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ON MAXIMALLY NON-FACTORIAL NODAL FANO THREEFOLDS

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Abstract. We classify non-factorial nodal Fano threefolds with 1 node and class group of rank 2.

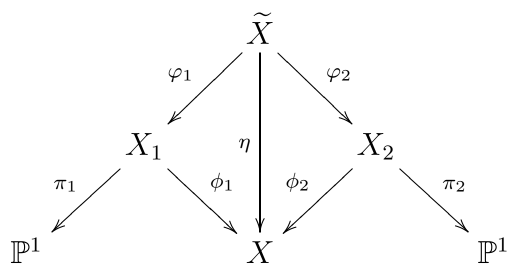
Let X be a Fano threefold that has at worst isolated ordinary double points (nodes). Then both the Picard group $\mathrm{Pic}(X)$ and the class group $\mathrm{Cl}(X)$ are torsion-free of finite rank, and $\mathrm{rk}\,\mathrm{Cl}(X) - \mathrm{rk}\,\mathrm{Pic}(X)$ is known as the *defect* of the threefold X [14, 19, 20, 32]. If the defect is zero, we say that X is *factorial* [7, 8]. Factoriality imposes significant constraints on the geometry of the Fano threefold [9, 11, 40, 50].

It is well known that the defect of X does not exceed the number of its singular points (see e.g. [31, Corollary 3.8]). If

$$\mathrm{rk}\,\mathrm{Cl}(X) - \mathrm{rk}\,\mathrm{Pic}(X) = |\mathrm{Sing}(X)|,$$

then we say that X is *maximally non-factorial*. This property is also called $\mathbb{Q}$-maximal non-factoriality; see [38, Proposition 6.13] and [39, Proposition A.14] for various ways to define it for a nodal Fano threefold X. By definition, if X has a single node, then X is maximally non-factorial if and only if it is non-factorial. Let us give the simplest example of a non-factorial threefold with one node.

Example. Let X be the quadric cone in $\mathbb{P}^4$ with one node. Then X is a maximally non-factorial nodal Fano threefold. Let $\eta\colon \widetilde{X} \to X$ be the blow up at the singular point of the threefold X, and let E be the η-exceptional surface. Then $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $E|_E \cong \mathcal{O}_E(-1,-1)$, and there exists the following commutative diagram:



where φ_1 and φ_2 are contractions of the surface E to curves such that $\varphi_2 \circ \varphi_1^{-1}$ is an Atiyah flop, both ϕ_1 and ϕ_2 are small projective resolutions, and both π_1 and π_2 are $\mathbb{P}^2$-bundles.

While maximally non-factorial nodal Fano threefolds are important in birational geometry because of their rich geometry, they also play a central role in the recent study of the derived categories of coherent sheaves for singular varieties (this in turn allows the study of the birational geometry with a very different toolset, such as stability conditions). Indeed, maximally non-factorial nodal Fano threefolds are very special from the perspective of derived categories of coherent sheaves, in particular their derived categories can often be separated into a smooth proper part and a singular part [31, 44, 55, 38, 39].

Let us explain the connection to derived categories in some more detail. In [31, 44, 55] the authors, inspired by the work of Kawamata [34], introduced and studied maximal non-factoriality

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Throughout this paper, all varieties are assumed to be projective and defined over $\mathbb{C}$.

of del Pezzo threefolds (a subset consisting of 8 of the 105 families of Fano threefolds). They proved that a del Pezzo threefold is maximally non-factorial if and only if its derived category admits a Kawamata semiorthogonal decomposition, that is an admissible semiorthogonal decomposition into a perfect part and derived categories of singular finite-dimensional algebras. It is thus natural to ask whether being a maximally non-factorial Fano threefold is a sufficient condition for the existence of a Kawamata decomposition. The proof of [44] relied on the classification of del Pezzo threefolds which are maximally non-factorial. Thus, in order to study Kawamata decompositions it is natural to have a classification of maximally non-factorial Fano threefolds.

A slightly weaker notion of categorical absorption of singularities was introduced in [38, 39]. By [38, Corollary 6.17], every maximally non-factorial Fano threefold with one ordinary double point admits a categorical absorption of singularities; the converse is also true, and holds for any number of nodes [38, Proposition 6.12]. A highlight of this theory in [39] is the deformation between the main components of the derived categories of one-nodal prime Fano threefolds of genus $2d+2$ and del Pezzo threefolds of rank one and degree d, for $d \in \{1,2,3,4,5\}$, which solves the so-called Fano threefold conjecture of Kuznetsov.

There is also a consequence of maximal non-factoriality to intermediate Jacobians. Namely, in some sense, a maximally non-factorial nodal Fano threefold X has a smooth projective intermediate Jacobian, so that singularities of X can be ignored from the Hodge theory perspective. The precise statement [39, Proposition A.16] is that a family of smooth Fano threefolds degenerating to a 1-nodal maximally non-factorial Fano threefold has a smooth projective family of intermediate Jacobians, i.e. no actual degeneration takes place in the middle degree cohomology.

On the other hand, maximally non-factorial Fano threefolds are rather rare among all nodal Fano threefolds. Motivated by the recent advances in derived categories of singular Fano threefolds, we pose the following problem.

Problem. *Classify all maximally non-factorial nodal Fano threefolds.*

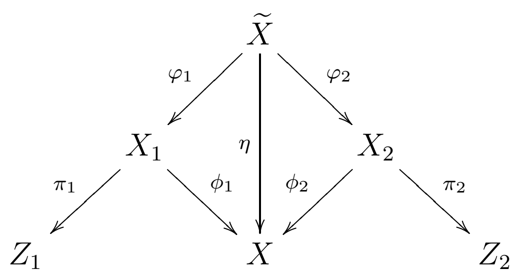
The goal of this paper is to partially solve this problem. Namely, we aim to classify maximally non-factorial nodal Fano threefolds of Picard rank one that have exactly one singular point (node). This case is particularly well behaved from the viewpoint of birational geometry, see the chain of equivalences in [38, Proposition 6.13] that applies only when one singular point is present.

