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# Low-Rank Sparse joint Representation for Moving Object Detection in Video

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**Abstract:** For the video sequences with fixed cameras, it is a reasonable assumption that the fixed background has low-rank characteristic, and the dynamic foreground has sparse characteristic. A new motion detection method based on low-rank and sparse joint representation is proposed in this paper. The ideas of the proposed method are described as follows: The noise of video sequence is removed by image preprocessing. T*he optical flow between continuous video sequences is estimated, which is used to generate a binary motion mask as a movement weight matrix. An optimization model with low-rank background and sparse foreground is established based on the idea of subspace learning theory. T*he background and foreground of each frame are obtained by using the ADMM-BCD iterative algorithm. Experimental results show that the proposed method is super to the other same sort of moving detection methods. The proposed method has perfect effect on slow moving target detection.

**Keywords:** robust principal component analysis (RPCA); subspace learning; background-foreground modeling; motion detection

# 基于低秩-稀疏联合表示的视频序列运动目标检测

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**摘要：**对于固定摄像机的视频序列，假设背景具有低秩特征，动态前景具有稀疏特性，提出了一种基于低秩稀疏联合表示的运动检测方法。思路如下：通过图像预处理降低视频序列的噪声；*估计连续帧之间的光流，生成二进制运动掩模作为运动权重矩阵；基于子空间学习理论，建立了低秩背景与稀疏前景的优化模型*；利用 ADMM-BCD 迭代算法得到视频背景和前景。实验结果表明，该方法优于其他同类运动检测方法，对慢速运动目标检测效果良好。

**关键词：**鲁棒主成分分析；子空间学习；背景-前景建模；运动检测

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## Introduction

Moving object detection as one of the most important and basic research fields in computer vision, which is widely used in vision application such as traffic monitoring, augmented reality and video surveillance and so on[1-3]. The accuracy, completeness and rapidity of moving objects detection have great influence on the subsequent analysis of moving objects. Therefore, the research of moving objects detection in video sequences is of

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great significance. The essence of motion detection is to separate the moving target from the fixed background. Recently, the application of robust principal component analysis based on low-rank and sparse decomposition has become a research hot spot in moving target detection. Among these methods[4-6], the most representative is the robust principal component analysis (RPCA) proposed by Wright[7], which assumes that the observed video can be divided into a low-rank matrix and a sparse matrix. And the two matrixes model the background and foreground, respectively. Although the RPCA performs well in most moving object detections cases[8], there are still many areas that need to improve. There are two main reasons. Firstly, due to the multiple iterative computation of SVD, the recovery of the low-rank matrix from the damaged observation matrix requires a larger amount of time cost. In order to reduce the computational complexity, GoDec[9] uses bilateral random projection to accelerate the low-rank approximation in RPCA. RMF[10], LMaFit[11] and ROSL[12] have been proposed for fast low-rank recovery, instead of minimizing the rank of low-rank matrix. These approaches represent low-rank matrix under some preset-rank subspaces. Secondly, the original PRCA has an excellent detection effect in video with static background and continuous foreground. Due to that, the video isn't as ideal as the above in practical applications, the background-foreground separation precision of the original RPCA needs to be further improved. In order to achieve more accurate separation of foreground and background components, a dense motion field is estimated for each frame and mapped into a weight matrix to assign the likelihood of pixels belonging to the background[13-14]. Robust PCA on graphs also strengthens the robustness of principal components to occlusions and missing values[15], which incorporates spectral graph regularization into the Robust PCA framework.

Motivated by subspace learning[12] and motion-assisted matrix restoration[13], a novel moving object detection model is proposed in this paper. The contributions are summarized as follows. First, the weight matrix is obtained by computing optical flow and a binary motion mask is created. The weight matrix is then assigned as the likelihood of pixels belonging to the background. Second, according to the theory of subspace learning representation, the low-rank matrix is represented by its subspace. And the rank of low-rank matrix is defined by its non-zero row of coefficients in its subspace. Third, the model based on the RPCA is extended with the above two preconditions. Finally, the ADMM-BCD[16-17] algorithm is extended to solve the proposed model. Experimental results confirm that the proposed method not only has perfect moving detection performance but also has low time complexity.

