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Optimal reinsurance with multivariate risks and dependence uncertainty

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Abstract

In this paper, we study the optimal reinsurance design from the perspective of an insurer with multiple lines of business, where the reinsurance is purchased by the insurer for each line of business, respectively. For the risk vector generated by the multiple lines of business, we suppose that the marginal distributions are fixed, but the dependence structure between these risks is unknown. Due to the unknown dependence structure, the optimal strategy is investigated for the worst-case scenario. We consider two types of risk measures: Value-at-Risk (VaR) and Range-Value-at-Risk (RVaR) including Expected Shortfall (ES) as a special case, and general premium principles satisfying certain conditions. To be more practical, the minimization of the total risk is conducted under some budget constraint. For the VaR-based model with only two risks, it turns out that the limited stop-loss reinsurance treaty is optimal for each line of business. For the model with more than two risks, we obtain two types of optimal reinsurance strategies if the marginals have convex or concave distributions on their tail parts by constraining the ceded loss functions to be convex or concave. Moreover, as a special case, the optimal quota-share reinsurance with dependence uncertainty has been studied. Finally, after applying our findings to two risks, some studies have been implemented to obtain both the analytical and numerical optimal reinsurance policies.

Keywords: Optimal reinsurance; Multivariate risks; Dependence uncertainty; Value-at-Risk; Expected Shortfall; Range-Value-at-Risk.

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1 Introduction

As an important tool for an insurance company to manage risk, reinsurance attracts much attention from both practitioners and researchers. The study of the optimal reinsurance design stems from the seminal paper of [Borch \(1960\)](#), which was followed by much research on minimizing some measures of the risk exposure of an insurer or maximizing the expected utility of financial wealth for a risk-averse insurer; see, e.g., [Arrow \(1963\)](#), [Heerwaarden *et al.* \(1989\)](#) and [Chi and Lin \(2014\)](#). Recently, many works have focused on the VaR- or ES-based reinsurance problem due to the prevalent use of VaR and ES to quantify the insurance risks and set regulatory capital. One can refer to the monograph [Albrecher *et al.* \(2017\)](#) for a comprehensive introduction on reinsurance and to the paper of [Cai and Chi \(2020\)](#) for a review of the risk measure-based reinsurance designs. On the other hand, other approaches of studying optimal reinsurance, e.g. by minimising the decision maker's ruin probability, have also been widely explored. For example, [Tan *et al.* \(2020\)](#) show that the optimal reinsurance exhibits a dual excess-of-loss form under a mean-TVaR premium principle.

Despite extensive studies on the reinsurance design for a single risk, the works on reinsurance with multivariate risks are very few. In practice, an insurance company is more likely to run multiple lines of business, and the reinsurance contract is purchased by the insurer for each line of business separately from one or more reinsurers. This motivates the study of the reinsurance problem with multivariate risks from a holistic perspective. In this paper, we are interested in minimizing some risk measures of the aggregate risk faced by each line of business after the corresponding reinsurance. Finding the optimal policy is not easy, as it involves both the marginal distributions of the risks and the dependence structure between these risks.

There are only a few papers on multivariate reinsurance designs. By considering the independent and identically distributed risks, [Denuit and Vermandele \(1998\)](#) show that the stop-loss reinsurance treaty is optimal for each line of business in the sense of a stop-loss order with expected premium principles. This result is extended to some positively dependent risks in [Cai and Wei \(2012\)](#). In [Zhu *et al.* \(2014\)](#), the optimization criterion is to minimize the insurer's capital requirement, showing that one- or two layer reinsurance policies are optimal for each line of business with general premium principles, which are valid for all the dependence structures. Without specifying any dependence structure between these risks, [Cheung *et al.* \(2014\)](#) investigate a min-max reinsurance problem by considering the worst-case scenario and using law-invariant convex risk measures to assess the risk and the expected premium principle, where the optimal reinsurance policy is stop-loss for each line of business. The fact that the worst-case dependence structure is comonotonic significantly simplifies the problem. [Bernard *et al.* \(2020\)](#) study the optimal coinsurance with multiple policyholders from the perspective of an insurer in the context of expected utility theory, showing that the optimal supply of insurance may not match the demand from the interdependent policyholders.

In this paper, we follow the setup in [Cheung *et al.* \(2014\)](#) to study the min-max reinsurance problem with multivariate risks without specifying the dependence structure. This setup is somehow reasonable as the estimation of the marginal distributions has high accuracy, but the estimation of the dependence structure has low accuracy and is sometimes even impossible. The misspecification of the dependence structure may

result in severe consequences in risk management; see, e.g., [McNeil *et al.* \(2015\)](#). Moreover, this assumption is also motivated from the context where data from different correlated products are separately collected and thus no dependence information is available; see [Embrechts *et al.* \(2013\)](#) and [Embrechts *et al.* \(2015\)](#). Due to the unknown dependence structure, we adopt a robust optimisation approach and consider the worst-case scenario. [Ben-Tal *et al.* \(2009\)](#) provide a standard reference of robust optimisation, while its various application can be found in references such as [Ben-Tal and Nemirovski \(2008\)](#), [Polak *et al.* \(2010\)](#), [Bertsimas *et al.* \(2011\)](#), [Gabrel *et al.* \(2014\)](#) and [Asimit *et al.* \(2017\)](#). [Benati and Conde \(2022\)](#) study a relatively robust optimization model combining expected returns with risk and regret measures. Most recently, the optimal reinsurance and insurance with model uncertainty have been studied in [Chi *et al.* \(2022\)](#), [Boonen and Jiang \(2024\)](#) and [Cai *et al.* \(2024\)](#).

Instead of considering the law-invariant convex risk measures in [Cheung *et al.* \(2014\)](#), we use VaR and RVaR to quantify the risk. The optimization criterion is to minimize the worst-case of VaR or RVaR of the total risk exposure after reinsurance for each line of business. To be more practical, we impose a constraint on the total budget of all lines of business.

As VaR and RVaR are not convex, our analysis is not easy and is based on the expressions of the worst case of VaR and RVaR in the risk aggregation with known marginal distributions and unknown dependence structure. One can find some of these expressions in [Makarov \(1981\)](#) and [Rüschendorf \(1982\)](#) for the case with two risks and [Wang *et al.* \(2013\)](#), [Embrechts *et al.* \(2013\)](#), [Bernard *et al.* \(2014\)](#) and [Jakobsons *et al.* \(2016\)](#) for the case with more than two risks. Our analysis mainly relies on an extension of the most recent expressions given in [Blanchet *et al.* \(2023\)](#), covering all the existing results in the literature where all the marginal distributions have decreasing or increasing densities. We extend these results to the cases of the marginals with convex or concave distributions on their tail parts, which is key in our analysis. Note that the worst-case dependence structure under the VaR or the RVaR is the combination of joint-mixability (see [Wang and Wang \(2016\)](#)), and mutual exclusivity (introduced in [Dhaene and Denuit \(1999\)](#)), which is in fact negatively dependent and completely different from the comonotonic dependence as shown in [Cheung *et al.* \(2014\)](#).

As a result of the limitation of convexity or concavity on the marginal distributions in the expressions on robust risk aggregation for VaR and RVaR, we constrain the feasible ceded loss functions to be convex or concave for the case with more than two risks. Similar constraints on the ceded loss functions can also be found in, e.g., [Cai *et al.* \(2008\)](#) and [Cheung \(2010\)](#). Moreover, we consider general premium principles satisfying several assumptions, including the expected principle as a special case. With the above-mentioned assumptions, we obtain the optimal reinsurance policy for each line of business for both VaR-based and RVaR-based reinsurance problems. One optimal ceded loss function is a piecewise linear function with three pieces, including the stop-loss function as a special case, and another optimal ceded loss function is a linear function with an upper limit, which extends the quota-share function. If there are only two risks, all the constraints on the ceded loss functions and the marginal distributions can be removed for the VaR-based reinsurance problem and the optimal reinsurance policy is a limited stop-loss for each line of business. For

the RVaR-based reinsurance problem, we exclude the case with two risks and the case for marginals with convex distributions on the tail parts due to the lack of expression for robust risk aggregation.

The quota-share reinsurance design with dependence uncertainty is also considered. It turns out that the optimal reinsurance policy for each line of business is either full coverage reinsurance or zero coverage reinsurance when the budget constraint is meaningless. This implies that the insurer may refuse to buy reinsurance contracts for some lines of business, which is somehow connected to the findings in [Bernard *et al.* \(2020\)](#), where the insurer refuses to sell insurance to some policyholders for the scenario of dependent risks.

The structure of the rest of the paper is as follows. In [Section 2](#), we set up our model and display some necessary notation. [Sections 3-4](#) are devoted to the study of the VaR-based and RVaR-based reinsurance design with multivariate risks, respectively. In [Section 5](#), we investigate a special case: quota-share reinsurance under dependence uncertainty and obtain the optimal reinsurance policies for each line of business. In [Section 6](#), we obtain both the analytical and numerical optimal reinsurance policies for the case with two risks using our findings in [Section 3](#). All the proofs are relegated to the Appendix.

2 Model description and notation

Assume an insurance company is running n lines of business over a time period. The total loss or claims faced by each line over this time period is denoted by $X_1, \dots, X_n$, which are n random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with cumulative distribution functions $F_i(x) = \mathbb{P}(X_i \leq x)$ and $0 < \mathbb{E}(X_i) < \infty$, respectively. The total loss of an insurer without reinsurance is $S_n = X_1 + \dots + X_n$. In order to protect from the potential huge loss for each line of business, the insurer applies reinsurance strategy to each line. Let $f_i(X_i)$ be the part of loss ceded to the reinsurer and $R_{f_i}(X_i) := X_i - f_i(X_i)$ be the part of risk retained by the insurer, where f_i and R_{f_i} are referred to as ceded loss function and retained loss function, respectively. To avoid the ex-post moral hazard, reinsurance contract for each line of business should be incentive compatible. Hence both f_i and R_{f_i} are non-negative and increasing on $[0, \infty)$ for $i = 1, \dots, n$, which is equivalent to the Lipschitz continuity, i.e., for each $i = 1, \dots, n$, $0 \leq f_i(y) - f_i(x) \leq y - x$, $0 \leq x \leq y$, and $f_i(0) = 0$. Hence the domain of the ceded loss functions in this paper is restricted to

$$\mathcal{D}^n = \{\mathbf{f} = (f_1, \dots, f_n) : 0 \leq f_i(y) - f_i(x) \leq y - x, 0 \leq x \leq y, f_i(0) = 0, i = 1, \dots, n\}.$$

As a compensation of risk transfer from the insurer to the reinsurer, the insurer pays the reinsurance premium to the reinsurer, which is charged by a premium principle π_i for i -th line of business. Let $\mathcal{X}_1$ be a subset of all non-negative random variables with finite mean defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{X} \subset \mathcal{X}_1$. Then $\pi_i : \mathcal{X} \rightarrow \mathbb{R}^+$. Similarly as in [Chi and Tan \(2013\)](#), we consider the general premium principles satisfying the following conditions: (i) Distribution invariance: For $Y, Z \in \mathcal{X}$, $\pi_i(Y) = \pi_i(Z)$ if $F_Y = F_Z$; (ii) Risk loading: $\pi_i(Y) \geq \mathbb{E}(Y)$ for $Y \in \mathcal{X}$; (iii) Reserving stop-loss order: For $Y, Z \in \mathcal{X}$, $\pi_i(Y) \leq \pi_i(Z)$ whenever $\mathbb{E}((Y - d)_+) \leq \mathbb{E}((Z - d)_+)$ for all $d \in \mathbb{R}$; and (iv) Continuity: For $Y_n, Y \in \mathcal{X}$, $n \geq 1$, $\lim_{n \rightarrow \infty} \pi_i(Y_n) =$

$\pi_i(Y)$ provided that $\lim_{n \rightarrow \infty} \text{ess-sup}|Y_n - Y| = 0$. Moreover, $\lim_{d \rightarrow \infty} \pi_i(Y \wedge d) = \pi_i(Y)$.

Throughout the paper, we assume that π_i satisfies (i) – (iii). We will see later that the existence of the optimal ceded loss functions is guaranteed by the continuity of π_i . Some examples of the practically used premium principles satisfying (i)–(iii) are expected principle, Wang’s principle, Dutch principle and many others. We refer to Wang *et al.* (1997) for Wang’s principle and Young (2004) for a summary of the commonly used premium principles.

The total risk exposure for the insurer after reinsurance is $S_n^{\mathbf{f}} = \sum_{i=1}^n (R_{f_i}(X_i) + \pi_i(f_i(X_i)))$. For a risk measure $\rho : \mathcal{X} \rightarrow \mathbb{R}$, the typical target of the optimal reinsurance problem from the perspective of an insurer is to find the optimal reinsurance policies $\mathbf{f} \in \mathcal{D}^n$ to minimize

$$\rho(S_n^{\mathbf{f}}(X_1, \dots, X_n))$$

under the budget constraint: $\sum_{i=1}^n \pi_i(f_i(X_i)) \leq P$, where $P > 0$ is the budget limit set by the insurer. We may add the constraint to the domain of $\mathbf{f}$ as below:

$$\mathcal{D}^n(P) = \left\{ \mathbf{f} \in \mathcal{D}^n : \sum_{i=1}^n \pi_i(f_i(X_i)) \leq P \right\}.$$

This problem involves the multivariate models of risks. In practice, it is hard to estimate the dependence structure of $X_1, \dots, X_n$ with high accuracy. The misspecification of the dependence structure may lead to severe consequences in risk management; see e.g., McNeil *et al.* (2015). This motivates us to consider the problem with dependence uncertainty, i.e., the marginal distributions are fixed but the dependence structure is unknown. We denote the uncertainty set by

$$\mathcal{E}_n(\mathbf{F}) = \{(X_1, \dots, X_n) : X_i \sim F_i, i = 1, \dots, n\},$$

where $\mathbf{F} = (F_1, \dots, F_n)$. We are interested in the optimal reinsurance strategies for the worst-case scenario under dependence uncertainty, i.e., finding the optimal reinsurance policies $\mathbf{f} \in \mathcal{D}^n(P)$ to minimize

$$\sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} \rho(S_n^{\mathbf{f}}(X_1, \dots, X_n)). \quad (1)$$

Due to the prevalent use of VaR and ES in practice for quantifying the insurance risks and setting the regulatory capital, VaR and ES-based optimal reinsurance problems have attracted great attention from academia. Define the left quantile of a distribution F and a random variable X with $X \sim F$ at $\alpha \in (0, 1]$ as

$$F^{-1}(\alpha) = \text{VaR}_\alpha(X) = \inf\{x : F(x) \geq \alpha\},$$

and the right quantile at $\alpha \in [0, 1]$ as

$$F_+^{-1}(\alpha) = \text{VaR}_\alpha^+(X) = \inf\{x : F(x) > \alpha\}$$

with the convention $\inf \emptyset = 0$. Moreover, for $\alpha \in [0, 1]$, and $X \sim F$, ES_α is given by

$$\text{ES}_\alpha(F) = \text{ES}_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 F^{-1}(t) dt.$$

One can refer to e.g., [McNeil *et al.* \(2015\)](#) and [Föllmer and Schied \(2016\)](#) for more discussions on VaR and ES. Proposed by [Cont *et al.* \(2010\)](#) as a family of robust risk measures, Range-Value-at-Risk (RVaR) is defined by for $0 \leq \beta < \beta + \alpha \leq 1$,

$$R_{\beta,\alpha}(F) = R_{\beta,\alpha}(X) = \frac{1}{\alpha} \int_\beta^{\beta+\alpha} F_+^{-1}(1-t) dt.$$

Note that RVaR connects VaR and ES as below. For $\alpha \in (0, 1)$,

$$\text{ES}_\alpha(X) = R_{0,\alpha}(X), \text{ VaR}_\alpha(X) = \lim_{\beta \downarrow 0} R_{1-\alpha,\beta}(X), \text{ and } \text{VaR}_\alpha^+(X) = \lim_{\beta \downarrow 0} R_{1-\alpha-\beta,\beta}(X).$$

In this paper, we mainly study the VaR-based and RVaR-based optimal reinsurance problems. Due to the limited analysis in robust risk aggregations for VaR and RVaR, in some cases, we have to constrain the ceded loss functions to be convex/concave, i.e., we consider the following smaller domains:

$$\mathcal{D}_1^n(P) = \{\mathbf{f} = (f_1, \dots, f_n) : \mathbf{f} \in \mathcal{D}^n(P), f_i \text{ is convex for } i = 1, \dots, n\},$$

$$\mathcal{D}_2^n(P) = \{\mathbf{f} = (f_1, \dots, f_n) : \mathbf{f} \in \mathcal{D}^n(P), f_i \text{ is concave for } i = 1, \dots, n\}.$$