Now, we are ready to present the main result of this paper. To do this, we suppose that

- the nodal Fano threefold X has one node,
- the rank of the Picard group $\mathrm{Pic}(X)$ is one,
- the rank of the class group $\mathrm{Cl}(X)$ is two.

Let $\eta\colon \widetilde{X} \to X$ be the blow up of the node of the threefold X, let E be the η-exceptional surface. Then $\widetilde{X}$ is smooth, $E \cong \mathbb{P}^1 \times \mathbb{P}^1$, $E|_E \simeq \mathcal{O}_E(-1,-1)$, and it follows from [16] that X uniquely determines the following Sarkisov link:

(★)



where φ_1 and φ_2 are contractions of the surface E to curves such that $\varphi_2 \circ \varphi_1^{-1}$ is an Atiyah flop, both ϕ_1 and ϕ_2 are small projective resolutions, and both π_1 and π_2 are extremal contractions [41].

Note that $-K_{X_1} \sim \phi_1^*(-K_X)$ and $-K_{X_2} \sim \phi_2^*(-K_X)$, so that

$$-K_{X_1}^3 = -K_{X_2}^3 = -K_X^3.$$

It follows from [43, 30] that X admits a smoothing $X \rightsquigarrow X_s$, where X_s is a smooth Fano threefold, $-K_X^3 = -K_{X_s}^3$, and the rank of the Picard group $\mathrm{Pic}(X_s)$ is 1. We also know from [14] that

$$(\maltese) \qquad h^{1,2}(\widetilde{X}) = h^{1,2}(X_1) = h^{1,2}(X_2) = h^{1,2}(X_s),$$

which imposes a significant constraint on the link ($\bigstar$). We set

$$d = -K_X^3, \quad h^{1,2} = h^{1,2}(X_s),$$

and

$$I = \max\big\{n \in \mathbb{Z}_{>0} \text{ such that } -K_{X_s} \sim nH \text{ for } H \in \mathrm{Pic}(X_s)\big\}.$$

Then I is the *index* of the Fano threefold X_s, which is also the index of the Fano threefold X [30].

In the remaining part of this paper, we prove the following theorem.

Theorem. *There are exactly 17 types of non-factorial Fano threefolds of Picard rank one with one node. All possibilities for ($\bigstar$), up to swapping the left and right sides of the diagram, are described in the table at the end of the paper.*

Each Sarkisov link in the table exists and can be described explicitly. For the reader's convenience, we provide the relevant references in the table. For the particular case of $-K_X^3 = 22$, the result is proved in [45]. A. Kuznetsov and Y. Prokhorov have independently obtained the same classification of non-factorial Fano threefolds of Picard rank one [37].

Remark. It should be pointed out that it follows from our classification that one-nodal maximally non-factorial degenerations of smooth Fano threefolds of Picard rank one (if any) have the same rationality as their smoothing (in the cases **2** and **7** in the table we need to assume that the smooth Fano threefolds are general). Indeed, this can be verified case by case, using the rationality results from [3, 10, 15, 23, 46, 53].

Observation. If X is a del Pezzo threefold ($I = 2$) of Picard rank one with $-K_X^3 \leqslant 32$, then the nodal Fano threefold X is never maximally non-factorial. This follows from from the defect computation [19, 20], see [44, Corollary 2.5]. Therefore, the only options for X when $I > 1$ are these two Fano threefolds:

- the nodal quadric threefold in $\mathbb{P}^4$ ($I = 3$, $-K_X^3 = 54$, the Sarkisov link **17**);
- a quintic del Pezzo threefold ($I = 2$, $-K_X^3 = 40$, the Sarkisov link **16**).

We prove the theorem by analyzing the possible links ($\bigstar$) in the following order:

(1) π_1 is a del Pezzo fibration, and π_2 is arbitrary;
(2) both π_1 and π_2 are birational;
(3) π_1 is a conic bundle and π_2 is arbitrary.

These cover all possible Mori fiber spaces arising in ($\bigstar$), up to swapping π_1 and π_2.

Note that all possibilities for the smooth Fano variety X_s are known and can be found in [25]. Using this classification, we list the possible values of $h^{1,2}$ as follows.

(d,I)	$(2,1)$	$(4,1)$	$(6,1)$	$(8,1)$	$(10,1)$	$(12,1)$	$(14,1)$,	$(16,1)$	$(18,1)$	$(22,1)$
$h^{1,2}$	52	30	20	14	10	7	5	3	2	0

(d,I)	$(8,2)$	$(16,2)$	$(24,2)$	$(32,2)$	$(40,2)$	$(54,3)$	$(64,4)$
$h^{1,2}$	21	10	5	2	0	0	0

Possibilities for (★) are studied in [1, 4, 17, 18, 21, 22, 26, 27, 28, 29, 30, 33, 45, 47, 48, 51, 52, 54]. Using some of these results, we immediately obtain the following corollary.

Corollary. *Suppose that π_1 is a fibration into del Pezzo surfaces. Then (★) is one of the links*

1, **2**, **3**, **4**, **5**, **6**, **8**, **9**, **10**, **12**, **15**, **16**, **17**

in the table in the end of the paper.

Proof. If π_1 is a fibration into del Pezzo surfaces of degree 6, the assertion follows from [21, 22], in which case we get the link **15**. In the remaining cases, the required assertion follows from [52]. □

Therefore, we may assume that neither π_1 nor π_2 is a fibration into del Pezzo surfaces.

