_0(2) < 1$$

and $$0 < \beta < 1-\alpha < 1.$$

The tools for the solution are:

(1) The auxiliary function $m_p(c)$ which is the solution of

$$P(c, np) = P \text{ for } 0 < P \leq 1$$

where $P(c, np)$ is the usual Poisson distribution and can be approximated by its above-mentioned approximation.

(2) The decreasing function of c defined as:

$$R(c, \alpha, \beta) = \{m_\beta(c)\}/\{m_{1-\alpha}(c)\}$$

which could be expressed as R(c) since α and β are given in this context.

(3) (a) For integral value of c we considered solving the above-mentioned inequalities rather than the equalities. These will imply that the smallest c is uniquely determined by

$$R(c-1) > \{p_0(2)/AQL\} > R(c) \qquad (I)$$

(b) Later on, the corresponding n satisfies the constraint that it should lie within the interval:

$$(\{m_\beta\ (c)/p_0(2)\}, \{m_{1-\alpha}\ (c)/AQL\}) \qquad (II)$$

The importance of this method lies in the fact that the function R(c) is not a function of n. By guess work and some small amount of trial and error we could locate the smallest c value. This c is then substituted in equation (II) above to find n [or probably an interval of n].

Note that if the interval does not include an integral value of n then increase c and go through the process from (3)(b) all over again. Also note that:

$$B(c;\ n,\ p) \doteq P(c,\ np) \doteq (1/c!) \int_{np}^{\infty} t^c\ e^{-t}\ dt$$

where the last term is the χ^2-probability:

$$Pr\{\chi^2 > 2np : \text{given } 2(c+1) \text{ d.f.}\}$$

This methodology of solving for n and c has been incorporated in the Fortran Routine called MATCH contained in the computer program called EQUIV which is responsible for the matching of the two-class attributes (as well as the other orthodox) plans to our "Ramp" plans. (These programs are listed in Appendix (B.4.)). An illustrative part of the results are shown in the tables in Appendix (C.3.) together with results for the "equivalent" variables cases (whose decision parameters were determined as discussed in the next two sections).

4.5.2 σ-known Variables Plans "Equivalence"

To match the σ-known variables plans to our "new" plans we would utilise the following standard univariate normal theory and its analytical relations corresponding to the $(1-\alpha)$ 100% and (β) 100% levels, that is:

$$\{k_\sigma - \mu_{(1-\alpha)}\} \cdot (n_\sigma)^{1/2} = \sigma \cdot \Phi^{-1}(1-\alpha)$$

and

$$\{k_\sigma - \mu_{(\beta)}\} \cdot (n_\sigma)^{1/2} = \sigma \cdot \Phi^{-1}(\beta)$$

where k_σ and n_σ are the usual parameters of the decision rules of the σ-known variables plans.

These equations solve for k_σ and n_σ, thus if $\alpha = 0.05$ and $\beta = 0.50$:

$$k_\sigma = \mu_{0.50}$$

and

$$n_\sigma = (1.6449\sigma)^2/(\mu_{0.50} - \mu_{0.95})^2$$

It should be noted that in practice this sample size has to be

integral-valued (and usually small), so that an exact integral value does not necessarily follow from these theoretical results but the size of such a problem of non-exactness gets smaller for larger sample sizes. Results for the matching of σ-known plans are shown also in the same tables of Appendix (C.3.).

4.5.3 "Equivalence" for Unknown-σ Variables Plans

For matching these S-plans to our new schemes given the two points of the O.C. curve we use the iterative approximating procedure for finding the parameters of the Non-Central t-distribution suggested by Hamaker (1976) and advocated by Wetherill and Kollerstrom (1979) and Bravo and Wetherill (1980) who all checked it as adequate. Our results using the same method compare very well with the Bravo and Wetherill (1980) published results. The method uses the σ-known results and is as summarised in the following argument:

In the σ-known plan based on m (sample size) and k (test constant) we accept if the sample mean $\bar{X}_m \leqslant A - k\,\sigma$. And if σ is unknown then we will have the same O.C. curve if n (the S-plan sample size) and h (its test constant) are adjusted such that the random variables $[\bar{X}_n + h \cdot S]$ and $[\bar{X}_m + k \cdot \sigma]$ are matched so as to have the same mean and same variance, where S is the usual unbiased sample standard deviation. As a result of equating mean and variance we get the following approximate solutions:

$$k = h \cdot (4n - 5)/(4n - 4) \qquad (1)$$

$$m = n/(1 + h^2/2) \qquad (2)$$

A numerical solution could be sought iteratively noting that n has to be greater than m with an incremental factor of $1 + h^2/2$. An initial value for h is h=k since the factor $(4n - 5)/(4n - 4)$ could be approximated, first time, by 1. From the second equation, (2) above, we evaluate an approximation for n by substituting for h=k. This value of n is then used in (1) to give a better h value. A final approximation of n is computed using (2).

We have found that this iterative process converges quickly to values of k and n and a solution existed for all cases. The method was used in the Fortran program, EQUIV, which gave the results partly displayed in Appendix (C.3.) for cases of S-plan "equivalents" together with the other cases.

4.5.4. All the equivalences and their procedures are incorporated in the FORTRAN program EQUIV mentioned above which is listed in the Appendix (B.4.). Our work has arrived at the conclusion that matching (as is recognised in many of the publications of the existing orthodox schemes, e.g. BS6002, MIL-STD-414 ..., etc) is in some cases very poor indeed. For further evidence in this respect we refer to Bravo and Wetherill (1980).

The results of our matchings as discussed above are meant to serve three purposes:

(a) Give illustrations of the "equivalents" for their own sake and show procedures for moving from a scheme to another through "equivalence".

(b) Show the performance of the sample size savings in the comparative

sense with and in relation to the two extremes (i.e. the σ-known variables and the two-class attribute schemes). A detailed study of this phenomenon is given in section (4.5.5) below, but some definite savings were made by our "new" schemes over the attributes case and competitively with the variables (unknown-σ) schemes.

(c) Make a basis for the analysis of the robustness property since under equivalence (matching) all schemes will have the same O.C. performance for the similar background assumption of a Normal distribution model of the process. This point is used in the robustness studies which are dealt with in Chapter 5.

4.5.5 Performance of the Sample Sizes under Equivalence of Different Schemes.

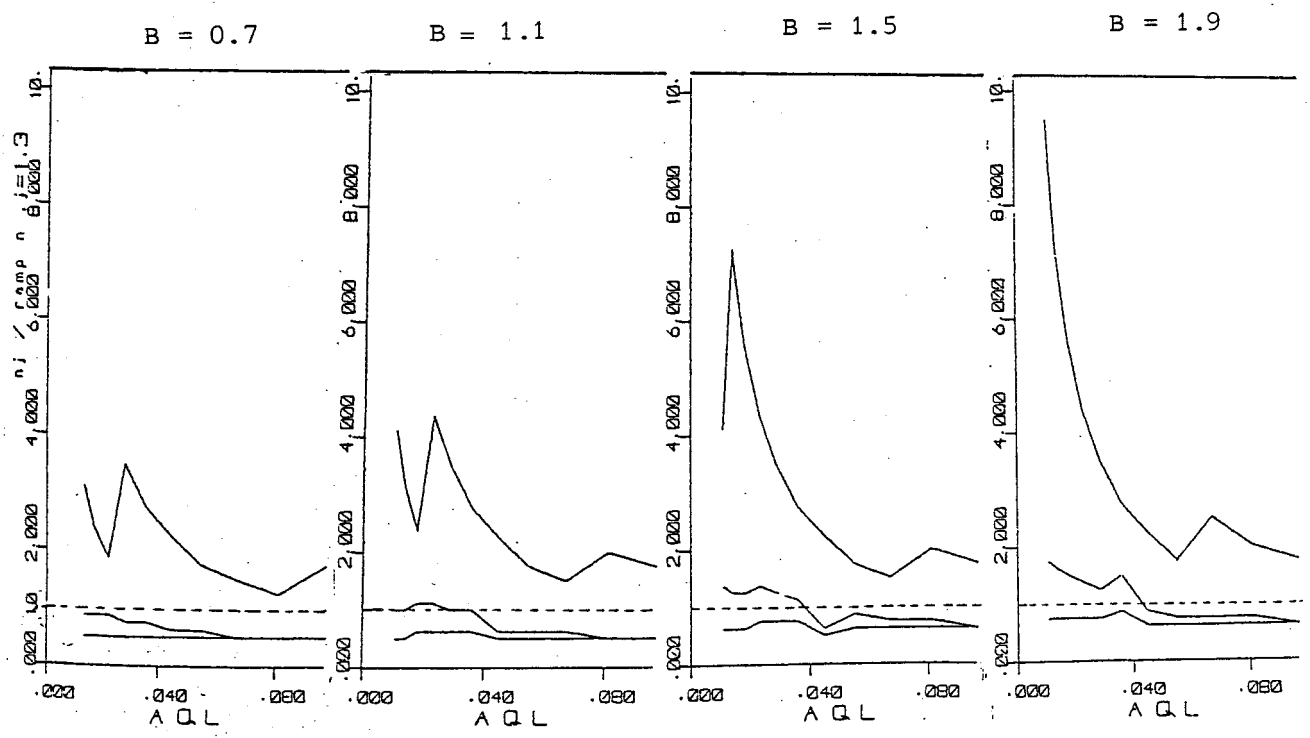
The results in the tables in Appendix(C.4.) about the behaviour of sample size show the sample sizes for the different schemes under equivalence. These results are reflected graphically in figure (4.5.5) below so as to give a more illuminating insight of the behaviour of each scheme in so far as sample size is concerned. The graphs show each plans performance by plotting the ratio of their sample sizes to their "equivalent" Ramp sample size against AQL's. The horizontal broken-line indicate such a curve for the case of the Ramp itself. The bottom curves are the cases of the orthodox by-variables plans, the lowest one refers to the σ-plan. For each B value there is a separate figure for different (Ramp) sample sizes. As a function of AQL each figure shows quite clearly that the lower the AQL the higher the gap between the Attribute curve and any other alternative case. It is beyond doubt that there is a large amount of savings in the

Ramp over the Attribute case. All the diagrams indicate how the Ramp could act as a transition between the attribute and the by-variables schemes. Moreover, and interestingly enough, there is evidence of substantial savings of the Ramp over the S-plan.

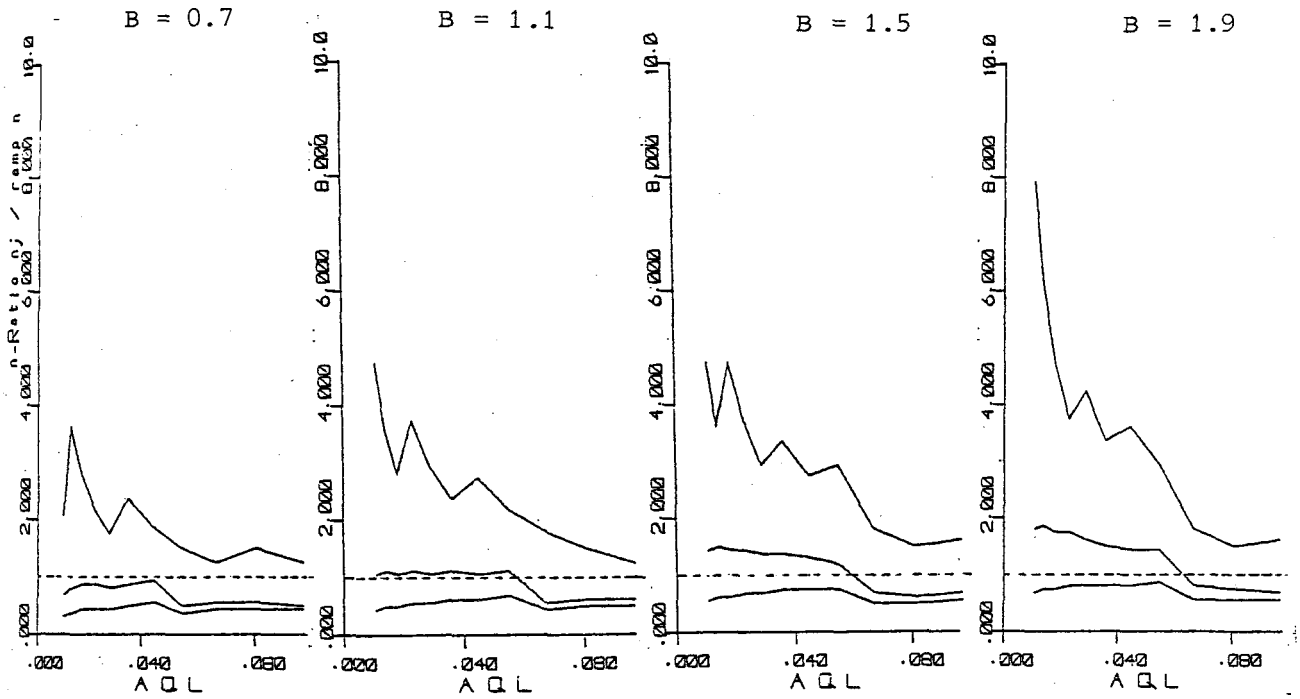
Figure (4.5.5): Sample size ratio n_j/n as a function of AQL (where n is the Ramp sample size and n_j is the alternative-plan sample size).

Note: The highest curve in each diagram is that of the attribute scheme, the second is for the S-plan and the bottom is for the σ-plan. The broken line is for the Ramp case.

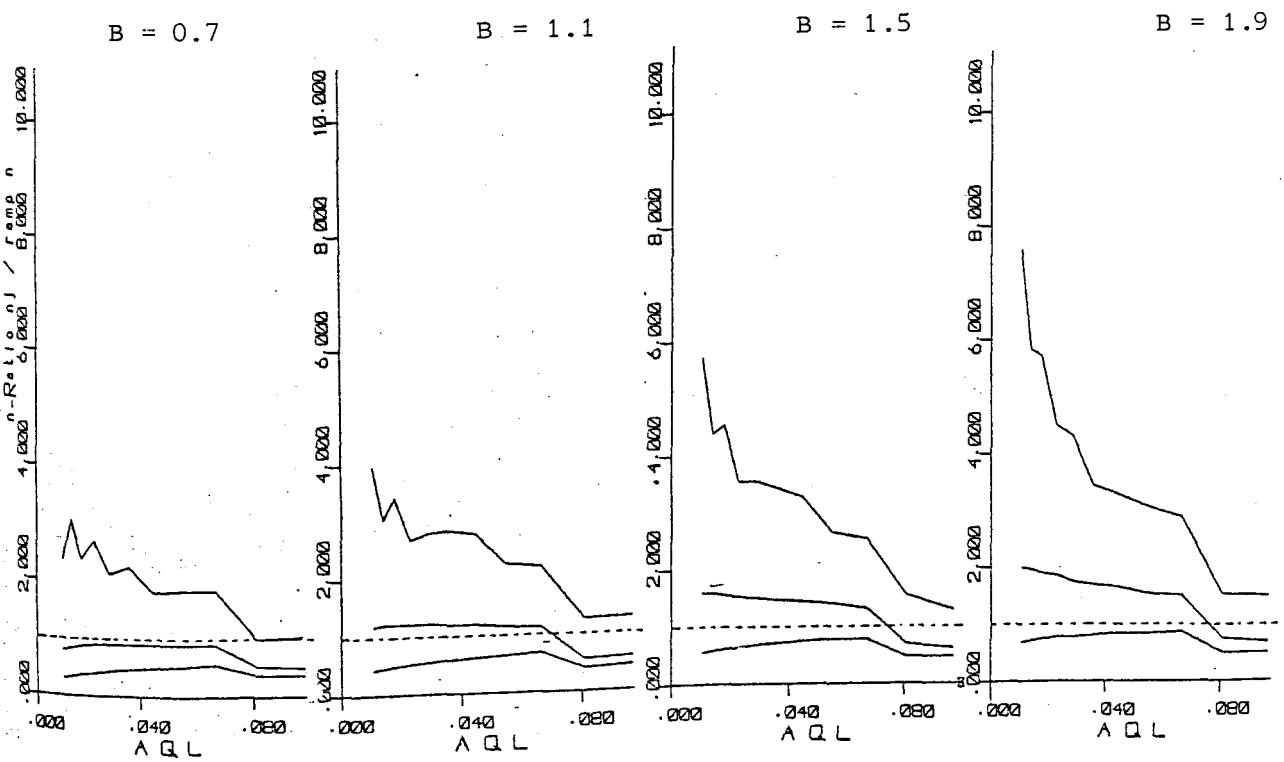
n = 8



n = 16

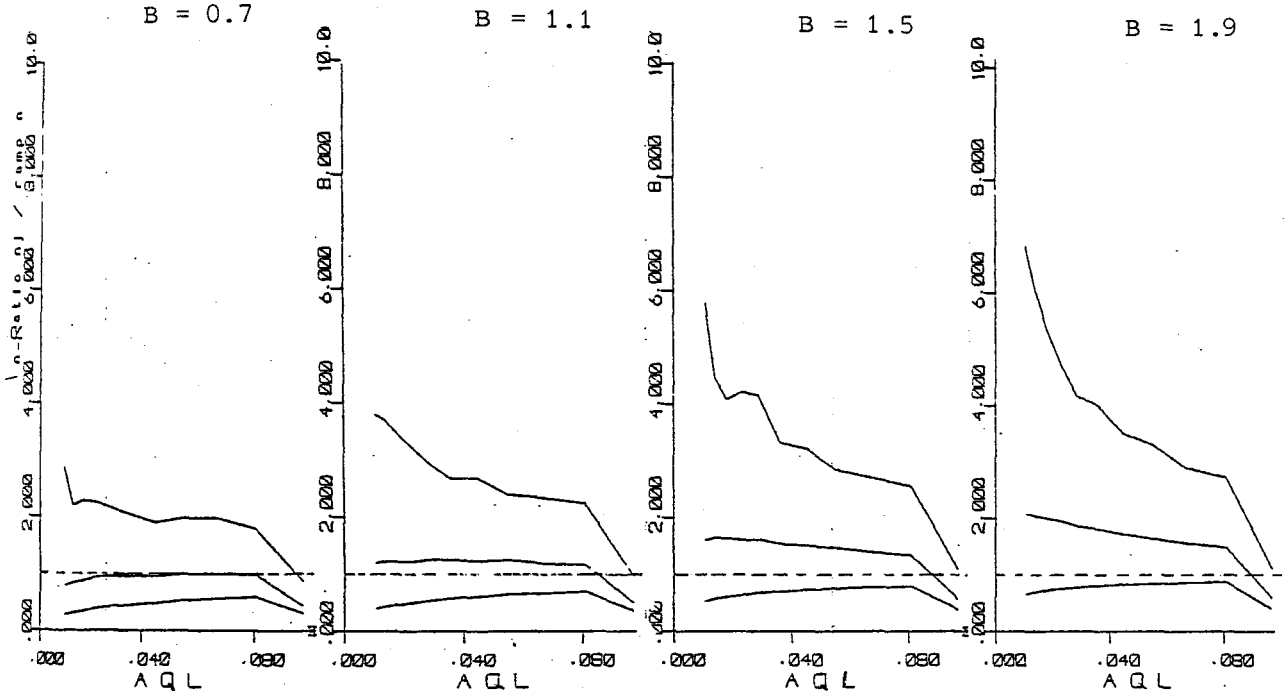

B = 0.7
B = 1.1
B = 1.5
B = 1.9
n-Ratio nJ / ramp n
10.0
8,000
6,000
4,000
2,000
.000
.000
.040
.080
AQL

n = 32


B = 0.7
B = 1.1
B = 1.5
B = 1.9
n-Ratio nJ / ramp n
10.000
8,000
6,000
4,000
2,000
.000
.000
.040
.080
AQL

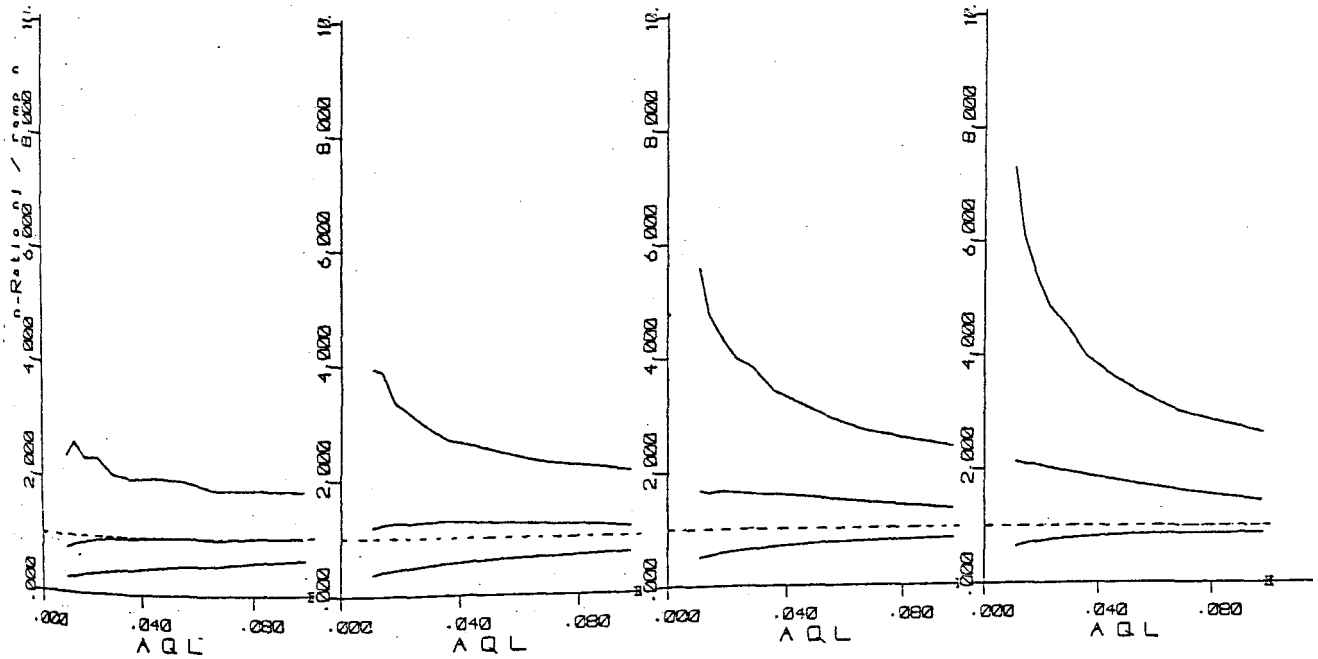
n = 64

B = 0.7 B = 1.1 B = 1.5 B = 1.9



n = 128

B = 0.7 B = 1.1 B = 1.5 B = 1.9



CHAPTER 5

ROBUSTNESS STUDIES

(Analytical and Simulated)

5.1 Definition of Robustness Concept and Work in the Area

5.1.1 General Note and Definition Problems

From the beginning "robustness" has been, and still is, a rather vague idea. Box and Anderson (1955) introduced "robustness" in the following context. 'Procedures are required which are "robust" (insensitive to changes in extraneous factors not under test) as well as powerful (sensitive to specific factors under test)'.