In the following, we define some special ceded loss functions which are the optimal ceded loss functions in our later results. We denote $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n)$, $\mathbf{c} = (c_1, \dots, c_n)$, $\mathbf{d} = (d_1, \dots, d_n)$. Limited stop-loss function is defined as $l_{a,b}(x) := (x-a)_+ - (x-b)_+$, with $0 \leq a \leq b \leq \infty$. Let $\mathbf{l}_{\mathbf{a},\mathbf{b}} = (l_{a_1,b_1}, \dots, l_{a_n,b_n})$ with the domain of the parameters

$$\mathcal{A}(P) = \{(\mathbf{a}, \mathbf{b}) : \mathbf{l}_{\mathbf{a},\mathbf{b}} \in \mathcal{D}^n(P), 0 \leq a_i \leq b_i \leq \infty, i = 1, \dots, n\}.$$

Moreover, define $h_{a,b,c,d}(x) := c(x-a)_+ + d(x-b)_+$, where $0 \leq a \leq b \leq \infty$ and $0 \leq c, d \leq c+d \leq 1$, which is a generalization of the stop-loss ceded function ($c = 1$). Let $\mathbf{h}_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}} = (h_{a_1,b_1,c_1,d_1}, \dots, h_{a_n,b_n,c_n,d_n})$ with the domain of the parameters

$$\mathcal{A}_1(P) = \{(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) : \mathbf{h}_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}} \in \mathcal{D}^n(P), 0 \leq a_i \leq b_i \leq \infty, 0 \leq c_i, d_i \leq c_i + d_i \leq 1, i = 1, \dots, n\}.$$

Define $g_{a,b}(x) := a \min(x, b)$, where $0 \leq a \leq 1$ and $0 \leq b \leq \infty$, including quota-share function as a special case ($b = \infty$). Let $\mathbf{g}_{\mathbf{a},\mathbf{b}} = (g_{a_1,b_1}, \dots, g_{a_n,b_n})$ with the domain of the parameters

$$\mathcal{A}_2(P) = \{(\mathbf{a}, \mathbf{b}) : \mathbf{g}_{\mathbf{a},\mathbf{b}} \in \mathcal{D}^n(P), 0 \leq a_i \leq 1, 0 \leq b_i \leq \infty, i = 1, \dots, n\}.$$

In the following sections, we will show that the optimal reinsurance design on the domains of $\mathcal{D}^n(P)$, $\mathcal{D}_1^n(P)$ and $\mathcal{D}_2^n(P)$ can be simplified as the problem on the following domains respectively:

$$\mathcal{L}^n(P) = \{\mathbf{l}_{\mathbf{a},\mathbf{b}} : (\mathbf{a}, \mathbf{b}) \in \mathcal{A}(P)\}, \quad \mathcal{H}^n(P) = \{\mathbf{h}_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}} : (\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \in \mathcal{A}_1(P)\},$$

$$\mathcal{G}^n(P) = \{\mathbf{g}_{\mathbf{a},\mathbf{b}} : (\mathbf{a}, \mathbf{b}) \in \mathcal{A}_2(P)\}.$$

In the rest of the paper, we tacitly suppose $X_i \sim F_i$, $i = 1, \dots, n$.

3 Optimal reinsurance under VaR

In order to compute $\sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} \text{VaR}_\alpha(S_n^{\mathbf{f}}(X_1, \dots, X_n))$, we need to extend a recent result in robust risk aggregation in [Blanchet et al. \(2023\)](#), which will be displayed in Proposition 1. We extend the results of marginals with monotone densities in Theorem 2 of [Blanchet et al. \(2023\)](#) to the marginals with convex/concave distributions on their tail parts, which would be crucial to find the optimal reinsurance policies later. Note that the convex/concave distributions allow for positive probability mass at the maximum/mimumum value of the distributions, which is crucial for us to construct the optimal ceded loss functions. For $\alpha \in (0, 1)$, we say that a distribution F is concave beyond its α -quantile if the distribution $\frac{(F(x)-\alpha)_+}{1-\alpha}$ is concave over $(F_+^{-1}(\alpha), \infty)$; a distribution F is convex beyond its α -quantile if the distribution $\frac{(F(x)-\alpha)_+}{1-\alpha}$ is convex on $(-\infty, F^{-1}(1))$ (which is equivalent to that $\frac{(F(x)-\alpha)_+}{1-\alpha}$ is convex on its support and $F(F_+^{-1}(\alpha)) = \alpha$ or 1). We denote $\mathcal{M}_{cx}^\alpha$ as the set of all distributions that are convex beyond their corresponding α -quantiles and $\mathcal{M}_{cv}^\alpha$ as the set of all distributions that are concave beyond their corresponding α -quantiles. In practice, α is chosen to be close to 1, such as $\alpha = 0.95$ and $\alpha = 0.975$. Many practically used distributions fall in either $\mathcal{M}_{cv}^\alpha$ or $\mathcal{M}_{cx}^\alpha$ for large α . For instance, exponential distribution, Pareto distribution and Gaussian distribution with $\alpha \geq 0.5$ fall in $\mathcal{M}_{cv}^\alpha$, and continuous uniform distribution and some Beta distribution fall in $\mathcal{M}_{cx}^\alpha$. Let

$$\Delta_n = \left\{ \boldsymbol{\gamma} \in (0, 1) \times [0, 1]^n : \sum_{i=0}^n \gamma_i = 1 \right\}, \quad \text{with } \boldsymbol{\gamma} = (\gamma_0, \gamma_1, \dots, \gamma_n).$$

For any $\alpha \in (0, 1)$, $\alpha \Delta_n = \{\boldsymbol{\gamma} \in (0, 1) \times [0, 1]^n : \sum_{i=0}^n \gamma_i = \alpha\}$.

Proposition 1. For $\alpha \in (0, 1)$ and $\mathbf{F} \in (\mathcal{M}_{cx}^\alpha)^n \cup (\mathcal{M}_{cv}^\alpha)^n$, we have

$$\sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} \text{VaR}_\alpha^+ \left(\sum_{i=1}^n X_i \right) = \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} \sum_{i=1}^n R_{\gamma_i, \gamma_0}(X_i).$$

Applying the above proposition, we next show that the optimal reinsurance problems with feasible ceded loss functions $\mathcal{D}^n(P)$, $\mathcal{D}_1^n(P)$ and $\mathcal{D}_2^n(P)$ can be simplified to the problems on the domain of $\mathcal{L}^n(P)$, $\mathcal{H}^n(P)$, $\mathcal{G}^n(P)$, respectively.

Theorem 1. Suppose $F_1^{-1}(\cdot), \dots, F_n^{-1}(\cdot)$ are all continuous over $(0, 1)$ and $\alpha \in (0, 1)$. We have the following conclusion.

(i) If $n = 2$ or $n - \sum_{i=1}^n F_i(F_i^{-1}(\alpha)) \leq 1 - \alpha$, or $\sum_{i=1}^n (F_i(F_i^{-1}(1) - \alpha) \leq 1 - \alpha$, then

$$\inf_{\mathbf{f} \in \mathcal{D}^n(P)} \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} \text{VaR}_\alpha(S_n^{\mathbf{f}}(X_1, \dots, X_n)) = \inf_{(\mathbf{a}, \mathbf{b}) \in \mathcal{A}(P)} \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} L(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma}),$$

where $L(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma}) = \sum_{i=1}^n \{R_{\gamma_i, \gamma_0}(X_i - l_{a_i, b_i}(X_i)) + \pi_i(l_{a_i, b_i}(X_i))\}$.

(ii) If $\mathbf{F} \in (\mathcal{M}_{cx}^\alpha)^n$, then

$$\inf_{\mathbf{f} \in \mathcal{D}_1^n(P)} \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} \text{VaR}_\alpha(S_n^{\mathbf{f}}(X_1, \dots, X_n)) = \inf_{(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \in \mathcal{A}_1(P)} \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} H(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \boldsymbol{\gamma}),$$

where $H(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \boldsymbol{\gamma}) = \sum_{i=1}^n \{R_{\gamma_i, \gamma_0}(X_i) - R_{\gamma_i, \gamma_0}(h_{a_i, b_i, c_i, d_i}(X_i)) + \pi_i(h_{a_i, b_i, c_i, d_i}(X_i))\}$.

(iii) If $\mathbf{F} \in (\mathcal{M}_{cv}^\alpha)^n$, then

$$\inf_{\mathbf{f} \in \mathcal{D}_2^n(P)} \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} \text{VaR}_\alpha(S_n^{\mathbf{f}}(X_1, \dots, X_n)) = \inf_{(\mathbf{a}, \mathbf{b}) \in \mathcal{A}_2(P)} \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} G(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma}),$$

where $G(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma}) = \sum_{i=1}^n \{R_{\gamma_i, \gamma_0}(X_i) - R_{\gamma_i, \gamma_0}(g_{a_i, b_i}(X_i)) + \pi_i(g_{a_i, b_i}(X_i))\}$.

Note that (i) of Theorem 1 shows that the limited stop-loss reinsurance policy is optimal for each line of business if $n = 2$ or the tail parts of the marginals are mutually exclusive. Comparing the optimal reinsurance solutions in Chi and Tan (2011) and Chi and Tan (2013) with (i) of Theorem 1, we can find that the optimality of limited stop loss is quite robust under the VaR risk measure. As we constrain the feasible ceded loss functions to be convex in (ii) of Theorem 1, the optimal reinsurance policy has the form of $h_{a, b, c, d}(x) = c(x - a)_+ + d(x - b)_+$ for each line of business, including the stop-loss policy as a special case. This ceded loss function is piecewise linear with three pieces. The reinsurer pays 0 for the part of risk that is smaller than a and covers $100c\%$ for the part of risk between a and b and covers $100(c + d)\%$ for the part of risk that is larger than b . It also implies that stop-loss might not be optimal in this case. In (iii) of Theorem 1, the optimal reinsurance policy has the form of $g_{a, b}(x) = a \min(x, b)$ for each line of business, which includes quota-share as a special case. We can call it *limited quota-share*. This implies a counter-intuitive fact that

in general the limited quota-share policy might be better than the quota-share policy from the point of view of the insurer. This can be explained in the following way: using the limited quota-share, the reinsurer may cover more than the quota-share policy for the part of the risk that is lower than the upper limit and the upper limit can be set to be large enough. Moreover, limited quota-share reinsurance contract may have lower price.

Next, we give the optimal reinsurance policies applying Theorem 1 and a condition on the existence of the optimal parameters.

Proposition 2. *The parameters of the optimal ceded loss functions $\mathbf{l}_{\mathbf{a},\mathbf{b}}$, $\mathbf{h}_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}$, $\mathbf{g}_{\mathbf{a},\mathbf{b}}$ in (i)-(iii) of Theorem 1 are determined as follows.*

(i) For $\mathbf{l}_{\mathbf{a},\mathbf{b}}$,

$$(\mathbf{a}^*, \mathbf{b}^*) \in \arg \inf_{(\mathbf{a},\mathbf{b}) \in \mathcal{A}(P)} \left\{ \inf_{\gamma \in (1-\alpha)\Delta_n} L(\mathbf{a}, \mathbf{b}, \gamma) \right\};$$

(ii) For $\mathbf{h}_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}$,

$$(\mathbf{a}^*, \mathbf{b}^*, \mathbf{c}^*, \mathbf{d}^*) = \arg \inf_{(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) \in \mathcal{A}_1(P)} \left\{ \inf_{\gamma \in (1-\alpha)\Delta_n} H(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \gamma) \right\};$$

(iii) For $\mathbf{g}_{\mathbf{a},\mathbf{b},\mathbf{c}}$,

$$(\mathbf{a}^*, \mathbf{b}^*) = \arg \inf_{(\mathbf{a},\mathbf{b}) \in \mathcal{A}_2(P)} \left\{ \inf_{\gamma \in (1-\alpha)\Delta_n} G(\mathbf{a}, \mathbf{b}, \gamma) \right\}.$$

Moreover, the existence of these parameters is guaranteed by the continuity of π_i , for $i = 1, \dots, n$.

For the case of $n = 2$, we can derive the optimal ceded loss functions by a different approach and obtain a simpler expression. This is crucial in the numerical study of the optimal ceded loss functions as the simpler form allows the transformation of the non-convex optimization problem into several convex optimization problems in Section 6. In light of Makarov (1981) and Rüschendorf (1982), we arrive at the following result.

Proposition 3. *For $n = 2$, suppose that F_1^{-1} and F_2^{-1} are continuous over $(0, 1)$, then*

$$\inf_{(f_1, f_2) \in D^2(P)} \sup_{(X_1, X_2) \in \mathcal{E}_2} \text{VaR}_\alpha(S_2^f(X_1, X_2)) = \inf_{(a_1, a_2, b_1, b_2) \in \mathcal{A}(P)} \inf_{t \in [0, 1-\alpha]} L_1(a_1, a_2, b_1, b_2, t),$$

where

$$L_1(a_1, a_2, b_1, b_2, t) = \text{VaR}_{\alpha+t}(X_1 - l_{a_1, b_1}(X_1)) + \text{VaR}_{1-t}(X_2 - l_{a_2, b_2}(X_2)) + \pi_1(l_{a_1, b_1}(X_1)) + \pi_2(l_{a_2, b_2}(X_2)).$$

Moreover, $(l_{a_1, b_1}, l_{a_2, b_2})$ is the optimal ceded loss function for the worst case scenario provided

$$(a_1, a_2, b_1, b_2) \in \arg \inf_{(a_1, a_2, b_1, b_2) \in \mathcal{A}(P)} \left\{ \inf_{t \in [0, 1-\alpha]} L_1(a_1, a_2, b_1, b_2, t) \right\}.$$

Note that in this section, we only consider a budget constraint. In fact, we can also add other constraints. For instance, we can impose a profit constraint: $\mathbb{E}(S_n^f(X_1, \dots, X_n)) \leq (1 - \theta)P_1$, where $P_1 > 0$ is the total

premium received by the insurer over the period and $0 < \theta < 1$ represents the profit rate required by the insurer. Our conclusions in this section are still valid if we change $\mathcal{D}^n(P)$ into

$$\left\{ \mathbf{f} \in \mathcal{D}^n : \sum_{i=1}^n \pi_i(f_i(X_i)) \leq P, \mathbb{E}(S_n^{\mathbf{f}}(X_1, \dots, X_n)) \leq (1 - \theta)P_1 \right\}$$

by enhancing (iii) of π_i to reserving stop-loss order for the difference: For $Y, Z \in \mathcal{X}$, $\pi_i(Y) - \mathbb{E}(Y) \leq \pi_i(Z) - \mathbb{E}(Z)$ whenever $\mathbb{E}((Y - d)_+) \leq \mathbb{E}((Z - d)_+)$ for all $d \in \mathbb{R}$.

4 Optimal reinsurance under Range-Value-at-Risk

In this section, we study the RVaR-based optimal reinsurance problem. RVaR is recognized as a bridge between VaR and ES by including these two risk measures as the limiting cases. Moreover, RVaR is a family of robust risk measures in the sense of [Cont et al. \(2010\)](#). Analogously to the analysis of VaR, we first extend Theorem 1 of [Blanchet et al. \(2023\)](#) from marginals with decreasing densities in the tail part to those with concave distributions in the tail part.

Proposition 4. *For any α, β with $0 \leq \beta < \beta + \alpha \leq 1$ and $\mathbf{F} \in (\mathcal{M}_{cv}^{1-\beta-\alpha})^n$,*

$$\sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} R_{\beta, \alpha} \left(\sum_{i=1}^n X_i \right) = \inf_{\gamma \in (\beta + \alpha)\Delta_n, \gamma_0 \geq \alpha} \sum_{i=1}^n R_{\gamma_i, \gamma_0}(F_i). \quad (2)$$

It turns out from the following results that the optimal reinsurance problem with domains $\mathcal{D}^n(P)$ and $\mathcal{D}_2^n(P)$ can be reduced to that with domains $\mathcal{L}^n(P)$ and $\mathcal{G}^n(P)$ respectively.

Theorem 2. *For any α, β with $0 \leq \beta < \beta + \alpha \leq 1$, we have the following two conclusions.*

(i) *If $\beta = 0$ or $n - \sum_{i=1}^n F_i((F_i)_+^{-1}(1 - \beta - \alpha)) \leq \beta + \alpha$, then*

$$\inf_{\mathbf{f} \in \mathcal{D}^n(P)} \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} R_{\beta, \alpha}(S_n^{\mathbf{f}}(X_1, \dots, X_n)) = \inf_{(\mathbf{a}, \mathbf{b}) \in \mathcal{A}(P)} \inf_{\gamma \in (\beta + \alpha)\Delta_n, \gamma_0 \geq \alpha} L(\mathbf{a}, \mathbf{b}, \gamma),$$

where $L(\mathbf{a}, \mathbf{b}, \gamma)$ is given in [Theorem 1](#).