# 1 The Proposed Method

## 1.1 Motivation

The application scene of the proposed method is a video sequence captured by a fixed camera. First, each frame of the video sequence to be observed is formed into a column in the vector matrix. Suppose we have a set of videos that need to handle $X=[X_1,X_2,...,X_i,...X_p],X_i\in \mathrm{R}^{M\times N},i=1,2,...p$, where $M$ is the height of image, $N$ is the width of image, and $p$ is the number of the frame. Due to the influence of weather and other extraneous factors, there will be some noise points in the video images captured by a fixed camera. These noise points will interfere with the detection of moving targets. Therefore, it is necessary to denoise for each video frame. The median filtering method is used in this paper. Median filtering can not only filter out the noise points in the video images, but also protect the edges of the images. Then we stack the processed video into an observed matrix $D=[d_1,d_2,...d_p]\in \mathrm{R}^{MN\times p}$.

The main idea of RPCA is to decompose the observed matrix into low-rank matrix and sparse matrix. Because the background in a video sequence is same and the foreground is sparse essentially. The task of background-foreground model can be completed by solving the following RPCA model:

$$\min_{B,F}\|B\|_*+\lambda\|F\|_1 \quad s.t \quad D=B+F \tag{1}$$

where $B\in \mathrm{R}^{MN\times p}$ and $F\in \mathrm{R}^{MN\times p}$ represent low-rank background matrix and sparse foreground matrix respectively. $\|\|_*$ is the nuclear norm, $\|\|_1$ is the $l_1$-norm and $\lambda$ is the coefficient to control the sparsity of $F$.

RPCA has achieved good research and applications in moving target detection. However, there still remain two crucial problems to be solved in this approach. First, the multiple iterative operations of SVD make RPCA spend a large amount of computation. Second, the initial RPCA is motion-unaware and may generate smearing-effect artifacts when handling slows motion or motionless foreground. We attempt to fill this research gap by utilizing the novel notion of weight matrix and subspace learning in the component of the background model.

## 1.2 The Proposed Model

Due to multiple iterations of SVD, ROSL[12] speeds the rank-minimization of a matrix by imposing the group sparsity of its coefficients under orthonormal subspace spanned by orthonormal bases. Its underlying idea is that, the rank of $B$ is upper bound determined by the number of non-zero rows of $\alpha$ given the subspace representation $B=C\alpha$. Then Eq. (1) can be shown as the following model:

$$\min_{B,F}\|\alpha\|_{row-1}+\lambda\|F\|_1 \quad s.t \quad D=C\alpha+F \tag{2}$$

where $\|\alpha\|_{row-1}=\sum_{i=1}^{k}\|\alpha_i\|_2$, the nuclear norm is equal to the group sparsity of $\|\alpha\|_{row-1}$ under orthonormal subspace $C$, where $B=C\alpha$.

In order to deal with the problem of foreground and background separation for slow motion and motionless foreground, motion-assisted matrix is incorporated into the primary model. Based on the above motivation, we proposed the following model:

$$\min_{B,F}\|\alpha\|_{row-1}+\lambda\|F\|_1 \quad s.t \quad W\circ D=W\circ(C\alpha+F) \tag{3}$$

where '$\circ$' denotes element-wise multiplication of two matrices, $W$ is a motion-assisted matrix. We assume the background is motionless in a video sequence taken by a fixed camera. Under this assumption, any area with motion actually does not belong to the background. Therefore, the weight matrix $W$ is constructed from motion information, the specific methods of $W$ in Section 1.3.