Proposition. *Suppose that π_1 and π_2 are birational. Then (★) is the link* **13** *in the table.*

Proof. Both Z_1 and Z_2 are (possibly singular) Fano threefolds, and $\operatorname{rk}\operatorname{Pic}(Z_1) = \operatorname{rk}\operatorname{Pic}(Z_2) = 1$.

Suppose Z_1 is smooth so that π_1 is a contraction of type $E1$ or $E2$ in [35, Theorem 1.32] and π_2 is a contraction of type $E1$–$E5$. Then possibilities for $h^{1,2}(Z_1)$ are listed in the two tables presented above. Using [18], we obtain all possible values of $h^{1,2}(X_1)$. Now, using (✠), in combination with the list of Sarkisov links in [18, Tables 1–7] we see, carrying out a case-by-case analysis, that $Z_1 \cong Z_2 \cong \mathbb{P}^3$, and π_1 and π_2 are the blow ups along smooth rational curves of degree 5. Alternatively, one can run a short computer program exhausting all the possibilities for Z_1 and Z_2 and reach the same conclusion.

Therefore, to show that (★) is the link **13** in the table it suffices to explain why the rational quintic curves are not contained in a quadric. Indeed, none of these curves are contained in a smooth quadric surface, because in that case one of the rulings of this quadric will be contracted in the anticanonical model, but birational morphisms ϕ_1 and ϕ_2 are small by construction. Furthermore, a degree 5 smooth rational curve in $\mathbb{P}^3$ is never contained in a singular quadric.

Thus, we may assume that both Z_1 and Z_2 are singular. Now, using [18, Tables 8–9], we get $-K_X^3 \in \{2, 4\}$. Hence, if $|-K_X|$ does not have base points, then X is one of the following threefolds:

(1) sextic hypersurface in $\mathbb{P}(1,1,1,1,3)$,
(2) quartic hypersurface in $\mathbb{P}^4$,
(3) complete intersection of a quadric cone and a quartic hypersurface in $\mathbb{P}(1,1,1,1,1,2)$.

Indeed, by the Riemann-Roch theorem [25, Corollary 2.1.14], $|-K_X|$ defines a finite map $\phi\colon X \to \mathbb{P}^N$ with $N = 3$ (respectively $N = 4$) when $-K_X^3 = 2$ (respectively $-K_X^3 = 4$). We have $\deg(\phi(X)) \cdot \deg(\phi) = -K_X^3$. If $-K_X^3 = 2$, then ϕ is a double cover of $\mathbb{P}^3$ ramified at a sextic hypersurface by Hurwitz's formula, thus giving the first case. If $-K_X^3 = 4$ we either get that $\deg(\phi) = 1$ and $\phi(X)$ is a quartic threefold or $\deg(\phi) = \deg(\phi(X)) = 2$ and we get the last case.

By studying the defect, in each of these cases, the threefold X is factorial as it follows from [6, 7, 8, 9, 50], contradicting our assumption.

Therefore $|-K_X|$ has base points, hence using [29, Theorem 1.1 (i)], we see that $-K_X^3 = 2$, and X is the complete intersection of a quadric cone and a sextic hypersurface in $\mathbb{P}(1,1,1,1,2,3)$ on variables $x_0, \ldots, x_5$. We can assume that the quadric cone is given by $x_0x_1 - x_2x_3 = 0$. Then the projection on x_0, x_2 coordinates gives (after a small resolution of the singularity) a fibration by del Pezzo surfaces of degree 1. Similarly, the projection on x_0, x_3 coordinates gives another such fibration. This implies that (★) is the Sarkisov link **1** in the table, so that π_2 is not birational, which contradicts our assumption. □

Thus, we may assume that π_1 is a conic bundle, and either π_2 is birational, or π_2 is a conic bundle. Then the surface Z_1 is smooth [41, (3.5.1)], which implies that $Z_1 = \mathbb{P}^2$, since X_1 has Picard rank two. Let d_1 be the degree of the discriminant curve of the conic bundle π_1. Then [49, 1.6 Main Theorem] implies $0 \leqslant d_1 \leqslant 11$; $d_1 = 0$ if π_1 is a $\mathbb{P}^1$-bundle. By [3], we get

$$(\spadesuit) \qquad h^{1,2}(X_1) = \frac{d_1(d_1-3)}{2},$$

so $d_1 \notin \{1,2\}$. Using (✠) and the list of possible values of $h^{1,2}$ presented in tables above, we get

$$d_1 \in \{0,3,4,5,7,8\}.$$

Using the Observation above, for the rest of the proof we will assume that $I = 1$. Therefore we have

$$(\diamondsuit) \qquad (d, h^{1,2}, d_1) \in \{(6,20,8),(8,14,7),(14,5,5),(18,2,4),(22,0,0),(22,0,3)\}.$$

Let D_2 be a Cartier divisor on X_2, let D_1 be its strict transform on X_1, and let H_1 be a sufficiently general surface in $|\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(1))|$. Then $D_1 \sim_{\mathbb{Q}} a(-K_{X_1}) - bH_1$ for some rational numbers a and b. Moreover, if $d_1 \neq 0$, then a and b are integers, because the conic bundle has no sections and the Picard group of the generic fiber is generated by its canonical class. If $d_1 = 0$, then $2a$ and $2b$ are integers, because the Picard group of the generic fiber is generated by the class of a section. On the other hand we have (e.g. see [12, Lemma A.3])

$$-K_{X_1} \cdot D_1^2 = -K_{X_2} \cdot D_2^2,$$
$$\big(-K_{X_1}\big)^2 \cdot D_1 = \big(-K_{X_2}\big)^2 \cdot D_2.$$

Moreover, we have [5, Proposition 6]

$$(\dagger) \qquad -K_{X_1}^3 = d, \;\; (-K_{X_1})^2 \cdot H_1 = 12 - d_1, \;\; -K_{X_1} \cdot H_1^2 = 2, \;\; H_1^3 = 0.$$

This gives

$$(\heartsuit) \qquad \begin{cases} da^2 - 2(12-d_1)ab + 2b^2 = -K_{X_2} \cdot D_2^2, \\ da - (12-d_1)b = \big(-K_{X_2}\big)^2 \cdot D_2. \end{cases}$$

Lemma. *Suppose that π_2 is birational. Then (★) is either the link* **11** *or the link* **14** *in the table.*

Proof. Let D_2 be the π_2-exceptional surface. Then $a = D_1 \cdot H_1^2 \geqslant 0$.