(ii) *If $\mathbf{F} \in (\mathcal{M}_{cv}^{1-\beta-\alpha})^n$, then*

$$\inf_{\mathbf{f} \in \mathcal{D}_2^n(P)} \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} R_{\beta, \alpha}(S_n^{\mathbf{f}}(X_1, \dots, X_n)) = \inf_{(\mathbf{a}, \mathbf{b}) \in \mathcal{A}_2(P)} \inf_{\gamma \in (\beta + \alpha)\Delta_n, \gamma_0 \geq \alpha} G(\mathbf{a}, \mathbf{b}, \gamma),$$

where $G(\mathbf{a}, \mathbf{b}, \gamma)$ is given in [Theorem 1](#).

[Theorem 2](#) shows that the form of the optimal reinsurance policy for each line of business for RVaR model is the same as the VaR model except that we do not have conclusion for RVaR model for the case of $n = 2$ and the case for the marginals with convex distributions on the tail parts. The results obtained in (i)

of Theorem 2 partially extend the results in Cheung *et al.* (2014), where convex risk measures and expected principles are considered. Moreover, it is worth pointing out that for convex risk measures, the multiple risks are comonotonic under the worst case dependence structure. Whereas for RVaR which is not convex except the case of ES, the dependence structure for the worst case scenario is the combination of mutual exclusivity and joint mixability, which is negatively dependent; see Blanchet *et al.* (2023) or Jakobsons *et al.* (2016) for the detailed discussion of the worst case dependence structure.

The next result gives the optimal reinsurance policies and a condition on their existence.

Proposition 5. *The parameters of the optimal ceded loss functions $\mathbf{l}_{\mathbf{a},\mathbf{b}}$ and $\mathbf{g}_{\mathbf{a},\mathbf{b}}$ in (i)-(ii) of Theorem 2 are determined as follows.*

(i) For $\mathbf{l}_{\mathbf{a},\mathbf{b}}$,

$$(\mathbf{a}^*, \mathbf{b}^*) \in \arg \inf_{(\mathbf{a}, \mathbf{b}) \in \mathcal{A}(P)} \left\{ \inf_{\gamma \in (\beta + \alpha)\Delta_n, \gamma_0 \geq \alpha} L(\mathbf{a}, \mathbf{b}, \gamma) \right\};$$

(ii) For $\mathbf{g}_{\mathbf{a},\mathbf{b}}$,

$$(\mathbf{a}^*, \mathbf{b}^*) = \arg \inf_{(\mathbf{a}, \mathbf{b}) \in \mathcal{A}_2(P)} \left\{ \inf_{\gamma \in (\beta + \alpha)\Delta_n, \gamma_0 \geq \alpha} G(\mathbf{a}, \mathbf{b}, \gamma) \right\}.$$

Moreover, the existence of these parameters is guaranteed by the continuity of π_i , $i = 1, \dots, n$.

5 Quota-share reinsurance under dependence uncertainty

In this section, we consider the simplest reinsurance form, quota-share reinsurance, where the ceded loss is proportional to the loss, i.e., $f_i(X_i) = w_i X_i$, $i = 1, \dots, n$, for some share levels $w_i \in [0, 1]$. This form of reinsurance is popular in almost all reinsurance branches, particularly due to its conceptual and administrative simplicity; see e.g., Albrecher *et al.* (2017). Note that in the previous sections, we suppose π_i satisfies (i)-(iii), which is complicated. For the quota-share reinsurance, we only need to impose a very simple assumption on π_i . Moreover, we will find a very interesting phenomenon: in some cases, the optimal reinsurance is either full reinsurance coverage or zero reinsurance coverage. We say that π satisfies homogeneity if $\pi(\lambda X) = \lambda \pi(X)$ for $\lambda > 0$ and $X \in \mathcal{X}$. In this section, we suppose that π satisfies distribution invariance, safety loading, and homogeneity. Note that many of the practical premium principles π satisfy the above three properties, such as, expected principle, standard deviation principle, Wang's principle and Dutch principle; see Young (2004) for more principles.

The total risk faced by the insurer after reinsurance is $S^{\mathbf{w}}(X_1, \dots, X_n) = \sum_{i=1}^n ((1 - w_i)X_i + w_i \pi_i(X_i))$ with $\mathbf{w} = (w_1, \dots, w_n)$. We denote the domain of $\mathbf{w}$ with budget constraint as

$$\Lambda_n(P) = \left\{ \mathbf{w} \in [0, 1]^n : \sum_{i=1}^n w_i \pi_i(X_i) \leq P \right\}.$$

Note that $\mathbf{0} \in \Lambda_n(P)$ and $\Lambda_n(P)$ is a convex and compact set, where its boundary is denoted as $\partial \Lambda_n(P)$. Applying Propositions 1 and 4, we arrive at the following results.

Proposition 6. For $\alpha \in (0, 1)$, if $n = 2$, or $\mathbf{F} \in (\mathcal{M}_{cx}^\alpha)^n \cup (\mathcal{M}_{cv}^\alpha)^n$ and $F_1^{-1}(\cdot), \dots, F_n^{-1}(\cdot)$ are all continuous over $(0, 1)$, then

$$\begin{aligned} & \inf_{\mathbf{w} \in \Lambda_n(P)} \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} \text{VaR}_\alpha(S^{\mathbf{w}}(X_1, \dots, X_n)) \\ &= \inf_{\mathbf{w} \in \partial \Lambda_n(P)} \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} \sum_{i=1}^n ((1-w_i)R_{\gamma_i, \gamma_0}(X_i) + w_i\pi_i(X_i)). \end{aligned}$$

Moreover, there exists optimal quota-share levels $\mathbf{w}^* \in \partial \Lambda_n(P)$.

Proposition 7. For any α, β with $0 \leq \beta < \beta + \alpha \leq 1$, if $\beta = 0$ or $\mathbf{F} \in (\mathcal{M}_{cv}^{1-\beta-\alpha})^n$, then

$$\begin{aligned} & \inf_{\mathbf{w} \in \Lambda_n(P)} \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} R_{\beta, \alpha}(S^{\mathbf{w}}(X_1, \dots, X_n)) \\ &= \inf_{\mathbf{w} \in \partial \Lambda_n(P)} \inf_{\boldsymbol{\gamma} \in (\beta+\alpha)\Delta_n, \gamma_0 \geq \alpha} \sum_{i=1}^n ((1-w_i)R_{\gamma_i, \gamma_0}(X_i) + w_i\pi_i(X_i)). \end{aligned}$$

Moreover, there exists optimal quota-share levels $\mathbf{w}^* \in \partial \Lambda_n(P)$.

Note that in this section, we only consider the quota-share reinsurance. However, in the past two sections, quota-share reinsurance for each line of business may not be optimal. Hence, the quota-share reinsurance leads to loss compared with the optimal reinsurance strategy obtained in the previous two sections. As an example, we next see the loss for case $n = 2$ for VaR. It follows from Theorem 1 and Proposition 6 that

$$\begin{aligned} & \inf_{\mathbf{w} \in \Lambda_2(P)} \sup_{(X_1, X_2) \in \mathcal{E}_2(\mathbf{F})} \text{VaR}_\alpha(S^{\mathbf{w}}(X_1, X_2)) - \inf_{\mathbf{f} \in \mathcal{D}^2(P)} \sup_{(X_1, X_2) \in \mathcal{E}_2(\mathbf{F})} \text{VaR}_\alpha(S_2^{\mathbf{f}}(X_1, X_2)) \\ &= \inf_{\mathbf{w} \in \partial \Lambda_2(P)} \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_2} \sum_{i=1}^2 ((1-w_i)R_{\gamma_i, \gamma_0}(X_i) + w_i\pi_i(X_i)) \\ &\quad - \inf_{(\mathbf{a}, \mathbf{b}) \in \mathcal{A}(P)} \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_2} \sum_{i=1}^2 \{R_{\gamma_i, \gamma_0}(X_i - l_{a_i, b_i}(X_i)) + \pi_i(l_{a_i, b_i}(X_i))\} \geq 0. \end{aligned}$$

We believe the above value, which is the loss of using quota-share reinsurance, is strictly larger than 0 in general.

Note that if $P \geq \sum_{i=1}^n \pi_i(X_i)$, then $\Lambda_n(P) = [0, 1]^n$. We denote $\Lambda_n^0 = \{\mathbf{w} \in [0, 1]^n : w_i \in \{0, 1\}, i = 1, \dots, n\}$. In this case, the optimal reinsurance policy for each line of business is either the full reinsurance coverage or zero reinsurance coverage. This means that the insurer will only take extreme decisions for each line of business and may refuse to buy reinsurance contracts for some lines of business. Interestingly, in [Bernard et al. \(2020\)](#), the insurer refuses to sell insurance to some policyholders due to their preference and risk exposures.

Corollary 1. If $P \geq \sum_{i=1}^n \pi_i(X_i)$, then there exist optimal quota-share levels $\mathbf{w}^*$ in both Propositions 6-7 such that $\mathbf{w}^* \in \Lambda_n^0$.

6 Examples

Our main results show that finding the optimal ceded loss functions for the worst case of reinsurance models with dependence uncertainty boils down to finding the minimiser of a deterministic function. In order to illustrate the effectiveness of our findings, this section will show both analytical and numerical solutions for some multi-risk optimal reinsurance problems with the existence of dependence uncertainty. Since VaR and ES are the prevalent risk measures for quantifying insurance risks and setting regulatory capitals, and ES has been investigated in [Cheung *et al.* \(2014\)](#), we will put our focus on VaR in this section.

6.1 Analytical Solutions

In this section, we will discuss the analytical solution of our main optimisation problem (1) with $\rho = \text{VaR}$ and $n = 2$. That is, we solve

$$\min_{\mathbf{f} \in \mathcal{D}^2(P)} \max_{(X_1, X_2) \in \mathcal{E}_2(\mathbf{F})} \text{VaR}_\alpha(S_2^{\mathbf{f}}(X_1, X_2)). \quad (3)$$

As stated by Proposition 3, the above optimisation problem (3) can be written as

$$\min_{\substack{(a_1, a_2, b_1, b_2) \\ \in \mathcal{A}(P)}} \min_{t \in [0, 1-\alpha]} L_1(a_1, a_2, b_1, b_2, t), \quad (4)$$

where

$$L_1(a_1, a_2, b_1, b_2, t) = \text{VaR}_{\alpha+t}(X_1 - l_{a_1, b_1}(X_1)) + \text{VaR}_{1-t}(X_2 - l_{a_2, b_2}(X_2)) + \pi_1(l_{a_1, b_1}(X_1)) + \pi_2(l_{a_2, b_2}(X_2)).$$

Example 1. In this example, we solve the optimisation problem (4) by assuming X_i is exponentially distributed with expected value $E(X_i) = \lambda_i > 0$ and distribution function $F_i(x) = 1 - e^{-x/\lambda_i}$, $x > 0$ for $i = 1, 2$. It is further assumed that both $\pi_1(\cdot)$ and $\pi_2(\cdot)$ follow the expected premium principle with an overhead expense $w_i \geq 0$ and a risk-loading factor $\delta_i > 0$, i.e., $\pi_i(X_i) = w_i + (1 + \delta_i)\mathbb{E}(X_i)$, for $i = 1, 2$. Note that $\{(u_1, u_2) : (l_{u_1, v_1}, l_{u_2, v_2}) \in \mathcal{D}^2(P), 0 \leq u_i \leq v_i, i = 1, 2\} = \emptyset$, if $w_1 + w_2 > P$. Therefore, we will only consider the non-trivial cases where $w_1 + w_2 \leq P$ for the rest of our discussion. We define

$$\mathcal{T} := \left\{ t \in [0, 1 - \alpha] : 1 - \alpha - \frac{1}{1 + \delta_1} \leq t \leq \frac{1}{1 + \delta_2} \right\}. \quad (5)$$

Proposition 8. Given $w_1 + w_2 \leq P$, Table 1 summarises the optimal solutions of problem (4)

$$(a_1^*, a_2^*, b_1^*, b_2^*, t^*) := \arg \min_{\substack{(a_1, a_2, b_1, b_2) \\ \in \mathcal{A}(P)}} \min_{t \in [0, 1-\alpha]} L_1(a_1, a_2, b_1, b_2, t)$$

for the following cases:

- (i) $(1 + \delta_1)\lambda_1 - (1 + \delta_2)\lambda_2 = 0$ and $\sum_{i=1}^2(w_i + \lambda_i) - (1 + \delta_1)(1 - \alpha)\lambda_1 \leq P$;
- (iia) $(1 + \delta_1)\lambda_1 - (1 + \delta_2)\lambda_2 > 0$, $(1 - \alpha)(1 + \delta_1) < 1$ and $\sum_{i=1}^2(w_i + \lambda_i) - (1 + \delta_1)(1 - \alpha)\lambda_1 \leq P$;
- (iib) $(1 + \delta_1)\lambda_1 - (1 + \delta_2)\lambda_2 > 0$, $(1 - \alpha)(1 + \delta_1) \geq 1$ and $w_2 + \lambda_2 \leq P$;
- (iiia) $(1 + \delta_1)\lambda_1 - (1 + \delta_2)\lambda_2 < 0$, $(1 - \alpha)(1 + \delta_2) < 1$ and $\sum_{i=1}^2(w_i + \lambda_i) - (1 + \delta_2)(1 - \alpha)\lambda_2 \leq P$;
- (iiib) $(1 + \delta_1)\lambda_1 - (1 + \delta_2)\lambda_2 < 0$, $(1 - \alpha)(1 + \delta_2) \geq 1$ and $w_1 + \lambda_1 \leq P$.

Case	Sub-case	Solution of $(a_1^*, a_2^*, b_1^*, b_2^*, t^*)$
(i)		$t^* \in \mathcal{T}$; $a_i^* = -\lambda_i \ln\left(\frac{1}{1+\delta_i}\right)$, for $i = 1, 2$; $b_1^* = -\lambda_1 \ln(1 - \alpha - t^*)$; $b_2^* = -\lambda_2 \ln t^*$
(ii)	(iia)	$t^* = 0$; $a_i^* = -\lambda_i \ln\left(\frac{1}{1+\delta_i}\right)$, for $i = 1, 2$; $b_1^* = -\lambda_1 \ln(1 - \alpha)$; $b_2^* = \infty$.
	(iib)	$t^* = 0$; $a_1^* = b_1^* = -\lambda_1 \ln(1 - \alpha)$; $a_2^* = -\lambda_2 \ln\left(\frac{1}{1+\delta_2}\right)$; $b_2^* = \infty$.
(iii)	(iiia)	$t^* = 1 - \alpha$; $a_i^* = -\lambda_i \ln\left(\frac{1}{1+\delta_i}\right)$, for $i = 1, 2$; $b_1^* = \infty$; $b_2^* = -\lambda_2 \ln(1 - \alpha)$.
	(iiib)	$t^* = 1 - \alpha$; $a_1^* = -\lambda_1 \ln\left(\frac{1}{1+\delta_1}\right)$; $b_1^* = \infty$; $a_2^* = b_2^* = -\lambda_2 \ln(1 - \alpha)$.

Table 1: Summary of optimal solutions for Example 1.

Note that for all other cases, a numerical approach will be sought to find $(a_1^*, a_2^*, b_1^*, b_2^*, t^*)$ later. The proof of Proposition 8 is lengthy and involves complicated computations and it is displayed in the supplementary materials in Section 10.

Table 1 has some interesting implications. Firstly, the division of the three cases (i)-(iii) is based on the comparison between values of $(1 + \delta_1)\lambda_1$ and $(1 + \delta_2)\lambda_2$. Recall that, for $i = 1, 2$, $\lambda_i = E(X_i)$ and δ_i is the risk-loading factor associated to the expected premium for reinsuring X_i . Therefore, $(1 + \delta_i)\lambda_i$ indicates the degree of expensiveness of reinsuring X_i before the optimal reinsurance has been found. When $(1 + \delta_1)\lambda_1 > (1 + \delta_2)\lambda_2$, i.e. the degree of expensiveness of reinsuring X_1 is relatively higher, it is the case (ii), in which $b_1^* < \infty$ while $b_2^* = \infty$. That is, when the degree of expensiveness of reinsuring X_1 is relatively higher, it is optimal to cap the reinsurance of X_1 at the level of $b_1^* - a_1^*$ while leaving the reinsurance of X_2 uncapped. Similarly, when $(1 + \delta_1)\lambda_1 < (1 + \delta_2)\lambda_2$, i.e. the degree of expensiveness of reinsuring X_2 is relatively higher, it is optimal to cap the reinsurance of X_2 at the level of $b_2^* - a_2^*$ while leaving the reinsurance of X_1 uncapped (i.e. as in case iii). If $(1 + \delta_1)\lambda_1 = (1 + \delta_2)\lambda_2$ (i.e. case i), the degree of expensiveness is the same for reinsuring both X_1 and X_2 . In this case, the solution is not unique, and whether the optimal reinsurance of X_i is capped at $b_i^* - a_i^*$ or uncapped will depend on the decision-maker's level of risk tolerance, α , and the risk-loading factors δ_1 and δ_2 .