## 1.3 Weight Matrix Construction From Motion Information

The optical flow based moving detection method uses the optical flow characteristics of the moving target which changed with time. We can initialize the contour based tracking algorithm by computing the displacement vector optical flow field, and effectively extract the moving target. Therefore, we use the optical flow method to obtain the motion weight matrix in this paper. First, the horizontal components $v_i^x$ and vertical components $v_i^y$ of two consecutive video frame $X_i$ and $X_{i-1}$ are obtained by optical flow method. Then, the motion weight matrix is obtained through the following formula:

$$w_{i,k}=\begin{cases}1, & \sqrt{(v_{i,k}^x)^2+(v_{i,k}^y)^2}<\tau \\ 0, & other\end{cases} \quad (4)$$

where $v_{i,k}^x$ is the horizontal component and $v_{i,k}^y$ is the vertical component of the motion vector at position $k$ computed between the frames $X_i$ and $X_{i-1}$, $\tau$ is the threshold of motion magnitude, which is computed adaptively as the average of all pixels in the motion field. The threshold $\tau$ is selected such that all pixels in X exhibiting motion larger than $\tau$ definitely belong to the foreground, otherwise it belongs to background. The motion weight matrix $W=[w_1,w_2,...w_p]\in \mathrm{R}^{MN\times p}$ is obtained from the Eq. (4).

### 1.4 Optimization Algorithm

Following, we solve the Eq. (3) by ADMM-BCD algorithm[16-17] in this work. The original constrained optimization problem of Eq. (3) is firstly converted to the minimization of the augmented Lagrangian function:

$$L_\mu(\alpha,C,F,Y)=\|\alpha\|_{row-1}+\lambda\|F\|_1+ \\ <Y,W\circ(D-C\alpha-F)>+ \\ \frac{\mu}{2}\|W\circ(D-C\alpha-F)\|_F^2 \quad (5)$$

where $\mu>0$ is the penalty parameter, $Y\in \mathrm{R}^{MN\times q}$ is the Lagrangian multipliers and $\|\ \|_F$ denotes the matrix Frobenius norm. We introduce an alternating direction strategy to optimize one variable while fixing others. The optimization problem is now divided into four major subproblems.

(1) Update $F^{n+1}$ by minimizing the augmented Lagrangian function as:

$$F^{n+1}=\arg\min_L \lambda\|F\|_1+(\mu^n/2)\|W\circ(D-C^n\alpha^n-F)\|_F^2+ \\ <Y^n,W\circ(D-C^n\alpha^n-F)>= \\ shrink(M\circ(D-C^n\alpha^n)+\frac{Y^n}{\mu^n},\frac{\lambda}{\mu^n}) \quad (6)$$

where $shrink(a,b)$ is the soft-threshold function defined as:

$$shrink(a,b)=\mathrm{sgn}(a)\times\max(|a|-b,0) \quad (7)$$

(2) Update $C^{n+1},\alpha^{n+1}$ by minimizing the augmented Lagrangian defined as:

$$(C^{n+1},\alpha^{n+1})=\arg\min_B\|\alpha\|_{row-1}+ \\ <Y^n,W\circ(D-C\alpha-F^{n+1})>+ \\ \frac{\mu^n}{2}\|W\circ(D-C\alpha-F^{n+1})\|_F^2 \quad (8)$$

Solving $C^{n+1}$ and $\alpha^{n+1}$ simultaneously with constraint $W\circ\ D+Y^n/\mu=W\circ(C\alpha+F)$ is a non-convex problem. So we apply block coordinate descent (BCD) to solve this problem. Suppose the subspace bases are $C=[C_1,...,C_t,...C_k]$ and $\alpha=[\alpha_1;...;...;\ \alpha_t;...;\alpha_k]$ ($k$ represents the dimension of subspace), the BCD scheme updates the pair $(C_t^{n+1},\alpha_t^{n+1})$ by keeping all the other indexes unchanged as:

$$\begin{cases}C_t^{n+1}=R_t^n(\alpha^n)^T \\ \alpha_t^{n+1}=\frac{1}{\|C_t^{n+1}\|_2^2}shrink((C_t^{n+1})^T R_t^n,\frac{1}{\mu^n})\end{cases} \quad (9)$$

where $R_t^i$ is the residual defined as

$$R_t^n=D-F^{n+1}-\sum_{j<t}D_j^{n+1}\alpha_j^{n+1}- \\ \sum_{j>t}D_j^n\alpha_j^n+Y^n/\mu^n \quad (10)$$