If $\pi_2(D_2)$ is a point, then it follows from [41, Theorem (3.3)] that one of the following cases holds:

(A) $D_2 = \mathbb{P}^2$ with normal bundle $\mathcal{O}(-1)$,
(B) $D_2 = \mathbb{P}^2$ with normal bundle $\mathcal{O}(-2)$,
(C) D_2 is an irreducible quadric surface in $\mathbb{P}^3$ with normal bundle $\mathcal{O}(-1)$.

A simple computation using the adjunction formula implies that $-K_{X_2} \cdot D_2^2 = -2$ and

$$\big(-K_{X_2}\big)^2 \cdot D_2 = \begin{cases} 4 \text{ in the case (A)}, \\ 1 \text{ in the case (B)}, \\ 2 \text{ in the case (C)}. \end{cases}$$

Now, solving ($\heartsuit$) for each triple $(d, h^{1,2}, d_1)$ listed in ($\diamondsuit$), we see that $2a$ is never a non-negative integer. This shows that $\pi_2(D_2)$ is not a point.

We see that Z_2 is a smooth Fano threefold of Picard rank 1, and $\pi_2(D_2)$ is a smooth curve in Z_2. Then it follows from [28, Theorem 7.14] and (✠) that (★) is one of the Sarkisov links **11** and **14**, which would complete the proof of the lemma.

Note, however, that the paper [28] has gaps [13, Remark 1.18]. For instance, the link in the Construction below contradicts [28, Theorem 7.4], and few examples constructed in [54] contradict [28, Proposition 7.2]. Keeping this in mind, let us complete the proof of the lemma without using [28, Theorem 7.14].

Set $C_2 = \pi_2(D_2)$. Let $d_2 = -K_{Z_2} \cdot C_2$, and let g_2 be the genus of the curve C_2. Then as π_2 is the blow up along a curve on a threefold, we have that

$$h^{1,2}(Z_2) + g_2 = h^{1,2} \in \{0, 2, 5, 14, 20\},$$

where the inclusion follows from ($\diamondsuit$). As a result, using the classification of smooth Fano threefolds [25, §12.2], we get $h^{1,2}(Z_2) \in \{0, 2, 3, 5, 7, 10, 14, 20\}$. In fact, we can say a bit more. Let $e = -K_{Z_2}^3$, let i be the index of the Fano threefold Z_2. Then

- $(e, i) = (64, 4) \iff Z_2 = \mathbb{P}^3$,
- $(e, i) = (54, 3) \iff Z_2$ is a smooth quadric threefold in $\mathbb{P}^4$.

Moreover, the possible values $h^{1,2}(Z_2) \leqslant 20$ can be listed as follows.

(e,i)	$(6,1)$	$(8,1)$	$(10,1)$	$(12,1)$	$(14,1)$,	$(16,1)$	$(18,1)$	$(22,1)$
$h^{1,2}(Z_2)$	20	14	10	7	5	3	2	0

(e,i)	$(16,2)$	$(24,2)$	$(32,2)$	$(40,2)$	$(54,3)$	$(64,4)$
$h^{1,2}(Z_2)$	10	5	2	0	0	0

This leaves not so many possibilities for the genus $g_2 = h^{1,2} - h^{1,2}(Z_2)$.

One the other hand, it follows from [25, Lemma 4.1.2] that

$$\begin{aligned} -K_{X_2} \cdot D_2^2 &= 2g_2 - 2, \\ (-K_{X_2})^2 \cdot D_2 &= d_2 + 2 - 2g_2, \\ -K_{X_2}^3 &= e - 2 + 2g_2 - 2d_2, \end{aligned}$$

so that ($\heartsuit$) gives

$$\begin{cases} da^2 - 2(12 - d_1)ab + 2b^2 = 2g_2 - 2, \\ da - (12 - d_1)b = d_2 + 2 - 2g_2, \\ d = e - 2 + 2g_2 - 2d_2. \end{cases}$$

Now, solving this system of equations for each triple $(d, I, h^{1,2}, d_1)$ listed in ($\diamondsuit$), and each possible triple $(e, i, g_2) = (e, i, h^{1,2} - h^{1,2}(Z_2))$, we obtain the following two cases:

(I) $d = 18$, $I = 1$, $h^{1,2} = 2$, $d_1 = 4$, $Z_2 = \mathbb{P}^3$, $d_2 = 24$, $g_2 = 2$, $a = 3$, $b = 4$;
(II) $d = 22$, $I = 1$, $h^{1,2} = 0$, $d_1 = 3$, Z_2 is a smooth quadric in $\mathbb{P}^4$, $d_2 = 15$, $g_2 = 0$, $a = 3$, $b = 4$;

In the case (I), ($\bigstar$) is the link **11** in the table. In the case (II), ($\bigstar$) is the link **14** in the table. $\square$

Therefore, we may assume that π_2 is also a conic bundle and $Z_2 = \mathbb{P}^2$. Let d_2 be the discriminant curve of the conic bundle π_2. Using ($\spadesuit$) and $h^{1,2}(X_1) = h^{1,2}(X_2)$ we obtain that either $d_1 = d_2$ or $d_1, d_2 \in \{0, 3\}$. Now, we let D_2 be a general surface in $|\pi_2^*(\mathcal{O}_{\mathbb{P}^2}(1))|$. Then ($\heartsuit$) simplifies as

$$\begin{cases} da^2 - 2(12 - d_1)ab + 2b^2 = 2, \\ da - (12 - d_1)b = 12 - d_2. \end{cases}$$

Solving these equations for each quadruple $(d, h^{1,2}, d_1)$ listed in ($\diamondsuit$), we get the following cases:

(1) $a = 0$, $b = -1$;
(2) $d = 14$, $I = 1$, $h^{1,2} = 5$, $d_1 = d_2 = 5$, $a = 1$, $b = 1$.