Secondly, the sub-case (iib) represents the scenario where $(1 + \delta_1)\lambda_1 > (1 + \delta_2)\lambda_2$ and $1 - \alpha \geq \frac{1}{1+\delta_1}$, i.e. α is relatively small and/or δ_1 is relatively large. In other words, this is a scenario where the decision-maker has a relatively high level of risk tolerance and/or the risk premium charged for having X_1 reinsured is relatively high. As a result, reinsuring X_1 becomes cost-ineffective and hence, it is optimal not to have X_1 reinsured,

i.e. $a_1^* = b_1^*$. A similar implication is delivered by the sub-case (iiib). When α is relatively small and/or δ_2 is relatively large, reinsuring X_2 now becomes cost-ineffective and thus, it is optimal not to have X_2 reinsured, i.e. $a_2^* = b_2^*$.

Unfortunately, due to the lack of convexity of the optimisation model (4), analytical solutions are not always available. For example, it will become more interesting to study the solutions when X_1 and X_2 follow heavy-tail distributions, but the analytical solution cannot be easily found in this case and numerical methods are necessary to be sought, which is explored in the next section.

6.2 Numerical Examples

In this section, we turn to solve the optimisation problem (4) using a numerical approach. In particular, we will solve the problem using Grid Search. Let us first discretise X_i into a group of observations with size m represented by $\mathbf{x}_i := (x_{i1}, x_{i2}, \dots, x_{im})^T$, for $i = 1, 2$. The observations X_{ij} , for $i = 1, 2$ and $j = 1, 2, \dots, m$, can be obtained by either using historical data or using Monte-Carlo simulations. The reinsurance indemnity function $l_{a_i, b_i}(X_i)$ will be represented by $\mathbf{y}_i := (y_{i1}, y_{i2}, \dots, y_{im})^T$, where y_{ij} is the reinsurance coverage associated to x_{ij} , for all $i = 1, 2$ and $j = 1, 2, \dots, m$. Without loss of generality, we can assume that x_{ij} is collected in an ascending order, i.e. $x_{i1} \leq x_{i2} \leq \dots \leq x_{im}$ for $i = 1, 2$. Also, because of the definition of $l_{a_i, b_i}(X_i)$, we know that $y_{i1} \leq y_{i2} \leq \dots \leq y_{im}$ is true for $i = 1, 2$.

Example 2. In this example, we solve the optimisation problem (4) by incorporating the assumptions introduced in Example 1. Furthermore, it is assumed that $\lambda_1 = 8000$, $\lambda_2 = 3000$, $\delta_1 = 0.8$ and $\delta_2 = 0.3$. Hence, this example demonstrates the case (ii) in Example 1. Samples of X_1 and X_2 are obtained by using Monte-Carlo simulations with sample size m assumed to be 200.

For any given value of $t \in [0, 1 - \alpha]$, the local optimum of the optimisation problem (4) can be found by solving the following linear minimisation problem:

$$\begin{aligned} \min_{(\mathbf{y}_1, \mathbf{y}_2) \in \mathbb{R}^m \times \mathbb{R}^m} \quad & x_{1q_1} - y_{1q_1} + x_{2q_2} - y_{2q_2} + \sum_{i=1}^2 \left(w_i + \frac{(1 + \delta_i)}{m} \mathbf{1}^T \mathbf{y}_i \right) \\ \text{s.t.} \quad & \mathbf{0} \leq \mathbf{A} \mathbf{y}_i \leq \mathbf{A} \mathbf{x}_i, \quad \forall i = 1, 2, \\ & \sum_{i=1}^2 \left(w_i + \frac{1 + \delta_i}{m} \mathbf{1}^T \mathbf{y}_i \right) \leq P, \end{aligned} \tag{6}$$

where $\mathbf{1}$ is a column vector of 1 with dimension m -by-1, $q_1 = \lceil n(\alpha + t) \rceil$, $q_2 = \lceil n(1 - t) \rceil$ and $\mathbf{A}$ is an m -by- m matrix such that

$$a_{kl} = \begin{cases} 1 & \text{if } k = l, \quad \forall l = 1, 2, \dots, m-1, \\ -1 & \text{if } k = l + 1, \quad \forall l = 1, 2, \dots, m-1, \\ 0 & \text{otherwise.} \end{cases}$$

As a result, the global optimum of the optimisation problem (4) can be found by using Global Search over

the value of $t \in [0, 1 - \alpha]$ as a grid. Note that, because of the definition of q_1 and q_2 , there are only finite number of grids over which the local optimisation (6) needs to be solved.

Figure 1 illustrates the numerical solution of optimal reinsurance function found in this example. The

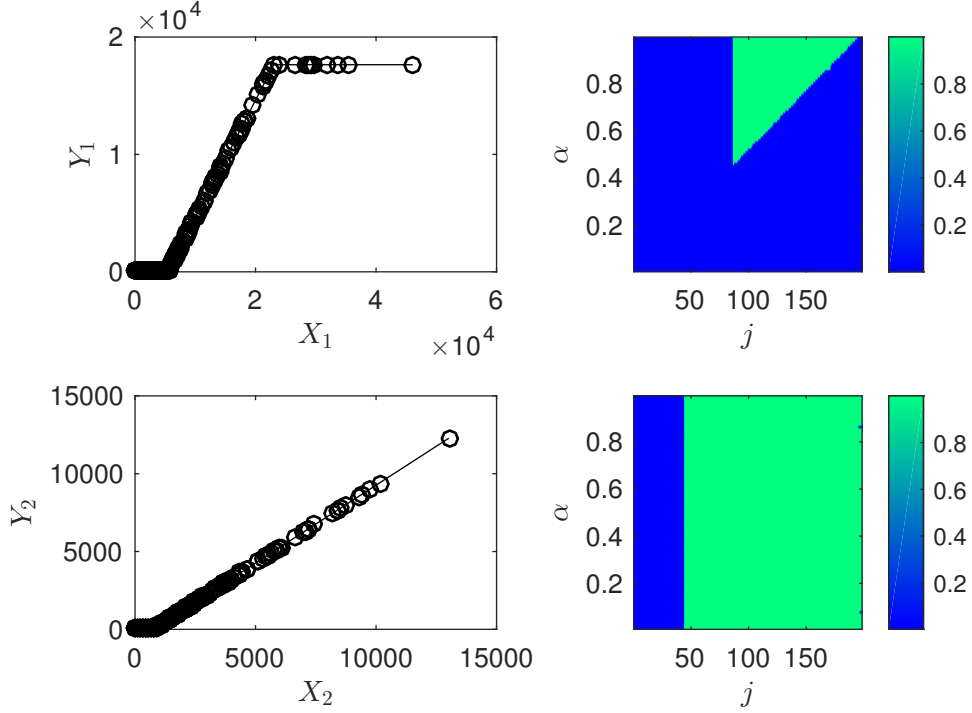


Figure 1: Illustration of numerical optimal reinsurance solution found in Example 2 using scatter plots and heat maps, where Y_i represents the reinsurance indemnity function $l_{a_i, b_i}(X_i)$ for $i = 1, 2$.

scatter plots on the left-hand side of the figure shows the numerical solutions of $l_{a_1^*, b_1^*}(X_1)$ and $l_{a_2^*, b_2^*}(X_2)$ when $\alpha = 0.95$, which corresponds to the sub-case (iia) of the analytical solution studied in Example 1. The shape of $l_{a_1^*, b_1^*}(X_1)$ and $l_{a_2^*, b_2^*}(X_2)$ is affirmed that $l_{a_1^*, b_1^*}(X_1)$ is a limited stop-loss reinsurance being capped at $\text{VaR}_{0.95}(l_{a_1^*, b_1^*}(X_1))$, while $l_{a_2^*, b_2^*}(X_2)$ is a standard stop-loss reinsurance with $b_2^* = \infty$.

It is also worth to show the changes in the shape of $l_{a_1^*, b_1^*}(X_1)$ and $l_{a_2^*, b_2^*}(X_2)$ when α changes, as this indicates how the insurer changes their decision on buying reinsurance when there level of risk concern changes. Therefore, we have solved for $l_{a_1^*, b_1^*}(X_1)$ and $l_{a_2^*, b_2^*}(X_2)$ for 99 different values of α (i.e. from $\alpha = 0.01$ to $\alpha = 0.99$ with a step-size of 0.01). In order to show a more compact visualisation of the 99 pairs of solutions, heat maps are used which can be found as the right-hand side panels in Figure 1. The heat map shows the gradient of $l_{a_i^*, b_i^*}(X_i)$ corresponds to each of the points X_{ij} where $j = 1, 2, \dots, 199$. Note that the gradient at various parts of $l_{a_i^*, b_i^*}(X_i)$ is either 0 or 1 due to the definition of the l function. As one can see in the heat maps, $l_{a_2^*, b_2^*}(X_2)$ always maintain the same shape, i.e. changing from a horizontal line to an upward-sloping line with a turning point at $a_2^* = -\lambda_2 \ln\left(\frac{1}{1+\delta_2}\right)$. By contrast, the shape of $l_{a_1^*, b_1^*}(X_1)$ changes as the value of α changes. When α is large (i.e. $\alpha > 1 - \frac{1}{1-\delta_1}$), there are two turning points as the gradients change from 0 to 1 then back to 0 again when X_{1j} reaches $\text{VaR}_\alpha(X_1)$. On the other hand, when

α is small, the gradients remain as 0 for all X_{1j} implying that $l_{a_1^*, b_1^*}(X_1) = 0$, which is indeed the conclusion made by the analytical solution of Example 1.

All computations are conducted on a desktop PC with an Intel Quad-Core i7-3770 CPU on a Linux 64-bit system. The computing times of solving the problem 4 for different values of α are listed in Table 2. It is noticed that the computing time increases as the value of α decreases, because the number of grids to

α	computing time (seconds)
0.9	977.97
0.7	4364.30
0.5	7748.17
0.3	9371.02
0.1	12390.15

Table 2: Computing times (in seconds) for different α .

be searched increases when α decreases. However, small values of α are less likely to be adopted by insurers.

Example 2 has shown that the proposed numerical approach is capable of solving the optimisation problem (4) effectively. It has then been applied to solve models where X_1 and X_2 follow a different distribution (e.g. a Pareto distribution and a LogNormal distribution). Similar results to those of Example 2 are obtained, and there is no significant difference in computing times taken. Therefore, we will refrain from reporting full details.

7 Conclusion

In the paper, we investigate the optimal reinsurance design with multivariate risks and dependence uncertainty. For the multivariate risks, the marginal distribution is fixed but the dependence structure is unknown. Hence we consider the worst-case scenario, i.e., the min max optimization problem. The aggregate risk is quantified by VaR and RVaR respectively. We obtain three types of optimal ceded loss functions: (i) limited stop-loss ($l_{a,b}$), (ii) limited quota-share ($g_{a,b}$) and (iii) a convex piece-wise linear function with three pieces ($h_{a,b,c,d}$) according to different risk measures and different constraints on the ceded loss functions and marginal distributions. Our results are valid for general premium principles. Our results are different from the ones in the literature in the sense that the worst-case dependence structure is negatively dependent. The quota-share optimal reinsurance is also considered under dependence uncertainty. The analytical solutions and the numerical studies for bivariate risks can illustrate the power of our theoretical results.

In this paper, our research focuses on the worst-case study of optimal reinsurance for multiple lines of business with completely unknown dependence structure of the multivariate risks. However, in practice, some additional information may be available on the dependence structure of the risks, such as the covariance matrix of the multivariate risks or positive/negative dependence among those risks. It will be more practical by incorporating those information in the uncertainty set, which makes the problem very difficult and is beyond the scope of this paper. Our future research may focus on cases with additional information on the

dependence structure and the numerical study and real data analysis for more than two risks.

8 Declaration of interest: none

9 Appendix

In Appendix, we provide all the proofs of the results in Sections 3-5.

9.1 Proofs in Section 3.

To prove Proposition 1, the following lemma is applied.

Lemma 1. For $\mathbf{F}^{(k)} = (F_1^{(k)}, \dots, F_n^{(k)})$, $k \geq 1$ and $\mathbf{F} = (F_1, \dots, F_n)$, suppose, for each i , $(F_i^{(k)})_+^{-1}(t) \rightarrow (F_i)_+^{-1}(t)$ uniformly over $t \in (\alpha, 1)$, then

$$\lim_{k \rightarrow \infty} \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F}^{(k)})} \text{VaR}_\alpha^+ \left(\sum_{i=1}^n X_i \right) = \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} \text{VaR}_\alpha^+ \left(\sum_{i=1}^n X_i \right).$$

Proof. By Theorem 4.6 of Bernard *et al.* (2014), we have

$$\sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F}^{(k)})} \text{VaR}_\alpha^+ \left(\sum_{i=1}^n X_i \right) = \sup_{U_1, \dots, U_n \sim U[0,1]} \text{ess-inf} \sum_{i=1}^n (F_i^{(k)})_+^{-1}(\alpha + (1 - \alpha)U_i),$$

and

$$\sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} \text{VaR}_\alpha^+ \left(\sum_{i=1}^n X_i \right) = \sup_{U_1, \dots, U_n \sim U[0,1]} \text{ess-inf} \sum_{i=1}^n (F_i)_+^{-1}(\alpha + (1 - \alpha)U_i).$$

It follows from the uniform convergence that

$$\begin{aligned} & \sup_{t_1, \dots, t_n \in (0,1)} \left| \sum_{i=1}^n (F_i^{(k)})_+^{-1}(\alpha + (1 - \alpha)t_i) - \sum_{i=1}^n (F_i)_+^{-1}(\alpha + (1 - \alpha)t_i) \right| \\ & \leq \sum_{i=1}^n \sup_{t \in (0,1)} \left| (F_i^{(k)})_+^{-1}(\alpha + (1 - \alpha)t) - (F_i)_+^{-1}(\alpha + (1 - \alpha)t) \right| \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$. Hence,

$$\begin{aligned} & \lim_{k \rightarrow \infty} \sup_{U_1, \dots, U_n \sim U[0,1]} \text{ess-inf} \sum_{i=1}^n (F_i^{(k)})_+^{-1}(\alpha + (1 - \alpha)U_i) \\ & = \sup_{U_1, \dots, U_n \sim U[0,1]} \text{ess-inf} \sum_{i=1}^n (F_i)_+^{-1}(\alpha + (1 - \alpha)U_i). \end{aligned}$$

This completes the proof. □

Proof of Proposition 1. We first focus on $\mathbf{F} \in (\mathcal{M}_{cx}^\alpha)^n$. For $i = 1, \dots, n$, and $k \geq 1$, if $F_i(F_i^{-1}(1)-) < 1$, let

$$F_i^{(k)}(x) = \begin{cases} F_i(x), & x < F_i^{-1}(1) - 1/k \\ F_i(x) + k(1 - F_i(F_i^{-1}(1)-))(x - F_i^{-1}(1) + 1/k), & F_i^{-1}(1) - 1/k \leq x < F_i^{-1}(1) \\ 1, & x \geq F_i^{-1}(1) \end{cases},$$

and if $F_i(F_i^{-1}(1)-) = 1$, let $F_i^{(k)} = F_i$. Clearly, each $F_i^{(k)}$ admits an increasing density beyond its α -quantile. Hence, in light of Theorem 2 of Blanchet *et al.* (2023),

$$\sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F}^{(k)})} \text{VaR}_\alpha^+ \left(\sum_{i=1}^n X_i \right) = \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} \sum_{i=1}^n R_{\gamma_i, \gamma_0}(F_i^{(k)}).$$

Note that $\sup_{t \in (0,1)} |(F_i^{(k)})_+^{-1}(t) - (F_i)_+^{-1}(t)| \leq 1/k$, which implies that $(F_i^{(k)})_+^{-1}(t) \rightarrow (F_i)_+^{-1}(t)$ uniformly on $(0,1)$. Consequently, by Lemma 1, we have

$$\begin{aligned} \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} \text{VaR}_\alpha \left(\sum_{i=1}^n X_i \right) &= \lim_{k \rightarrow \infty} \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F}^{(k)})} \text{VaR}_\alpha \left(\sum_{i=1}^n X_i \right) \\ &= \lim_{k \rightarrow \infty} \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} \sum_{i=1}^n R_{\gamma_i, \gamma_0}(F_i^{(k)}) \\ &= \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} \sum_{i=1}^n R_{\gamma_i, \gamma_0}(F_i), \end{aligned}$$

where the last inequality is implied by the fact that $\sup_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} |R_{\gamma_i, \gamma_0}(F_i^{(k)}) - R_{\gamma_i, \gamma_0}(F_i)| \leq 1/k$. Therefore, the claim holds for $\mathbf{F} \in (\mathcal{M}_{cx}^\alpha)^n$.

