A New Construction of Enhanced Cross Z-Complementary Sets With Maximum Zero Correlation Zone

Zhen-Ming Huang (黃振銘)[§¶], Cheng-Yu Pai (白承祐)[§†], Zilong Liu (劉子龍)[‡], and Chao-Yu Chen (陳昭羽)[§†]
[§]Institute of Computer and Communication Engineering, National Cheng Kung University, Taiwan
[†]Department of Electrical Engineering, National Cheng Kung University, Taiwan
[‡]School of Computer Science and Electronic Engineering, University of Essex, U.K.
[¶]Department of Engineering Science, National Cheng Kung University, Taiwan
Email: {n98101012, cy_pai}@gs.ncku.edu.tw, zilong.liu@essex.ac.uk, super@mail.ncku.edu.tw

***Abstract*—Recently, the concept of enhanced cross Z-complementary sets (E-CZCS) has been proposed for training sequence design in generalized spatial modulation (GSM). Based on generalized Boolean functions, we present a new construction of E-CZCSs having maximum zero correlation zone (ZCZ) width. Based on the proposed E-CZCSs, numerical simulation results indicate that the resultant training sequences lead to superior channel estimation performance in broadband GSM systems.**

I. Introduction

Sequence sets with good correlation properties are desired for a wide range of communication applications, such as channel estimation [1], synchronization [2], peak-to-mean power control [3]–[6], and interference suppression [7]–[9]. Among many sequence families, complementary sequences have attracted tremendous research attention over the past decades. There are Golay complementary sets (GCSs) whereby a collection of GCSs may be called a mutually orthogonal complementary set (MOCS) [10]. By definition, every GCS exhibits non-trivial zero aperiodic autocorrelation sums of its component sequences and any two distinct GCSs in an MOCS enjoy zero aperiodic cross-correlation sums for all time-shifts. An optimal MOCS is called a complete complementary code (CCC) [11] if the maximum set size is attained. In fact, a special case of GCSs is known as Golay complementary pairs (GCPs), in memory of the discovery of Marcel J. E. Golay, where every GCP consists of two component sequences only [12]. Later, Fan *et al.* proposed in [13] the Z-complementary code set (ZCCS) which can be regarded as a generalization of MOCSs and CCCs, where the aperiodic correlation sums exhibit zero correlation zone (ZCZ) properties.

In 2020, Liu *et al.* introduced the concept of the cross Z-complementary pair (CZCP) as well as its relevant sparse training matrix design for optimal channel estimation in broadband spatial modulation (SM) systems [14]. Unlike the conventional GCPs/ZCPs, CZCPs are proposed to deal with the cross-channel interference when the two sequences in a pair are sent over two non-orthogonal channels. To this end, a unique feature of a CZCP is its cross-channel ZCZ. Many research efforts have been made in recent years for CZCPs with large ZCZ widths and more flexible lengths [15]–[21]. However, the ZCZ width of each CZCP is at most half of its sequence length. To overcome this shortcoming, the cross Z-complementary set (CZCS) with larger ZCZ width is developed [22].

In SM, it is noted that there is only one radio frequency (RF) chain and hence only a single transmit antenna (TA) is allowed to be active at each transmission instance [23]. An extension of SM, called generalized spatial modulation (GSM), provides higher spectral efficiency by allowing several antennas to be active simultaneously [24], [25]. Specifically, the GSM transmitter is configured with fewer RF chains than the total number of TAs. This feature enables GSM to offer flexibility among spectral efficiency, the cost of RF chains, and energy efficiency.

Recently, a subclass of ZCCSs with symmetric ZCZ properties, symmetrical Z-complementary code set (SZCCS), has been proposed for training sequence design in GSM systems [26]. However, their proposed GSM training framework consists of an additional overhead for mitigating inter-antenna interference (IAI) incurred by zero-padding. Thus, their method suffers from a reduced training efficiency. For more efficient training design, CZCP mates are studied by us in [27]. Recently, we further investigated a new class of sequence sets, called enhanced cross Z-complementary sets (E-CZCSs), for the design of optimal training sequences in GSM [28]. The proposed E-CZCS can be regarded as an extension of CZCP mates and CZCSs, and can achieve optimal GSM channel estimation performance with higher training efficiency.

In [28], the constructions of E-CZCSs were proposed. The ZCZ widths of the constructed E-CZCSs are less than or equal to half of their sequence length. In this paper, a new construction of E-CZCSs based on generalized Boolean function is proposed to have a larger ZCZ width. The constructed E-

*The work of Z.-M. Huang, C.-Y. Pai, and C.-Y. Chen was supported by the National Science and Technology Council, Taiwan, R.O.C., under Grants NSTC 112–2221–E–006–122–MY3 and NSTC 113–2927–I–006–501. The work of Z. Liu was supported in part by the UK Engineering and Physical Sciences Research Council under Grants EP/X035352/1 and EP/Y000986/1, by the Royal Society under Grant IEC\R3\223079, and by the Research Council of Norway under Grant 311646/070.

CZCSs can achieve the maximum ZCZ width that is equal to their sequence length. The GSM training framework employing the proposed E-CZCSs can achieve the minimum mean square error (MSE) of the channel estimation over frequency-selective channels.