(3) Update the Lagrange Multiplier $Y^{n+1}$ as:

$$Y^{n+1}=Y^n+\mu W\circ(D-C^{n+1}\alpha^{n+1}-F^{n+1}) \quad (11)$$

(4) Update the penalty parameter $\mu$ as:

$$\mu^{n+1}=\max(\mu^n\rho,\mu_{\max}) \quad (12)$$

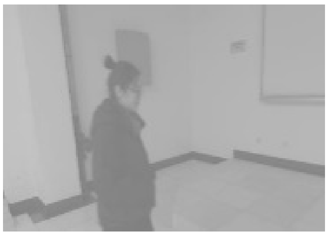
In the proposed method, the parameter $\lambda$ is as $\lambda=1/\sqrt{\max(MN,p)}$.

## 2 Experimental Results and Discussion

In order to illustrate the performance of the proposed method, we compared the proposed algorithm with the recent LSADM-RPCA algorithm[18] to deal with the standard video sequence and the video sequence captures by us. The experimental algorithm is implemented in the MATLAB R2014b environment. And hardware environment of the experiment is the Lenovo laptop of the 32Bit operation system, and the processor is Intel (R) Core (TM) 2 Duo CPU T5750@ 2.00GHz 2.00 GHz.

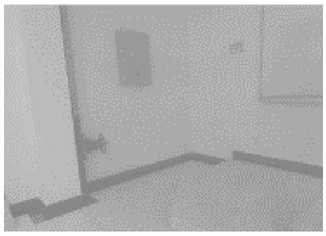
## 2.1 Motion Detection of Captured Video

We took two groups of videos of single target and double target movement respectively, and the two groups of videos are 40 frames and the size of each frame is 300×431. The parameter settings of the algorithm are as follows: $\rho = 1.2$, $\mu_0 = 1\,000 / norm$ ($norm$ is the largest singular value of video matrix), $\lambda = 0.002\,8$, $n_{\max} = 500$. We can see the results of the experiment as shown in Fig. 1~4.



(a) The 10th captured frame of our own videos

(b) The standard background



(c) The recovery backgrounds by the proposed algorithm

(d) The recovery backgrounds by RPCA method

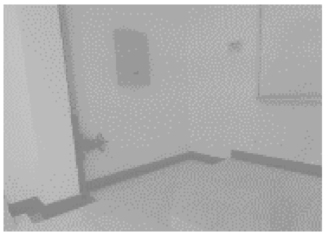
(e) The result of foreground detection by the proposed method

(f) The result of foreground detection by RPCA method

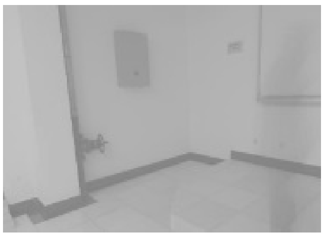
Fig. 1 The detection result of the proposed method and RPCA method on single-target video

(a) The 30th captured frame of our own videos

(b) The standard background



(c) the recovery backgrounds by the proposed algorithm



(d) The recovery backgrounds by RPCA method

(e) The result of foreground detection by the proposed method

(f) The result of foreground detection by RPCA method

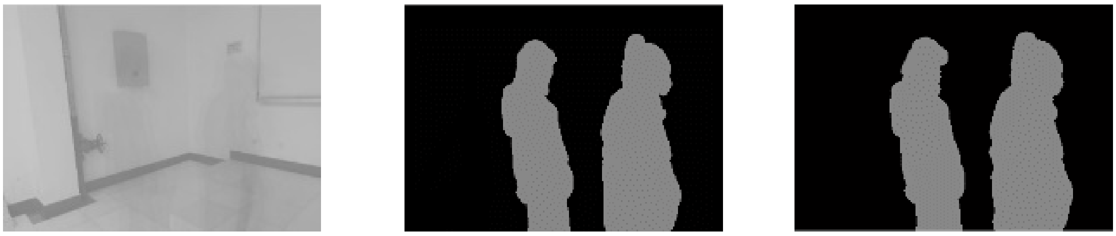
Fig. 2 The detection result of the proposed method and RPCA method on single-target video