We next focus on the case $\mathbf{F} \in (\mathcal{M}_{cv}^\alpha)^n$. For $i = 1, \dots, n$ and $k \geq 1$, if $q_i := F_i((F_i)_+^{-1}(\alpha)) - \alpha > 0$, let

$$F_i^{(k)}(x) = \begin{cases} F_i(x), & x < (F_i)_+^{-1}(\alpha) \\ F_i(x) - q_i + kq_i(x - (F_i)_+^{-1}(\alpha)), & (F_i)_+^{-1}(\alpha) \leq x < (F_i)_+^{-1}(\alpha) + 1/k \\ F_i(x), & x \geq (F_i)_+^{-1}(\alpha) + 1/k \end{cases},$$

and if $q_i = 0$, let $F_i^{(k)} = F_i$. Clearly, each $F_i^{(k)}$ admits a decreasing density beyond its α -quantile. By Theorem 2 of Blanchet *et al.* (2023), we have

$$\sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F}^{(k)})} \text{VaR}_\alpha^+ \left(\sum_{i=1}^n X_i \right) = \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} \sum_{i=1}^n R_{\gamma_i, \gamma_0}(F_i^{(k)}).$$

The rest of the proof is the same as the proof of $\mathbf{F} \in (\mathcal{M}_{cx}^\alpha)^n$. Hence we omit it. $\square$

Proof of Theorem 1. By Lemma 4.5 of Bernard *et al.* (2014), the continuity of $F_1^{-1}, \dots, F_n^{-1}$ over

$(0, 1)$ implies that for $\alpha \in (0, 1)$,

$$\sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} \text{VaR}_\alpha^+(S_n^{\mathbf{f}}(X_1, \dots, X_n)) = \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} \text{VaR}_\alpha(S_n^{\mathbf{f}}(X_1, \dots, X_n)).$$

We first focus on case (i). In light of Theorem 2 of [Blanchet et al. \(2023\)](#), we have

$$\begin{aligned} & \inf_{\mathbf{f} \in \mathcal{D}^n(P)} \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} \text{VaR}_\alpha(S_n^{\mathbf{f}}(X_1, \dots, X_n)) \\ &= \inf_{\mathbf{f} \in \mathcal{D}^n(P)} \inf_{\gamma \in (1-\alpha)\Delta_n} \sum_{i=1}^n \{R_{\gamma_i, \gamma_0}(X_i - f_i(X_i)) + \pi_i(f_i(X_i))\} \\ &= \inf_{\gamma \in (1-\alpha)\Delta_n} \inf_{\mathbf{f} \in \mathcal{D}^n(P)} \sum_{i=1}^n \{R_{\gamma_i, \gamma_0}(X_i - f_i(X_i)) + \pi_i(f_i(X_i))\}. \end{aligned}$$

Next we show that $\mathcal{D}^n(P)$ can be replaced by $\mathcal{L}^n(P)$. Evidently,

$$\inf_{\mathbf{f} \in \mathcal{D}^n(P)} \sum_{i=1}^n \{R_{\gamma_i, \gamma_0}(X_i - f_i(X_i)) + \pi_i(f_i(X_i))\} \leq \inf_{(\mathbf{a}, \mathbf{b}) \in \mathcal{A}(P)} L(\mathbf{a}, \mathbf{b}, \gamma). \quad (7)$$

Next, we show the converse inequality. Let $a_i = F_i^{-1}(1 - \gamma_0 - \gamma_i) - f_i(F_i^{-1}(1 - \gamma_0 - \gamma_i))$ and $b_i \in [\text{VaR}_{1-\gamma_0-\gamma_i}(X_i), \infty]$ be the solution of $g_i(s) = 0$, where

$$g_i(s) := \int_{1-\gamma_0-\gamma_i}^1 l_{a_i, s}(F_i^{-1}(t)) dt - \int_{1-\gamma_0-\gamma_i}^1 f_i(F_i^{-1}(t)) dt.$$

Note that b_i may take the value of ∞ . The existence of b_i is guaranteed by the continuity of g_i over $[a_i, \infty]$ and the fact that $g_i(F_i^{-1}(1 - \gamma_0 - \gamma_i)) \leq 0$ and $g_i(\infty) \geq 0$. Note that there exists some $x_i \geq b_i$ such that $l_{a_i, b_i}(x) \leq f_i(x)$ over $(-\infty, F_i^{-1}(1 - \gamma_0 - \gamma_i)) \cup (x_i, \infty)$ and $l_{a_i, b_i}(x) \geq f_i(x)$ over $(F_i^{-1}(1 - \gamma_0 - \gamma_i), x_i)$. This implies there exists some $t_i \geq 1 - \gamma_0 - \gamma_i$ such that $l_{a_i, b_i}(F_i^{-1}(t)) \leq f_i(F_i^{-1}(t))$ over $(0, 1 - \gamma_0 - \gamma_i) \cup (t_i, 1)$ and $l_{a_i, b_i}(F_i^{-1}(t)) \geq f_i(F_i^{-1}(t))$ over $(1 - \gamma_0 - \gamma_i, t_i)$. Clearly, $\int_s^1 l_{a_i, b_i}(F_i^{-1}(t)) dt \leq \int_s^1 f_i(F_i^{-1}(t)) dt$ for all $s \in [t_i, 1)$. Using $\int_{1-\gamma_0-\gamma_i}^1 l_{a_i, b_i}(F_i^{-1}(t)) dt = \int_{1-\gamma_0-\gamma_i}^1 f_i(F_i^{-1}(t)) dt$, we have

$$\int_s^1 l_{a_i, b_i}(F_i^{-1}(t)) dt \leq \int_s^1 f_i(F_i^{-1}(t)) dt \quad (8)$$

for all $s \in (0, 1)$. The choice of a_i and b_i is to guarantee the validity of (8) and $\int_{1-\gamma_0-\gamma_i}^1 l_{a_i, b_i}(F_i^{-1}(t)) dt = \int_{1-\gamma_0-\gamma_i}^1 f_i(F_i^{-1}(t)) dt$. By Theorem 2.57 in [Föllmer and Schied \(2016\)](#), we have that for any $u \in \mathbb{R}$,

$$\mathbb{E}((l_{a_i, b_i}(X_i) - u)_+) \leq \mathbb{E}((f_i(X_i) - u)_+).$$

Hence it follows from the property (iii) of π_i that $\pi_i(l_{a_i, b_i}(X_i)) \leq \pi_i(f_i(X_i))$. Moreover, combination of (8)

and the fact that $g_i(b_i) = 0$ yields

$$R_{\gamma_i, \gamma_0}(l_{a_i, b_i}(X_i)) \geq R_{\gamma_i, \gamma_0}(f_i(X_i)).$$

Consequently,

$$\sum_{i=1}^n \{R_{\gamma_i, \gamma_0}(X_i - f_i(X_i)) + \pi_i(f_i(X_i))\} \geq L(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma}).$$

We next verify $(\mathbf{a}, \mathbf{b}) \in \mathcal{A}(P)$ for our constructed $(\mathbf{a}, \mathbf{b})$. Using the fact that $\mathbf{f} \in \mathcal{D}^n(P)$, we have

$$\sum_{i=1}^n \pi_i(l_{a_i, b_i}(X_i)) \leq \sum_{i=1}^n \pi_i(f_i(X_i)) \leq P.$$

Hence, for any $\mathbf{f} \in \mathcal{D}^n(P)$, there exists $(\mathbf{a}, \mathbf{b}) \in \mathcal{A}(P)$ such that

$$\sum_{i=1}^n \{R_{\gamma_i, \gamma_0}(X_i - f_i(X_i)) + \pi_i(f_i(X_i))\} \geq L(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma}),$$

which combined with (7) implies that (7) is an equality. This completes the proof of (i).

We next consider case (ii). Note that for $\mathbf{f} \in \mathcal{D}_1^n(P)$, we have $R_{f_i}(0) = 0$, and $R_{f_i}(x)$ is a concave, continuous and increasing function on $[0, \infty)$. If $R_{f_i} = 0$, the distribution of $R_{f_i}(X_i)$ is convex beyond its α -quantile. Next we suppose $R_{f_i} \neq 0$. Direct computation gives

$$\mathbb{P}(R_{f_i}(X_i) \leq x) = \mathbb{P}(X_i \leq R_{f_i}^{-1}(x)) = F_i(R_{f_i}^{-1}(x)), \quad x \geq 0,$$

where $R_{f_i}^{-1}(x) = \sup\{t \geq 0 : R_{f_i}(t) \leq x\}$ for $x \geq 0$. Note that $R_{f_i}^{-1}$ is convex on $[0, R_{f_i}(\infty)]$. Hence $\frac{(F_i(R_{f_i}^{-1}(x)) - \alpha)_+}{1 - \alpha}$ is convex on $(-\infty, R_{f_i}(F_i^{-1}(1)))$. Hence in light of Proposition 1,

$$\begin{aligned} & \inf_{\mathbf{f} \in \mathcal{D}_1^n(P)} \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{f})} \text{VaR}_\alpha(S_n^{\mathbf{f}}(X_1, \dots, X_n)) \\ &= \inf_{\mathbf{f} \in \mathcal{D}_1^n(P)} \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} \sum_{i=1}^n \{R_{\gamma_i, \gamma_0}(X_i - f_i(X_i)) + \pi_i(f_i(X_i))\} \\ &= \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} \inf_{\mathbf{f} \in \mathcal{D}_1^n(P)} \sum_{i=1}^n \{R_{\gamma_i, \gamma_0}(X_i - f_i(X_i)) + \pi_i(f_i(X_i))\}. \end{aligned}$$

Next we show that $\mathcal{D}_1^n(P)$ can be replaced by $\mathcal{H}^n(P)$. We will only show

$$\inf_{\mathbf{f} \in \mathcal{D}_1^n(P)} \sum_{i=1}^n \{R_{\gamma_i, \gamma_0}(X_i - f_i(X_i)) + \pi_i(f_i(X_i))\} \geq \inf_{(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \in \mathcal{A}_1(P)} H(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \boldsymbol{\gamma}). \quad (9)$$

as the inverse inequality is trivial. Let

$$a_i = \begin{cases} F_i^{-1}(1 - \gamma_0 - \gamma_i) - \frac{f_i(F_i^{-1}(1 - \gamma_0 - \gamma_i))}{(f_i)'_+(F_i^{-1}(1 - \gamma_0 - \gamma_i))}, & (f_i)'_+(F_i^{-1}(1 - \gamma_0 - \gamma_i)) \neq 0 \\ F_i^{-1}(1 - \gamma_0 - \gamma_i), & (f_i)'_+(F_i^{-1}(1 - \gamma_0 - \gamma_i)) = 0 \end{cases},$$

$b_i = f_i(F_i^{-1}(1 - \gamma_0 - \gamma_i))$, $c_i = (f_i)'_+(F_i^{-1}(1 - \gamma_0 - \gamma_i))$, and $d_i \in [0, 1 - c_i]$ be the solution of $g_i(s) = 0$, where

$$g_i(s) := \int_{1-\gamma_0-\gamma_i}^1 (s + c_i)(F_i^{-1}(t) - F_i^{-1}(1 - \gamma_0 - \gamma_i)) + F_i^{-1}(1 - \gamma_0 - \gamma_i) dt - \int_{1-\gamma_0-\gamma_i}^1 f_i(F_i^{-1}(t)) dt.$$

Note that g_i is continuous on $[0, 1]$ with $g_i(1 - c_i) \geq 0$ and $g_i(0) \leq 0$. Hence the existence of d_i is guaranteed. The existence of b_i is guaranteed by the continuity of g_i over $[a_i, \infty]$ and the fact that $g_i(F_i^{-1}(1 - \gamma_0 - \gamma_i)) \leq 0$ and $g_i(\infty) \geq 0$. Note that there exists some $x_i \geq b_i$ such that $h_{a_i, b_i, c_i, d_i}(x) \leq f_i(x)$ over $(-\infty, F_i^{-1}(1 - \gamma_0 - \gamma_i)) \cup (x_i, \infty)$ and $h_{a_i, b_i, c_i, d_i}(x) \geq f_i(x)$ over $(F_i^{-1}(1 - \gamma_0 - \gamma_i), x_i)$. This implies there exists some $t_i \geq 1 - \gamma_0 - \gamma_i$ such that $h_{a_i, b_i, c_i, d_i}(F_i^{-1}(t)) \leq f_i(F_i^{-1}(t))$ over $(0, 1 - \gamma_0 - \gamma_i) \cup (t_i, 1)$ and $h_{a_i, b_i, c_i, d_i}(F_i^{-1}(t)) \geq f_i(F_i^{-1}(t))$ over $(1 - \gamma_0 - \gamma_i, t_i)$. Using $\int_{1-\gamma_0-\gamma_i}^1 h_{a_i, b_i, c_i, d_i}(F_i^{-1}(t)) dt = \int_{1-\gamma_0-\gamma_i}^1 f_i(F_i^{-1}(t)) dt$, we have

$$\int_s^1 h_{a_i, b_i, c_i, d_i}(F_i^{-1}(t)) dt \leq \int_s^1 f_i(F_i^{-1}(t)) dt \quad (10)$$

for all $s \in (0, 1)$. The choice of a_i, b_i, c_i, d_i is to ensure that (10) holds and $\int_{1-\gamma_0-\gamma_i}^1 h_{a_i, b_i, c_i, d_i}(F_i^{-1}(t)) dt = \int_{1-\gamma_0-\gamma_i}^1 f_i(F_i^{-1}(t)) dt$. For the rest of the proof, using the same argument as in the proof of (i), we can show that (9) holds. This completes the proof of (ii).

Finally, we focus on case (iii). Note that for $\mathbf{f} \in \mathcal{D}_2^n(P)$, we have $R_{f_i}(0) = 0$, and $R_{f_i}(x)$ is a convex, continuous and increasing function on $[0, \infty)$. Note that the distribution of $R_{f_i}(X_i)$ is concave beyond its α -quantile if $R_{f_i} = 0$. We next suppose $R_{f_i} \neq 0$. Direct computation gives

$$\mathbb{P}(R_{f_i}(X_i) \leq x) = \mathbb{P}(X_i \leq R_{f_i}^{-1}(x)) = F_i(R_{f_i}^{-1}(x)), \quad x \geq 0,$$

where $R_{f_i}^{-1}(x) = \sup\{t \geq 0 : R_{f_i}(t) \leq x\}$ for $x \geq 0$. Note that $R_{f_i}^{-1}$ is concave and continuous on $[0, \infty)$. Hence $\frac{(F_i(R_{f_i}^{-1}(x)) - \alpha)_+}{1 - \alpha}$ is concave on its support. Hence in light of Proposition 1,

$$\begin{aligned} & \inf_{\mathbf{f} \in \mathcal{D}_2^n(P)} \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} \text{VaR}_\alpha(S_n^{\mathbf{f}}(X_1, \dots, X_n)) \\ &= \inf_{\mathbf{f} \in \mathcal{D}_2^n(P)} \inf_{\gamma \in (1-\alpha)\Delta_n} \sum_{i=1}^n \{R_{\gamma_i, \gamma_0}(X_i - f_i(X_i)) + \pi_i(f_i(X_i))\} \\ &= \inf_{\gamma \in (1-\alpha)\Delta_n} \inf_{\mathbf{f} \in \mathcal{D}_2^n(P)} \sum_{i=1}^n \{R_{\gamma_i, \gamma_0}(X_i - f_i(X_i)) + \pi_i(f_i(X_i))\}. \end{aligned}$$

Next we show that $\mathcal{D}_2^n(P)$ can be replaced by $\mathcal{G}^n(P)$. Analogously as case (ii), we will only show

$$\inf_{\mathbf{f} \in \mathcal{D}_2^n(P)} \sum_{i=1}^n \{R_{\gamma_i, \gamma_0}(X_i - f_i(X_i)) + \pi_i(f_i(X_i))\} \geq \inf_{(\mathbf{a}, \mathbf{b}) \in \mathcal{A}_2(P)} G(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma}). \quad (11)$$