Box and Tiao (1964-b) and later Barnard (1974-a) put forward the case for the distinction between "criterion robustness" and "inference robustness". They see "criterion robustness" as concerned with the effects of departures from the assumptions on the null distribution when the test criterion under investigation is taken as given; with "inference robustness" consideration is also given to questioning the appropriateness of the test criterion on a sample to sample basis. Our analysis of the problem takes the line of "criterion robustness" for two reasons. Firstly, we are studying test criteria which are given in the sense that their schemes have existed for a long time and are well established so that it is only practical to look at their limitations and warn against them in the short run while in the long run consideration of "new", better and appealingly simple, alternatives could be sought. Secondly, the approach

by directly comparing methods forces us to take the test criterion as given especially since these have some optimal properties when assumptions are correct, which renders a study in terms of "inference robustness", that might entail a change in the criterion, not recommendable.

At a different level, the literature has got two streams of thought, one deals with large sample sizes and asymptotic studies while the other looks at small sample sizes. In his paper Huber (1972) gave a review of recent theoretical works on robustness and their different schools of thought, citing many references in the area. In the paper there is a discussion of the notions of robustness through asymptotic studies.

Huber seems to encourage problem-oriented or local goals since he argues that:

(a) a small asymptotic variance over some neighbourhood of one shape, in particular the normal one (Huber (1964)), and;

(b) the distribution of the estimate should change little under arbitrary small variations of the underlying distribution and uniformly with n;

are the important criteria. Like Anscombe (1960) he advocates the view-point that robustness is a kind of insurance problem whereby one is willing to lose the premium (here, loss of efficiency of, say, 5% to 10%) to guarantee and safeguard against ill effects caused by small deviations from the ideal model.

For finite small sampling robustness Huber (1972) admits that the

appropriate robustness criteria are much more difficult to lay down. This and other factors make our robustness studies difficult and the theoretical literature in the area is not very helpful although directive. In the next section we have a review of some of the work on robustness of relevance to our problem to act as a broad guide and a base from which to build.

The discussion, especially on the theoretical ideas on robustness criteria shows that there are differences in emphasis but still quite an agreement on the notion created by Box and Anderson (1955) mentioned above.

All the complex robustness requirements reviewed by Huber (1972) are not easy to maintain. This is especially true if we are to handle cases of robustness of estimators not only of locations or dispersion but effectively of the whole distribution. This is because we need to estimate the quality in terms of the proportions in the tails as in our case of the triplets (p_0, p_1, p_2). Under the circumstances we chose to start thinking in terms of the Box and Anderson general notion and relate to Huber's ideas ((a) and (b), above) with special reference to (b).

Our strategy will have the following features. When studying the impact of the (non-normal) distributional changes we will vary the distribution form to encompass reasonable changes in the process while controlling and fixing the variance. As mentioned earlier one standard measure of quality in acceptance sampling and which is common between all distributions is the triplet (p_0, p_1, p_2). Varying the variance alone (under Normality), when studying variance effects, will have the effect of changing these three components for a given mean or any other central value. The changes in the distribution form serves two purposes:

(1) Indicate which departures from normality are most crucial for each sampling scheme.

(2) Check which of the plans under study here can restrict the range of variation of P_a over a reasonable range of non-normal distributions. [This could be facilitated by a standardised measure of robustness to define such a range (see section (5.2.1)) below].

5.1.2 Some Review of Work Related to the Problem

Past work relevant to our robustness studies was done by Wald and Wolfowitz (1946), Rao, Subrahamaniam and Owen (1972) [who investigated Owen's work (1964)], and Pearson, D'Agostino and Bowman (1977).

The work of Wald and Wolfowitz (1946) tried to control the overall combined proportion in the tails but did not go beyond that to try to control the level in each tail. To only guarantee that with probability P the event that the proportion between $(\bar{X} - k_1.s)$ and $(\bar{X} + k_2.s)$ is at least (1-p) would not necessarily guarantee that each of the tails carries the right amount of the proportion defective p, say p/2 .

This last point constituted a motivation for the work of Owen (1964) who developed certain methods to tie up the constants of the by-variables sampling plans (with given limits of specifications) such that each tail has p/2 proportion. In turn Owen's work was investigated in a robustness sense by Rao, Subrahamaniam and Owen (1972). They looked into the effect of non-normality on Owen's techniques and controlling the proportions of p/2 in each tail. Their conclusion is that Owen's limits when subjected

to "very moderately Non-Normal " distributions, showed that non-normality effects are:

- more pronounced as n increases,
- increasingly felt as p decreases and
- are dependent on relative magnitude and sign of the standardised cumulants $\sqrt{\beta}_1$ and $(\beta_2 - 3)$.

Pearson et al (1977) showed that the error rates of the t-test were approximately linear in $\sqrt{\beta}_1$ and β_2. This point together with the latest point (above) showed the significance of the factors of skewness and peakedness in investigating any non-normality or indeed any robustness studies. Partly for this reason we chose the specific alternative distributional models (discussed in the next sections) to model the non-normality of our production process. We also add the consideration of variance changes from assumed values within normality and make studies on such effects within non-normal distributions. The models show varying degrees of peakedness and assymmetry together with other real considerations of a process like variance changes. We note in passing that our results agreed with Owen's results, especially for the first two points mentioned above. In common with the work of Pearson, D'Agnostino and Bowman (1977) our choice of the distributional models was primarily made so that we could study the performance of our plans under non-normal models with prior knowledge of the direction of non-normality in terms of the degrees of symmetry and/or kurtosis. In their work they suggested specific regions of non-normality related directly to the Normality point (0,3) in the general sub-classification of the (β_1, β_2) plane. These regions were $(\sqrt{\beta}_1=0, \beta_2 > 3)$, $(\sqrt{\beta}_1=0, \beta_2 < 3)$, $\sqrt{\beta}_1 < 0$ and $\sqrt{\beta}_1 > 0$.

Relating our work to all the above-mentioned works, Owen's investigation is very relevant. Yet, it is limited in comparison with our requirements in the following two senses:

(i) It considers a specific case in which p, the proportion defective is equally balanced in the two tails while our study is interested in one-sided defective control. Moreover, they limit non-normality to the close neighbourhood of normality.

(ii) It was shown that for low values of p the effect of non-normality is increasingly noticeable. Though this does not tell us specificallly about cases of smaller proportions in the lower tails there is evidence (Pearson et al (1964)) that two-sided control is more stable than one-sided control, and so we do not seem likely to have satisfactory results if we judge by Owen's results.

With reference to the investigation of Owen's methods for robustness made by Rao et al (1972), who used Gayen's (1949) approach which is highly tied to $\sqrt{\beta}_1$ and $(\beta_2 - 3)$ as parameters of departure from normality, we would highlight as significant their conclusions that the variation in $(\beta_2 - 3)$ is compensated for by the effect of $\sqrt{\beta}_1$. Also the correction afforded by $(\beta_2 - 3)$ tends to increase P (the statistical confidence that no more than p_1, say, is below $(\bar{x} - k_1.s)$ and no more than p_2, say, is above $(\bar{x} + k_2.s)$). Moreover, they remarked that for large p values (< 0.5) the values of P are most robust to departures from normality. We took non-normal cases as below in such a way as to reflect skewness in a separate model from that reflecting kurtosis.

5.2 Forms of Distribution Models used for Robustness

In accord with the previous discussion, (and also for other motivating factors which will be mentioned in the following sections), we decided to consider testing our plans with each of the following distributions as alternative, non-normal, process models:

(1) Contaminated Normal Distribution (defined as a mixture of two normals): this deals with real effects that could possibly be experienced quite often, e.g., in a product of a six-headed production unit or when a batch contains an output which is supposed to come from a homogeneous grade but came from two (or more) processes with heterogeneous grades of products.

(2) Lognormal Distribution: this reflects a skewness of the process. Though it is skewed to the right one can imagine the impact of the left-hand skewness.

(3) Two-point Distribution: this is expected to show the effects of extremely polarised 2-class processes. It may not be a realistic case to consider from a process point of view but it is a simple extreme case of a mixture of distributions.

(4) Uniform Distribution: as a heavy-tailed distribution model it can reflect the kurtosis requirements of a testing non-normal process. It can arise as a reasonable approximation to the product from a batch which is selected before delivery.

The distributions (1) and (2) were dealt with by simulation while

(3) and (4) were amenable to analytical handling as is shown below in the section relevant to each of these distribution models. The following sections of detailed study of the performance of our plans' credibility give clues about the effects of the distinctions between these four models and the normal one. Two major features of these which are already highlighted are skewness and kurtosis.

One other feature is the contamination of the distribution of the process which besides reflecting changes in skewness and kurtosis give more reality to mimicking the batches behaviour. Batches experience mixed distributions when their contained items come from different lines of production or even the same line but under different exogenous factors.

5.2.1 A Device for Measuring O.C. Sensitivity to Assumptions

The distributional changes will reflect themselves on the resultant O.C. curves. And we would like to be able to see the direction and magnitude of the changes of the O.C. resulting from these violations of the distributional assumptions. This will give an idea about how robust are the plans and a definitive measure is needed. The standardised measure ROB_{ij} below would show direction and magnitude of the resultant changes on O.C. performance:

$$ROB_{ij}(p_0) \doteq \frac{P^*_{a_i}(p_0) - P_{a_j}(p_0)}{\sqrt{P^*_{a_i}(p_0).\{1-P^*_{a_i}(p_0)\}/n}}$$

where $P^*_{a_i}(.)$ is the accept probability under the assumed type i distribution.

$P_{a_j}(.)$ is accept probability when distribution type i is violated by type j.

p_0 is the proportion defective at which the discrepancy is measured.

n is the sample size for the "equivalent" attribute plan.

Here, $\sqrt{n}$ is meant to crudely adjust for the steepness of the O.C. curve such that a compressed O.C. (as when n,the equivalence 2-class sample size,is high) is compared on the same basis with a flatter O.C. (as when that n is low). This is to facilitate the comparison of large-sample-size equivalents with small-sample-size ones irrespective of the biasedness due to n. We should mention that the $\sqrt{(P_a(1-P_a))}$ factor in the measure relates systematic differences to the standard error for a single Bernoulli trial. With this the absolute differences in P_a are turned into standardised and, therefore, comparable ones.

Distribution type i is the $N(\mu_i,1)$. Distribution j is any of the alternative models with the same proportion defective p_0 but some shape other than that of the normal. In another context, namely the study of the variance-effect robustness , type j could be $N(\mu,\sigma^2)$ where the variance σ^2 is not equal to 1.

ROB could, for example, be evaluated at a particular value of p_0 above the indifference point or alternatively, for a pre-determined set of p_0's in which case a modification is needed, e.g. taking the average. However, one p_0-value of particular interest in Acceptance Sampling is the AQL, which will receive the special consideration in this research. Note that above the indifference point (as compared with below it) the sign (read, direction) changes.

The measure ROB is irrelevant when the plans under study are

"equivalent" since equivalence by definition implies a given O.C. performance for all the concerned plans.

P_a is chosen to be 95% throughout the rest of this work. This choice is made for the following reason. Sampling schemes should encourage submission of good quality material, hence the top-end of the curve is most likely to be used.

5.2.2 Matching the Distribution Models

Before assessing the robustness we should agree on a frame or basis for matching all of the distributional models so as to make them comparable. The most sensible basis for this is matching the three proportions of quality (p_0, p_1, p_2) but unfortunately it is not strictly applicable. For the discrete distributions especially the ones with relatively few values this basis is particularly unreasonable. An example here is the case of the two-point distribution (considered in section (5.2.2.3) below. Under such circumstances we shift the basis to matching the mean and variance but under all circumstances fix p_0.

Since the mean and variance are very effective in matching the discrete distributions we will let the basis of comparison be the mean and variance such that the proportion defective matches. The necessary procedures for this are shown below. The matchings are done for each of the distributional models as in the next sections.

5.2.2.1. Normal model: Here, for given p_0 and p_1 (hence p_2) we have:

$$A_\sigma - \mu = \sigma \cdot k_0 \qquad (1)$$

$$B_\sigma - \mu = \sigma \cdot k_1 \qquad (2)$$

where $\Phi(k_o) = p_0$ and $\Phi(k_1) = p_0 + p_1$.

Solving (1) and (2) we get the mean, μ, and standard deviation, σ, as:

$$\mu = \frac{k_0 \cdot B_\sigma - k_1 \cdot A_\sigma}{k_0 - k_1}$$

and

$$\sigma = \left(\frac{B_\sigma - A_\sigma}{k_1 - k_0}\right)$$

These mean and standard deviation could act as the matching points for purposes of comparing the performances of the plans. This device of translating the (p_0, p_1, p_2) into a mean-and-variance copes spendidly with the transition from the continuous to the discrete distribution models.

The evaluation of P_a (the acceptance probability) for the normal model has already been handled and its results were given by Chapter 4 which matches the O.C. curves under the assumption of normality. What the rest of this chapter poses to accomplish is to use the test criteria and parameters of Chapter 4 (under equivalence) and subject them to the following non-normal models. A note of how each of the schemes under study performed is made therewith.

5.2.2.2. Uniform model: for a specific (p_0, p_1, p_2) point our uniform distribution theory tells us that;

$$\frac{A - C_1}{C_2 - C_1} = p_0 \quad , \quad \frac{B - A}{C_2 - C_1} = p_1 \quad \text{and} \quad \frac{C_2 - B}{C_2 - C_1} = p_2 \qquad \text{(I)}$$

where $[C_1, C_2]$ is the domain of definition of the uniform distribution. If we define $G_j = p_j/p_1$ for $j = 0,2$; then from (I) above we get:

$$C_1 = - G_0 \cdot B + (G_0 + 1) \cdot A$$

and

$$C_2 = (G_2 + 1) \cdot B - G_2 \cdot A$$

And knowledge of the mean and variance of the uniform in (C_1, C_2) gives us:

$$\text{Mean} = \{B \cdot (1 + G_2 - G_0) + A \cdot (1 - G_2 + G_0)\}/2$$

$$\text{Variance} = \{(B - A)(1 + G_2 + G_0)\}^2/12.$$

This mean and variance of the uniform may not be needed if the matching with the discrete distributions is made via the normal model only. If we feel that we need to match the two distributions on the (p_0, p_1, p_2) basis we have to let the variance loose and unrestricted.

5.2.2.2.1 The P_a for the Ramp plan under the uniform distribution, given the test criteria T, could be evaluated as follows.

$$P_a = \Pr\{ \sum_{i=1}^{n} Q(x_i) \geq n.T : \text{ given } n\} \qquad (*)$$

Consider the n random variates $Q(X_i)$, $i = 1,2,\ldots,n$. The convolution of these r.v.'s will be composed of three variables r_0, r_1 and r_2 (corresponding to the numbers of defectives, marginals and effectives, respectively).

r_0 and r_2 have spike probabilities p_0 and p_2 respectively, while each of the r_1 marginals have a continuous uniform in (0,1). Denoting the convolution of these r_1 r.v.'s by $S(r_1)$ then

$$S(r_1) = \sum_{j=1}^{r_1} Q(x_j), \quad \text{for } 0 < x_j < 1$$

$$= 0 \qquad \text{elsewhere.}$$

Considering the fact that all the r_0 defectives score a sum of 0, and that all the r_2 in the sample score a sum of r_2, then the acceptance condition as in the argument in (*) above will give:

$$S(r_1) \geq n.T - r_2 \qquad (**)$$

For the given n and T we have $S(r_1)$ representing a sum of r_1 uniforms in (0,1) and the right-hand-side of (**), above, is a constant for a given combination of r_1, r_2 in n. In other words, in evaluating P_a for a given combination of r_1, r_2 in n we are dealing with a cumulative distribution of a sum of r_1 uniforms being greater than $(n.T - r_2)$. If we define F(t) as the distribution function for the mean of r_1 uniforms then the probability of the event (**) above is available by substituting the pivotal quantity $(n.T - r_2)/r_1$ for t in F(t). And as clear r_0, r_1 and r_2 are random variables and their distribution depends on the triplet (p_0, p_1, p_2) [a trinomial r.v.]. Consequently, we find that (*) developes into the following composite form:

$$P_a = \sum_{r_1=0}^{n-r_0} \sum_{r_0=0}^{n} \Pr\{S(r_1)/r_1 \geq (nT-r_2)/r_1 : \text{given } r_1, r_2\}$$

$$\times p_0^{r_0} p_1^{r_1} p_2^{r_2} \, n!/(r_0!r_1!r_2!) \qquad (+)$$

(1) In case of $r_1=0$, i.e. no marginals, this P_a turns out to be the conditional Binomial exactly as:

$$(p_0 + p_2)^n \cdot \sum_{r_2=[nT]}^{n} \frac{n!}{r_2!(n-r_2)!} \left(\frac{p_2}{p_0 + p_2} \right)^{r_2} \left(\frac{p_0}{p_0 + p_2} \right)^{(n-r_2)}$$

(put in this way for computational purposes).

(2) In case of $0 < r_1 \leqslant 10$, we calculated the exact P_a replacing the factor $Pr\{S(r_1)/r_1 \geqslant (nT-r_2)/r_1 : r_0, r_1\}$ by the complement of the cumulative uniform distribution, F(.), that is:

$$Pr\{S(r_1)/r_1 \geqslant t: r_1, r_2\} = 1 - F(t) \quad .$$

According to Kendall and Stuart (1963):

$$F(t) = \frac{r_1^{r_1}}{(r_1 - 1)!} \sum_{i=0}^{k} (-1)^i \frac{r_1!}{i!(r_1 - i)!} \frac{(t - i/r_1)^{r_1}}{r_1} \quad , \quad \frac{k}{r_1} \leqslant t \leqslant \frac{k+1}{r_1}$$

and $k=0,1....,r_1-1$. A suitable FORTRAN routine was developed to evaluate this F function (listed as UNFRMF in the Appendix (B.5.) within the program UNIFRM.FOR).

(3) Cases of $r_1 > 10$ were computed via the Normal Approximation to the Uniform F using the relation

$$F(t) = \Phi \left(\frac{t - 0.5}{s} \right)$$

where $s = 1/\sqrt{(12.r_1)}$.

(Some work on this Normal Approximation with some other more precise versions of it is reported in Appendix (A.1.)).

5.2.2.2.2 Results of Robustness under Uniform Model

The following comments describe the results of Uniform robustness as shown in the tables in Appendix (C.3.) (on the right side of the tables in a column under the heading of "Uniform Robustness").

For any B value and given AQL then as Ramp n increases robustness P_a decreases. This is more so for small AQL's than for large ones. In terms of n the robustness P_a of the Ramp scheme, given AQL, decreases as B increases for small AQL. For medium AQL it decreases, flattens and then increases. For large AQL this increases. All this is a reflection of the level of contribution from the continuous (variable) part of the r.v., Q(X). The more the continuous part (i.e. the larger p_1) the less are the chances for acceptance (and so, low robustness level).

In comparative terms the Ramp competes with the S-plan from earlier AQL's than with the σ-plan (if at all). The σ-plan is more enduring than the S-plan under uniformity. Coupled with the Property "Q" this makes the σ-plan more recommendable than the S-plan, which only rivals the σ-plan on the variance-effect robustness.

Most interesting is the result that the Ramp plans are more robust than any other by-variables plan for any AQL $\geq$ 0.0445 for as small a Ramp n as 4. Moreover, here, the Ramp is better than the S-plan for earlier AQL, AQL $\geq$ .02875. It should be added that if a fair amount of flexibility is allowed then in cases of AQL < 0.0445 we can say that with respect of robustness (under Uniformity) the Ramp is not so markedly worse than the other by-variables plans, and is definitely better than them for AQL $\geq$ 0.0445 as mentioned.

For $n \geq 8$ a different picture appears. The Ramp Uniform-robustness gets worse compared with other schemes as n increases. There are some cases where they are comparable but these cases are the exception rather than the rule. However the discrepancies between the Ramp and by-variables (which are always in favour of the σ-plan) could be thought of as not markedly large for n as small as 8. For values of n greater than 8 it is quite noticeable that the S-plan is marginally better than the Ramp, and as Ramp n gets larger and larger the Ramp falls far behind the rest but then all plans become very poor indeed and prove not to be all that useful except for the 2-class attribute schemes of course.

For high Ramp n values, low AQL's are markedly in favour of the orthodox by-variables when compared with Ramp for B values greater than $B = 0.5$. The robustness levels in this context are low anyway especially for small AQL.

As could be seen the cases where the Ramp plan competes with the σ-plan (under Uniformity) are less than with the S-plan and are negligible. We can generalise that under Uniformity the by-variables plans for $n > 8$ are better than the Ramp; and that for high n the only winner is the Ramp with small B (close to 0) since the high B values and the other by-variables seem to fail to give any reasonbly acceptable level of protection and are proving to be useless.

If a maximum difference of 0.3 between values of P_a under uniformity is acceptable as reasonable then we can strongly recommend replacing the 2-class attribute by their equivalent Ramp plans if $B < 0.5$ in the following cases:

AQL $\geq$ 0.01785 for Ramp n = 16

AQL $\geq$ 0.0287 for Ramp n = 32

AQL $\geq$ 0.0445 for Ramp n = 64

This is because of the savings in sample size, as the discussion on sample savings of section (4.5.5) of Chapter 4 shows. A general conclusion in the light of the analytical results of the FORTRAN program UNIFRM.FOR - (listed in Appendix (B.5.), and which studies the robustness of the schemes under the Uniform distribution model) - is that the Ramp plans are only robust enough when used for fat-tailed symmetrical distributions when B is small. Also significant is the point that the by-variables plans are infelxible to overcome their weak robustness as far as heavy-tailed distributions are concerned.