In the case (1), the composition $\varphi_2 \circ \varphi_1^{-1}$ is biregular. This contradicts our initial assumption. So, the case (2) holds. Then (★) is the link **7** in the table, which proves the theorem.

Let us conclude this paper by showing that the Sarkisov link **7** in the table is always obtained using the following:

Construction ([48, § 3.4 Case 4^o]). Let $\overline{E} = \{z_1 = z_2 = 0\} \subset \mathbb{P}^2_{x_1,y_1,z_1} \times \mathbb{P}^2_{x_2,y_2,z_2}$, and let

$$\overline{X} = \big\{z_1 f(x_1, y_1, z_1; x_2, y_2, z_2) = z_2 g(x_1, y_1, z_1; x_2, y_2, z_2)\big\},$$

where f and g are sufficiently general polynomials of bi-degrees $(1, 2)$ and $(2, 1)$, respectively. Then $\overline{X}$ is a singular Verra threefold (a bidegree $(2, 2)$ threefold in $\mathbb{P}^2 \times \mathbb{P}^2$) with 5 nodes. Note that $\overline{E} \cong \mathbb{P}^1 \times \mathbb{P}^1$, $\overline{E} \subset \overline{X}$ and

$$\mathrm{Sing}(\overline{X}) = \big\{z_1 = z_2 = f = g = 0\big\} \subset \overline{E}.$$

Let $\rho\colon \mathbb{P}^2_{x_1,y_1,z_1} \times \mathbb{P}^2_{x_2,y_2,z_2} \dashrightarrow \mathbb{P}^4_{x,y,z,t,w}$ be the rational map given by

$$\big([x_1 : y_1 : z_1], [x_2 : y_2 : z_2]\big) \mapsto \big[x_1z_2 : y_1z_2 : x_2z_1 : y_2z_1 : z_1z_2\big].$$

Then ρ is birational, and the inverse map ρ^{-1} is given by $[x : y : z : t : w] \mapsto ([x : y : w], [z : t : w])$. Let $\xi\colon W \to \mathbb{P}^2_{x_1,y_1,z_1} \times \mathbb{P}^2_{x_2,y_2,z_2}$ be the blow up along the surface $\overline{E}$ and let $\mathscr{E}$ be its exceptional divisor. Let $\overline{G}_1 = \{z_1 = 0\}$ and $\overline{G}_2 = \{z_2 = 0\}$, and let G_1 and G_2 be the proper transforms on W of $\overline{G}_1$ and $\overline{G}_2$. Then we have the following commutative diagram:

$$\begin{array}{ccc} & W & \\ \swarrow^{\xi} & & \searrow^{\theta} \\ \mathbb{P}^2_{x_1,y_1,z_1} \times \mathbb{P}^2_{x_2,y_2,z_2} & \overset{\rho}{\dashrightarrow} & \mathbb{P}^4_{x,y,z,t,w} \end{array}$$

where θ blows down G_1 and G_2 to the lines $\ell_1 = \{z = t = w = 0\}$ and $\ell_2 = \{x = y = w = 0\}$. Note that $\theta(\mathscr{E})$ is the hyperplane $\{w = 0\}$ — the unique hyperplane containing the lines ℓ_1 and ℓ_2. Set $V = \rho(\overline{X})$. Then V is a smooth cubic threefold in $\mathbb{P}^4_{x,y,z,t,w}$. Moreover, we have

$$V = \big\{f(x, y, w; z, t, w) = g(x, y, w; z, t, w)\big\} \subset \mathbb{P}^4_{x,y,z,t,w}.$$

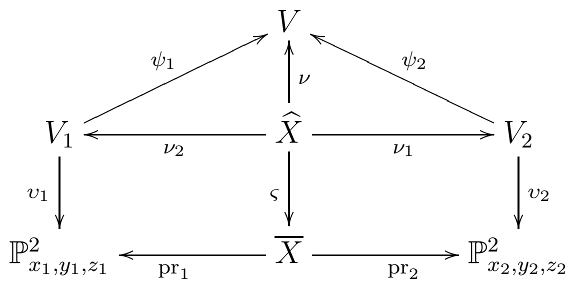
Now, let $\widehat{X}$ be the strict transform of the threefold $\overline{X}$ on W, let $\varsigma\colon \widehat{X} \to \overline{X}$ be the morphism induced by ξ, and let $\nu\colon \widehat{X} \to V$ be the morphism induced by θ. Then $\widehat{X}$ is smooth, ς is a small projective resolution, and we have the following commutative diagram:

$$\begin{array}{ccc} & \widehat{X} & \\ \swarrow^{\varsigma} & & \searrow^{\nu} \\ \overline{X} & \overset{\rho|_{\overline{X}}}{\dashrightarrow} & V \end{array}.$$

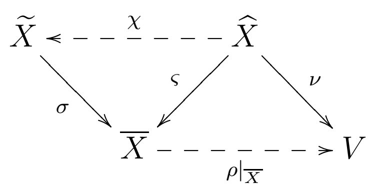
Note that ν is the blow up of the cubic threefold V along the lines ℓ_1 and ℓ_2. Let $\widehat{E} = \mathscr{E}|_{\widehat{X}}$. Then

- the induced map $\varsigma|_{\widehat{E}}\colon \widehat{E} \to \overline{E}$ is the blow up at the points in $\mathrm{Sing}(\overline{X})$,
- $\widehat{E}$ is isomorphic to a smooth cubic surface,
- $\nu(\widehat{E})$ is the hyperplane section $\{w = 0\} \cap V$.