Let

$$a_i = \begin{cases} \frac{f_i(F_i^{-1}(1-\gamma_0-\gamma_i))}{F_i^{-1}(1-\gamma_0-\gamma_i)}, & F_i^{-1}(1-\gamma_0-\gamma_i) \neq 0 \\ 1, & F_i^{-1}(1-\gamma_0-\gamma_i) = 0 \end{cases},$$

and $b_i \in [F_i^{-1}(1-\gamma_0-\gamma_i), \infty]$ be the solution of $g_i(s) = 0$, where

$$g_i(s) := \int_{1-\gamma_0-\gamma_i}^1 g_{a_i, s}(F_i^{-1}(t)) dt - \int_{1-\gamma_0-\gamma_i}^1 f_i(F_i^{-1}(t)) dt.$$

For the rest of the proof, using the same argument as in the proof of (i), we can show that (11) holds. The inverse inequality of (11) is trivial. This establishes the claim of case (iii). $\square$

Proof of Proposition 2. It is clear that $\mathbf{l}_{\mathbf{a}^*, \mathbf{b}^*}$, $\mathbf{h}_{\mathbf{a}^*, \mathbf{b}^*, \mathbf{c}^*, \mathbf{d}^*}$ and $\mathbf{g}_{\mathbf{a}^*, \mathbf{b}^*}$ are the optimal ceded loss functions for cases (i)-(iii), respectively. We next focus on the existence of these optimal parameters under the continuity of π_1 . We only prove the existence of the optimal parameters for case (i) as other two cases follow in a similar way. First note that

$$R_{\gamma_i, \gamma_0}(X_i - l_{a_i, b_i}(X_i)) + \pi_i(l_{a_i, b_i}(X_i)) = R_{\gamma_i, \gamma_0}(X_i) - R_{\gamma_i, \gamma_0}(l_{a_i, b_i}(X_i)) + \pi_i(l_{a_i, b_i}(X_i)).$$

For $0 \leq \varepsilon \leq T \leq 1$, define

$$\overline{\Delta}_n^{\varepsilon, T} = \left\{ (\gamma_0, \gamma_1, \dots, \gamma_n) \in [\varepsilon, T] \times [0, T]^n : \sum_{i=0}^n \gamma_i = T \right\}. \quad (12)$$

Clearly, for $0 < \varepsilon < 1 - \alpha$, $R_{\gamma_i, \gamma_0}(X_i)$ is continuous with respect to $\boldsymbol{\gamma}$ over $\overline{\Delta}_n^{\varepsilon, 1-\alpha}$ and

$$R_{\gamma_i, \gamma_0}(l_{a_i, b_i}(X_i)) = \frac{1}{\gamma_0} \int_{\gamma_i}^{\gamma_i + \gamma_0} l_{a_i, b_i}(F_i^{-1}(t)) dt$$

is continuous with respect to $(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma})$ over $\mathcal{A}(P) \times \overline{\Delta}_n^{\varepsilon, 1-\alpha}$. Hence, by the continuity of π_i , $L(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma})$ is a continuous function of $(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma})$ over $\mathcal{A}(P) \times \overline{\Delta}_n^{\varepsilon, 1-\alpha}$ for $\varepsilon > 0$ and $\mathcal{A}(P)$ is a closed set. Therefore there exists $(\mathbf{a}_\varepsilon^*, \mathbf{b}_\varepsilon^*, \boldsymbol{\gamma}_\varepsilon^*) \in \mathcal{A}(P) \times \overline{\Delta}_n^{\varepsilon, 1-\alpha}$ such that

$$(\mathbf{a}_\varepsilon^*, \mathbf{b}_\varepsilon^*, \boldsymbol{\gamma}_\varepsilon^*) \in \arg \inf_{(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma}) \in \mathcal{A}(P) \times \overline{\Delta}_n^{\varepsilon, 1-\alpha}} L(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma}).$$

Let $n_0 \in \mathbb{N}$ such that $n_0 \geq \frac{1}{1-\alpha}$. We next discuss the limit of $(\mathbf{a}_{1/k}^*, \mathbf{b}_{1/k}^*, \boldsymbol{\gamma}_{1/k}^*), k \geq n_0$. To this end, we need

to extend the domain of L . Note that

$$\begin{aligned} \lim_{\gamma_0 \downarrow 0} R_{\gamma_i, \gamma_0}(X_i - l_{a_i, b_i}(X_i)) &= \lim_{\gamma_0 \downarrow 0} \frac{1}{\gamma_0} \int_{\gamma_i}^{\gamma_i + \gamma_0} \text{VaR}_{1-t}(X_i - l_{a_i, b_i}(X_i)) dt \\ &= \text{VaR}_{1-\gamma_i}(X_i - l_{a_i, b_i}(X_i)). \end{aligned}$$

We define $R_{\gamma_i, 0}(X_i - l_{a_i, b_i}(X_i)) = \text{VaR}_{1-\gamma_i}(X_i - l_{a_i, b_i}(X_i))$. Notice that $R_{0,0}(X_i - l_{a_i, b_i}(X_i)) = \infty$ for $b_i < \infty$ and $\text{ess-sup} X_i = \infty$. Then $R_{\gamma_i, \gamma_0}(X_i - l_{a_i, b_i}(X_i))$ is a continuous function of $(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma})$ over $\mathcal{A}(P) \times \overline{\Delta}_n^{0, 1-\alpha}$ if we allow $R_{0,0}(X_i - l_{a_i, b_i}(X_i)) = \infty$ in the above scenario. It further implies that $L(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma})$ is a continuous function of $(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma})$ over $\mathcal{A}(P) \times \overline{\Delta}_n^{0, 1-\alpha}$ if we allow $L(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma}) = \infty$ in the above mentioned scenario. There exists a subsequence $(\mathbf{a}_{1/k_l}^*, \mathbf{b}_{1/k_l}^*, \boldsymbol{\gamma}_{1/k_l}^*), l \geq n_0$ such that $(\mathbf{a}_{1/k_l}^*, \mathbf{b}_{1/k_l}^*, \boldsymbol{\gamma}_{1/k_l}^*)$ converges to $(\mathbf{a}^*, \mathbf{b}^*, \boldsymbol{\gamma}^*) \in \mathcal{A}(P) \times \overline{\Delta}_n^{0, 1-\alpha}$ as $l \rightarrow \infty$. By the continuity of L ,

$$L(\mathbf{a}^*, \mathbf{b}^*, \boldsymbol{\gamma}^*) = \lim_{l \rightarrow \infty} L(\mathbf{a}_{1/k_l}^*, \mathbf{b}_{1/k_l}^*, \boldsymbol{\gamma}_{1/k_l}^*) = \inf_{(\mathbf{a}, \mathbf{b}) \in \mathcal{A}, \boldsymbol{\gamma} \in (1-\alpha)\Delta_n} L(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma}) = \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} L(\mathbf{a}^*, \mathbf{b}^*, \boldsymbol{\gamma}).$$

This implies

$$(\mathbf{a}^*, \mathbf{b}^*) \in \arg \inf_{(\mathbf{a}, \mathbf{b}) \in \mathcal{A}(\mathbf{P})} \left\{ \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} L(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma}) \right\}.$$

Hence we establish the existence of the optimal parameters for case (i). $\square$

Proof of Proposition 3. By the result in Makarov (1981) and the continuity of F_1^{-1} and F_2^{-1} , we have

$$\sup_{(X_1, X_2) \in \mathcal{E}_2(\mathbf{F})} \text{VaR}_\alpha(S_2^f(X_1, X_2)) = \inf_{t \in [0, 1-\alpha]} \{R_{f_1}(\text{VaR}_{\alpha+t}(X_1)) + R_{f_2}(\text{VaR}_{1-t}(X_2)) + \pi_1(f_1(X_1)) + \pi_2(f_2(X_2))\}.$$

Hence

$$\begin{aligned} &\inf_{(f_1, f_2) \in \mathcal{D}^2(P)} \sup_{(X_1, X_2) \in \mathcal{E}_2(\mathbf{F})} \text{VaR}_\alpha(S_2^f(X_1, X_2)) \\ &= \inf_{t \in [0, 1-\alpha]} \inf_{(f_1, f_2) \in \mathcal{D}^2(P)} \{\text{VaR}_{\alpha+t}(X_1 - f_1(X_1)) + \text{VaR}_{1-t}(X_2 - f_2(X_2)) + \pi_1(f_1(X_1)) + \pi_2(f_2(X_2))\}. \end{aligned}$$

Let $a_1 = F_1^{-1}(\alpha+t) - f_1(F_1^{-1}(\alpha+t))$ and $a_2 = F_2^{-1}(1-t) - f_2(F_2^{-1}(1-t))$. Moreover, let $b_1 \in [F_1^{-1}(\alpha+t), \infty]$ be the solution of $g_1(s) = 0$, where $g_1(s) = \int_{\alpha+t}^1 l_{a_1, s}(F_1^{-1}(u)) du - \int_{\alpha+t}^1 f_1(F_1^{-1}(u)) du$ and $b_2 \in [F_2^{-1}(1-t), \infty]$ be the solution of $g_2(s) = 0$, where $g_2(s) = \int_{1-t}^1 l_{a_2, s}(F_2^{-1}(u)) du - \int_{1-t}^1 f_2(F_2^{-1}(u)) du$. Using the above a_i, b_i we observe that $\text{VaR}_{\alpha+t}(X_1 - f_1(X_1)) = \text{VaR}_{\alpha+t}(X_1 - l_{a_1, b_1}(X_1))$ and $\text{VaR}_{1-t}(X_2 - f_2(X_2)) = \text{VaR}_{1-t}(X_2 - l_{a_2, b_2}(X_2))$, and for $s \in (0, 1)$,

$$\int_s^1 l_{a_i, b_i}(F_i^{-1}(t)) dt \leq \int_s^1 f_i(F_i^{-1}(t)) dt, \quad i = 1, 2,$$

which, with the help of Theorem 2.57 in Föllmer and Schied (2016) and property (iii) of π_i , implies that

for $i = 1, 2$, $\pi_i(l_{a_i, b_i}(X_i)) \leq \pi_i(f_i(X_i))$. Using the similar argument as in the proof of Theorem 1, we can establish the claim.

9.2 Proofs in Section 4.

To prove Proposition 4, we need the following lemma.

Lemma 2. For $\mathbf{F}^{(k)} = (F_1^{(k)}, \dots, F_n^{(k)})$, $k \geq 1$ and $\mathbf{F} = (F_1, \dots, F_n)$, if for each i , $(F_i^{(k)})^{-1}(t) \rightarrow F_i^{-1}(t)$ uniformly for $t \in (1 - \beta - \alpha, 1)$ with $0 \leq \beta < \beta + \alpha \leq 1$, then

$$\lim_{k \rightarrow \infty} \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F}^{(k)})} R_{\beta, \alpha} \left(\sum_{i=1}^n X_i \right) = \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} R_{\beta, \alpha} \left(\sum_{i=1}^n X_i \right).$$

Proof. First note that

$$\sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F}^{(k)})} R_{\beta, \alpha} \left(\sum_{i=1}^n X_i \right) = \sup_{U_1, \dots, U_n \sim U[0,1]} R_{\beta/(\beta+\alpha), \alpha/(\beta+\alpha)} \left(\sum_{i=1}^n (F_i^{(k)})^{-1}(1 - \beta - \alpha + (\beta + \alpha)U_i) \right)$$

and

$$\sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} R_{\beta, \alpha} \left(\sum_{i=1}^n X_i \right) = \sup_{U_1, \dots, U_n \sim U[0,1]} R_{\beta/(\beta+\alpha), \alpha/(\beta+\alpha)} \left(\sum_{i=1}^n (F_i)^{-1}(1 - \beta - \alpha + (\beta + \alpha)U_i) \right).$$

Using the fact

$$\begin{aligned} u_k &:= \sup_{x_1, \dots, x_n \in (0,1)} \left| \sum_{i=1}^n (F_i^{(k)})^{-1}(1 - \beta - \alpha + (\beta + \alpha)x_i) - \sum_{i=1}^n (F_i)^{-1}(1 - \beta - \alpha + (\beta + \alpha)x_i) \right| \\ &\leq \sup_{x_1, \dots, x_n \in (0,1)} \sum_{i=1}^n \left| (F_i^{(k)})^{-1}(1 - \beta - \alpha + (\beta + \alpha)x_i) - (F_i)^{-1}(1 - \beta - \alpha + (\beta + \alpha)x_i) \right| \\ &\rightarrow 0 \text{ as } k \rightarrow \infty, \end{aligned}$$

we have

$$\lim_{k \rightarrow \infty} \left| \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F}^{(k)})} R_{\beta, \alpha} \left(\sum_{i=1}^n X_i \right) - \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} R_{\beta, \alpha} \left(\sum_{i=1}^n X_i \right) \right| \leq \lim_{k \rightarrow \infty} u_k = 0.$$

This completes the proof. $\square$

Proof of Proposition 4. For $i = 1, \dots, n$ and $k \geq 1$, if $q_i := F_i((F_i)_+^{-1}(1 - \beta - \alpha)) - (1 - \beta - \alpha) > 0$, let

$$F_i^{(k)}(x) = \begin{cases} F_i(x), & x < (F_i)_+^{-1}(1 - \beta - \alpha) \\ F_i(x) - q_i + kq_i(x - (F_i)_+^{-1}(1 - \beta - \alpha)), & (F_i)_+^{-1}(1 - \beta - \alpha) \leq x < (F_i)_+^{-1}(1 - \beta - \alpha) + 1/k \\ F_i(x), & x \geq (F_i)_+^{-1}(1 - \beta - \alpha) + 1/k \end{cases},$$

and if $q_i = 0$, let $F_i^{(k)} = F_i$. Clearly, each $F_i^{(k)}$ admits a decreasing density beyond its $1 - \beta - \alpha$ -quantile. Hence, in light of Theorem 1 of [Blanchet et al. \(2023\)](#), we have

$$\sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F}^{(k)})} R_{\beta, \alpha} \left(\sum_{i=1}^n X_i \right) = \inf_{\gamma \in (\beta + \alpha)\Delta_n, \gamma_0 \geq \alpha} \sum_{i=1}^n R_{\gamma_i, \gamma_0}(F_i^{(k)}).$$

Moreover, by the construction of $F_i^{(k)}$, it follows that $\sup_{t \in (1 - \beta - \alpha, 1)} |(F_i^{(k)})^{-1}(t) - F_i^{-1}(t)| \leq 1/k \rightarrow 0$ as $k \rightarrow \infty$. Hence, by Lemma 2, we have

$$\lim_{k \rightarrow \infty} \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F}^{(k)})} R_{\beta, \alpha} \left(\sum_{i=1}^n X_i \right) = \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} R_{\beta, \alpha} \left(\sum_{i=1}^n X_i \right).$$

One can easily check

$$\lim_{k \rightarrow \infty} \inf_{\gamma \in (\beta + \alpha)\Delta_n, \gamma_0 \geq \alpha} \sum_{i=1}^n R_{\gamma_i, \gamma_0}(F_i^{(k)}) = \inf_{\gamma \in (\beta + \alpha)\Delta_n, \gamma_0 \geq \alpha} \sum_{i=1}^n R_{\gamma_i, \gamma_0}(F_i).$$

Combination of the above results establishes the claim. $\square$

Proof of Theorem 2. We first show case (i). By Theorem 1 of [Blanchet et al. \(2023\)](#), we have

$$\begin{aligned} & \inf_{\mathbf{f} \in \mathcal{D}^n(P)} \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} R_{\beta, \alpha} (S_n^{\mathbf{f}}(X_1, \dots, X_n)) \\ &= \inf_{\mathbf{f} \in \mathcal{D}^n(P)} \inf_{\gamma \in (\beta + \alpha)\Delta_n, \gamma_0 \geq \alpha} \sum_{i=1}^n \{R_{\gamma_i, \gamma_0}(X_i - f_i(X_i)) + \pi_i(f_i(X_i))\} \\ &= \inf_{\gamma \in (\beta + \alpha)\Delta_n, \gamma_0 \geq \alpha} \inf_{\mathbf{f} \in \mathcal{D}^n(P)} \sum_{i=1}^n \{R_{\gamma_i, \gamma_0}(X_i - f_i(X_i)) + \pi_i(f_i(X_i))\}. \end{aligned}$$

Analogously as in the proof of Theorem 1, we can show

$$\inf_{\mathbf{f} \in \mathcal{D}^n(P)} \sum_{i=1}^n \{R_{\gamma_i, \gamma_0}(X_i - f_i(X_i)) + \pi_i(f_i(X_i))\} = \inf_{(\mathbf{a}, \mathbf{b}) \in \mathcal{A}(P)} L(\mathbf{a}, \mathbf{b}, \gamma).$$

This completes the proof of (i).