5.2.2.3 The Two-Point Model

As mentioned earlier the matching of this discrete distribution could not be done directly in terms of the (p_0, p_1, p_2) basis. It could be handled through the mean and variance basis such that the proportion defective is fixed as p_0. Depending on the position of the mean (or in fact X_2, the second point of the distribution) the proportion $(1-p_0)$ is either p_1 or p_2. It is p_1 if B is less then X_2, otherwise it is p_2.

The mean, M, and variance , V, are

$$M = (1-p_0).X_2 + p_0.X_1$$

$$V = (1-p_0).(X_2-M)^2 + p_0.(X_1-M)^2$$

We have fixed X_1 at 0, and with the variance made equal to 1 we have

$$M = (1-p_0) \cdot X_2$$

and $$X_2 = \frac{1}{\sqrt{(p_0(1-p_0)}}$$

5.2.2.3.1. Probability of Acceptance in Two-Point with Variance = 1 and given p_0:

Suppose in a sample of n r.v.'s from a Two-point distribution we have r_0 defectives, then we get the following binomial density:

Probability	p_0	$1-p_0$
No. of r.v.'s	r_0	$n-r_0$
Value of Q(x)	X_1	X_2

where as before $X_1 = 0$ and $X_2 = \frac{1}{\sqrt{p_0(1-p_0)}}$

And so,

$$\frac{1}{n} \cdot \Sigma X = \frac{n - r_0}{\sqrt{p_0(1-p_0)}} \cdot$$

The P_a for each of the schemes under study are shown below:

(a) Under 2-class Attribute there is no difference from that under normality (and indeed any other model). So, the P_a is the same as in each of the plans in this category.

(b) Under the Ramp: the r.v. statistic is

$$\bar{Q}(X) = \frac{n - r_0}{n} \qquad \text{if } X_2 \geq B \tag{1}$$

$$= \frac{n - r_0}{n\sqrt{p_0(1-p_0)}} \text{ if } X_2 < B \tag{2}$$

We spot that r_0 is the r.v. and in order to satisfy the acceptance criterion (i.e. $\bar{Q}(X) \geq T$) we have to have:

$$r_0 < n(1 - T) \qquad \text{for case (1)}$$

and $$r_0 \leq n\{1 - T\, B\, \sqrt{p_0(1-p_0)}\,\} \qquad \text{for case (2)}$$

(c) Under S-plan: if n and K are the sample and the test criterion in the S-plan respectively, and the mean and variance are

$$\bar{X} = \frac{n - r_0}{n\sqrt{(p_0(1-p_0))}}$$

$$S^2 = \frac{1}{p_0(1-p_0)} \cdot \frac{r_0(n-r_0)}{n(n-1)} \qquad \text{respectively,}$$

Then the acceptance condition (namely that $\frac{\bar{X}}{S} > K$), will imply that

$$\frac{n - r_0}{n\sqrt{(p_o(1-p_0))}} \quad \sqrt{\frac{p_0(1-p_0).n(n-1)}{r_0(n-r_0)}} > K$$

This gives the condition as:

$$r_0 < \frac{n(n-1)}{\{n(K^2 + 1) - 1\}}$$

[And because of the integral requirements on r_o we may have an erratic P_a for small n].

(d) Under the σ-plan with test parameters n and k the mean is as in (c) above. And as the variance is 1, the acceptance rule of $\bar{X} > k$ implies

$$\frac{n - r_o}{n\sqrt{(p_0(1-p_0))}} > k$$

Therefore,

$$r_o < n\{1 - k\sqrt{(p_0(1-p_0))}\}$$

[This could only make sense if $k^2 < \frac{1}{p_0(1-p_0)}$]

5.2.2.3.2. Results of Robustness under the Two-Point Model

The results revealed that except for low AQL's with small Ramp n values the by-variables plans (Ramp included) tend to have an acceptance probability greater than the target 95% (ranging from 99% to 100%). However, the Ramp is closer to the target than other plans. The cases where P_a's are reasonably close to 0.95 indicate that the Ramp is the best in the sense of being closest to the target more especially for small n.

Given both AQL and Ramp n, small B values in Ramp tend to give a P_a closer to the target than the high B-values do. As n gets larger P_a for all by-variables tend to 1.

5.2.2.4 The Lognormal and the Contaminated Normal

The discussion of matching these two distributions so as to have a unit variance and p_0 proportion defective is left for later. They are treated by simulation, and their matching is described in sections (5.3.5.1)

and (5.3.5.2). In the next few sections we will set the general guide lines of the simulation exercise and a broad layout of its planning.

5.3 Simulation for Robustness Study of Lognormal and Contaminated Normal

5.3.1. Aims of the Simulation, Design of the Generators and Preparations

As mentioned in Chapter 2, the distribution of the mean quality function for the Ramp (and indeed also the Cumulative Normal and the Cumulative Logistic) plan was not derivable analytically. Now, not only do we have this complexity but, moreover, we have more intractable alternative background distributions than the Normal one used then. Here, we are referring to the Lognormal and the Contaminated Normal models as non-normal alternatives.

We would like to compare the sensitivity of the different schemes to the changes in the background assumptions from Normality to a Contaminated Normal or a Lognormal, both of which come from the Normal model whose pseudo-random variables generator is described and justified as in the next paragraph.

Some planning and timings of the generation of the normal r.v.'s needed for generation of the Lognormal and the Contaminated Normal processes was carried out. In this respect the work reported by Atkinson and Pearce (1976) and their review comments were found invaluable. They studied the performances of different methods of generation of the random normals together with their properties. In the light of their comments we used Brent's Algorithm (1974) as recommended by them because we were trying to

insure against a slow generation of random numbers on the Computer of the University of Essex (a DEC-System 10). Brent's algorithm reduces the number of calls to the random number generator using 1.37 random numbers on average per one normal. In doing so it takes advantage of the theoretical results that if a uniform $U_{n-1} \leq U_n$ then $(U_n - U_{n-1})/(1 - U_{n-1})$ is also uniform in (0,1).

Moreover, Brent's Method is basically a Forsythe method but an improvement over it and Forsythe's method is amongst the best according to Atkinson and Pearce.

We used Brent's Grand algorithm together with our local random number generator (RAN(0)) to create a supply of normal pseudo r.v.'s. A block of a 1000 r.v.'s is generated at a time, shuffled and then stored on disc ready for use in the simulations. Generation of such a block costs 60 milliseconds and so it was found that this strategy of blocks of 1000 is cheaper especially that writing and reading from disc costs virtually nothing. The justification of all this and others is given by the following disucssion describing some preliminary experiments on which outcome we based our simulations plans.

We have carried out some necessary experiments on costs using a stored block of a 1000 pseudo-random normal variates at a time. We selected random samples of some (arbitrary) sizes of 30, 40 and 50 from this array of a 1000 variates and timed the selection process in milliseconds. The selection is made by two methods which were compared for time: a "crude" selection (whereby any of the chosen random normal variates could possibly be re-selected in the sample) and a "refined" selection (in which no re-selection of any random variate is permitted). The experiment is to give us an idea on the costs of the operations of the desirable refined selection and to indicate how the overall plan of the simulation exercise would be affected by these costs. For the sample sizes above we found that for crude selection, on the average, the absolute times taken were

(56, 69, 101) in milliseconds respectively. For refined selection the respective scores were (96, 129, 187). In relative terms we have (1.9, 1.8, 2.0) for crude and (3.2, 3.2, 3.7) for refined selection. Though as is obvious from these results refined selection is less expensive the larger n is, but generally it is expensive. Still, it is very important to have refined selection for the real representation of the process under study and for the genuine standard statistical considerations. One indirect way of achieving a refined selection is to sequentially use an already shuffled and stored supply of random variates, since shuffling will ensure randomisation and the sequential use of the r.v.'s is a neat way of ensuring no re-selection. So we decided to adopt the strategy of separating the generation and shuffling processes from the simulation main computations in such a style that the former is done once and for all. Having found that writing on and reading from a disk is very cheap indeed then according to this strategy all our needs of the normal variates were generated and shuffled once and for all. And as we need them and through no costly operations we can read them sequentially since they were well shuffled already. Advantages of this strategy are enormous. To name some we have the same large set of normals that can be used for Normal, Contaminated Normals and Lognormal processes and therefore routine correlation between these processes arise, hence facilitating comparisons by producing differences between methods with smaller variance than in case of uncorrelated streams. Details of the shuffling procedures are given in the Computer program CONTAM (listed in Appendix (B.6.)). Another advantage is that this helps to speed-up the simulation operations as well as improve its efficiency. In line with the recommended simulation practices these operations are described together with the recording techniques of acceptance probability and the precision levels in the next few sections. Also mentioned there is the use of the technique of the Monte Carlo control-variate for reduction of the variation of the estimations.

The simulation (experiments) will give a P_a under non-normality (of Lognormal and Contaminated-Normal) for given proportion p_0 of defective, for predetermined A and B and the given plans parameters and decision rules. There will be N' simulation samplings the aggregate effect of which is to give a point (p_0, P_a) on the O.C. curve where P_a is estimated by:

$$\hat{P}_a = \sum_{i=1}^{N'} u_i/N'$$

and where u in its simplest form (that of the attribute plan) is 1 or 0 indicating an acceptance or a rejection of the i-th sample simulated. [For other forms of u see below].

5.3.2. Variance-Reduction Monte Carlo Methods (Control-Variate)

Monte Carlo techniques suggest in comparison with Crude simulation of a certain sampling variance (of e.g. the usual PQ/N, in our case) some more sophisticated techniques that can favourably reduce the sampling error of estimation (Hammersley and Handscomb (1964)). One suitable technique of those is what is known as the control-variate method. The basic idea of this method is that instead of directly estimating a parameter, T, by an estimator t_1 (by the crude simulation) a new "strongly positively correlated" estimator t_2 (which can mimic t_1 and absorb most of its variation) and whose statistical expectation is numerically (or theoretically) known, T_2 say, could be used in the estimation process. The idea of this method is the improvement in efficiency (less labour and more precision). The procedure is to sample t_1 amd t_2 simultaneously (by using the same random numbers) and then use the estimator $[(t_1-t_2) + T_2]$ for estimating T. The reduction in variation comes from the fact that:

$$\mathrm{Var}(t_1 - t_2 + T_2) = \mathrm{Var}(t_1) + \mathrm{Var}(t_2) - 2\,\mathrm{Cov}(t_1, t_2)$$

which is smaller than $\mathrm{Var}(t_1)$ (the variance of crude estimation) if $2\,\mathrm{Cov}(t_1, t_2) > \mathrm{Var}(t_2)$.

In our case we let T be P_{a_j}, T_2 be P_{a_1} and $(t_1 - t_2)$ be the sampled difference between P_{a_1} and $\hat{P}_{a_j}$; and the whole argument applies. In this way we can take advantage of the Monte Carlo control-variate since we can estimate the difference between P_{a_1} and the alternative scheme and the true value for P_{a_1} (the attribute case) is known.

As will be shown in the details of the plan we will economise by re-using the same random numbers whenever possible, and the advantages of that are:

(1) It ensures positive correlation between the estimates of the P_a for all the "equivalent" plans. This means that the difference in P_a's is almost completely caused by the effect of the distribution change only [as is generally recommended in the literature, e.g., see Hammersley and Handscomb (1964)].

(2) Where applicable it ensures more precision. When a larger sample (attribute) can be proportioned into at least two smaller sample sizes of any other equivalent plan, then this plan will be simulated at least as twice to the original attribute one. And with more samplings smaller sample plans will as a by-product of such a strategy experience more precise estimation.

5.3.3 The Basic Simulation Operations

If for simplicity we use the subscript j to refer to the j-th scheme (let j=1 refer to the attribute plan) then at the i-th simulation sampling we select a sample [of pseudo-random Lognormal or Contaminated-normal variates] of the attribute plan. Then using the Attribute test parameters and decision rules we record u_{i1} =1 or 0 depending on whether the sample is accepted or rejected, respectively. For the attribute case the true P_a could be assessed analytically via the binomial model but the simulation result checks the simulation performance. Similarly let u_{ij} stand for the same idea as u_{i1} with the only difference that it now records the overall average performance of as many sub-samples as are allowed within the attribute i-th sample. This is crucial to the idea of positive correlation needed in the simulation using a control variate. It relates directly to the idea of re-using the same sample discussed above (in section 5.3.3).

Noting that generally $n_1 \geq n_j$ for all j, then for $j > 1$ define $d_{ij} = u_{ij} - u_{i1}$ where u_{ij} assumes the value 1, $\frac{s-1}{s}$, ..., $\frac{1}{s}$ or 0 where s = integral part of $\left(\frac{n_1}{n_j}\right)$. Record is kept of d_{ij} and d_{ij}^2. Also recorded are the sums and sums of squares of d_{ij} and u_{i1}.

The estimate of P_a for the j-th plan is:

$$\hat{P}_{a_j} = P_{a_1} + \bar{d}_j$$

where, P_{a_1} is the true attribute acceptance probability,

and $\bar{d}_j$ is the mean of d_{ij} over i.

The variance of this estimate is given by:

$$\mathrm{Var}(\hat{P}_{a_j}) = \mathrm{Var}(\bar{d}_j)$$

5.3.4. Precision

The number of samplings for simulation purposes were made to be a standard of 2000 samples for the largest sample size (generally the Attribute sample size). This leaves us with a sampling error of less than or equal to 0.01. This is even improved upon by the fact that any attribute sample is sub-divided into as many smaller samples as is possible by using the fact that plans have an n less than in the attribute case.

5.3.5. Matching the Simulated Processes and Transforming the r.v.'s

To start with let us use some notation for the following sections in order to clearly show the procedures of matching the two simulated processes (the Lognormal and the Contaminated-Normal). Let,

$N(\mu,\sigma^2)$ be the Normal distribution function with mean, μ, and standard deviation, σ;

$LN(M,V)$ be the Lognormal function with mean M and variance V.

$CN(M_c,M_o,G,V_c,V_o)$ be the Contaminated-Normal which is a mixture of two normals; an "original" Normal $N(M_o,V_o)$ and a "contaminant" Normal $N(M_c,V_c)$, mixed in the proportions (1-G) and G, respectively, and where G (such that $0 < G << 1$) is the contamination-fraction.

$\Phi(Z)$ denotes the standard Cumulative Normal probability at Z.

5.3.5.1 The Contaminated Normal Process

If in terms of the above notation we let the two components of the mixed contaminated Normals be $N(M_o,\sigma_o^2)$ as the original Normal and $N(M_c,\sigma_c^2)$ as contaminating Normal. Let G be the contamination fraction then the Contaminated process is defined by the following distribution

$$CN(M_c,M_o,\sigma_c,\sigma_o) = (1-G).\ N(M_o,\sigma_o^2) + G.\ N(M_c,\sigma_c^2)$$

And if we assume $\sigma_c = \sigma_o = \sigma$ then this distribution will have a mean of $\{(1-G).M_o + G.M_c\}$ and a variance of

$$(1-G).(\sigma^2 + M_o^2) + G.(\sigma^2 + M_c^2) - \{(1-G).M_o + G.M_c\}^2$$

Implementing the constraint that the overall variance should be equal to 1 defines a relation between M_o, M_c and σ. However, if we then set M_c=0 we will be able to determine M_o and σ from the following constraint when the value of the AQL is given:

$$AQL = (1-G).\Phi(-M_o/\sigma) + G.\Phi(-M_c/\sigma)$$

This will determine the distribution of the contaminated process. A simulation of such a process is derivable from the basic Normal r.v.'s generated in the manner described before. Such a simulation has been incorporated in the Fortran program CONTAM listed with full details and comments in Appendix (B.6.).

If we are considering variance-hererogeneity in the two Normals then the algebra is similar to the above except that $\sigma_c = D.\ \sigma_o$ will be

another relation to add to the above system and solve simultaneously. Here D is a "heterogeneity" constant to inflate the variance of the contaminant Normal.

We select a Normal deviate in the manner discussed earlier and then a new and independent uniform deviate is generated. If this new uniform deviate is less than G (the contamination fraction) the selected Normal deviate is then transformed into a contaminant Normal (by rescaling for variance and adding mean M_c) and similarly for an original variate which occurs when that uniform is greater than or equal to G.

Of course, it is sensible and realistic to make G << 0.5.

5.3.5.2 The Lognormal Process

The Lognormal distribution is only defined in the domain of $(0,\infty)$. The parameters linking the background Normal and the mean and variance of the Lognormal appear only in exponential type functions and require numerical methods for the determination of the appropriate values of the specification limit and mean, for the parameters especially the variance are inflexibly exponential.

We would exploit the fact that if Y is distributed as $N(u,E^2)$ then $W = \exp(Y)$ is distributed as LN(M,V) where the Lognormal mean is

$$M = \exp(u + E^2/2)$$

and the variance is

$$V = \exp(2u).\{\exp(2E^2) - \exp(E^2)\}.$$

We know that the Normal theory of $N(u,E^2)$ gives us the lower specification limit A as

$$A = u + K.E$$

to ensure a proportion defective (left tail) of p_0; but such A as set to be 0 is not convenient for the Lognormal distribution because of the domain of definition. We re-defined the lower cut-off point, since the conventional setting at zero is no longer convenient, at C where

$$\ell n\ C = u_0 + K.E \qquad (1)$$

where C is a constant (> 0) such that if Y is $N(u_0 + \ell n\ C,\ E^2)$

then $W = \exp(Y)$ is $LN(M,V)$ such that

$$M = \exp(u_0 + \ell n\ C + E^2/2) \qquad (2)$$

and

$$V = \exp(2(u_0 + \ell n\ C).\{\exp(2E^2) - \exp(E^2)\} \qquad (3)$$

and

$$K = \Phi^{-1}(p_0) \qquad (4)$$

If we fix E and u_0 by solving for them using (1), (3) and (4) simultaneously such that V = 1 in (3), then we will have defined a Lognormal distribution such that we have:

(i) same proportion of p_0 below C

(ii) a unit variance, V = 1

as would be needed for the matching.

Unfortunately, we would not have the p_1 to match. This is a characterising feature of all our alternative distribution forms. In all of these one can not fix the variance and p_0 and then be in a position to fix p_1 (and consequently p_2) as well.

For solving for E, mentioned above, we have an exponential function to solve for which Newton-Raphson iterative technique is used.

These results are used to create the Lognormal process using X, the normal r.v. ({where X is distributed as N(0,1}) by the following simple algorithm:

1. $Y' = X \cdot E + u_0 + \ell n\ C$

2. $Y = \exp(Y')$

This Y is a Lognormal variate with mean M and variance V = 1 and a proportion, p_0, below C (which is +ve, and is determined above). For this Y to represent and mimic the actual process we require the displacement:

3. $W = Y - C$

Thus, W is ready as a Lognormally distributed process for studying the robustness performance of the different plans under test.

5.3.6 Simulation Results of Contaminated-Normal and the Lognormal Distribution Effect

5.3.6.1 Contamination Robustness

We dealt with two cases of Contamination processes; one is when

$\sigma_c = \sigma_o$ (i.e. the variances in the Contaminant and the Original processes (both Normally distributed) are equal). The second case is when $\sigma_c > \sigma_o$. But under all these cases the overall variance is held to be 1. (Results are displayed in the tables of Appendix (F.1.)).

Case (i): $\sigma_c = \sigma_o$

It would be clearly seen in the tables that for the two mixed normals (agreeing in variances but with different central tendencies) the Ramp plans are frequently marginally better (in robustness) than their equivalent orthodox by-variables plans. But for all practical reasons the discrepancy is immaterial which will consequently lead to these plans being comparable. We believe that the real reason behind this is the established belief that the mixture of two normals is not markedly different from Normal, especially for as a small γ (the contamination fraction) as 10% (Tukey (in Olkin et al [Eds] (1960)). The noticeable fact is that regardless of n, B and AQL our P_a results are indistinguishable from those under normality.

Case (ii): $\sigma_c > \sigma_o$

The tables in Appendix (F.1.), show the performance under a process which is a mixture of two normals with the contaminant normal of not only an inferior mean but of an "out-of-control" variance. The variance of the contaminant distribution is inflated by a reasonable constant. If σ_o and σ_c' are the standard deviations of the original and the contaminant processes respectively then we can have:

$$\sigma_c = d.\sigma_o$$

where d is an arbitrary constant such that $d > 1$. (Note that $d=1$ is in fact case (i) discussed above).

We only looked at two cases of d; a case of $d = 1.2$ which is a reasonable choice and a case of $d = 2$ which is a bit wild and just enough to establish our case. The results of robustness for $d = 1.2$ did not give any basically different results from those of $d = 1$.

However, in the case of $d = 2$ (and where computations are mathematically feasible) there is some evidence of some instances of small losses of robustness by all the variables plans (including Ramp). But more important is that the Ramp is slightly more discriminatory and more robust than the orthodox variables plans.

5.3.6.2 The Lognormal Robustness

Similar to previous distributions we impose the constraint that the overall variance is constrained to be 1, and the proportion defective matched as p_0. The results of the Lognormal robustness appear in Appendix (F.1.) in the first column of the tables in the appendix. Discussions of the trends of these results follow here below.

The Ramp behaviour under the Lognormal process varies wildly with B. Rightfully this is not strictly true for the orthodox schemes. As B increases the Ramp P_a robustness decreases. This is expected of the Ramp because of the earlier argument that as $B \to 0$ the Ramp tends to the 2-class attribute scheme whereby "perfect" robustness is maintained. Note that irrespective of anything, the Lognormality results in the orthodox schemes show how comparable (though sometimes appearing very poor) are the

s-plan and σ-plan schemes under this distribution-effect.

Comparing the Ramp with the orthodox by-variables schemes under Lognormality, the Ramp gives better performance for small B and AQL but gives way to them for large AQL's. This is generally so for any n (with a difference of intensity or degree as would be revealed in the next paragraphs).

For Small AQL

If we consider small values of B then large Ramp n values appear to be more indicative of a better performance of the Ramp scheme than the orthodox variables schemes. Though for small Ramp n this is still true yet it is not as significant and acute as for large Ramp n. This is as far as comparison between the schemes is concerned. In another wider context than this comparison we find that: for large n values under Lognormality all the variables schemes (Ramp of $B \gg 0$ included) seem to give way as unreliably weak schemes (in the Lognormal robustness). (Note that the Ramp with B close to 0 could then be of possible comparability as opposed to the "ideal-robust" (i.e. the 2-class attribute schemes), since the latter is the limiting case of the Ramp as $B \to 0$.