(a) The 10th captured frame of our own videos　(b) The standard background　(c) The recovery backgrounds by the proposed algorithm

(d) The recovery backgrounds by RPCA method　(e) The result of foreground detection by the proposed method　(f) The result of foreground detection by RPCA method

Fig. 3 The detection result of the proposed method and RPCA method on double-target video

(a) The 20th captured frame of our own videos　(b) The standard background　(c) The recovery backgrounds by the proposed algorithm



(d) The recovery backgrounds by RPCA method　(e) The result of foreground detection by the proposed method　(f) The result of foreground detection by RPCA method

Fig. 4 The detection result of the proposed method and RPCA method on double-target video

Tab. 1 shows the time cost and accuracy rate comparison of the proposed method with the reference method. From the experimental results of the two groups of video, it can be seen that the proposed method is greatly reduced in time cost than that of RPCA. In single-target video, the proposed method is 0.72% lower than the accuracy rate of RPCA, but it is within the acceptable range. Due to moving in the double-target video is slower than that of single-target, the RPCA method can’t detect the slow moving target very well. The accuracy rate of the RPCA is 6.52% lower than the proposed method. From the recovery results of the background of RPCA method, it can be seen that there are obvious artifacts in the background recovery which makes the effect of foreground extraction not to be very ideal.

## 2.2 Motion Detection of Standard Video Library

Two groups of standard videos are used to perform the proposed method and the reference RPCA method, which are the Lobyy video and the Hall video. As for the motion detection of the Lobyy video, the Lobyy video is a single-target video with 36 frames, and the size of each frame is 128×160. The parameter settings of the algorithm are as follows: $n_{\max}=500$ , $\lambda=0.0070$ , $\mu_0=1000/norm$ ($norm$ is the largest singular value of the Lobyy video matrix), $\rho=1.2$ , $n_{\max}=500$ . The results of the experiment as shown in Fig. 5 and Fig. 6.

The Lobyy's background is relatively static. The only variable factor may be the change of light, and the moving target is a person from right to left. From the detection results of the Lobyy video, we can see that the proposed method can be basically consistent with the detection effect of RPCA algorithm for background stationary single moving target video. But it is noticed that RPCA method takes 9.371 6 seconds and the proposed method only takes 1.621 4 seconds when the same detection result is achieved.

As for the motion detection of the Hall video, the Hall video is a double-target video with 31 frames, and the size of each frame is 144×176. The parameter settings of the algorithm are as follows: $\rho=1.2$ , $\mu_0=1000/norm$ ($norm$ is the largest singular value of hall video matrix), $\lambda=0.0063$ , $n_{\max}=500$ . The results of the experiment as shown in Fig. 7~8.

Tab. 1 Time cost and accuracy rate comparison of the proposed method with the RPCA method

| videos | The proposed method | | RPCA | |
|---|---|---|---|---|
| | Time/s | Accuracy rate/% | Time /s | Accuracy rate/% |
| Single-target video | 23.34 | 98.26 | 165.78 | 98.98 |
| Double-target video | 21.62 | 85.95 | 167.27 | 79.43 |