For the proof of case (ii), it follows from Proposition 4 that

$$\begin{aligned} & \inf_{\mathbf{f} \in \mathcal{D}_2^n(P)} \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} R_{\beta, \alpha} (S_n^{\mathbf{f}}(X_1, \dots, X_n)) \\ &= \inf_{\mathbf{f} \in \mathcal{D}_2^n(P)} \inf_{\gamma \in (\beta + \alpha)\Delta_n, \gamma_0 \geq \alpha} \sum_{i=1}^n \{R_{\gamma_i, \gamma_0}(X_i - f_i(X_i)) + \pi_i(f_i(X_i))\} \\ &= \inf_{\gamma \in (\beta + \alpha)\Delta_n, \gamma_0 \geq \alpha} \inf_{\mathbf{f} \in \mathcal{D}_2^n(P)} \sum_{i=1}^n \{R_{\gamma_i, \gamma_0}(X_i - f_i(X_i)) + \pi_i(f_i(X_i))\}. \end{aligned}$$

Analogously as in the proof of Theorem 1, we have

$$\inf_{\mathbf{f} \in \mathcal{D}_2^n(P)} \sum_{i=1}^n \{R_{\gamma_i, \gamma_0}(X_i - f_i(X_i)) + \pi_i(f_i(X_i))\} = \inf_{(\mathbf{a}, \mathbf{b}) \in \mathcal{A}_2(P)} G(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma}),$$

which completes the proof of case (ii). $\square$

Proof of Proposition 5 It is clear that $\mathbf{l}_{\mathbf{a}^*, \mathbf{b}^*}$ and $\mathbf{g}_{\mathbf{a}^*, \mathbf{b}^*}$ are the optimal ceded loss functions for cases (i)-(ii), respectively. We next focus on the existence of the optimal parameters for case (i). First note

$$R_{\gamma_i, \gamma_0}(X_i - l_{a_i, b_i}(X_i)) + \pi_i(l_{a_i, b_i}(X_i)) = R_{\gamma_i, \gamma_0}(X_i) - R_{\gamma_i, \gamma_0}(l_{a_i, b_i}(X_i)) + \pi_i(l_{a_i, b_i}(X_i)).$$

Clearly, $R_{\gamma_i, \gamma_0}(X_i)$ is continuous with respect to $\boldsymbol{\gamma}$ over $\overline{\Delta}_n^{\beta+\alpha}$ and

$$R_{\gamma_i, \gamma_0}(l_{a_i, b_i}(X_i)) = \frac{1}{\gamma_0} \int_{\gamma_i}^{\gamma_i + \gamma_0} l_{a_i, b_i}(\text{VaR}_{1-t}(X_i)) dt$$

is continuous with respect to $(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma})$ over $\mathcal{A}(P) \times \overline{\Delta}_n^{\alpha, \beta+\alpha}$, where the definition of $\overline{\Delta}_n^{\alpha, \beta+\alpha}$ is given in (12). Hence $L(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma})$ is continuous over $\mathcal{A}(P) \times \overline{\Delta}_n^{\alpha, \beta+\alpha}$. Therefore there exists $(\mathbf{a}^*, \mathbf{b}^*, \boldsymbol{\gamma}^*) \in \mathcal{A}(P) \times \overline{\Delta}_n^{\alpha, \beta+\alpha}$ such that

$$L(\mathbf{a}^*, \mathbf{b}^*, \boldsymbol{\gamma}^*) = \inf_{(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma}) \in \mathcal{A}(P) \times \overline{\Delta}_n^{\alpha, \beta+\alpha}} L(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma}) = \inf_{\boldsymbol{\gamma} \in \overline{\Delta}_n^{\alpha, \beta+\alpha}} L(\mathbf{a}^*, \mathbf{b}^*, \boldsymbol{\gamma}).$$

Hence the existence of $(\mathbf{a}^*, \mathbf{b}^*)$ for case (i) is established. The proof for case (ii) is similar to case (i) and hence we omit it. $\square$

9.3 Proof of Section 5.

Proof of Propositions 6-7. By Proposition 1, we have

$$\begin{aligned} & \inf_{\mathbf{w} \in \Lambda_n(P)} \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} \text{VaR}_\alpha(S^{\mathbf{w}}(X_1, \dots, X_n)) \\ &= \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} \inf_{\mathbf{w} \in \Lambda_n(P)} \sum_{i=1}^n ((1-w_i)R_{\gamma_i, \gamma_0}(X_i) + w_i\pi_i(X_i)). \end{aligned}$$

Note that $\sum_{i=1}^n ((1-w_i)R_{\gamma_i, \gamma_0}(X_i) + w_i\pi_i(X_i))$ is a linear function for each w_i . Hence for each $\boldsymbol{\gamma} \in (1-\alpha)\Delta_n$, we have

$$\inf_{\mathbf{w} \in \Lambda_n(P)} \sum_{i=1}^n ((1-w_i)R_{\gamma_i, \gamma_0}(X_i) + w_i\pi_i(X_i)) = \inf_{\mathbf{w} \in \partial\Lambda_n(P)} \sum_{i=1}^n ((1-w_i)R_{\gamma_i, \gamma_0}(X_i) + w_i\pi_i(X_i)),$$

which implies

$$\begin{aligned}
& \inf_{\mathbf{w} \in \Lambda_n(P)} \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} \text{VaR}_\alpha(S^{\mathbf{w}}(X_1, \dots, X_n)) \\
&= \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} \inf_{\mathbf{w} \in \partial\Lambda_n(P)} \sum_{i=1}^n ((1-w_i)R_{\gamma_i, \gamma_0}(X_i) + w_i\pi_i(X_i)) \\
&= \inf_{\mathbf{w} \in \partial\Lambda_n(P)} \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} \sum_{i=1}^n ((1-w_i)R_{\gamma_i, \gamma_0}(X_i) + w_i\pi_i(X_i)).
\end{aligned}$$

Note that $\inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} \sum_{i=1}^n ((1-w_i)R_{\gamma_i, \gamma_0}(X_i) + w_i\pi_i(X_i))$ is a continuous function with respect to $\mathbf{w}$ and $\partial\Lambda_n(P)$ is compact. Hence there exists optimal quota-share levels $\mathbf{w}^* \in \partial\Lambda_n(P)$. This completes the proof of Proposition 6. Analogously, we can prove Proposition 7. $\square$

Proof of Corollary 1. By Proposition 6, we have

$$\begin{aligned}
& \inf_{\mathbf{w} \in [0,1]^n} \sup_{(X_1, \dots, X_n) \in \mathcal{E}_n(\mathbf{F})} \text{VaR}_\alpha(S^{\mathbf{w}}(X_1, \dots, X_n)) \\
&= \inf_{\mathbf{w} \in [0,1]^n} \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} \sum_{i=1}^n ((1-w_i)R_{\gamma_i, \gamma_0}(X_i) + w_i\pi_i(X_i)) \\
&= \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} \inf_{\mathbf{w} \in [0,1]^n} \sum_{i=1}^n ((1-w_i)R_{\gamma_i, \gamma_0}(X_i) + w_i\pi_i(X_i)) \\
&= \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} \inf_{\mathbf{w} \in \Lambda_n^0} \sum_{i=1}^n ((1-w_i)R_{\gamma_i, \gamma_0}(X_i) + w_i\pi_i(X_i)) \\
&= \inf_{\mathbf{w} \in \Lambda_n^0} \left\{ \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} \sum_{i=1}^n ((1-w_i)R_{\gamma_i, \gamma_0}(X_i) + w_i\pi_i(X_i)) \right\}.
\end{aligned}$$

Hence there exist optimal quota-share levels $\mathbf{w}^* \in \Lambda_n^0$. We can similarly show the same conclusion for Proposition 7. $\square$

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10 Supplementary materials

In order to show Proposition 8, we need the following result.

Proposition 9. Recall that $\pi_i(\cdot)$ for $i = 1, 2$ are assumed to satisfy the axioms of distribution invariance, risk loading and stop-loss ordering preserving. If $\mathcal{A}(P) \neq \emptyset$, for any $t \in [0, 1 - \alpha]$, the followings are true:

$$(i) \quad \pi_1(l_{u_1, v_1}(X_1)) + \pi_2(l_{u_2, v_2}(X_2)) \leq \pi_1(l_{a_1, b_1}(X_1)) + \pi_2(l_{a_2, b_2}(X_2));$$

$$(iii) \quad (a_1, a_2, b_1, b_2) \in \mathcal{A}(P);$$

$$(iii) \quad L_1(u_1, u_2, v_1, v_2, t) \leq L_1(a_1, a_2, b_1, b_2, t), \quad \forall (a_1, a_2, b_1, b_2) \in \mathcal{A}(P).$$

where $v_1 = \text{VaR}_{\alpha+t}(X_1)$, $v_2 = \text{VaR}_{1-t}(X_2)$, $u_1 = v_1 - l_{a_1, b_1}(v_1)$ and $u_2 = v_2 - l_{a_2, b_2}(v_2)$.

Furthermore, we have

$$\min_{\substack{(a_1, a_2, b_1, b_2) \\ \in \mathcal{A}(P)}} \min_{t \in [0, 1 - \alpha]} L_1(a_1, a_2, b_1, b_2, t) = \min_{\substack{(u_1, u_2) \\ \in \mathcal{A}_v(P, v_1, v_2)}} \min_{t \in [0, 1 - \alpha]} L_2(u_1, u_2, t),$$

where $L_2(u_1, u_2, t) = \sum_{i=1}^2 \left(u_i + \pi_i(l_{u_i, v_i}(X_i)) \right)$ and

$$\mathcal{A}_v(P, v_1, v_2) := \{(u_1, u_2) : (l_{u_1, v_1}, l_{u_2, v_2}) \in \mathcal{D}^2(P), \ 0 \leq u_i \leq v_i, \ i = 1, 2\}.$$

Proof of Proposition 9. By applying the Theorem 3.1 in Chi and Tan (2013), there exists admissible limited stop-loss function $l_{u_1, v_1}(X_1)$ and $l_{u_2, v_2}(X_2)$ such that, for any $t \in [0, 1 - \alpha]$,

$$\text{VaR}_{\alpha+t}(X_1 - l_{u_1, v_1}(X_1)) + \pi_1(l_{u_1, v_1}(X_1)) \leq \text{VaR}_{\alpha+t}(X_1 - l_{a_1, b_1}(X_1)) + \pi_1(l_{a_1, b_1}(X_1)), \quad \forall (a_1, b_1) \in \mathcal{A}(P),$$

and

$$\text{VaR}_{1-t}(X_1 - l_{u_1, v_1}(X_1)) + \pi_1(l_{u_1, v_1}(X_1)) \leq \text{VaR}_{1-t}(X_1 - l_{a_1, b_1}(X_1)) + \pi_1(l_{a_1, b_1}(X_1)), \quad \forall (a_1, b_1) \in \mathcal{A}(P).$$

As a result, for any $t \in [0, 1 - \alpha]$,

$$L_1(u_1, u_2, v_1, v_2, t) \leq L_1(a_1, a_2, b_1, b_2, t) \quad \forall (a_1, a_2, b_1, b_2) \in \mathcal{A}(P). \quad (13)$$

Furthermore, for any given $\hat{t} \in [0, 1 - \alpha]$, the Theorem 3.1 in [Chi and Tan \(2013\)](#) also implies

$$\begin{aligned} \min_{\substack{(a_1, a_2, b_1, b_2) \\ \in \mathcal{A}(P)}}} L_1(a_1, a_2, b_1, b_2, \hat{t}) &= \min_{\substack{(u_1, u_2, v_1, v_2) \\ \in \mathcal{A}(P)}}} L_1(u_1, u_2, v_1, v_2, \hat{t}) \\ &= \min_{\substack{(u_1, u_2) \\ \in \mathcal{A}_v(P, v_1, v_2)}}} L_2(u_1, u_2, \hat{t}). \end{aligned}$$

By building on the above, we are now ready to show

$$\min_{\substack{(a_1, a_2, b_1, b_2) \\ \in \mathcal{A}(P)}}} \min_{t \in [0, 1 - \alpha]} L_1(a_1, a_2, b_1, b_2, t) = \min_{\substack{(u_1, u_2) \\ \in \mathcal{A}_v(P, v_1, v_2)}}} \min_{t \in [0, 1 - \alpha]} L_2(u_1, u_2, t)$$

in two steps. In the first step, we show

$$\min_{\substack{(a_1, a_2, b_1, b_2) \\ \in \mathcal{A}(P)}}} \min_{t \in [0, 1 - \alpha]} L_1(a_1, a_2, b_1, b_2, t) = \min_{\substack{(u_1, u_2, v_1, v_2) \\ \in \mathcal{A}(P)}}} \min_{t \in [0, 1 - \alpha]} L_1(u_1, u_2, v_1, v_2, t).$$

Let us denote S_{ab}^* to be the set of optimal solutions that solve the optimisation problem

$$\min_{\substack{(a_1, a_2, b_1, b_2) \\ \in \mathcal{A}(P)}}} \min_{t \in [0, 1 - \alpha]} L_1(a_1, a_2, b_1, b_2, t),$$

while S_{uv}^* denotes the set of optimal solutions that solve

$$\min_{\substack{(u_1, u_2, v_1, v_2) \\ \in \mathcal{A}(P)}}} \min_{t \in [0, 1 - \alpha]} L_1(u_1, u_2, v_1, v_2, t).$$

Assume there exists a solution $(u_1^*, u_2^*, v_1^*, v_2^*, t^*) \in S_{uv}^*$ but $(u_1^*, u_2^*, v_1^*, v_2^*, t^*) \notin S_{ab}^*$, there must exist a solution $(\hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2, \hat{t}) \in S_{ab}^*$, such that

$$L_1(\hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2, \hat{t}) < L_1(u_1^*, u_2^*, v_1^*, v_2^*, t^*). \quad (14)$$

If $\hat{t} = t^*$, (14) contradicts with (13). If $\hat{t} \neq t^*$, (13) implies that there must exist another solution $(\hat{u}_1, \hat{u}_2, \hat{v}_1, \hat{v}_2, \hat{t})$ that is admissible to the optimisation problem

$$\min_{\substack{(u_1, u_2, v_1, v_2) \\ \in \mathcal{A}(P)}}} \min_{t \in [0, 1 - \alpha]} L_1(u_1, u_2, v_1, v_2, t)$$

such that

$$L_1(\hat{u}_1, \hat{u}_2, \hat{v}_1, \hat{v}_2, \hat{t}) \leq L_1(\hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2, \hat{t}) < L_1(u_1^*, u_2^*, v_1^*, v_2^*, t^*),$$

which contradicts with the assumption that $(u_1^*, u_2^*, v_1^*, v_2^*, t^*) \in S_{uv}^*$. Therefore, if $(u_1^*, u_2^*, v_1^*, v_2^*, t^*) \in S_{uv}^*$, $(u_1^*, u_2^*, v_1^*, v_2^*, t^*) \in S_{ab}^*$ must also hold.

On the other hand, let us assume there exists a solution $(a_1^*, a_2^*, b_1^*, b_2^*, t^*) \in S_{ab}^*$ but $(a_1^*, a_2^*, b_1^*, b_2^*, t^*) \notin S_{uv}^*$. Then, there must exist a solution $(u_1', u_2', v_1', v_2', t') \in S_{uv}^*$ such that

$$L_1(u_1', u_2', v_1', v_2', t') < L_1(a_1^*, a_2^*, b_1^*, b_2^*, t^*).$$

Since the solution $(u_1', u_2', v_1', v_2', t') \in S_{uv}^*$ is also admissible to the optimisation problem

$$\min_{\substack{(a_1, a_2, b_1, b_2) \\ \in \mathcal{A}(P)}} \min_{t \in [0, 1-\alpha]} L_1(a_1, a_2, b_1, b_2, t),$$

it contradicts with the assumption that $(a_1^*, a_2^*, b_1^*, b_2^*, t^*) \in S_{ab}^*$. Therefore, if $(a_1^*, a_2^*, b_1^*, b_2^*, t^*) \in S_{ab}^*$, $(a_1^*, a_2^*, b_1^*, b_2^*, t^*) \in S_{uv}^*$ must also be true. By combining the above, the first part of the proof is complete.