Now, we extend the last commutative diagram to the following commutative diagram:



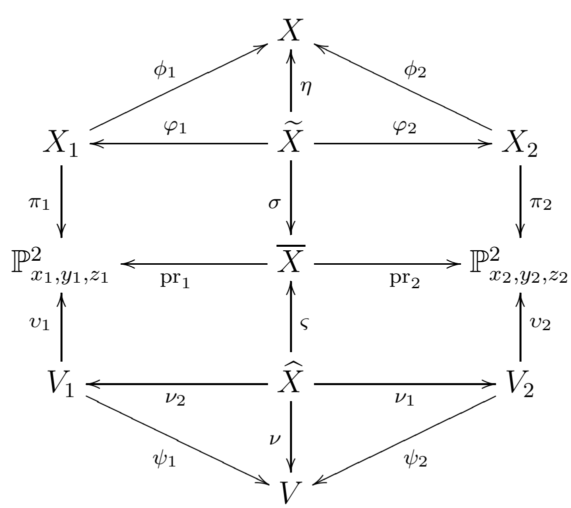
Here ψ_1 and ψ_2 are the blow ups along the lines ℓ_1 and ℓ_2, respectively, ν_1 and ν_2 are the blow ups along the strict transforms of the lines ℓ_1 and ℓ_2, respectively, both υ_1 and υ_2 are standard conic bundles [46], and pr_1 and pr_2 are the natural projections. Let Δ_1 and Δ_2 be the discriminant curves of the conic bundles υ_1 and υ_2, respectively. Then Δ_1 and Δ_2 are quintic curves with at most nodal singularities. Since ς is a flopping contraction, there exists a composition of flops $\chi\colon \widehat{X} \dashrightarrow \widetilde{X}$ of the 5 curves contracted by ς (this is the only projective flop which exists because the relative Picard number of ς equals 1). Then $\widetilde{X}$ is smooth and projective, and we have another commutative diagram:



where σ is a small resolution. Let $E = \chi(\widehat{E})$. Then χ induces a morphism $\widehat{E} \to E$ that blows down all five curves contracted by ς, which implies that σ induces an isomorphism $E \cong \overline{E} \cong \mathbb{P}^1 \times \mathbb{P}^1$. Note that $E|_E \sim \mathcal{O}_E(-1,-1)$, and there exists a birational morphism $\eta\colon \widetilde{X} \to X$ that blows down the surface E to an ordinary double point of the threefold X. We have $-K_X^3 = -K_{\widetilde{X}}^3 - 2 = 14$ and

$$1 = \operatorname{rk}\operatorname{Pic}(X) < \operatorname{rk}\operatorname{Cl}(X) = 1 + \big|\operatorname{Sing}(X)\big| = 2.$$

Therefore, the threefold X is a non-factorial nodal Fano threefold that has one node. We complete the picture with the following commutative diagram



where ϕ_1 and ϕ_2 are two small resolutions such that the composition $\phi_1^{-1} \circ \phi_2$ is an Atiyah flop, both φ_1 and φ_2 are contractions of the surface E to curves, π_1 and π_2 are standard conic bundles whose discriminant curves are Δ_1 and Δ_2, respectively. Note that X is irrational as it is birational to a smooth cubic threefold [15], and

$$h^{1,2}(X_1) = h^{1,2}(X_2) = h^{1,2}(\widetilde{X}) = h^{1,2}(\widehat{X}) = h^{1,2}(V) = 5.$$

Instead of using the Verra threefold $\overline{X}$ containing $\overline{E}$, we can construct the nodal threefold X using the birational map ρ^{-1}, and the smooth cubic threefold V containing the lines ℓ_1 and ℓ_2.

Now consider link **7** in the table: $Z_1 = Z_2 = \mathbb{P}^2$, and both π_1 and π_2 are conic bundles with discriminant curves of degree 5. Let C_1 and C_2 be the curves contracted by ϕ_1 and ϕ_2, respectively.

Recall that we denote by H_1 (respectively D_2) the pullback of the ample generator by π_1 from Z_1 (respectively by π_2 from Z_2), and D_1 is the divisor corresponding to D_2 on X_1 under flop. Then it follows from the calculations above (see case (2) before the Construction) that $D_1 \sim -K_{X_1} - H_1$. We have

$$-1 = \big(-K_{X_1} - H_1\big)^3 = D_1^3 = D_2^3 - \big(D_2 \cdot C_2\big)^3 = -\big(D_2 \cdot C_2\big)^3,$$

where we used (†) in the first equality, and [12, Lemma A.3] in the third one. It follows that $D_2 \cdot C_2 = 1$. Similarly, we get $H_1 \cdot C_1 = 1$.

Let $h_1 = \varphi_1^*(H_1)$ and $h_2 = \varphi_2^*(D_2)$. A simple computation using $D_2 \cdot C_2 = 1$ implies that

$$\varphi_1^*(D_1) \sim h_2 + E.$$

Thus we can express the canonical class $-K_{\widetilde{X}}$ in terms of h_1 and h_2 as follows

$$-K_{\widetilde{X}} \sim -\varphi_1^*(K_{X_1}) - E \sim \varphi_1^*(H_1 + D_1) - E \sim h_1 + h_2.$$

Note that $-K_{\widetilde{X}}^3 = 12$, $h^{1,2}(\widetilde{X}) = 5$ and $\operatorname{rk}\operatorname{Pic}(\widetilde{X}) = 3$, which implies that $-K_{\widetilde{X}}$ is not ample, because smooth Fano threefolds with these invariants do not exist [42, Table 3].