For large B values and large Ramp n values though as mentioned earlier the orthodox schemes are better yet the differences are not markedly noticeable; and for what matters all appear to be very poor and bad for lognormality distributed processes. Therefore, for skewed processes never mix high B values with high n values.

For large AQL a similar analogy reveals that for small B values and

large Ramp n values the orthodox variables are markedly more robust than the Ramp, while if Ramp n is small this is still true but there is not much of a distinction as is the case for large Ramp n.

(Note the persistent result that the S-plan and the σ-plan are comparable (even when poor!) under the understanding that the process variance is 1).

The results show the clear pattern that for any Ramp scheme cases of say, B < 0.9 the performance of the Ramp is better than other variables schemes (and than other Ramp with B values > 0.9). This confirms our intuition about the relation of the Ramp with the other variables plans. The comparative pattern of the behaviour between the three variables schemes is summarised by the following figure which shows that the rate of decrease of p_a with respect to B is higher for the Ramp than for the σ-plan and S-plan, hence the conspicuous intersection on the diagram (Figure (5.3.1.)):

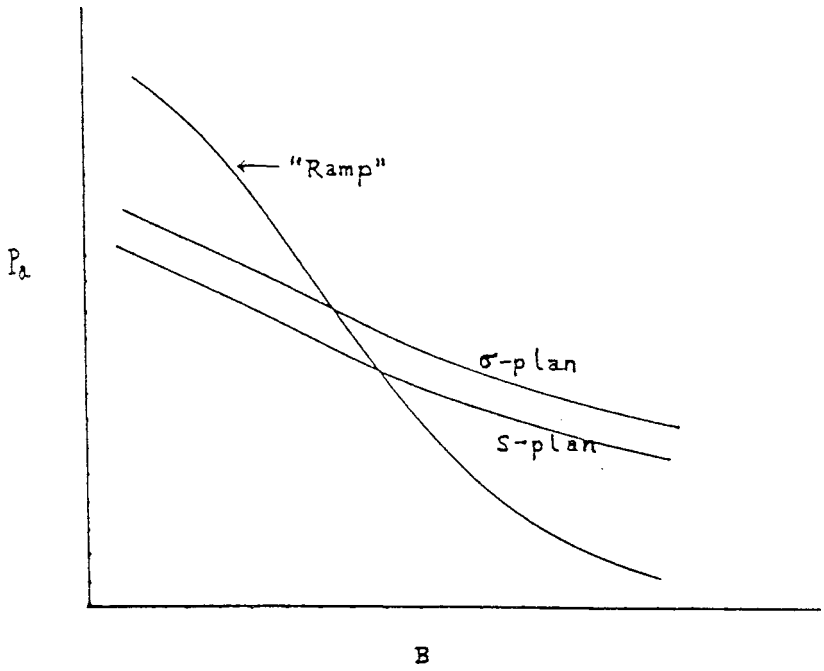


Figure (5.3.1): The Lognormal Robustness comparison between schemes for given AQL (as a function of B).

In the results as Ramp n increases the Ramp P_a-curve tends to dominate the σ-plan and S-plan $P_a$curves (but as the discussion above revealed, all these 3 curves move towards the point of origin indicating a poor performance by all variables schemes as they move).

The point of intersection when it occurs is of special significance in the comparative study of robustness. Note that for small AQL, e.g. AQL = 0.0139, at n = 16 the intersection of Ramp with σ-plan curve is close to B = 1.3. Below this value of B the Ramp is dominant and above it it is dominated by other by-variables schemes. As n increases this is still true for small AQL but by then all the plans seem to perform poorly. So for maintaining the same level of P_a the intersection point will have to move to the right if n is higher (i.e. a lower B value).

This is also true for large AQL too with the exception that as n increases there is a tendency for a shift of the intersection point to move to the left.

Another pattern in the results is that for small B a given B and AQL the P_a varies with n as in the following figure (Figure (5.3.2.)). The curves are dominated by the Ramp P_a-curve. This indicates the resilience of the Ramp plan under Lognormality. The "orthodox" by-variables tend to lose more on robustness for large Ramp n than does the Ramp.

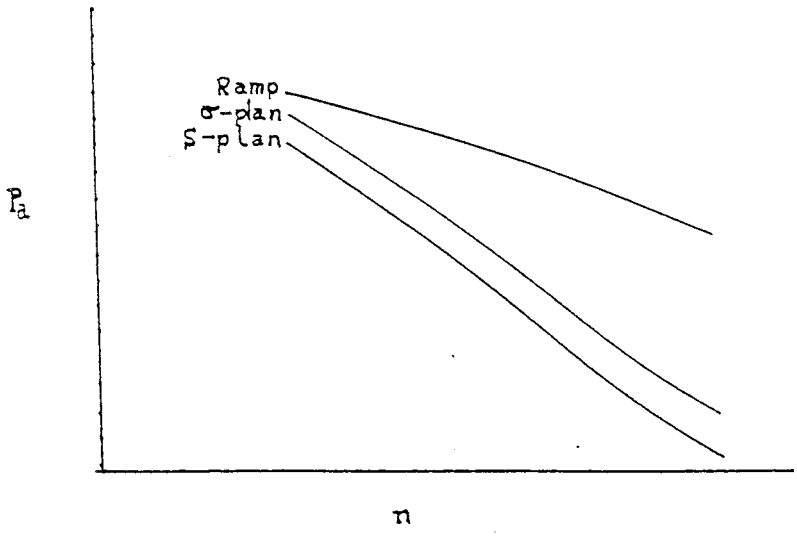


Figure (5.3.2): The P_a performance against n under the Lognormal Model for the three by-variables schemes.

5.4 The Variance-Effect Robustness

Here, we will only deal with the changes in the variance of the process but still under normality assumption.

5.4.1 The Variance-Effect for Each of the Equivalent Schemes

It is interesting to note that the change in variance in the Normal process will imply a change in the scale of the Ramp Quality function and by the same factor (as shown in section (2.2.3.1(a)) of Chapter 2). Not only that but considering the shape of the quality function which is imune to scale effects at the two tails the changes in scale (if variance changes) would only be reflected on B. In this context, of variance robustness, this point will be an advantageous basis in the evaluation of the robustness P_a. Simply, a change of twice the variance will mean a change in B to half B.

As for the Two-class Attribute scheme the variance changes are absolutely immaterial. The S-plan is not affected directly by the value of σ^2, it evaluates a sample-counterpart S^2 and, so, effectively has a built-in flexibility to accommodate such variance changes. It can be said that the S-plan is robust as far as the variance-effect is concerned.

The only other scheme that is very responsive to variance changes is the σ-plan since it assumes knowledge of σ. The evaluation of such changes on the P_a are analytically manageable. Analogous to the Ramp this scheme will experience the scaling effects on K of changing the variance, as on B. Intuitively, and because of the partial effect on the Ramp system (i.e. only on the marginal zone) as opposed to the total

effect on the σ-plan system we would expect the Ramp to do better than the σ-plan when subjected to the same variance change.

5.4.2 The Strategy for the Study

It would have been a natural strategy to ask the direct question "what is the effect on P_a of changing σ from the standard 1 to a pre-determined set of σ values?" But since we already know that a change in σ is a rescaling of B in the Ramp plan (and of K_σ in the σ-plan) and also, since we are constrained by a tabular grid distribution of the Ramp statistic we have changed the strategy. We would rather be investigating the effects on P_a when the planned Ramp B value, say B*, is rescaled by σ such that we move to a new "True" B value, say B^T within the given grid points of B. The question asked becomes: "what are the effects on P_a if σ changes to the "True" σ, say $\sigma^T = B^*/B^T$?" This strategy simplifies the computations and avoids unnecessary interpolation between the B values when evaluating P_a since it uses the tabulated B grid. The results shown on table (5.4.3) in Appendix (F.2.) answer such a question.

5.4.3 Results of Variance Robustness

The table (5.4.3) shows the variance-effect robustness for Ramp and σ-plan. For a given Ramp n and a proportion defective AQL there is a sub-table. B* values appear on the left of every sub-table in brackets; while the true B^T values are shown at the heads of the columns. For each B there are two rows. The first row is for the Ramp plan with the first entry representing the Ramp test criterion t. (If t is negative it means that there is a problem of "non-Existence" and hence the entries "NE" indicate such a problem). The second row for each B is for the (equivalent)

σ-plan, and similarly this row is prefixed by value of K_σ, the test criterion for the σ-plan. This gives us a double entry for each (B^*, B^T) cell; the top P_a is the Ramp performance and the bottom is that for the (equivalent) σ-plan.

To conserve space the true value of σ, i.e. σ^T, is not shown on the table since it is easy to evaluate as shown above. ($\sigma^T = B^*/B^T$ which in the sub-table below correspond to dividing the row B value (shown in brackets) by the column B value). In fact the tables below do not need any display of the σ values, and are easy to follow as the following summary and conclusions of these sub-tables show.

In these tables it is clear that the Ramp shows remarkable resilience to changes in variance, while the σ-plan the unfavourable dependency on the "known" value σ. This shows in the tables simply by the fact that almost every double entry the Ramp P_a is closer to the goal P_a (of 0.95) in these sub-tables than its σ-plan counterpart. The few entries where this is not strictly true show a tie between Ramp and σ-plan. So we should emphasise that no σ-plan behaves better than the Ramp plan in the Variance-effect robustness.

The conclusive evidence in these sub-tables indicates one clear result that under Variance-effect the Ramp is more robust than the σ-plan. A blurred but still noticeable result in terms of n is that as n increases the stability of the Ramp becomes more pronounced: the σ-plans start to collapse more quickly for these large values of n than for smaller n values.

N.B.

The following table is useful in giving the true changes in σ for the tables of the Variance-effect robustness result given in Appendix (F.2.):

Value of True $\sigma = B^*/B$

{B* : is the planned value, and B is the True value)

B :	0.5	0.7	0.9	1.1	1.3	1.5	1.7	1.9
B*								
0.5 :	1	0.714	0.556	0.455	0.385	0.333	0.294	0.263
0.7 :	1.400	1	0.778	0.636	0.538	0.467	0.412	0.368
0.9 :	1.800	1.286	1	0.818	0.692	0.600	0.529	0.474
1.1 :	2.200	1.571	1.222	1	0.846	0.733	0.647	0.579
1.3 :	2.600	1.857	1.444	1.182	1	0.867	0.765	0.684
1.5 :	3.000	2.143	1.667	1.364	1.154	1	0.882	0.789
1.7 :	3.400	2.429	1.889	1.545	1.308	1.133	1	0.895
1.9 :	3.800	2.714	2.111	1.727	1.462	1.267	1.118	1

CHAPTER 6

EXTENSIONS AND DEVELOPMENTS

6.1 Extensions and Developments of the New Schemes

With the level of robustness decided upon in Chapter 5 we make use of our new schemes to exploit their potentialities. Their basic concepts are extended to mixed (or "attri-var") dependent plans. We will show how we would use the whole of the new variables Ramp schemes to fit the system of mixed dependent plans maintaining their best properties and acquiring new ones as well. We are not to broaden the scope of this chapter and will confine it to the extent of substantiating our claims about the potentials and adaptability of our "new" schemes. Besides, we will give the theoretical results and framework of the operating characteristics of the developments suggested in this chapter, together with an example to show our attri-var procedures in relation to an example of the Schilling and Dodge plans. Some simulation of these two plans has been made for comparison.

The motivating factor here is that the Schilling and Dodge (1969) formulation of the mixed dependent plans has a number of favourable properties, so we can build on their results. On the other hand one certain important weakness of their procedure is that it is not safe-guarded against non-normality effects when accepting, first-time, the batch in the by-variables-sample stage. One argues that rejection occurs only after re-sampling i.e. on the attributes basis according to the set-up of their plans. Acceptance, according to their scheme can possibly occur

on the variables first sample. And as a consequence of the reliance of the variables plan (especially the σ-plans) on normality assumption, and how non-robust they are, the acceptance decision may not be very solidly based.

They rightfully detected this fault and suggested as a remedy that one should accept (on first sample i.e. variables basis) only the "very clear cases". One thinks that this could imply an increase in the ASN (average sample number) by a factor equal to the proportion of the occurrence of the "not very clear cases" times the attribute sample size. After all it is the minimisation of the ASN which motivated the attri-var schemes. With the call for easy-to-operate schemes in acceptance sampling such statements like the "not very clear cases" are a complication.

With the established favourable property of robustness of our Ramp schemes we can trade-off more robustness for more (or conceivably no) premium at all. With this discussion in the background we could propose three developments in the attri-var area. In the order below, each development is an evolutionary progress from its predecessor. This culminates in a "new" "attri-var" scheme based on the single Ramp scheme advocated in the previous chapters.

6.1.1 Development I

The mixed dependent plans as discussed by Schilling and Dodge (1969) have the following procedure. For given parameters n_1, n_2 (first and second sample sizes), c_1, c_{12} (attribute acceptance numbers on first and combined n_1+n_2 sample, respectively) and A (the acceptance specification limit); take a first sample, n_1. If on the basis of the by-variablesplan of n_1

and test constant k, one had the sample mean, $\bar{X}$, such that $\bar{X} > A + k$ then one accepts the batch, otherwise test the n_1 sample on by-attribute basis accepting if the number of defectives (i.e. item below specification A) is $\leq c_1$; and if not accepted then resample a further n_2. (Note one is never rejecting on the basis of n_1). Now, if re-sampling takes place a dichotomous decision is made on the basis of the combined $(n_1 + n_2)$ sample as follows: accept if number of effectives $\geq n_1 + n_2 - c_{12}$, but reject otherwise.

Now, our development I has two versions, (Ia) and (Ib), both of which use our idea of marginal quality (between the limits A and B) and the idea that the decision to resample is done on the variables-basis (viz; when the batch quality is marginal) rather than on the attribute (n_1, c_1) basis used by Schilling and Dodge. But version (Ia) has more ASN than (Ib) due to its allowing more chance of resampling than (Ib).

6.1.1.1. Version (Ia): This allows rejection and acceptance on the first sample but only in a precisely defined "very clear cases", otherwise resample. Its procedures are:

Take a sample n_1

Accept if $\bar{X} > A + K_{1,\sigma}$.

Reject if $\bar{X} < A + K_{0,\sigma}$.

Resample otherwise.

In the Combined $n_1 + n_2$ Sample

Accept when the number of combined effectives $\geq n_1 + n_2 - c_{12}$.

Reject otherwise.

Where the test constants $K_{0,\sigma}$ and $K_{1,\sigma}$ are such that $A + K_{0,\sigma} = \Phi^{-1}(p_0)$ and $A + K_{1,\sigma} = \Phi^{-1}(p_0 + p_1)$ and c_{12} is the maximum of permissible number of defectives in the usual by-attributes plan of sample size equal to $n_1 + n_2$. [And $\Phi(.)$ is the standard Normal Distribution Function].

This development is very useful and recommendable because it gives a second chance only for the marginal cases to be re-examined under more information. So, it is not only a producer-oriented plan (as is the case of Schilling and Dodge's scheme) but considers producer and consumer interests simultaneously and impartially.

The mentioned development necessitates different computations but we can use Schilling and Dodge's conditional results and the approximates given by Elder and Muse (1982) as well as the identity:

$$P_n(r \text{ defectives: } \mu,\sigma) = P_n(r \mid \text{"V1"}, \mu,\sigma).Pr(\text{"V1"} \mid \mu,\sigma) + P_n(r \mid \text{"V3"}, \mu,\sigma).Pr(\text{"V3"} \mid \mu,\sigma) + P_n(r \mid \text{"V2"}, \mu,\sigma).Pr(\text{"V2"} \mid \mu,\sigma)$$

where,

"V1" is the event of $\bar{X} < A + K_{0,\sigma}$

"V2" is the event of $A + K_{0,\sigma} \leq \bar{X} \leq A + K_{1,\sigma}$

"V3" is the event of $\bar{X} > A + K_{1,\sigma}$.

The left-hand-side is computable through the binomial distribution with parameters n and $p_0 = \Phi(A + K_{0,\sigma})$. On the right-hand-side the first and the second terms could be evaluated using Schilling and Dodge (1969) conditional probabilities [or the approximation of Elder and Muse (1982)] with the necessary adjustments. The third term is deduced as a solution of this whole equation. This third term is of direct interest to us. It is the probability of re-sampling (in version (Ia)) on which the contribution to the probability of acceptance from the attribute combined basis is conditional.

P_a, the acceptance probability, under this scheme is

$$P(\text{accept: } \mu,\sigma) = P_{n_1}(\text{"V3"} \mid \mu,\sigma) + \sum_{r=0}^{c_{12}} P_M(r,\text{"V2"} \mid \mu,\sigma)$$

where $M = n_1 + n_2$,

and "V1", "V2", "V3" are as above.

6.1.1.2. Version (Ib): It is the same as version (Ia) except that there is a chance of assessing the first n_1 sample on its attribute characteristics whenever $A + K_{0,\sigma} \leq \bar{X} \leq A + K_{1,\sigma}$ before resampling. Here, if d_1 (the number of defectives in the n_1 sample) is less than c_1 we accept the lot, otherwise resample and carry on as in version (Ia) above.

If, for convenience, we developed the following notation:

$P_n(E)$ is the probability that event E occurs in a sample of n,

$P_n(E1,E2)$ joint probability of events E1 and E2 in sample of n,

$B(j;n)$ probability of j defectives in a sample of n;

then the acceptance probability is:

$$P_a = P_{n_1}("V3") + \sum_{r=0}^{c_1} \sum_{j=0}^{c_2-r} P_{n_1}(r,"V2").B(j;n_2)$$

NOTE: The modification in version (Ib) on (Ia) is a technical device meant to reduce the average sample number (ASN).

6.1.2 Development II

This uses our previous results of single sample quality functions of the new designs. Here, for the variables part of mixed plans we suggest using our "equivalent" variables schemes; thus:

- Take a first n_1 sample $X_1, X_2, \ldots, X_{n_1}$.
- Evaluate the quality functions for each X_i, and find $\bar{Q}(x)$, the sample quality mean.
- For the relevent test criterion, t, and parameters of our plan:

Accept if $\bar{Q}(x) > t$, otherwise use "attribute" test whereby if d_1, the first sample number of defectives, is such that $d_1 \geq c_1$ reject and if $d_1 < c_1$ resample a second n_2. On resampling, if the combined total defectives $(d_1 + d_2) < c_{12}$ accept, and otherwise reject, where d_2 is the sample number of defectives in the second sample.

The acceptance probability here is:

$$P_a = P_{n_1}\{\bar{Q}(x,n_1) \geq t(X,n_1)\} + \sum_{l=0}^{c_1} \sum_{j=0}^{c_2-l} P_{n_1}\{l,\bar{Q}(x_i,n_1) < t(X_i,n_1)\}.B(j;n_2)$$

where (in addition to the notation defined above, we have:

x_i, X_i are the observation and population r.v. of the process respectively.

$\bar{Q}$ is the sample mean of the observed quality functions.

6.1.3 Development III

The basic idea in this development is that we will be using our B-values (used in our basic single sampling plans) to cater for both the by-variables and the by-attribute counterparts of the mixed plans. Then, we will not be talking about the plans as "mixed" in the sense that we are mixing attributes with by-variables schemes but that we are just mixing different B and n values within the context of the "new" by-variables scheme.

The algorithmic procedures of this plan are as follows:

(i) Stretch B towards the extreme (i.e. approach the variables schemes), call this B_1.

(ii) Use a sample size n_1 with the test criterion t_1. Denote the drawn sample by $(x_1, x_2, \ldots, x_{n_1})$

(iii) Evaluate the quality functions for each x, and find $\bar{Q}(\underline{x},n_1,B_1)$.

(iv) Compare:

If $\bar{Q}(\underline{x}, n_1, B_1) \geq t_1(\underline{X}, n_1, B_1)$ then accept, otherwise resample a further n_2 items.

[NOTE: no rejection is allowable yet].

(v) If not accepted in (iv), then collapse B_1 to a smaller value, say B_2 (i.e. approach the attribute scheme within the variables one). And using the combined sample of $m=n_1 + n_2$ evaluate the statistic $\bar{Q}(\underline{x}_c, m, B_2)$ and compare:

if $\bar{Q}(\underline{x}_c, m, B_2) \geq t_2(\underline{X}^{(m)}, m, B_1)$ accept, and reject otherwise.

The probability of acceptance in this case is:

$$P_a = P_{n_1}\{\bar{Q}(\underline{x}, n_1, B_1) \geq t_1(\underline{X}^{(m)}, n_1, B_1)\}$$

$$+ P_m\{\bar{Q}(\underline{x}_c, m, B_2) \geq t_2(\underline{X}^{(m)}, m, B_2)\}$$

$$* P_{n_1}\{\bar{Q}(\underline{x}, n_1, B_1) < t_1(\underline{X}^{(n_1)}, n_1, B_1)\}$$

Here, B_1 and B_2 are pre-determined values that depend on our certainty of the knowledge of the model of the background distribution or the variance. Large values of B_2 imply that we are reasonably certain of the model. This relates and ties in well with the discussion in section (2.2.3.2) about the decision levels and risks of non-robustness.

It is implied in this approach that B_1 would be greater than B_2.

6.1.4. An Example

In this section we cite an example to show how our attri-var plans

could represent those of Schilling and Dodge and to reflect the operating procedures of both schemes.

To select an attri-var plan from our scheme we need the AQL and the producer's and consumer's risks. There is an arbitrary element in the choice of the first and second sample sizes and also in the test criteria used to decide when to re-test. In our case we set $B_1 = 1.9$ to produce a sensitivity close to that of the by-variables plans, and choose n_1 and T_1' to give a rough equivalence with the first stage acceptance by the Dodge and Schilling procedure.

A second test value T_1'' is used to give an approximate equivalence with the rejection stage using the first sample only in Dodge and Schilling. The decision to re-test uses the two criteria T_1' and T_1''. The choice of second sample size is set at $n_2 = 3n_1$ so that $n_1 + n_2 = 4n_1$, and in the case of the Dodge and Schilling plans the second sample is set equal to the difference between their first sample size and the attribute (second) sample size equivalent to $4n_1$, using $B_2 = 0.5$ and the corresponding T_2 for the AQL. A literal reproduction of the Dodge and Schilling scheme would require $B_2 = 0$ but this is an extreme case and we have seen that $B_2 = 0.5$ gives many of the advantages of attribute plans but with a smaller sample size.