Now let us move to the second step of the proof and show

$$\min_{\substack{(u_1, u_2, v_1, v_2) \\ \in \mathcal{A}(P)}} \min_{t \in [0, 1-\alpha]} L_1(u_1, u_2, v_1, v_2, t) = \min_{\substack{(u_1, u_2) \\ \in \mathcal{A}_v(P, v_1, v_2)}} \min_{t \in [0, 1-\alpha]} L_2(u_1, u_2, t).$$

We now denote S_u^* to be the set of optimal solutions that solve

$$\min_{\substack{(u_1, u_2) \\ \in \mathcal{A}_v(P, v_1, v_2)}} \min_{t \in [0, 1-\alpha]} L_2(u_1, u_2, t).$$

Assume the optimal solution $(u_1^*, u_2^*, v_1^*, v_2^*, t^*) \in S_{uv}^*$ but $(u_1^*, u_2^*, t^*) \notin S_u^*$, there must exist a solution $(\hat{u}_1, \hat{u}_2, \hat{t}) \in S_u^*$, such that

$$L_2(\hat{u}_1, \hat{u}_2, \hat{t}) < L_2(u_1^*, u_2^*, t^*). \quad (15)$$

Because of the translation invariance property of VaR, it is not difficult to see that

$$L_1(\hat{u}_1, \hat{u}_2, \hat{v}_1, \hat{v}_2, \hat{t}) = L_2(\hat{u}_1, \hat{u}_2, \hat{t}) \quad (16)$$

where, by definition, $\hat{v}_1 = \text{VaR}_{\alpha+\hat{t}}(X_1)$ and $\hat{v}_2 = \text{VaR}_{1-\hat{t}}(X_2)$. Similarly, it also holds that

$$L_1(u_1^*, u_2^*, v_1^*, v_2^*, t^*) = L_2(u_1^*, u_2^*, t^*), \quad (17)$$

where, by definition, $v_1^* = \text{VaR}_{\alpha+t^*}(X_1)$ and $v_2^* = \text{VaR}_{1-t^*}(X_2)$. By combining (15), (16) and (17), we obtain

$$L_1(\hat{u}_1, \hat{u}_2, \hat{v}_1, \hat{v}_2, \hat{t}) < L_1(u_1^*, u_2^*, v_1^*, v_2^*, t^*). \quad (18)$$

Note that $(\hat{u}_1, \hat{u}_2, \hat{v}_1, \hat{v}_2, \hat{t})$ is admissible to the optimisation problem

$$\min_{\substack{(u_1, u_2, v_1, v_2) \\ \in \mathcal{A}(P)}} \min_{t \in [0, 1-\alpha]} L_1(u_1, u_2, v_1, v_2, t),$$

and thus, (18) contradicts with the assumption that $(u_1^*, u_2^*, v_1^*, v_2^*, t^*) \in S_{uv}^*$. Therefore, if $(u_1^*, u_2^*, v_1^*, v_2^*, t^*) \in S_{uv}^*$, it must also holds that $(u_1^*, u_2^*, t^*) \in S_u^*$.

On the other hand, let us now show that if $(u_1^*, u_2^*, t^*) \in S_u^*$, $(u_1^*, u_2^*, v_1^*, v_2^*, t^*) \in S_{uv}^*$ must also be true. Assume that $(u_1^*, u_2^*, v_1^*, v_2^*, t^*) \notin S_{uv}^*$, then there must exist a solution $(\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2, \tilde{t}) \in S_{uv}^*$ such that

$$L_1(\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2, \tilde{t}) < L_1(u_1^*, u_2^*, v_1^*, v_2^*, t^*), \quad (19)$$

where by definition $\tilde{v}_1 = \text{VaR}_{\alpha+\tilde{t}}(X_1)$ and $\tilde{v}_2 = \text{VaR}_{1-\tilde{t}}(X_2)$. If we apply again the translation invariance property of VaR, (19) implies the following

$$L_2(\tilde{u}_1, \tilde{u}_2, \tilde{t}) < L_2(u_1^*, u_2^*, t^*). \quad (20)$$

Since $(\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2, \tilde{t})$ is admissible to the optimisation problem

$$\min_{\substack{(u_1, u_2) \\ \in \mathcal{A}_v(P, v_1, v_2)}} \min_{t \in [0, 1-\alpha]} L_2(u_1, u_2, t),$$

(20) contradicts with the assumption that $(u_1^*, u_2^*, t^*) \in S_u^*$. That is, we have shown that if $(u_1^*, u_2^*, t^*) \in S_u^*$, $(u_1^*, u_2^*, v_1^*, v_2^*, t^*) \in S_{uv}^*$ must also be true, and the second part of the proof is also complete.

Overall, we have shown that

$$\min_{\substack{(a_1, a_2, b_1, b_2) \\ \in \mathcal{A}(P)}} \min_{t \in [0, 1-\alpha]} L_1(a_1, a_2, b_1, b_2, t) = \min_{\substack{(u_1, u_2, v_1, v_2) \\ \in \mathcal{A}(P)}} \min_{t \in [0, 1-\alpha]} L_1(u_1, u_2, v_1, v_2, t)$$

and

$$\min_{\substack{(u_1, u_2, v_1, v_2) \\ \in \mathcal{A}(P)}} \min_{t \in [0, 1-\alpha]} L_1(u_1, u_2, v_1, v_2, t) = \min_{\substack{(u_1, u_2) \\ \in \mathcal{A}_v(P, v_1, v_2)}} \min_{t \in [0, 1-\alpha]} L_2(u_1, u_2, t),$$

which lead to the conclusion that

$$\min_{\substack{(a_1, a_2, b_1, b_2) \\ \in \mathcal{A}(P)}} \min_{t \in [0, 1-\alpha]} L_1(a_1, a_2, b_1, b_2, t) = \min_{\substack{(u_1, u_2) \\ \in \mathcal{A}_v(P, v_1, v_2)}} \min_{t \in [0, 1-\alpha]} L_2(u_1, u_2, t).$$

Proof of Proposition 8 It is not difficult to show that

$$\begin{aligned} \mathbb{E}(l_{u_i, v_i}(X_i)) &= \int_{u_i}^{v_i} (x - u_i) f_i(x) dx + \int_{v_i}^{\infty} (v_i - u_i) f_i(x) dx \\ &= \lambda_i e^{-u_i/\lambda_i} - \lambda_i e^{-v_i/\lambda_i} \end{aligned}$$

Moreover, v_i is given by

$$v_1 = F_1^{-1}(\alpha + t) = -\lambda_1 \ln(1 - \alpha - t), \quad \text{for } i = 1,$$

and

$$v_2 = F_2^{-1}(1 - t) = -\lambda_2 \ln t, \quad \text{for } i = 2.$$

As a result, $L_2(u_1, u_2, t)$ becomes

$$\begin{aligned} L_3(u_1, u_2, t) &= \sum_{i=1}^2 \left(u_i + w_i + (1 + \delta_i) \mathbb{E}(l_{u_i, v_i}(X_i)) \right) \\ &= \sum_{i=1}^2 \left(u_i + w_i + (1 + \delta_i) \lambda_i e^{-u_i/\lambda_i} \right) - (1 + \delta_1) \lambda_1 (1 - \alpha) + ((1 + \delta_1) \lambda_1 - (1 + \delta_2) \lambda_2) t. \end{aligned} \quad (21)$$

We are now ready to apply Proposition 9 and solve the optimisation problem (4) in two steps. The first step considers the inner minimiser and denote

$$\tilde{t} := \arg \min_{t \in [0, 1-\alpha]} L_3(u_1, u_2, t),$$

while the second step solves the outer minimiser

$$\min_{\substack{(u_1, u_2) \\ \in \mathcal{A}_v(P, v_1, v_2)}} L_3(u_1, u_2, \tilde{t}).$$

As we can see in (21), $L_3(u_1, u_2, t)$ is linear in t . Thus, we will discuss the solution in the following three cases.

- (i) If $(1 + \delta_1) \lambda_1 - (1 + \delta_2) \lambda_2 = 0$, the solution of the inner minimiser ($\tilde{t}$) is not unique and does not affect the

value of $L_3(u_1, u_2, t)$. Furthermore, the objective function of the outer minimiser $L_3(u_1, u_2, \tilde{t})$ becomes

$$L_3^{(i)}(u_1, u_2, \tilde{t}) = \sum_{i=1}^2 \left(u_i + w_i + (1 + \delta_i) \lambda_i e^{-u_i/\lambda_i} \right) - (1 + \delta_1) \lambda_1 (1 - \alpha).$$

Differentiating $L_3^{(i)}(u_1, u_2, \tilde{t})$ with respect to u_1 and u_2 respectively, we obtain

$$\frac{dL_3^{(i)}(u_1, u_2, \tilde{t})}{du_i} = 1 - (1 + \delta_i) e^{-u_i/\lambda_i}, \quad \text{for } i = 1, 2. \quad (22)$$

Setting (22) to 0 gives

$$u_i = -\lambda_i \ln \left(\frac{1}{1 + \delta_i} \right) > 0, \quad \text{for } i = 1, 2. \quad (23)$$

Recall that the feasible region for u_i is $(u_1, u_2) \in \mathcal{A}_v(P, v_1, v_2)$, which lead to the following budget constraint

$$\sum_{i=1}^2 (w_i + \lambda_i) - (1 + \delta_1)(1 - \alpha) \lambda_1 \leq P \quad (24)$$

and feasible set $\mathcal{T}$ given by (5). Note that $\mathcal{T}$ is non-empty, because $1 - \alpha - \frac{1}{1 + \delta_1} < 1 - \alpha$ and $0 < \frac{1}{1 + \delta_2}$ hold. Therefore, if (24) holds, the feasibility set is not empty and the optimal solution

$$(u_1^*, u_2^*, t^*) := \arg \min_{\substack{(u_1, u_2) \\ \in \mathcal{A}_v(P, v_1, v_2)}} \min_{t \in [0, 1 - \alpha]} L_3(u_1, u_2, t)$$

is given by

$$\begin{cases} u_i^* = -\lambda_i \ln \left(\frac{1}{1 + \delta_i} \right), & \text{for } i = 1, 2, \\ t^* = t_0, & \text{where } t_0 \in \mathcal{T}. \end{cases}$$

Otherwise, a numerical approach shall be employed to solve the optimisation problem.

- (ii) If $(1 + \delta_1) \lambda_1 - (1 + \delta_2) \lambda_2 > 0$, $L_3(u_1, u_2, t)$ is an increasing function in t . Thus, $\tilde{t} = 0$, and $L_3(u_1, u_2, \tilde{t})$ becomes

$$L_3^{(ii)}(u_1, u_2, \tilde{t}) = \sum_{i=1}^2 \left(u_i + w_i + (1 + \delta_i) \lambda_i e^{-u_i/\lambda_i} \right) - (1 + \delta_1) \lambda_1 (1 - \alpha).$$

Therefore, differentiating $L_3^{(ii)}(u_1, u_2, \tilde{t})$ with respect to u_1 and u_2 respectively will lead to the same first order conditions and solutions as in (22) and (23). Note that $v_2 = \infty$ when $t = \tilde{t} = 0$. That is,

the feasibility constraint of u_2 , $u_2 \in [0, v_2)$, will always hold. Provided that $\mathcal{A}_v(P, v_1, v_2) \neq \emptyset$,

$$u_2^* = -\lambda_2 \ln \left(\frac{1}{1 + \delta_2} \right) \quad \text{and} \quad u_2^* \in (0, v_2).$$

On the other hand, $v_1 = -\lambda_1 \ln(1 - \alpha) < \infty$ when $t = \tilde{t} = 0$. Thus, it is necessary to consider the following two sub-cases and check the budget constraint in each sub-case:

(ia) If $(1 - \alpha)(1 + \delta_1) < 1$, it holds that

$$-\lambda_1 \ln \left(\frac{1}{1 + \delta_1} \right) < -\lambda_1 \ln(1 - \alpha) = v_1$$

and thus,

$$u_1^* = -\lambda_1 \ln \left(\frac{1}{1 + \delta_1} \right) \in (0, v_1),$$

which leads to the same budget constraint as in (24). That is, $\mathcal{A}_v(P, v_1, v_2) \neq \emptyset$ if (24) holds and the optimal solution (u_1^*, u_2^*, t^*) is given by

$$\begin{cases} u_i^* = -\lambda_i \ln \left(\frac{1}{1 + \delta_i} \right), & \text{for } i = 1, 2, \\ t^* = 0. \end{cases}$$

Otherwise, a numerical approach will be sought to solve the optimisation problem.

(ib) If $(1 - \alpha)(1 + \delta_1) \geq 1$, we have

$$0 \leq u_1 \leq v_1 = -\lambda_1 \ln(1 - \alpha) \leq -\lambda_1 \ln \left(\frac{1}{1 + \delta_1} \right).$$

One may notice that the first order condition (22) is an increasing function in u_i and is equal to 0 when $u_i = -\lambda_i \ln \left(\frac{1}{1 + \delta_i} \right)$. Therefore,

$$\frac{dL_3^{(ii)}(u_1, u_2, \tilde{t})}{du_1} < 0, \quad \forall u_1 \in \left[0, -\lambda_1 \ln \left(\frac{1}{1 + \delta_1} \right) \right)$$

Consequently, $u_1^* = v_1$, i.e. there will be no reinsurance purchased for X_1 , which leads to the budget constraint of

$$w_2 + \lambda_2 \leq P \tag{25}$$

Therefore, if (25) holds, the optimal solution (u_1^*, u_2^*, t^*) is given by

$$\begin{cases} u_1^* = -\lambda_1 \ln(1 - \alpha), \\ u_2^* = -\lambda_2 \ln\left(\frac{1}{1 + \delta_2}\right), \\ t^* = 0. \end{cases}$$

Otherwise, a numerical approach will be sought to solve the optimisation problem.

- (iii) Finally, if $(1 + \delta_1)\lambda_1 - (1 + \delta_2)\lambda_2 < 0$, $L_3(u_1, u_2, t)$ is now a decreasing function in t . Thus, $\tilde{t} = 1 - \alpha$, and $L_3(u_1, u_2, \tilde{t})$ becomes

$$L_3^{(iii)}(u_1, u_2, \tilde{t}) = \sum_{i=1}^2 \left(u_i + w_i + (1 + \delta_i)\lambda_i e^{-u_i/\lambda_i} \right) - (1 + \delta_2)\lambda_2(1 - \alpha).$$

Again, as in case (ii), differentiating $L_3^{(iii)}(u_1, u_2, \tilde{t})$ with respect to u_1 and u_2 respectively will lead to the same first order conditions and solutions as in (22) and (23). Note that $v_1 = \infty$ when $t = \tilde{t} = 1 - \alpha$. That is, the feasibility constraint of u_1 , $u_1 \in [0, \infty)$, will always hold. Provided that $\mathcal{A}_v(P, u_1, u_2) \neq \emptyset$,

$$u_1^* = -\lambda_1 \ln\left(\frac{1}{1 + \delta_1}\right).$$

On the other hand, $v_2 = -\lambda_2 \ln(1 - \alpha) < \infty$ when $t = \tilde{t} = 1 - \alpha$. Again, it is necessary to discuss two sub-cases and check the budget constraint:

- (iiia) If $(1 - \alpha)(1 + \delta_2) < 1$, it holds that

$$-\lambda_2 \ln\left(\frac{1}{1 + \delta_2}\right) < -\lambda_2 \ln(1 - \alpha) = v_2$$

and thus,

$$u_2^* = -\lambda_2 \ln\left(\frac{1}{1 + \delta_2}\right) \in (0, v_2),$$

which leads to the following budget constraint

$$\sum_{i=1}^2 (w_i + \lambda_i) - (1 + \delta_2)\lambda_2(1 - \alpha) \leq P. \quad (26)$$

That is, $\mathcal{A}_v(P, v_1, v_2) \neq \emptyset$ if (26) holds and the optimal solution (u_1^*, u_2^*, t^*) is given by

$$\begin{cases} u_i^* = -\lambda_i \ln\left(\frac{1}{1 + \delta_i}\right), & \text{for } i = 1, 2, \\ t^* = 1 - \alpha. \end{cases}$$

Otherwise, a numerical approach will be sought to solve the optimisation problem.

(iiib) If $(1 - \alpha)(1 + \delta_2) \geq 1$, we have

$$0 \leq u_2 \leq v_2 = -\lambda_2 \ln(1 - \alpha) \leq -\lambda_2 \ln\left(\frac{1}{1 + \delta_2}\right).$$

Similar to what we have mentioned in sub-case (iib), the first order condition (22) is an increasing function in u_2 and is equal to 0 when $u_2 = -\lambda_2 \ln\left(\frac{1}{1 + \delta_2}\right)$. Therefore,

$$\frac{dL_3^{(iii)}(u_1, u_2, \tilde{t})}{du_2} < 0, \quad \forall u_2 \in \left[0, -\lambda_2 \ln\left(\frac{1}{1 + \delta_2}\right)\right)$$

Consequently, $u_2^* = v_2$, i.e. there will be no reinsurance purchased for X_2 , which leads to the budget constraint of

$$w_1 + \lambda_1 \leq P \tag{27}$$

Therefore, if (27) holds, the optimal solution (u_1^*, u_2^*, t^*) is given by

$$\begin{cases} u_1^* = -\lambda_1 \ln\left(\frac{1}{1 + \delta_1}\right), \\ u_2^* = -\lambda_2 \ln(1 - \alpha), \\ t^* = 1 - \alpha. \end{cases}$$

Otherwise, a numerical approach will be sought to solve the optimisation problem.