(a) The 20th frame of the Lobyy video

(b) The standard background

(c) The recovery backgrounds of the Lobyy video by the proposed method

(d) The recovery backgrounds of the Lobyy video by RPCA

(e) The result of foreground detection by the proposed method

(f) The result of foreground detection by RPCA

Fig. 5 The detection effect of the proposed method and RPCA on the Lobyy video

(a) The 30th frame of the Lobyy video

(b) The standard background

(c) The recovery backgrounds of the Lobyy video by the proposed method

(d) The recovery backgrounds of the Lobyy video by RPCA

(e) The result of foreground by the proposed method

(f) The result of detection foreground detection by RPCA

Fig. 6　The detection effect of the proposed method and RPCA on the Lobyy video

(a) The 10th frame of the Hall video

(b) The standard background

(c) The recovery backgrounds of the Hall video by the proposed method

(d) The recovery backgrounds of the Hall video by RPCA

(e) The result of foreground detection by the proposed method

(f) The result of foreground detection by RPCA

Fig. 7　The detection effect of the proposed method and RPCA on the Hall video

(a) The 25th frame of the Hall video

(b) The standard background

(c) The recovery backgrounds of the Hall video by the proposed method

(d) The recovery backgrounds of the Hall video by RPCA

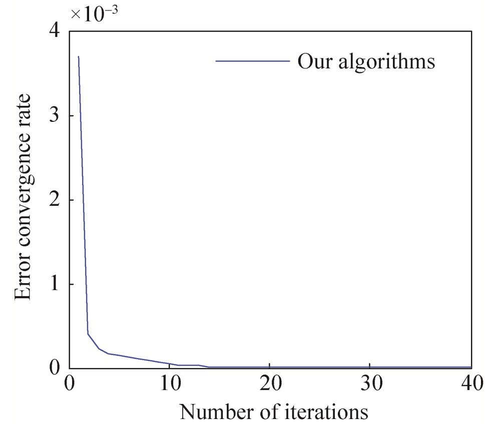
(e) The result of foreground detection by the proposed method

(f) The result of foreground detection by RPCA

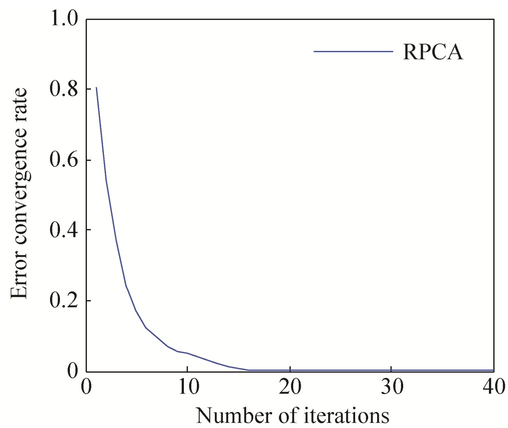
Fig. 8 The detection effect of the proposed method and RPCA on the Hall video

Compared with the Lobyy, the Hall's background is relatively complex. Besides the change of light, there is also the reflection of the ground. The moving targets are the crowd moving in the hall. From the detection results of the Hall video, we can see that the detection result of the 10th frame which using the proposed method and the RPCA method are the same. But the moving targets marked with red and yellow boxes in the 25th frame video shown in Figure 8, we can see that the two methods can also detect moving target marked by red boxes. Due to the moving target of the yellow box moves slowly, the RPCA does not have a good detection. But the proposed method can detect moving targets accurately.

At last, we compared of error convergence between the proposed method with RPCA method. In order to quantitatively analyze the experimental result, we use RPCA and our proposed algorithm to detect moving objects in the Hall video. When the same iteration is set as 40, the error convergence is shown in Fig. 9. From Fig. 9, we can see that both methods can achieve satisfactory accuracy, but the proposed method performed better than the reference algorithm. It can rapidly reach the ideal accuracy with the same background condition. The proposed method can accelerate the convergence rate of error and reduce the calculation time.




(a) Implemented by the proposed method




(b) Implemented by RPCA method

Fig. 9 The error convergence curve with the number of iteration for the Hall video

## 3 Conclusions

In this paper, a new background-foreground motion detection model is proposed, which uses motion assistance and subspace representation. By estimating the optical flow between consecutive video sequences to generate a binary motion mask as a movement weight matrix, weight matrix helps to obtain the relevant background and foreground pixel components of prior knowledge. Based on subspace learning idea, the low-rank matrix B is denoted by the subspace representation $B = C\alpha$. The rank of B depends on the non-zero row of $\alpha$, which can effectively accelerate the computation speed. We combine subspace representation with motion weight to improve the primary RPCA model. Finally, the background and foreground of each frame are obtained by using the ADMM-BCD iterative algorithm. The algorithm solves the problem of inaccurate detection of target detection with slow motion and long computing time in RPCA. Experimental results show that the proposed method is superior to the RPCA, and can be used to extract one or more moving targets. At the same time, the proposed method is the same as some improved RPCA, in which the target detection precision is not high in the complex background such as illumination changes, fluctuations in the water, swaying leaves and so on. The future work will improve the accuracy of the method with complex background conditions.

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