The choice of T_1' and T_1'' is made as if the acceptance and rejection at the first stage was made with sample size n_1, $B_1 = 1.9$, and the AQL's equal to 0.77 and 1.64 times the desired AQL. T_2 is chosen to correspond with the true AQL and sample size $n_1 + n_2$ and $B_2 = 0.5$.

To illustrate the techniques consider the following example.

Example

Suppose we have the following pool of randomly selected data measuring the yield point for certain steel castings (in 1000's) drawn from the same batch which is to be inspected for acceptance.

50.8	50.5	49.9	50.0	50.0	49.7	50.6	48.6	49.9	51.2
52.5	51.4	50.9	50.7	50.1	49.4	50.1	51.7	53.0	49.9
52.9	51.2	49.3	51.9	52.1	51.4	50.7	50.2	50.1	49.8
53.2	49.3	51.3	49.7	51.1	51.0	49.9	47.9	51.0	50.2
51.0	51.3	50.7	51.1	49.7	50.3	50.3	50.7	51.0	51.2
50.6	50.2	50.1	49.2	49.5	50.0				

Suppose that their order of appearance is immaterial, and for convenience we will pick the first samples from the first sequence of measurements and the second sample from what is left.

Let the specified minimum yield point for these castings (shown in units of 1000's), be L = 48.7psi. The batch submitted for inspection is believed to have a "known" standard deviation of σ = 1.0psi. Suppose the specified Schilling and Dodge attri-var plan to be used here had AQL = 1.39%, a first sample size $m_1 = 6$, a first acceptance number $r_1 = 1$ and an acceptability (by-variables) constant k = 1.5537; while for the second sample $m_2 = 50$ and $r_2 = 2$.

Then for Schilling and Dodge we have the following table of decision rules and results, where $w = L + k\sigma = 48.7 + 1.5537\ (1.0) = 50.25$:

Decision Steps	Results
1. Determine the parameters.	For variables stage: $m_1=6, w=50.25$ For transitional stage: $c_1=1$ For final combined stage: $c_2=2$ $m_1+m_2=6+50=56$.
2. First sample m_1 specimens from the batch. Compute $\bar{x}$.	$\bar{x} = 50.233$.
3. Accept and terminate if $\bar{x} > w$ otherwise continue.	$\bar{x}$ is not $> w$, so next stage.
4. If $\bar{x} \leqslant w$ test the m_1 specimens (by-attributes) enumerating the number of defectives, d_1 in m_1.	$d_1 = 0$.
5. If $\bar{x} \leqslant w$ and $d_1 > c_1$ reject and terminate, otherwise continue to next stage.	Since d_1 is not $> c_1$, go to next stage.
6. Second sample of $m_2=50$ is drawn and number of defectives, d_2, in m_2 is observed. (If $d_1+d_2 \leqslant c_2$ accept, otherwise reject).	$d_2 = 1$ (since one defective item of 47.9 is noted) in m_2, implying $d_1+d_2=0+1=1$ as the overall combined number of defectives in m_1+m_2. Hence d_1+d_2 is not $> c_2(=2)$, and the batch is accepted.

Our plan to correspond to this one will be based on rough equivalence as stated above. The equivalence tables give us the following parameters. For first stage : $n_1 = 8$, $B_1 = 1.9$, AQL' = 1.39 (.77) = 1.07% giving $T_1' = 0.7330$ while AQL" = 1.39 (1.64) = 2.275% giving $T_1'' = 0.6463$. For second sampling $n_2 = 3$ (8) = 24 (so that $n_1 + n_2 = 32$), with $B_2 = 0.5$ and $T_2 = 0.9264$. (Note that 0.77 and 1.64 are arbitrary constants to deflate and inflate AQL respectively.)

The values of X_{1i} (i.e. the first sample values are standardised by the transformation $Z_{1i} = \frac{X_{1i} - L}{\sigma} = \frac{X_{1i} - 48.7}{1.0}$ before being evaluated by the Ramp score function:

$$Q_{B_1}(Z_{1i}) = \begin{cases} 0 & \text{if } Z_{1i} \leqslant 0 \\ Z_{1i}/B_1 & \text{if } 0 < Z_{1i} < B_1 \\ 1 & \text{elsewhere,} \end{cases}$$

and similarly for B_2 and X_{2i} (the second sample values).

On the following table we have the decision rules and the results for the Ramp attri-var version. The middle column of the tables shows how the calculations were made especially for the scores.

Decision Steps	Necessary Computation	Results
1. Parameters		First and transitional stage: $n_1=8$, $B=1.9, T_1'=0.7330, T_1''=0.6463$ Final combined stage: $n_1+n_2=32$ $B_2=0.5, T_2=0.9264$
2. First n_1 sample and their $Q_{B_1}(Z_{1i})$ scoring functions. Compute $\bar{Q}_{B_1}$.	Z_{1i}: 2.1 1.8 1.8 1.2 1.3 1.0 1.9 -0.1 Q_{B_1}: 1.0 .95 .95 .63 .68 .53 1.0 0.0 $\bar{Q}_{B_1} = \frac{\sum_{i=1}^{n_1} Q(Z_{1i})}{n_1} = \frac{5.74}{8} = 0.7175$	$\bar{Q}_{B_1} = 0.7175$
3. If $\bar{Q}_{B_1} > T_1'$ accept and terminate, otherwise continue		$\bar{Q}_{B_1}$ is not $> T_1' = 0.7330$
4. If $\bar{Q}_{B_1} \leqslant T_1'$ then compare with T''_1.		So no acceptance yet, so compare with T_1''.
5. If $T_1'' < \bar{Q}_{B_1} \leqslant T_1'$ then second sampling, while if $\bar{Q}_{B_1} \leqslant T_1''$ reject and terminate.		$\bar{Q}_{B_1}=.7175$ is not $> T_1'=.7330$ and is not $\leqslant T_1''=.6463$ (i.e. $T_1'' < \bar{Q}_{B_1} \leqslant T_1'$ hence second sample).
6. Second sample of n_2 is taken, scoring functions and $\bar{Q}_{B_2}$ evaluated and on the basis of (n_1+n_2), B_2 and T_2, compare: if $\bar{Q}_{B_2} > T_2$ accept and otherwise reject.	Some voluminous necessary computations were made similar to above, and whose summary is given as: $\bar{Q}_{B_2}(Z_{2i}) = \frac{\sum_{i=1}^{n_1+n_2} Q_{B_2}(Z_{2i})}{n_1+n_2} = \frac{31.0}{32} = 0.9688$	$\bar{Q}_{B_2}=.9688$ is $> T_2=0.9264$ hence acceptance of the batch.

Simulation of the Example

This same example was run under a variety of simulated processes each for 1000 simulation runs. The simulated processes were the normal, uniform and lognormal (all having a unit variance). For each case two processes with relatively good and poor quality respectively were studied. The performances of the above plan of Schilling and Dodge and our plan are recorded and reported below for:

(i) The average sample number, ASN.

(ii) The probabilities of acceptance, P_a, and of rejection, P_r.

Note that in the following tables the "relatively good quality" and the "relatively bad quality" refer to the proportions defective $p_0 = 0.0139$ and $p_0 = 0.1601$ respectively.

We used the abbreviation "Ramp" to stand for our attri-var, and "S & D" for Schilling and Dodge attri-var plan.

		Normal Model		Uniform Model		Lognormal Model	
		S & D	Ramp	S & D	Ramp	S & D	Ramp
Relatively	ASN	6	9	6	15	6	15
good	P_a	0.939	0.978	0.620	0.570	0.408	0.511
quality (p=.0139)	P_r	0.061	0.022	0.380	0.430	0.592	0.489
Relatively	ASN	20	11	6	12	6	9
bad	P_a	0.075	0.039	0.182	0.107	0.054	0.006
quality (p=.1601)	P_r	0.925	0.961	0.818	0.893	0.946	0.994

In this example it is clear that the probability of second sampling is greater for the Ramp plan than for that of "S & D". The conditions for a decision on the first sample are more stringent in the former than the latter. We could not exactly match the condition for re-testing in S & D, and the overall matching of the two attri-var plans is therefore approximate. This partly explains why the value of ASN is the value of m_1; the plan has more contribution from the second sampling than in the case of S & D. Second sampling is not taking place for S & D in the cases cited in the table above except for relatively bad quality under Normal Model where ASN for S & D is 20 and is higher than that of the Ramp.

The Ramp has a better discrimination than S & D plan for both good and bad quality under both Normality and Lognormality. It is also better under the Uniform model for bad quality batches. The word "better" here refers to the probability of having greater acceptance of good quality and greater rejection of bad quality.

6.2 Property "Q" in the Attri-var Schemes

If a final decision on the batch is reached on the basis of the first (i.e. variables) sample of the attri-var plan then the plan definitely has property "Q" only if the σ-known normality assumption holds. This is because here we will effectively be dealing with a variables scheme under known σ and normality assumption. The proof of Property "Q" holding in this case is dealt with by Farlie (1981). (Verification of Property "Q" is given in Appendix (A.2.)).

By the definition of Property "Q" a plan is questioned in its acceptance stage. So no matter how an attri-var scheme of known-σ schemes is designed then for the purposes of Property "Q" we would generally postulate that the attri-var has Property "Q" if-and-only-if all its acceptance phases have Property "Q". So, as a consequence of this, the attri-var plan will have Property "Q" if the decision of accepting the batch is reached on the basis of the attribute quality.

The advantage of our developments II and III is that the acceptance is made on the basis of the average quality $\bar{Q}(x)$ which is monotonic, and bounded by 0 and 1; and no matter how extreme are the values of X_i the quality function records them as 0's or 1's, depending on whether the values are at the low or high extreme.

CHAPTER 7

CONCLUSIONS (WITH RECOMMENDATIONS)

All the variables plans have a more robust performance for small n than for large n. Large n performances only show some degree of robustness for high AQL's. This is a general phenomenon that is true irrespective of the alternative (non-normal) distribution. The exception implied above (i.e. cases of large n for small AQL) is due to the failure of the by-variables schemes to perform well under the heavy-tailed model represented by the Uniform Distribution. So the recommendations for heavy-tailed processes can be as in the next two paragraphs.

For large n values all by-variables schemes (including the Ramp) are not robust for detecting small AQL's, and so one of the following strategies can be taken:

(a) use a smaller n with any of the orthodox by-variables scheme,

(b) use Ramp with the (same) large n choosing a value of B that is reasonably close to 0. The argument on "reasonableness" could be shifted to Property "Q", or any other relevant property,

or (c) use attribute scheme (probably for a higher n) which amounts to using a Ramp with B = 0.

For small n values all the by-variables are reasonably robust. So, choose any of them (Ramp included) according to the different other properties (e.g. Property "Q", ease of operation and/or variance importance as relevent).

With the finding that the Ramp is reasonably competitive to the S-plan in the properties of sample saving and robustness coupled with the improvement of the Ramp over the S-plan in the Property "Q" we feel that the conclusion that the Ramp could replace the S-plan is a quite credible one. We are aware of the favourable quality of the S-plan that it is immune to bad variance changes effects, but so relatively is the Ramp in this respect as long as we assume these variance changes (when they occur) are not very high. Such an assumption would be realistic because most (and probably all) processes under inspection are controlled in variance so reasonably that the variance changes are constrained at low levels. The value of variance though very crucial to the credibility of the σ-plan is very much less so for Ramp. The efficiency of the by-attribute and the S-plan schemes is not affected in this respect.

Property "Q" favours the by-attributes schemes, σ-plan and the Ramp in that order. The S-plan shows its weakness in this respect (Farlie (1981)).

The ease with which the Ramp could be operated stems from the fact that the variance exact computations are cumbersome for floor operators. They are crucial to the orthodox by-variables plans but not to the Ramp. Of course, the attributes scheme is again the best in this respect. In fact the only objection to the attribute schemes is the sample size needed.

The flexibility of B to enable acceptance sampling to encompass the by-attribute qualities as well as the by-variables ones in one system gives more reason for it replacing that of the awkward dichotomous system

of inspection sampling. We are aware that a lot of more work has to be made to exploit this flexibility quality. It is outside the objectives of this research to do so since we are only interested in giving evidence and persuasion towards its credibility and feasibility. We believe we have done so.

The fundamental comment and recommendation that can sum up all the findings of this research are given in the following points. The attributes scheme is the best in all facets other than sample sizes. All other facets including sample savings indicate that with a small B value it is always safe for the Ramp to act as one of the alternatives to by-variables schemes, and indeed to the by-attributes.

The Ramp is generally very reliable for destructive sampling (or any other small sample requirements sampling) for the following reasoning:

1. For small sample size of, e.g. 4 (as compared to 2,3,4 or 5 in other variables plans) we have a reasonbly comparable P_a under heavy-tailed (e.g. Uniform) distribution model.

2. For any sample size the Ramp is affected very less dramatically than small-sample σ-known by variance changes.

3. In Lognormal robustness the Ramp in "small sample" area is better than either of the S-plan and the σ-plan schemes except for very high B-values.

4. The knowledge of σ is, usually, not reliably good for small samples. This limits the power of the σ-plan and we have seen

how competitive are the Ramp with the S-plan schemes.

The following table can give clues to the performance of the different schemes under study:

	Ramp	S-plan	σ-plan
Small (Ramp) n	Robust and sometimes better than S-plan when comparing sample sizes too.	Robust and can give way to Ramp and certainly does to σ-plan in n .(worst Property "Q").	Robust and reliable, best except for limitation of its variance sensitivity.
Large (Ramp) n	more robust only for small B values.	(Poor for detecting small AQL)	

Table 6.1: General and Robustness trends of preferability among the By-variables schemes under "Equivalence".

The ideas of the schemes dealt with in the single sampling discussions were usefully and forcefully extended to fit in with the natural process of progress of acceptance sampling in the direction of multiple sampling. Some suggestions of extensions and developments of the "new" schemes (represented by the Ramp) were made along the lines of argument of the proponents of "attri-var" (or "mixed") schemes as in the work of, e.g., Schilling and Dodge (1969). The successful aim of establishing these extensions and developments is to show the potentialities of the "new" schemes, and more specifically to give concrete evidence that the "new"

scheme could be the basis for all acceptance sampling including double (and hence multiple) sampling. This can unify the methodologies of acceptance sampling on the same basic ground. The suggested extensions and developments satisfy the qualities of the existing "mixed" schemes and even improve on them. For the same reasons as in single sampling, they showed consistency in being purely "by-variables", and still one can achieve with them whatever one could achieve with any other attri-var scheme.

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APPENDIX (A.1.)

APPROXIMATING THE DISTRIBUTION OF THE MEAN OF n UNIFORMS IN [0,1] BY THE NORMAL CURVE AND MIXTURE OF NORMALS

We found that the convention of approximating the normal by a sum of 12 uniforms is a reasonable one but the level of accuracy is unsatisfactory if the desired absolute error is to be $< 2.4 \times 10^{-3}$, for n=12. Moreover, the distribution of the mean of n uniforms is composed of n polynomials of degree (n-1), and as n gets larger their computations get more cumbersome and expensive compared with the normal distribution whose computations do not necessarily increase with n. In our research we deal with large values of n. And for all these reasons we would do better with some normal approximations of such uniform distributions. Some methods are suggested below and assessed.

(1) "Straightforward" Normal Approximation

This simply matches the means and variances of the two distributions. Though, generally speaking this is not a bad approximator to the distribution of n uniforms but it tends to overestimate it most of the time towards the middle. Numerical evidence showed an absolute error of $\leq 5.8 \times 10^{-3}$ for n=5. For our purposes this is not satisfactory and indeed not for any serious purposes.

(2) "Adjusted" Normal Method

The fact that method (1) located the maximum absolute error at the first quartile suggested that we could adjust the standard deviation

$\frac{1}{\sqrt{12n}}$ by a factor c defined by matching the first quartiles. Interpolation was used to determine the value of c as follows. Instead of $X_u = 0.5 + X_N \cdot \frac{1}{\sqrt{12n}}$ used as a transformation of the normal, X_N, into a uniform mean, X_u, we use:

$$X'_u = 0.5 + X_N \cdot \frac{c}{\sqrt{12n}}$$

where the fraction:

$$c = (0.25 - P_u(P_{-1}))/(P_u(Q_1) - P_u(P_{-1}))$$

and where P_{-1} indicates the percentile immediately before Q_1, the first quartile; and $P_u(.)$ denotes the cumulative distribution of the mean of n uniforms in [0,1] up to the given cut-off point.

Results tabulated below showed a significant improvement in general. In more detailed assessment we found that except for regions below the first and above the last decile the error is $\leq 3 \times 10^{-3}$ for n=5. This also indicates that in these regions the "straightforward" method is more efficient than this "adjusted" method, while the latter supersedes the former in the rest of the regions, i.e., in the central part. Unfortunately, we could not switch from the "straightforward" normal approximation to the other (or vice versa) because the transition will not be smooth. Moreover, the approximation in the tails by the two methods is still unsatisfactory.

(3) Mixture of "Decile-Adjusted" Normals

A relatively better approximator would be a mixture of two adjusted normals:

$$\rho \cdot \Phi(X_N \cdot \sigma - \delta(1-\rho)) + (1-\rho) \cdot \Phi(X_N \cdot \sigma + \delta\rho)$$

where

$$\sigma = \frac{c}{\sqrt{12n}}$$ (c is decided by the quantile

and $\delta = 0.5\ \sigma$.

This has the same characteristics of the "adjusted" normal, method (2) above, but significantly better especially towards the middle.

In the empirical results given on the tables we could be a bit concerned about the fit in the tails. It is apparent in the data below that this weighting factor ρ was useful in bringing down the inflated values around the middle of the distribution. Yet, it adversely clustered all the discrepancy along the tails where maximum absolute error is about 4.3×10^{-3}.

However, if the adjustment is done around the first decile rather than the quartile we would spread the discrepancy almost evenly throughout the whole distribution. This is why we recommend the mixture of two "decile adjusted" normals as an approximation of the distribution of the mean of n uniforms. As for n we could be safe with n=10 since the maximum error is nearly 10^{-3}. For n=8 it is just under 2×10^{-3}.

n		5		6		7		8		9		10	
		tails	centre	tails	centre	tails	centre	tails	centre	tails	centre	tails	centre
Method (1)		2.4	5.8	2.3	4.8	2.3	4.1	2.0	3.6	1.7	3.2	1.5	2.9
Method (2)		4.34	2.8	3.6	2.5	3.1	1.9	2.7	1.7	2.4	1.5	2.2	1.4
Method (3) (ρ=0.1)	quartile adjusted	4.33	2.8	3.6	2.4	3.1	1.9	2.7	1.6	2.4	1.4	2.1	1.3
	decile adjusted	2.9	2.4	2.3	2.1	2.0	1.8	1.7	1.6	1.4	1.4	1.3	1.3

Table: Maximum absolute error (x 10^3) for n=5(1)10 in the tails and centre

Distribution of n=5 uniforms and its approximation by

A. "straightforward" normal distribution

B. "Adjusted" normal distribution.

Percentiles	B			A		
X_N	X'_u	P'_u	Error	X_u	P_u	Error
.010 -2.327	0.191	.0066	-.330E-02	0.200	.0082	-.175E-02
.020 -2.054	0.228	.0159	-.410E-02	0.235	.0185	-.142E-02
.030 -1.881	0.251	.0256	-.434E-02	0.257	.0291	-.805E-03
.040 -1.751	0.268	.0356	-.432E-02	0.274	.0398	-.116E-03
.050 -1.645	0.282	.0458	-.417E-02	0.288	.0505	.572E-03
.060 -1.555	0.294	.0560	-.395E-02	0.299	.0612	.123E-02
.070 -1.476	0.304	.0663	-.370E-02	0.309	.0718	.184E-02
.080 -1.405	0.314	.0765	-.343E-02	0.319	.0823	.240E-02
.090 -1.341	0.322	.0868	-.315E-02	0.327	.0929	.291E-02
.100 -1.282	0.330	.0971	-.287E-02	0.335	.1033	.337E-02
.110 -1.227	0.337	.1073	-.260E-02	0.342	.1137	.378E-02
.120 -1.175	0.344	.1176	-.234E-02	0.348	.1241	.414E-02
.130 -1.126	0.351	.1279	-.210E-02	0.355	.1344	.446E-02
.140 -1.080	0.357	.1381	-.186E-02	0.361	.1447	.474E-02
.150 -1.036	0.363	.1483	-.163E-02	0.366	.1549	.498E-02
.160 -0.994	0.368	.1585	-.142E-02	0.372	.1651	.519E-02
.170 -0.954	0.373	.1687	-.122E-02	0.377	.1753	.536E-02
.180 -0.915	0.379	.1789	-.103E-02	0.382	.1854	.550E-02
.190 -0.878	0.384	.1891	-.856E-03	0.387	.1956	.561E-02
.200 -0.841	0.388	.1993	-.690E-03	0.391	.2056	.569E-02
.210 -0.806	0.393	.2094	-.536E-03	0.396	.2157	.575E-02
.220 -0.772	0.398	.2196	-.392E-03	0.400	.2257	.578E-02
.230 -0.739	0.402	.2297	-.259E-03	0.405	.2357	.578E-02
.240 -0.706	0.406	.2398	-.137E-03	0.409	.2457	.577E-02
.250 -0.674	0.411	.2499	-.242E-04	0.413	.2557	.573E-02
.260 -0.643	0.415	.2600	.781E-04	0.417	.2656	.567E-02
.270 -0.612	0.419	.2701	.170E-03	0.421	.2755	.559E-02
.280 -0.582	0.423	.2802	.252E-03	0.425	.2854	.550E-02
.290 -0.553	0.427	.2903	.325E-03	0.429	.2953	.538E-02
.300 -0.524	0.431	.3003	.387E-03	0.432	.3052	.525E-02
.310 -0.495	0.434	.3104	.441E-03	0.436	.3150	.510E-02
.320 -0.467	0.438	.3204	.484E-03	0.440	.3249	.493E-02
.330 -0.439	0.442	.3305	.519E-03	0.443	.3347	.475E-02
.340 -0.412	0.445	.3405	.545E-03	0.447	.3445	.455E-02
.350 -0.385	0.449	.3505	.562E-03	0.450	.3543	.434E-02
.360 -0.358	0.453	.3605	.571E-03	0.454	.3641	.412E-02
.370 -0.331	0.456	.3705	.572E-03	0.457	.3738	.388E-02
.380 -0.305	0.460	.3805	.565E-03	0.461	.3836	.364E-02
.390 -0.279	0.463	.3905	.550E-03	0.464	.3933	.338E-02
.400 -0.253	0.466	.4005	.529E-03	0.467	.4031	.311E-02
.410 -0.227	0.470	.4105	.501E-03	0.471	.4128	.283E-02
.420 -0.202	0.473	.4204	.466E-03	0.474	.4225	.254E-02
.430 -0.176	0.477	.4304	.425E-03	0.477	.4322	.225E-02
.440 -0.151	0.480	.4403	.379E-03	0.481	.4419	.195E-02
.450 -0.125	0.483	.4503	.327E-03	0.484	.4516	.163E-02
.460 -0.100	0.487	.4602	.270E-03	0.487	.4613	.132E-02
.470 -0.075	0.490	.4702	.209E-03	0.490	.4709	.995E-03
.480 -0.050	0.493	.4801	.143E-03	0.494	.4806	.668E-03
.490 -0.025	0.497	.4900	.734E-04	0.497	.4903	.336E-03
.500 -0.000	0.500	.5000	-.112E-07	0.500	.5000	-.112E-07