Combining $\pi_1 \circ \varphi_1$ and $\pi_2 \circ \varphi_2$, we obtain a morphism $\widetilde{X} \to \mathbb{P}^2 \times \mathbb{P}^2$. Let $\overline{X}$ be its image, and let $\sigma \colon \widetilde{X} \to \overline{X}$ be the induced morphism.

Claim. *The threefold $\overline{X} \subset \mathbb{P}^2 \times \mathbb{P}^2$ is a divisor of bidegree $(2,2)$ with terminal singularities, containing a linearly embedded surface $\mathbb{P}^1 \times \mathbb{P}^1$, and σ is a small resolution.*

Therefore X is obtained by taking a small resolution of the singular Verra threefold $\overline{X}$ containing a divisor $\mathbb{P}^1 \times \mathbb{P}^1$ as in Construction above.

Proof. The threefold $\overline{X}$ is a divisor of bidegree (e_1, e_2) in $\mathbb{P}^2 \times \mathbb{P}^2$, with $e_1, e_2 > 0$ because $\overline{X}$ dominates both factors. We have

$$12 = (h_1 + h_2)^3 = \deg(\sigma)\deg(\overline{X}) = 3\deg(\sigma)(e_1 + e_2)$$

This implies that either $\deg(\sigma) = 1$ and $e_1 + e_2 = 4$, in which case $e_1 = e_2 = 2$ because the two projections give rise to conic bundle structures on $\widetilde{X}$, or $\deg(\sigma) = 2$ and $e_1 + e_2 = 2$ so that $e_1 = e_2 = 1$ because $e_1, e_2 > 0$. In other words

- either $\overline{X}$ is a divisor of degree $(2,2)$, and σ is birational,
- or $\overline{X}$ is a divisor of degree $(1,1)$, and σ is generically two-to-one.

In the former case, σ is crepant, and it follows from the subadjunction formula that the threefold $\overline{X}$ is normal. In the latter case, the threefold $\overline{X}$ is also normal, because there are only two isomorphism classes of irreducible $(1,1)$ divisors in $\mathbb{P}^2 \times \mathbb{P}^2$: one is smooth and the other has one node.

Set $\overline{E} = \sigma(E)$. Let $\mathrm{pr}_1 \colon \overline{X} \to \mathbb{P}^2$ and $\mathrm{pr}_2 \colon \overline{X} \to \mathbb{P}^2$ be the projections to the first and the second factors of the fourfold $\mathbb{P}^2 \times \mathbb{P}^2$, respectively. Then $\mathrm{pr}_1(\overline{E})$ and $\mathrm{pr}_2(\overline{E})$ are lines by $H_1 \cdot C_1 = D_2 \cdot C_2 = 1$, so we can choose coordinates $([x_1 : y_1 : z_1], [x_2 : y_2 : z_2])$ on $\mathbb{P}^2 \times \mathbb{P}^2$ such that

$$\overline{E} = \big\{z_1 = z_2 = 0\big\}.$$

Since $\overline{E} \subset \overline{X}$, we see that $\overline{X}$ is singular. Note also that σ induces an isomorphism $E \cong \overline{E} = \mathbb{P}^1 \times \mathbb{P}^1$.

Divisor classes h_1, h_2 and E generate the group $\mathrm{Pic}(\widetilde{X})$. We have

$$h_1^2 \cdot h_2 = h_1 \cdot h_2^2 = 2, \quad h_1 \cdot h_2 \cdot E = 1, \quad h_1^2 \cdot E = h_2^2 \cdot E = 0.$$

Assume that σ contracts a divisor $F \sim a_1 h_1 + a_2 h_2 + a_3 E$. Then we have

$$\begin{aligned} 2a_2 &= F \cdot h_1^2 = 0, \\ 2a_1 &= F \cdot h_2^2 = 0, \\ 2a_1 + 2a_2 + a_3 &= F \cdot h_1 \cdot h_2 = 0, \end{aligned}$$

which gives $a_1 = 0$, $a_2 = 0$, $a_3 = 0$. This shows that σ does not contract any divisors.

The Stein factorization of σ is the following commutative diagram:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\alpha} & \widehat{X} \\ & {\scriptstyle\sigma}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & \overline{X} & \end{array}$$

where α is a birational morphism, and β is either an isomorphism or a (ramified) double cover. Since σ does not contract divisors and $-K_{\widetilde{X}}$ is not ample, we see that α is a flopping contraction, and $\widehat{X}$ has terminal Gorenstein singularities. We must show that β is an isomorphism.

Suppose β is a double cover. Its Galois involution induces a birational involution $\tau \in \mathrm{Bir}(\widetilde{X})$. Then τ induces an action τ_* on $\mathrm{Pic}(\widetilde{X}) = \mathrm{Cl}(\widehat{X})$ such that $\tau_* h_1 \sim h_1$, $\tau_* h_2 \sim h_2$, and

$$\tau_*(E) \sim b_1 h_1 + b_2 h_2 + b_3 E$$

for some integers b_1, b_2, b_3. Then

$$\begin{aligned} 2b_2 &= \tau_*(E) \cdot h_1^2 = E \cdot h_1^2 = 0, \\ 2b_1 &= \tau_*(E) \cdot h_2^2 = E \cdot h_2^2 = 0, \\ 2b_1 + 2b_2 + b_3 &= \tau_*(E) \cdot h_1 \cdot h_2 = E \cdot h_1 \cdot h_2 = 1, \end{aligned}$$

which gives $b_1 = 0$, $b_2 = 0$, $b_3 = 1$, so $\tau_*(E) \sim E$, which gives $\tau(E) = E$, since E is η-exceptional.

Since $\tau(E) = E$ and σ induces an isomorphism $E \cong \overline{E}$, we see that the surface $\overline{E}$ is contained in the branch divisor of the double cover β. On the other hand, $\overline{E}$ can not be equal to this branch divisor by degree reasons, thus the branch divisor is reducible. This implies that $\widehat{X}$ has non-isolated singularities, which is impossible, since $\widehat{X}$ has terminal singularities. Thus, we see that β is an isomorphism.