Distribution of n=6 uniforms and approximations by

A. "Straightforward" normal distributions

B. "Adjusted" normal distributions.

Percentiles		B			A		
	X_N	X'_u	P'_u	Error	X_u	P_u	Error
.010	-2.327	0.220	.0072512	-.275E-02	0.226	.00856	-.143E-02
.020	-2.054	0.252	.0166059	-.339E-02	0.258	.01884	-.115E-02
.030	-1.881	0.273	.0264119	-.359E-02	0.278	.02935	-.648E-03
.040	-1.751	0.289	.0364147	-.359E-02	0.294	.03990	-.942E-04
.050	-1.645	0.302	.0465193	-.348E-02	0.306	.05045	.455E-03
.060	-1.555	0.313	.0566807	-.332E-02	0.317	.06098	.980E-03
.070	-1.476	0.322	.0668744	-.313E-02	0.326	.07147	.147E-02
.080	-1.405	0.331	.0770860	-.291E-02	0.334	.08192	.193E-02
.090	-1.341	0.338	.0873067	-.269E-02	0.342	.09234	.234E-02
.100	-1.282	0.346	.0975303	-.247E-02	0.349	.10272	.272E-02
.110	-1.227	0.352	.1077527	-.225E-02	0.355	.11306	.307E-02
.120	-1.175	0.358	.1179710	-.203E-02	0.362	.12337	.338E-02
.130	-1.126	0.364	.1281831	-.182E-02	0.367	.13365	.365E-02
.140	-1.080	0.370	.1383874	-.161E-02	0.373	.14389	.390E-02
.150	-1.036	0.375	.1485829	-.142E-02	0.378	.15410	.411E-02
.160	-0.994	0.380	.1587688	-.123E-02	0.383	.16428	.429E-02
.170	-0.954	0.385	.1689446	-.106E-02	0.388	.17444	.444E-02
.180	-0.915	0.390	.1791099	-.890E-03	0.392	.18456	.457E-02
.190	-0.878	0.394	.1892648	-.735E-03	0.397	.19466	.467E-02
.200	-0.841	0.399	.1994089	-.591E-03	0.401	.20474	.475E-02
.210	-0.806	0.403	.2095426	-.457E-03	0.405	.21479	.480E-02
.220	-0.772	0.407	.2196658	-.334E-03	0.409	.22482	.483E-02
.230	-0.739	0.411	.2297788	-.221E-03	0.413	.23483	.483E-02
.240	-0.706	0.415	.2398818	-.118E-03	0.417	.24482	.482E-02
.250	-0.674	0.419	.2499750	-.250E-04	0.421	.25479	.479E-02
.260	-0.643	0.422	.2600587	.587E-04	0.424	.26474	.474E-02
.270	-0.612	0.426	.2701333	.133E-03	0.428	.27467	.467E-02
.280	-0.582	0.430	.2801989	.199E-03	0.431	.28458	.459E-02
.290	-0.553	0.433	.2902561	.256E-03	0.435	.29449	.449E-02
.300	-0.524	0.437	.3003050	.305E-03	0.438	.30437	.438E-02
.310	-0.495	0.440	.3103459	.346E-03	0.442	.31424	.425E-02
.320	-0.467	0.444	.3203793	.379E-03	0.445	.32410	.411E-02
.330	-0.439	0.447	.3304054	.405E-03	0.448	.33395	.395E-02
.340	-0.412	0.450	.3404246	.425E-03	0.451	.34378	.379E-02
.350	-0.385	0.454	.3504371	.437E-03	0.455	.35360	.361E-02
.360	-0.358	0.457	.3604432	.443E-03	0.458	.36342	.342E-02
.370	-0.331	0.460	.3704433	.443E-03	0.461	.37322	.322E-02
.380	-0.305	0.463	.3804375	.438E-03	0.464	.38301	.302E-02
.390	-0.279	0.466	.3904262	.426E-03	0.467	.39280	.280E-02
.400	-0.253	0.470	.4004097	.410E-03	0.470	.40257	.258E-02
.410	-0.227	0.473	.4103880	.388E-03	0.473	.41234	.235E-02
.420	-0.202	0.476	.4203616	.362E-03	0.476	.42210	.211E-02
.430	-0.176	0.479	.4303305	.331E-03	0.479	.43186	.186E-02
.440	-0.151	0.482	.4402951	.295E-03	0.482	.44161	.161E-02
.450	-0.125	0.485	.4502554	.255E-03	0.485	.45135	.135E-02
.460	-0.100	0.488	.4602118	.212E-03	0.488	.46109	.109E-02
.470	-0.075	0.491	.4701643	.164E-03	0.491	.47082	.825E-03
.480	-0.050	0.494	.4801130	.113E-03	0.494	.48055	.554E-03
.490	-0.025	0.497	.4900583	.583E-04	0.497	.49027	.279E-03
.500	-0.000	0.500	.5000000	-.261E-07	0.500	.50000	-.261E-07

APPENDIX (A.2.)

VERIFICATION OF PROPERTY Q IN ATTRI-VAR PLANS UNDER KNOWN σ NORMALITY

Let x_i and z_i denote the first (variables) sample of size n_1 and the second (attributes) sample of size n_2 respectively. Define $x_{(i)}$, $z_{(i)}$ as their respective order statistics. Let y_i denote the amalgamated samples with size $n_1 + n_2$ and likewise define $y_{(i)}$.

Under normality $N(\mu,\sigma^2)$ with known σ, the decision rule "Accept if $\bar{Q}(x) > t_1$ where t_1 is the given decision constant" has property "Q" since any increment in any $x_{(i)}$ (i.e. a better sample and consequently better scoring $Q(x_{(i)})$ since the function $Q(.)$ is an increasing function) will lead to acceptance if the lesser sample is accepted. This is because $\bar{Q}$ will not decrease:

$$\bar{Q}(x) = \sum_{i=1}^{n} a_i \cdot Q(x_{(i)})$$

$$= \sum_{i=1}^{n-1} a_i \cdot Q(x_{(i)}) + a_n \cdot Q(x_{(n)})$$

where $a_i = \frac{1}{n} > 0$ and $Q(x)$ is a monotonic increasing function. Hence, property Q holds.

On taking the amalgamated sample of $\{y_i\}$ then under the attribute decision rule : "Accept if number of defectives < c" (i.e. "Accept if number of effectives $\geq n_1 + n_2 - c$") we have "property Q". This is because any increase, say α, in any of the overall order statistics $y_{(i)}$ would never decrease the actual number of effectices, thus guaranteeing an acceptance if acceptance was granted before the increment.

$$\text{Number of effectives} = \sum_{i=1}^{n_1+n_2} H(y_i)$$

where $H(y) = 0$ if $y \leq$ specification limit, A

$= 1$ if $y > A$.

Here, also property Q holds since an increment in y can never decrease the corresponding H.

APPENDIX (D.1.)

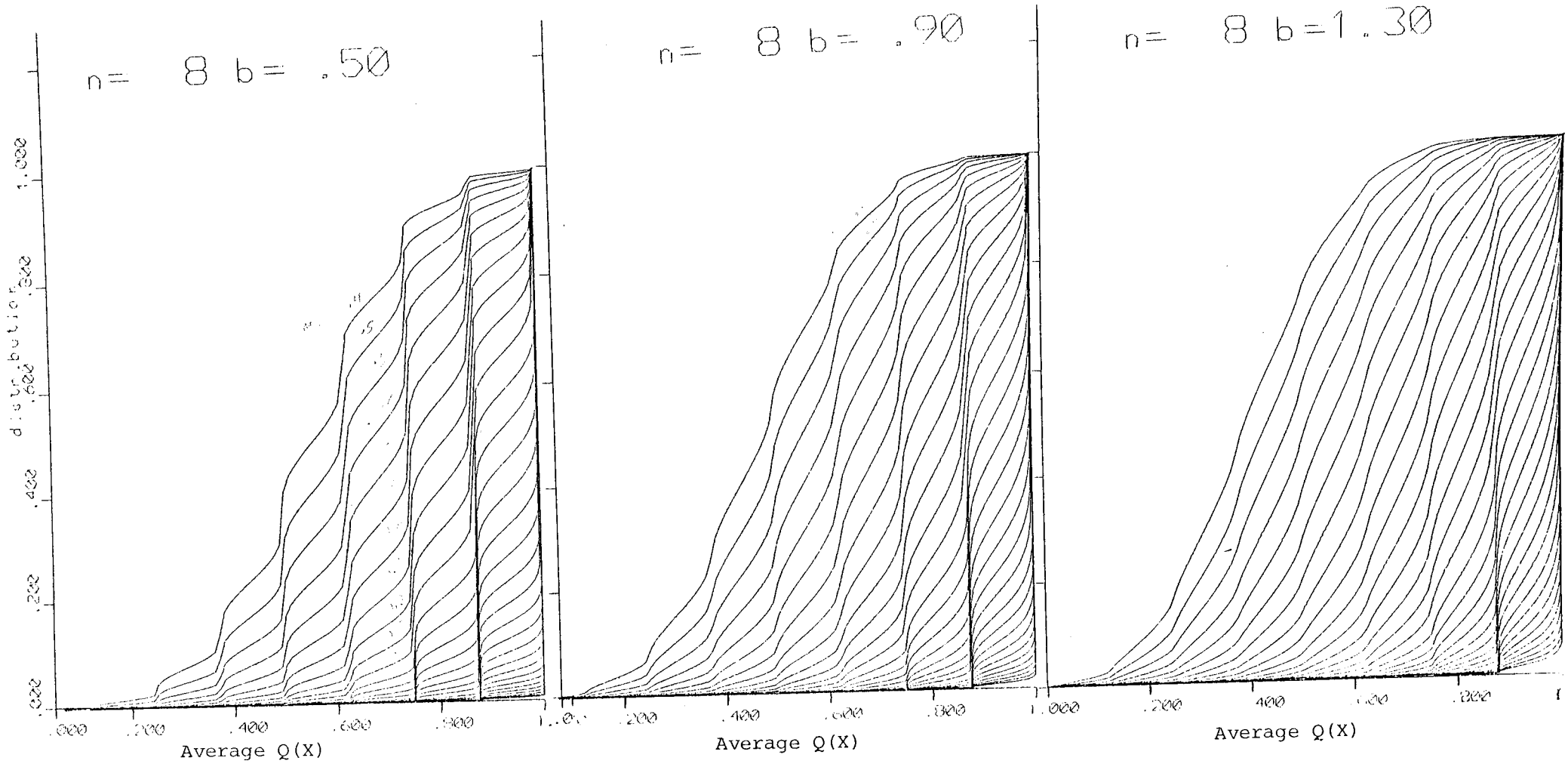
GRAPHS OF SOME REPRESENTATIVE CASES OF THE AVERAGE Q(X)

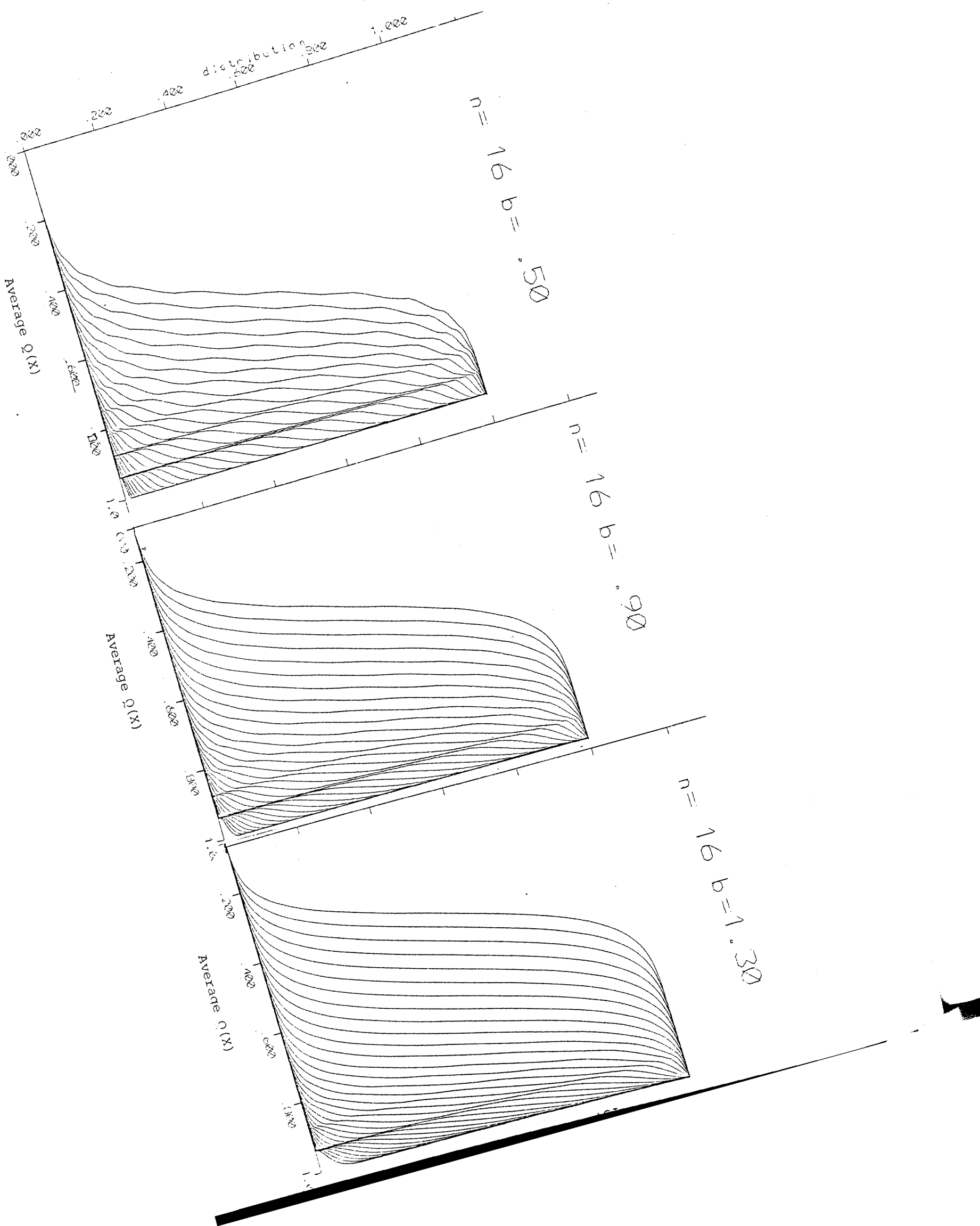
The following graphs are to represent the convoluted distributions for some cases of n = 8, 16, 32 and 64

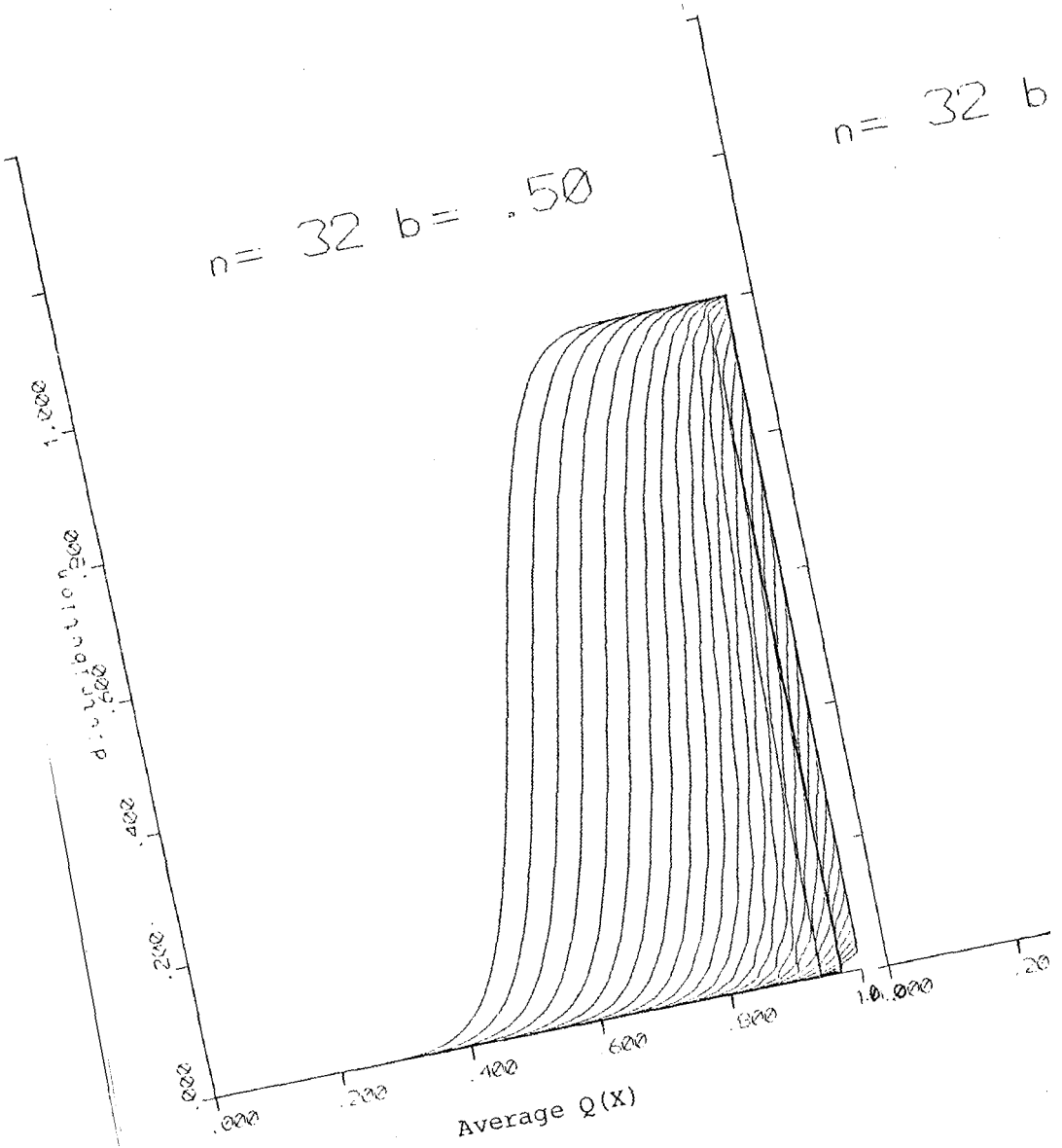
$$B = 0.5\ (0.4)\ 1.3.$$

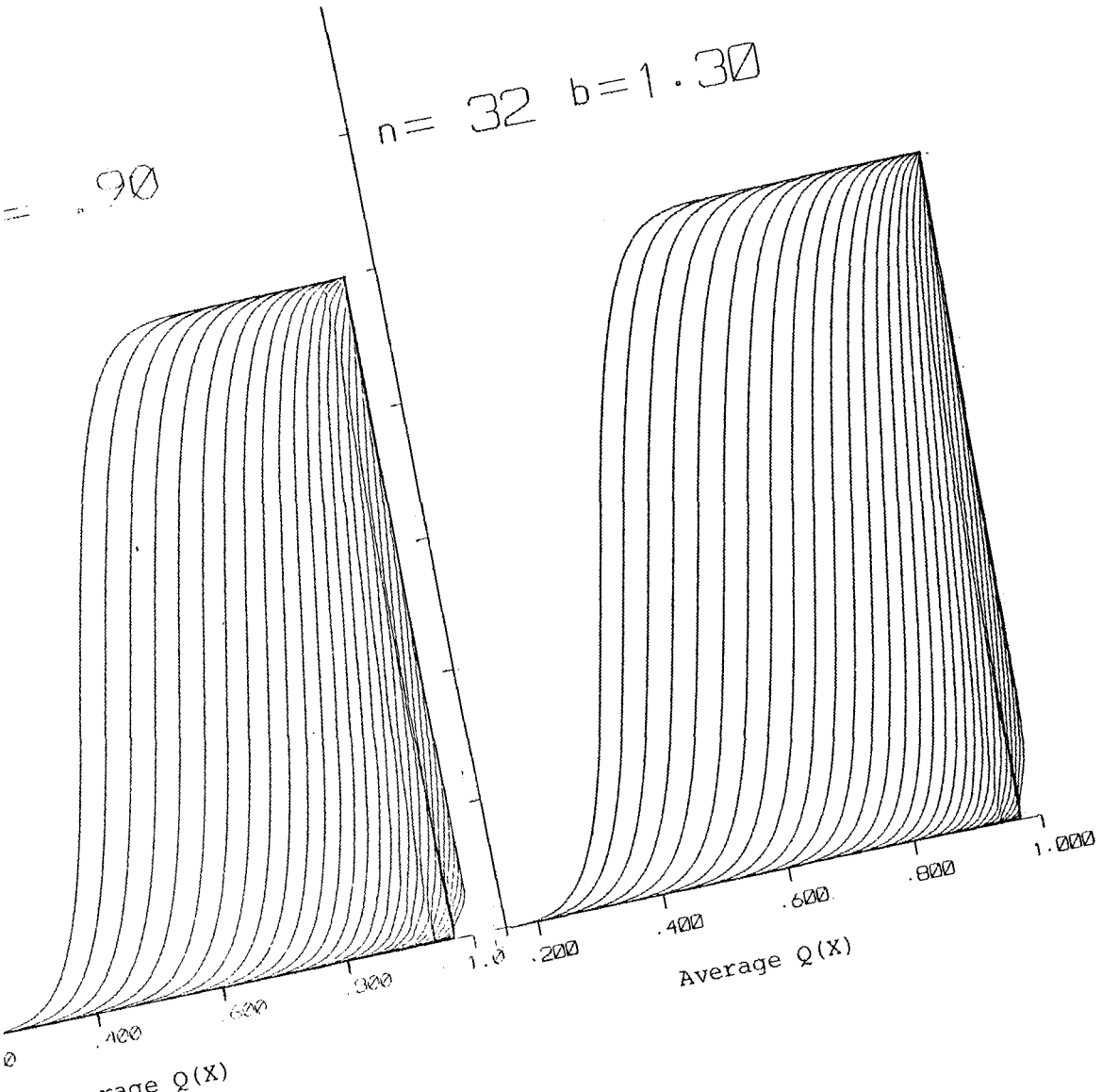
On each separate figure we have 32 distribution functions. From top to bottom these represent the cumulative distribution for cases of p_0 corresponding to the following μ-values:

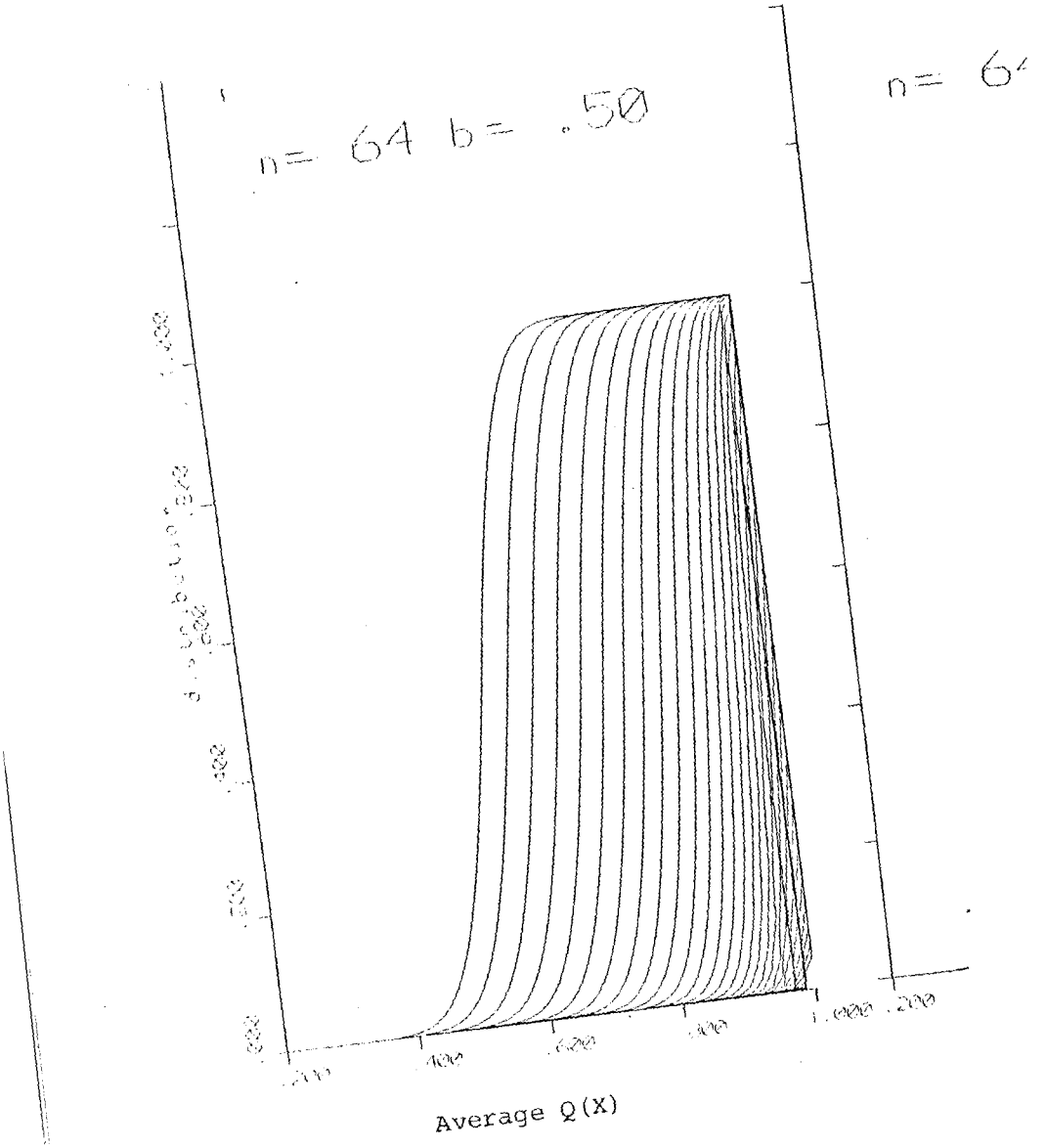
$$\mu = 0.4\ (0.1)\ 3.4.$$

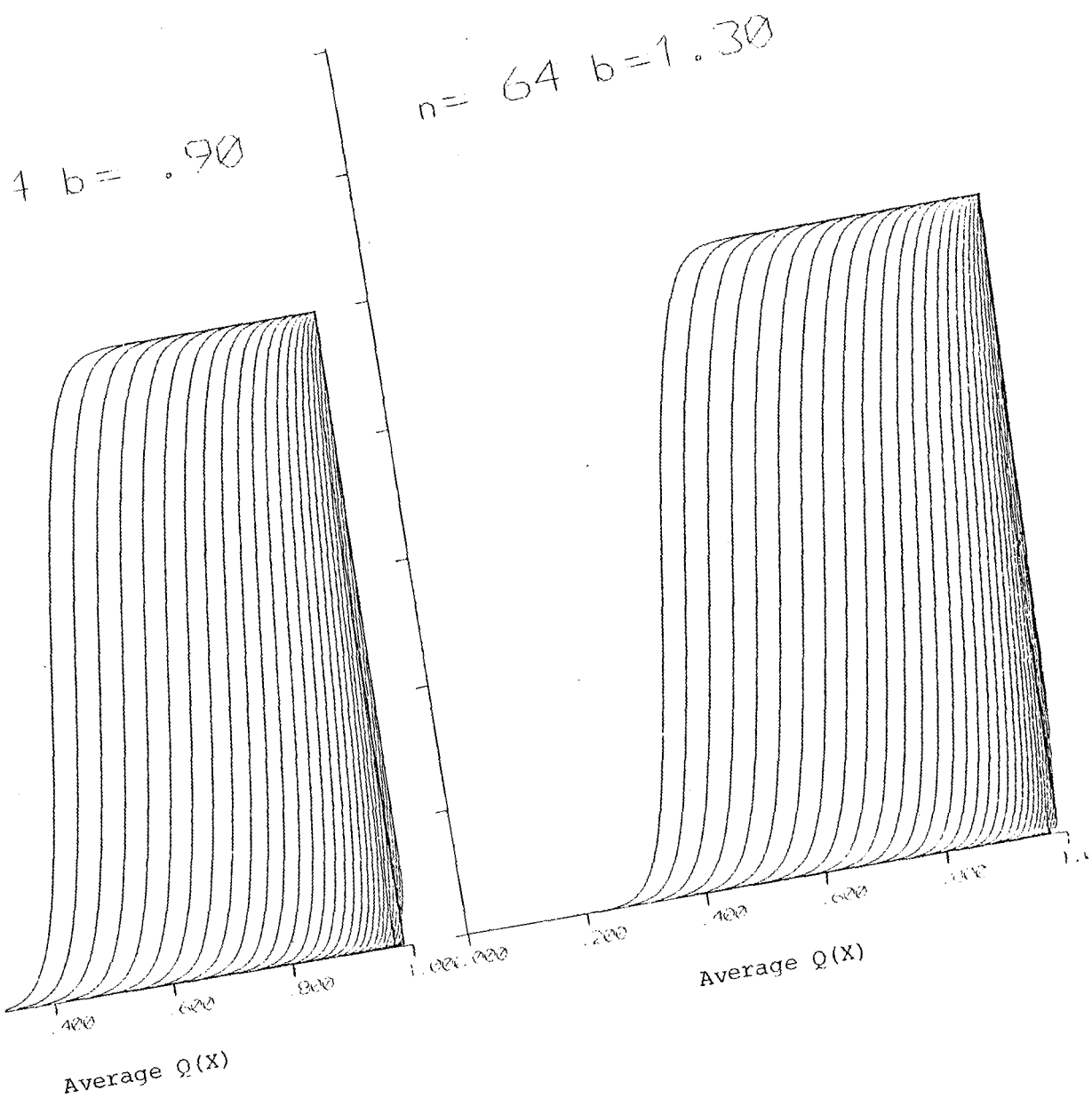

n= 8 b= .50
n= 8 b= .90
n= 8 b=1.30
Average Q(X)
Average Q(X)
Average Q(X)


n= 16 b= .50
n= 16 b= .90
n= 16 b=1.30
Average Q(X)
distribution
.000
.200
.400
.600
.800
1.000


n= 32 b= .50
n= 32 b
Average Q(X)
.000
.200
.400
.600
.800
1.000
1.200


= .90
n= 32 b=1.30
.100
.200
.300
1.0
.200
.400
.600
.800
1.000
Average Q(X)
Average Q(X)


n= 64 b= .50
n= 64
Average Q(X)


n= 64 b=1.30
1 b= .90
Average Q(X)
Average Q(X)

APPENDIX (D.2.)

GRAPHS TO REFLECT σ-PLAN O.C. RESPONSE TO SOME CHANGES IN σ

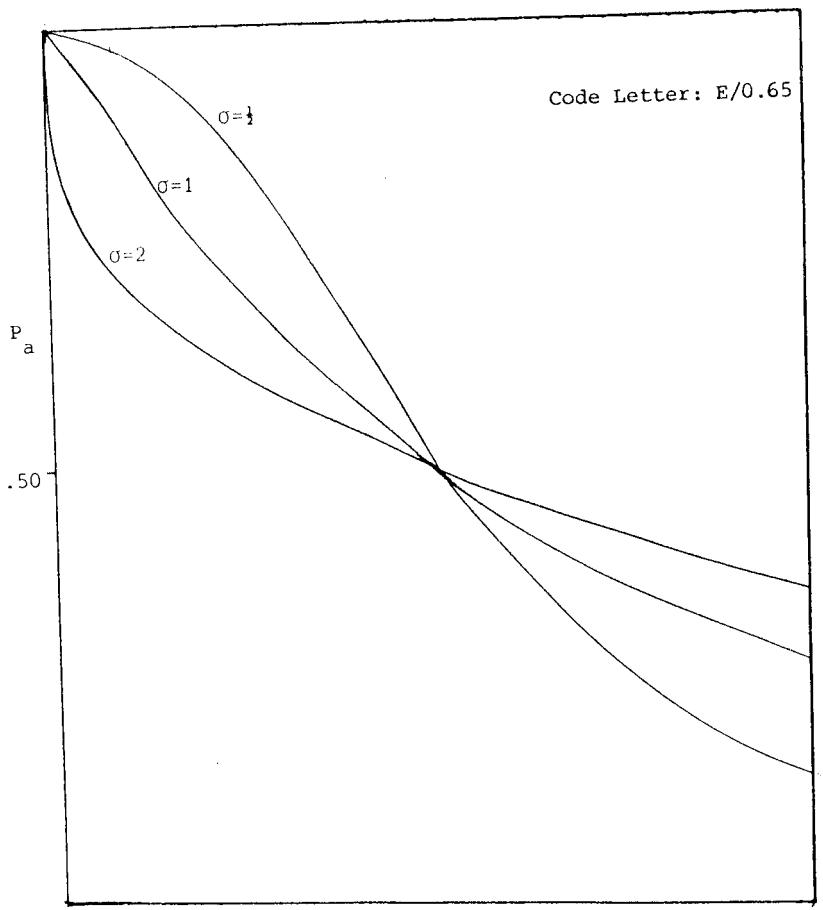
Some representative code letters (namely E, F, H and L from BS6002) are chosen each with one case of AQL as shown. In the diagrams the code letter E/0.65 means the code letter E for AQL = 0.65%.

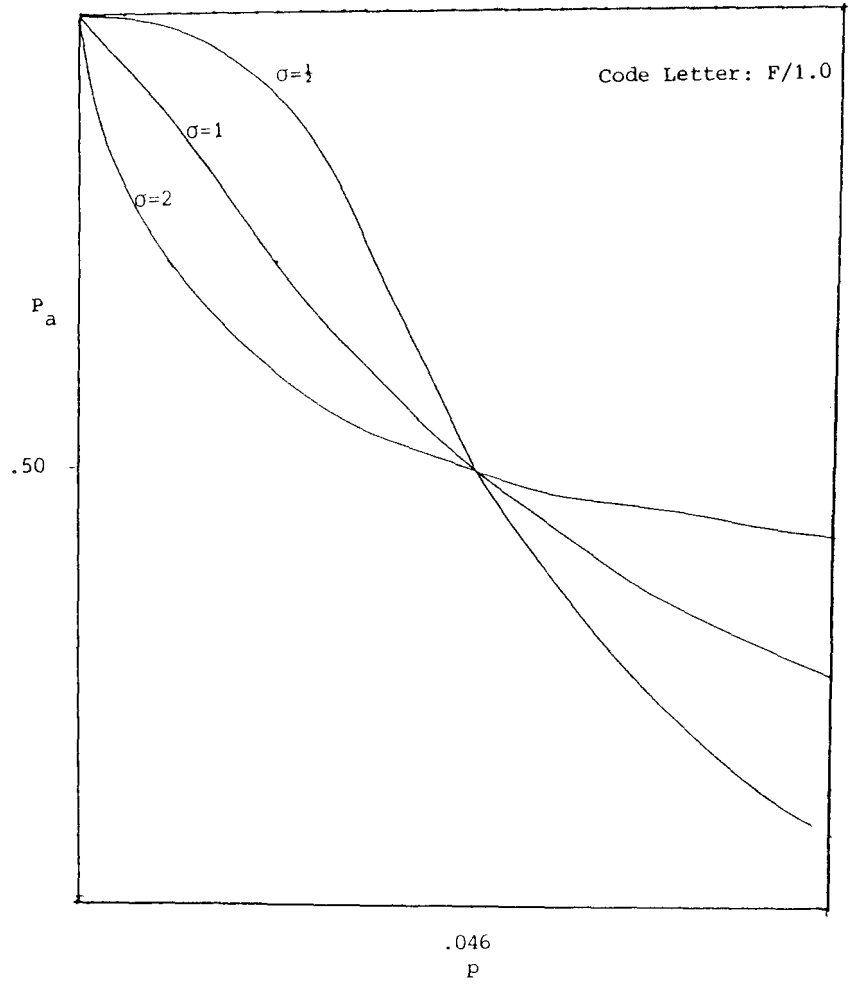
The σ changes are shown on each curve each passing through the indifference point of 50% acceptance.

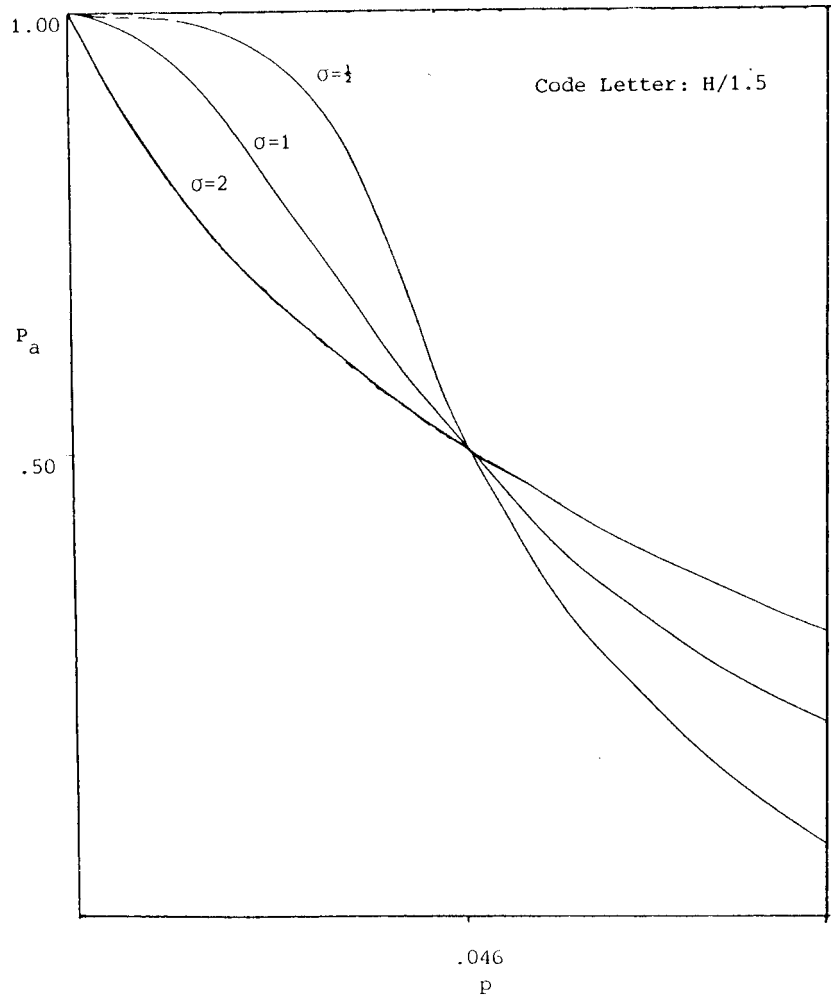
Below is a table of the test parameters of the plans for the four code letters:

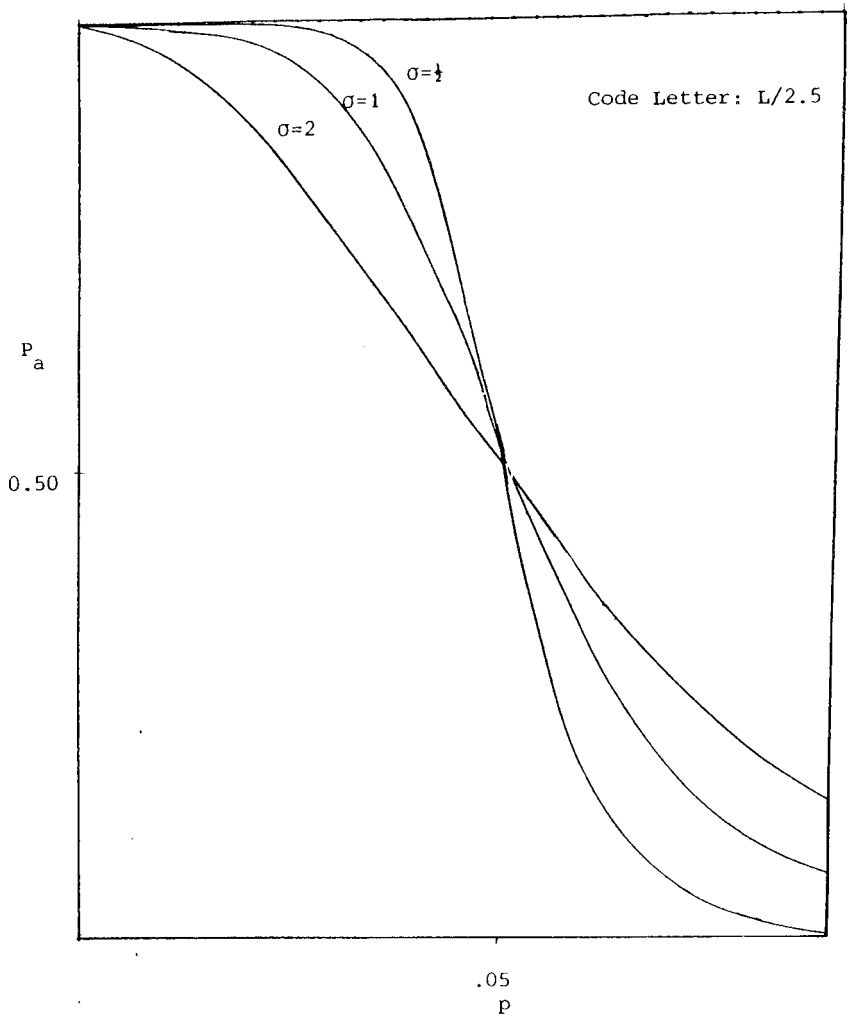
Code Letter	E/0.65	F/1.0	H/1.5	L/2.5
(k_σ, n)	(1.69,3)	(1.69,4)	(1.68,8)	(1.65,32)

where k_σ,n are the test criterion and sample size respectively.


Code Letter: E/0.65
σ=½
σ=1
σ=2
P
a
.50
.046
p


Code Letter: F/1.0
σ=½
σ=1
σ=2
P a
.50
.046
p


1.00
.50
P_a
σ=½
σ=1
σ=2
Code Letter: H/1.5
.046
p


0.50
P_a
σ=½
σ=1
σ=2
Code Letter: L/2.5
.05
p

APPENDIX (E.1.)

THE JOHNSON SYSTEMS AND THE DISTRIBUTION OF THE RAMP Q(X)

1. A Note on the Distribution Fitting Systems

Some work has been done on applying some transformations of systems of non-normal univariate curves to approximate the sampling distributions of random variables whose moments are known but whose exact density and/or distribution functions are unknown (or difficult to find). Pearson (1963) has found that there is often remarkable similarity in shape if the first four moments are identical. For many practical reasons the equivalence is adequate though in the strict mathematical sense they may not be exactly equivalent. Any system of distributions which can be easily transformed into a normal (or any other well known and manageable) distribution has certain obvious practical advantages.

The literature on fitting systems is wide ranging and has expanded rapidly. Reference works of Patil and Joshi (1968) and Johnson and Kotz (1969, 1970) are invaluable to statisticians. Also useful is the monograph by Ord (1972) with ample useful information on methods of analysis and problems involved in selecting and fitting an appropriate model.

Johnson (1949) proposed new positive steps to develop the basic claim that the first four moments of any distribution are sufficient to determine its shape and define it reasonably well. The approximations of Pearson type of curves to densities and distributions was discussed by Solomon and Stephens (1978).