We see that $\overline{X}$ is a singular divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ of degree $(2, 2)$, containing $\overline{E}$ and σ is a small resolution. □

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Table describing all possibilities for the Sarkisov link (★).

№	d	I	$h^{1,2}$	$\pi_1\colon X_1 \to Z_1$	$\pi_2\colon X_2 \to Z_2$	References
1	2	1	52	$Z_1 = \mathbb{P}^1$, π_1 is a fibration into del Pezzo surfaces of degree 1.	$Z_2 = \mathbb{P}^1$, π_2 is is a fibration into del Pezzo surfaces of degree 1.	[23, 24, 29], [52, (2.5.2)].
2	6	1	20	$Z_1 = \mathbb{P}^1$, π_1 is a fibration into del Pezzo surfaces of degree 2.	Z_2 is a del Pezzo threefold of degree 1 that has one singular double point, π_2 is the blow up at the singular point.	[10, Proposition 5.6], [23, 24], [47, Example 4.3], [52, (2.7.3)].
3	8	1	14	$Z_1 = \mathbb{P}^1$, π_1 is a fibration into cubic surfaces.	$Z_2 \cong \mathbb{P}^2$, π_2 is a conic bundle with septic discriminant curve.	[10, Proposition 5.9], [47, Example 4.6], [52, (2.9.4)].
4	10	1	10	$Z_1 = \mathbb{P}^1$, π_1 is a fibration into cubic surfaces.	Z_2 is a smooth del Pezzo threefold of degree 2, π_2 is the blow up along a smooth rational curve that has anticanonical degree 2.	[10, Example 1.11], [48, § 3.12 Case 11^o], [52, (2.9.3)].
5	12	1	7	$Z_1 = \mathbb{P}^1$, π_1 is a fibration into quartic del Pezzo surfaces.	$Z_2 \cong \mathbb{P}^3$, π_2 is the blow up along a smooth curve of degree 8 and genus 7.	[28, Proposition 6.5], [52, (2.11.5)].
6	14	1	5	$Z_1 = \mathbb{P}^1$, π_1 is a fibration into quartic del Pezzo surfaces.	Z_2 is a smooth cubic threefold, π_2 is the blow up at a smooth conic.	[28, Proposition 6.5], [48, § 3.13 Case 12^o], [52, (2.11.4)].
7	14	1	5	$Z_1 = \mathbb{P}^2$, π_1 is a conic bundle with quintic discriminant curve.	$Z_2 = \mathbb{P}^2$, π_1 is a conic bundle with quintic discriminant curve.	[48, § 3.4 Case 4^o], Construction and Claim in this paper.
8	16	1	3	$Z_1 = \mathbb{P}^1$, π_1 is a fibration into quintic del Pezzo surfaces.	Z_2 is a smooth quadric in $\mathbb{P}^4$, π_2 is the blow up along a smooth curve of degree 7 and genus 3.	[28, Proposition 6.5], [52, (2.13.4)].

9	16	1	3	$Z_1 = \mathbb{P}^1$, π_1 is a quadric bundle	$Z_2 = \mathbb{P}^1$, π_2 is a fibration into quartic del Pezzo surfaces.	[2, Example 4.9], [52, (2.3.8)], [52, (2.11.2)].
10	18	1	2	$Z_1 = \mathbb{P}^1$, π_1 is a fibration into quintic del Pezzo surfaces.	Z_2 is a smooth complete intersection of two quadrics in $\mathbb{P}^5$, π_2 is the blow up along a twisted cubic.	[28, Proposition 6.5], [52, (2.13.3)].
11	18	1	2	$Z_1 \cong \mathbb{P}^2$, π_1 is a conic bundle with quartic discriminant curve.	$Z_2 = \mathbb{P}^3$, π_2 is the blow up along a smooth curve of degree 6 and genus 2.	[4, Example 4.8], [28, Theorem 7.14], Lemma in this paper.
12	22	1	0	$Z_1 = \mathbb{P}^1$, π_1 is a fibration into quintic del Pezzo surfaces.	$Z_2 \cong \mathbb{P}^2$, π_2 is a $\mathbb{P}^1$-bundle.	[45, (IV)], [52, (2.13.1)].
13	22	1	0	$Z_1 = \mathbb{P}^3$, π_1 is the blow up along a smooth rational curve of degree 5 that is not contained in a quadric.	$Z_2 = \mathbb{P}^3$, π_1 is the blow up along a smooth rational curve of degree 5 that is not contained in a quadric.	[18, Proposition 2.11], [45, (I)].
14	22	1	0	$Z_1 \cong \mathbb{P}^2$, π_1 is a conic bundle with cubic discriminant curve.	Z_2 is a smooth quadric threefold, π_2 is the blow up along a smooth rational quintic curve.	[28, Theorem 7.14], [45, (II)], Lemma in this paper.
15	22	1	0	$Z_1 \cong \mathbb{P}^1$, π_1 is a fibration into sextic del Pezzo surfaces.	$Z_2 \cong V_5$, π_2 is the blow up along a rational quartic curve.	[28, Proposition 6.5], [45, (III)].
16	40	2	0	$Z_1 = \mathbb{P}^1$, π_1 is a quadric bundle.	$Z_2 = \mathbb{P}^2$, π_2 is a $\mathbb{P}^1$-bundle.	[26, Theorem 3.5], [52, (2.3.2)].
17	54	3	0	$Z_1 = \mathbb{P}^1$, π_1 is a $\mathbb{P}^2$-bundle.	$Z_2 = \mathbb{P}^1$, π_2 is a $\mathbb{P}^2$-bundle.	Example in this paper.

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