Like Pearson, Johnson has mainly used $\sqrt{\beta_1}$ and β_2 (the standardized third and fourth moments to determine the general shape leaving the mean and variance (the standardising first and second moments) for the obvious secondary role of shifting and scaling the domain. This is why a "β_1, β_2" plane is being used naturally as the main tool of analysis in choosing the appropriate system from among the exhaustive set of systems of transformations (a typical set is discussed below). The central idea is the transformation of the r.v. under consideration into a Normal one. He established a set of standard systems that can approximate almost any continuous univariate distribution. The system proposed by Johnson (1949) consist of the following three basic families of distributions, each with different form of transformation (which assume Z is the standard Normal r.v. and X is the r.v. to be transformed):

(1) Bounded System (S_B): $Z = \gamma + \delta \cdot \ln((X-\xi)/(\xi+\lambda-X)),\ \xi < X < \xi+\lambda$

(2) Lognormal System (S_L): $Z = \gamma + \delta \cdot \ln(X-\xi),\ X > \xi$

(3) Unbounded System (S_U): $Z = \gamma + \delta \cdot \sinh^{-1}((X-\xi)/\lambda)$

Hill et al (1976) suggested, for the sake of completeness, the explicit inclusion of the following special cases:

(4) the Normal Curve itself (S_N);

(5) A special case [which he called (S_T)] of S_B on the line $\beta_2 = \beta_1 + 1$.

For the first four moments of X to match those of any required distribution it is necessary to find out which of the above transformations

is needed and thereafter evaluate the parameters γ, δ, λ and ξ.

We should note that fitting by moments is not always desirable but in a number of situations it gives an adequate, though not necessarily the "best", solution. A need to fit a distribution by moments could arise in a purely theoretical context or else in finding an empirical fit to some data obtained from a random sample. In the empirical or sampling area we have the "generalised Lambda distribution" due to Ramberg et al (1979), the Pearson (β_1, β_2)-plane systems for which Johnson, Nixon and Amos (1963) provided tables, and we also have the Johnson (1949) systems. These are all examples of empirically fitting distributions to model some sampling data.

The Johnson systems mentioned above relate uniquely to the (β_1, β_2) plane as the picture on (Fig (1)) shows. If we evaluate and plot the (β_1, β_2) point of each system the S_N is the point (0,3), the S_L is a line from this point (0,3) - (emphasising that the Normal system is a special case of the Lognormal) - and separating the two other regions of the Bounded and the Unbounded systems. The S_L as a major boundary is characterised by the locus of (β_1, β_2) defined by the following parametric equations:

$$\beta_1 = (w-1)(w+2)^2 \quad , \quad (\sqrt{\beta_1} > 0)$$

$$\beta_2 = w^4 + 2w^3 + 3w^2 - 3$$

where $w = \exp(\delta^{-1})$.

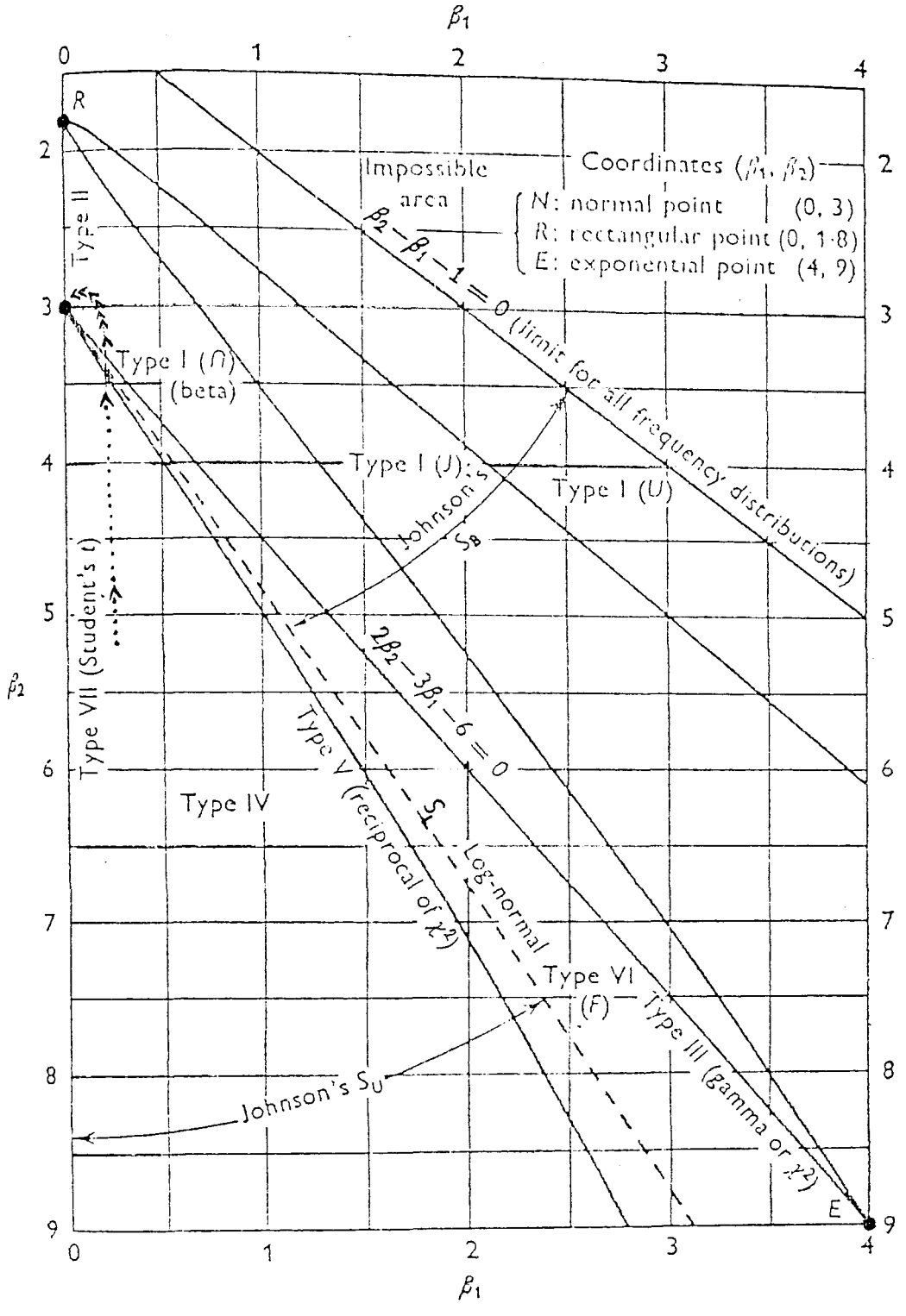


Fig. (1)

The β_1 v β_2 plane showing the map of systems of Pearson types and Johnson distributions. (Showing also, by "⁝", is an example of Ramp distribution points).

Of these regions the S_U lies below the Lognormal line while the S_B is above it. A region above the Bounded System is the impossible or inaccessible region. Johnson and Kotz (1970) claimed that it can be proved analytically that for any (β_1, β_2) point below or above the Lognormal line there is an appropriate S_U or S_B distribution respectively, and that the S_B, S_U and S_L cover the whole possible (β_1, β_2) plane uniquely. In other words there is just one appropriate distribution corresponding to each (β_1, β_2) point.

The general procedure for the use of the Johnson System is summarised by the following steps (which assume the availability of the first four moments):

(i) evaluate the β_1 and β_2 which on the plane determines the exact system according to the above standard mapping procedure;

(ii) transform into a normal r.v. (using the appropriate selected form as given by (i) above) via the appropriate estimation procedure of the parameters ξ, λ, γ and δ. This estimation procedure is probably the most difficult part of the whole operation and where most of the developments in this area are taking place.

In the case of empirical moments:

The details of (i) are obvious though computational problems are possible. As for (ii) there are many detailed procedures. Johnson (1949) suggests matching the percentiles for S_B and gave useful formulae for the

two cases of knowing only one or both end points. And if neither of the two end points is given he advocated a solution of a non-linear system of equations to solve. Aitchison and Brown (1957) gave procedures for evaluating the parameters of the S_L.

A two-way look-up table is Johnson's method for estimating the parameters of the S_U from the sample estimates. Mage (1980) and Bukac (1972) gave procedures of (ii) for S_B based on symmetrical points. Ord (1972) discussed different methods for (ii) in the estimation of δ, γ, ξ andλ. Slifker and Shapiro (1980) introduced a selection rule which depends on four percentiles for (i) and (ii) without the need of solving any simultaneous equations or looking-up special tables. According to Slifker and Shapiro (1980) the major snags of almost all these procedures are:

- the high variability of the estimates of the higher moments,
- the estimates of these moments are highly biased for small samples (see Johnson and Lowe (1979),
- the moment estimators are greatly affected by outliers.

In case of Availability of the theoretical parametric moments:

Originally the Johnson System of curves was mainly used for fitting to empirical data, but use has been extended to theoretical distributions. Leslie (1959) claimed that the generating function for some distributions though explicitly derivable in an exact form but still not amenable to inversion integral. Again, as in this research, only the moment generating function but not the compact form of the distribution is

obtainable, especially that the distribution is a mixture of discrete and continuous r.v.'s. When the first four moments are available parametrically the snag of sampling precision mentioned above are ruled out completely for the better. Necessarily, all the procedures meant to fit empirical sampling data are also applicable to parametric moments and at an advantage.

Despite all these favourable circumstances the computations in the estimation of the parameters γ, δ, ξ and λ are tedious and sometimes not very stable numerically. That explains why well written and validated computer routines are vital for enhancing these systems. Hill et al (1976) published a well-received FORTRAN Algorithm to select the appropriate system, estimate these parameters and fit the selected system to the Normal r.v. though not without inadequacies on the boundaries.

2. The Johnson Fit and the Case of the Distribution of the Ramp Q(X)

(a) Some comments on our application:

When applying the Johnson fits to approximate our Ramp distributions we need the moment generating function (m.g.f.). This m.g.f. for the Ramp case has been derived as shown in sub-section (b) below.

The moments were evaluated by the lengthy and cumbersome computations entailed by this specific derived m.g.f.. Hill's Algorithm was then used to locate the (β_1, β_2) points corresponding to these moments wherever the algorithm is appropriate.

Some effort was made to isolate and then represent the continuous part of the distribution only. In the figure (Fig (1)) we have shown some example points of the distribution for the continuous part of the Ramp average Q(X) for one sample size. The different points of (β_1, β_2) plotted and shown by a "$\hat{}$" in the figure correspond to distributions for different B-values. The values of n and p_0 were 32 and 0.05 respectively.

The arrow indicates the direction of the motion of the plotted (β_1, β_2) points in terms of an increasing B, (i.e. allowing more continuity and less discreteness). In other words as B increases the corresponding (β_1, β_2) of the distribution approach (0,3) (the normality point on the plane) as would be expected especially for a large n such as n=32. Though smaller n values have a similar pattern these tend to shift the (β_1, β_2) points towards the boundaries between S_B and S_U where Hill's algorithm is expected to show some instability. Since we are interested

in small n values (as the main attraction of by-variables sampling plans) the (β_1, β_2) points tend to be awkwardly placed for the Hill's algorithm which we were to use in fitting the Johnson systems to our distributions.

The case plotted above can be handled by Hill's algorithm for the two following reasons which help to explain the difficulties encountered for other cases:

1. The reasonably large value for n, namely n=32.

2. The reasonably high B values (here, in the above case the least value of B is 0.845).

This latter reason is critical since small B values would only allow low levels of contribution to the distribution from the continuous part to be approximated by the Johnson system. In fact we need to consider small B values in our research.

It should be mentioned that even if we work only with the continuous part then, for any parameters, if p_0 and p_2 absorb most of the distribution the residual continuous part would be too scanty for approximation purposes. That is because, then, we will be trying to approximate a distribution which is predominantly a discrete distribution with effectively zero probability density between the discrete values. We will definitely need to work with p_2 of a substantial size because batches, in acceptance sampling, are expected to be of a reasonably good quality, that is a high proportion have quality characteristics equal to 1.

b) The Moment Generating Function of $\sum_{i=1}^{n} Q(x_i)$ for the Ramp case:

If $\{X_i\}_{i=1,2,\ldots,n}$ is a random sample from a $N(\mu_0,1)$ distribution then the random variable $\sum_{i=1}^{n} Q(X_i)$ of the Ramp defined in Chapter 1, will have a derived moment generating function (m.g.f.) as obtained below.

According to the definition of $Q(X_i)$ for the Ramp case we would be dealing with a continuous and a discrete part of the random variable $\sum_{i=1}^{n} Q(X_i)$ as defined by:

$$\sum_{i=1}^{n} Q(X_i) = \sum_{i=1}^{r} y_i + s \qquad (1)$$

where $y_i = x_i/B$, $A < x_i < B$

$= 0$, elsewhere

and s, r are integers (assuming there are r marginal ("continuous") items in the sample).

The first term in (1) is the continuous variable part, while s and (n-r-s) constitute the discrete part.

Obviously, the continuous part has a double-truncated normal distribution and, hence, the m.g.f. for it is $\psi_c(t)$:

$$\psi_c(t) = \exp\{\mu_0 t/B + t^2/(2B^2)\}.\{\Phi(B-\mu_0-t/B) - \Phi(-\mu_o-t/B)\}/p_1 \qquad (2)$$

where $p_1 = \Phi(B-\mu_0) - \Phi(-\mu_0)$

$\Phi(.)$ is the Cumulative Normal Distribution

and μ_0 is the mean such that the proportion below A = 0 is equal to p_0.

The m.g.f. for the r.v. $\sum_{i=1}^{r} y_i + s$ (where r and s are given values) is

$$G(t) = e^{st} \cdot \psi_c(t) \qquad (3)$$

But since s varies together r (or n-r-s) according to their corresponding probability distributions the m.g.f. of the variable $\Sigma y_i + S$ (where S is a r.v.) would be looked upon as a probabilistic occurrence of G.

If we denote the overall m.g.f. by $M(\sum_{i=1}^{n} Q(X_i), t)$ and suppose $\sum_{i=1}^{r} y_i$ has a probability density function (p.d.f.) of f(.). Then

$\sum_{i=1}^{n} y_i + s$ (where s is fixed) has a p.d.f. f(.), too. Hence, $\Sigma y_i + S$ with S a random variable has a p.d.f. composed of the terms of

$\sum_{s} f(.) \Pr\{S=s\}$.

The overall m.g.f. is therefore:

$$M(\Sigma Q,t) = \int_{-\infty}^{\infty} e^{t(\Sigma y+S)} \left(\sum_{s} f(.).\ Pr\{S=s\}\right) dt$$

$$= \sum_{s,r} Pr\{S=s,\ R=r\} \int_{-\infty}^{\infty} e^{t(\Sigma y+S)}\ f(.)\ dt$$

$$= \sum_{s,r} Pr\{S=s,\ R=r\}\ .\ G(t)$$

$$= \sum_{s,r} Pr\{S=s,\ R=r\}\ e^{st}\ [\psi_c(t)]^r\ ,\ \text{using (3) above.}$$

and since $Pr\{S=s,\ R=r\} = \dfrac{n!}{r!s!(n-r-s)!}\ (1-p_1-p_2)^{n-r-s}\ p_1^r\ p_2^s$

then

$$M(\Sigma Q,t) = \sum_r \sum_s \frac{n!}{r!s!(n-r-s)!}\ (1-p_1-p_2)^{n-r-s}\ p_1^r\ p_2^s\ e^{st}\ [\psi_c(t)]^r$$

$$= \sum \sum \frac{n!}{r!s!(n-r-s)!}\ (1-p_1-p_2)^{n-r-s}\ p_2^s\ e^{st}$$

$$\cdot\ e^{r(t^2/(2b^2)+\mu_0 t/b)}\ .\ \{\Phi(b-\mu_0-\frac{t}{B}) - \Phi(-\mu_0-{}^t/B)\}^r$$

Since $p_1 = \Phi(b-\mu_0) - \Phi(-\mu_0)$

if we define $\psi^*(t) = \psi_c(t).p_1$ then

$$M(\Sigma Q(X),t) = \sum_{s=0}^{n-r} \sum_{r=0}^{n} \frac{n!}{s!n-s!} p_2^s e^{st} \frac{(n-s)!}{r!(n-r-s)!} p_0^{n-s-r} \{\psi^*(t)]^r$$

$$= \sum_{s=0}^{n-r} \sum_{r=1}^{n} \frac{n!}{s!(n-s)!} (p_2 e^t)^s \frac{(n-s)!}{r!(n-s-r)!} p_0^{n-s-r} [\psi^*(t)]^r +$$

$$\sum_{s=0}^{n} \frac{n!}{s!(n-s)!} p_2^s e^{st} p_0^{n-s}$$

Hence:

$$M(\Sigma Q(X),t) = [p_2 e^t + p_0 + \psi^*(t)]^n$$

This includes the case of r=0 where spike probabilities rather than a continuous curve are prevalent. To exclude this case we derive the conditional m.g.f., $M_c(t)$ excluding r=0 (i.e. zero marginals).

$$M_c(t) = \frac{[p_2 e^t + p_0 + \psi^*(t)]^n - [p_2 e^t + p_0]^n}{1 - (1-p_1)^n}$$

The moments:

Letting $\alpha = 1 - (1-p_1)^n$ then

$$\alpha M_c(t) = [p_2 e^t + p_0 + \psi^*(t)]^n - [p_2 e^t + p_0]^n \quad \ldots\ldots \qquad (I)$$

where

$$\psi^*(t) = e^{(\frac{\mu_0}{b} t + \frac{t^2}{2b^2})} \{\Phi(b-\mu_0-\frac{t}{b}) - \Phi(-\mu_0 - \frac{t}{b})\}$$

Operating on (I) by differentiating w.r.t. and setting t=0 we get:

$$\alpha M_c'(t)\Big/_{t=0} = n\{p_2 + \psi^{*\prime}(0) - p_2(p_0 + p_2)^{n-1}\}$$

$$\alpha M_c''(t)\Big/_{t=0} = n\{(p_2 + \psi^{*\prime\prime}(0)) - p_2(p_0 + p_2)^{n-1}$$

$$+ (n-1)\ \{(p_2 + \psi^{*\prime}(0))^2 - p_2^2(p_0 + p_2)^{n-2}\}\}$$

$$\alpha M_c'''(t)\Big/_{t=0} = n\{(p_2 + \psi^{*\prime\prime\prime}(0)) - p_2(p_0 + p_2)^{n-1}$$

$$+ (n-1)\ \{3(p_2 + \psi^{*\prime}(0))(p_2 + \psi^{*\prime\prime}(0)) - 3p_2^2(p_0 + p_2)^{n-2}$$

$$+ (n-2)\ \{(p_2 + \psi^{*\prime}(0))^3 - p_2^3(p_0 + p_2)^{n-3}\}\}$$

$$\alpha M_c^{iv}(t)\Big/_{t=0} = n\{(p_2 + \psi^{*iv}(0)) - p_2(p_0 + p_2)^{n-1}$$

$$+ (n-1)\ \{4(p_2 + \psi^{*\prime}(0))(p_2 + \psi^{*\prime\prime\prime}(0)) - 4p_2^2(p_0 + p_2)^{n-2} + 3(p_2 + \psi^{*\prime\prime}(0))^2$$

$$+ (n-2)\ \{6(p_2 + \psi^{*\prime}(0))^2(p_2 + \psi^{*\prime\prime}(0)) - 6p_2^2(p_0 + p_2)^{n-3}$$

$$+ (n-3)\ \{(p_2 + \psi^{*\prime}(0))^4 - p_2^4(p_0 + p_2)^{n-4}\}\}\}$$

These results are used to evaluate the following 3rd and 4th moments:

$$\sqrt{\beta_1} = [M_c'''(0) - 3M_c'(0).M_c''(0) + 2\{M_c'(0)\}^3]/(M_c''(0) - (M_c'(0))^2)^{3/2}$$

$$\beta_2 = [M_c^{iv}(0) - 4M_c'(0).M_c'''(0) + 6M_c''(0).\{M_c'(0)\}^2 - 3\{M_c'(0)\}^4]/[M_c''(0) - M_c^2(0)]$$

Note that since $\psi^*(t) = p_1\,\psi_c(t) = e^{(\frac{\mu_o}{b}t + \frac{t^2}{2b^2})}\{\Phi(b - \mu_0 - \frac{t}{b}) - \Phi(-\mu_0 - \frac{t}{b})\}$

$$\psi^{*\prime}(0) = \frac{1}{b}\{\mu_0 p_1 - [\phi(b-\mu_0) - \phi(\mu_0)]\}$$

$$\psi^{*\prime\prime}(0) = \frac{1}{b^2}\{(\mu_0^2 + 1)\ p_1 - 2\mu_0[\phi(b-\mu_0) - \phi(\mu_0)]$$

$$- [(b-\mu_0)\ \phi(b-\mu_0) + \mu_0\ \phi(\mu_0)]\}$$

$$\psi^{*\prime\prime\prime}(0) = \frac{1}{b^3}\{\mu_0^3 + 3\mu_0)\ p_1 - (3\mu_0^2 + 3)[\phi(b-\mu_0) - \phi(\mu_0)]$$

$$- 3\mu_0[(b-\mu_0)\ \phi(b-\mu_0) + \mu_0\ \phi(\mu_0)]$$

$$- \{[(b-\mu_0)^2 - 1]\ \phi(b-\mu_0) - [\mu_0^2 - 1]\ \phi(\mu_0)\}\}$$

$$\psi^{*iv}(0) = \frac{1}{b^4}\{\mu_0^4 + 6\mu_0^2 + 3)\ p_1$$

$$- (4\mu_0^3 + 12\mu_0)\ [\phi(b-\mu_0) - \phi(\mu_0)]$$

$$- (6\mu_0^2 + 6)\ [(b-\mu_0)\ \phi(b-\mu_0) + \mu_0\ \phi(\mu_0)]$$

$$- 4\mu_0\ \{[(b-\mu_0)^2 - 1]\ \phi(b-\mu_0) - [\mu_0^2 - 1]\ \phi(\mu_0)\}$$

$$- \{[(b-\mu_0)^3 - 3(b-\mu_0)]\ \phi(b-\mu_0) + [\mu_0^3 - 3\mu_0]\ \phi(\mu_0)\}\}$$

where $\phi(.)$ is the standard normal p.d.f.

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