II. Background and Definitions

For two sequences $\boldsymbol{v}_0 = (v_{0,0}, v_{0,1}, \ldots, v_{0,L-1})$ and $\boldsymbol{v}_1 = (v_{1,0}, v_{1,1}, \ldots, v_{1,L-1})$ with length L over $\mathbb{Z}_q = \{0, 1, \ldots, q-1\}$, we define the aperiodic cross-correlation function of $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ at integer shift μ as follows:

$$\phi(\boldsymbol{v}_0, \boldsymbol{v}_1; \mu) = \begin{cases} \sum_{k=0}^{L-1-\mu} \xi^{g_{0,k+\mu} - g_{1,k}}, & 0 \le \mu \le L-1; \\ \sum_{k=0}^{L-1+\mu} \xi^{g_{0,k} - g_{1,k-\mu}}, & -L+1 \le \mu < 0 \end{cases} \tag{1}$$

where $\xi = e^{2\pi\sqrt{-1}/q}$. When $\boldsymbol{v}_0 = \boldsymbol{v}_1$, $\phi(\boldsymbol{v}_0, \boldsymbol{v}_0; \mu)$ is called the aperiodic autocorrelation function of $\boldsymbol{v}_0$ which is denoted by $\phi(\boldsymbol{v}_0; \mu)$.

Let $\mathcal{V} = \{V^0, V^1, \ldots, V^{M-1}\}$ be a set of M sequence sets where each constitute set consists of N sequences with length L, i.e., $V^s = \{\boldsymbol{v}_0^s, \boldsymbol{v}_1^s, \ldots, \boldsymbol{v}_{N-1}^s\}$, $0 \le s \le M-1$.

Definition 1 (Z-Complementary Code Set): [13] For $V^{s_1}, V^{s_2} \in \mathcal{V}$ with $0 \le s_1, s_2 \le M-1$, if the set $\mathcal{V}$ meets the following condition:

$$\begin{aligned} \phi(V^{s_1}, V^{s_2}; \mu) &\triangleq \sum_{n=0}^{N-1} \phi(\boldsymbol{v}_n^{s_1}, \boldsymbol{v}_n^{s_2}; \mu) \\ &= \begin{cases} 0, & \text{for } 0 < |\mu| < Z,\ s_1 = s_2; \\ 0, & \text{for } |\mu| < Z,\ s_1 \ne s_2, \end{cases} \end{aligned} \tag{2}$$

then it is called an (M, N, L, Z)-ZCCS. When $Z = L$, the set $\mathcal{V}$ is called a *mutually orthogonal complementary set* and when $M = N$, a MOCS is referred to as a *complete complementary code*, denoted by (M, N, L)-MOCS and (M, L)-CCC, respectively.

Let $\mathcal{I}_1 \triangleq \{1, 2, \ldots, Z\}$ and $\mathcal{I}_2 \triangleq \{L-Z, L-Z+1, \ldots, L-1\}$ be two distinct intervals with $Z \le L$.

Definition 2 (Symmetrical Z-Complementary Code Set): [26] A set of M sequence sets $\mathcal{V}$ is called an (M, N, L, Z)-SZCCS, if and only if

$$\begin{aligned} \phi(V^{s_1}, V^{s_2}; \mu) &= \sum_{n=0}^{N-1} \phi(\boldsymbol{v}_n^{s_1}, \boldsymbol{v}_n^{s_2}; \mu) \\ &= \begin{cases} 0, & \text{for } |\mu| \in \mathcal{I}_1 \cup \mathcal{I}_2,\ s_1 = s_2; \\ 0, & \text{for } |\mu| \in \mathcal{I}_1 \cup \mathcal{I}_2 \cup \{0\},\ s_1 \ne s_2 \end{cases} \end{aligned} \tag{3}$$

where $V^{s_1}, V^{s_2} \in \mathcal{V}$ with $0 \le s_1, s_2 \le M-1$.

A. Enhanced Cross Z-Complementary Sets

Definition 3 (Enhanced Cross Z-Complementary Set): [28] Consider a set of M sequence sets $\mathcal{G} = \{G^0, G^1, \ldots, G^{M-1}\}$ where each constitute set consists of N sequences with length L, i.e., $G^s = \{\boldsymbol{g}_0^s, \boldsymbol{g}_1^s, \ldots, \boldsymbol{g}_{N-1}^s\}$, $0 \le s \le M-1$. Then, the set $\mathcal{G}$ is called an (M, N, L, Z)-E-CZCS, if it satisfies the following two conditions:

$$\begin{aligned} \text{(C1):}\quad \phi(G^{s_1}, G^{s_2}; \mu) &= \sum_{n=0}^{N-1} \phi(\boldsymbol{g}_n^{s_1}, \boldsymbol{g}_n^{s_2}; \mu) \\ &= \begin{cases} 0, & \text{for } |\mu| \in (\mathcal{I}_1 \cup \mathcal{I}_2) \cap \mathcal{I},\ s_1 = s_2; \\ 0, & \text{for } |\mu| \in \mathcal{I}_1 \cup \mathcal{I}_2 \cup \{0\}, s_1 \ne s_2; \end{cases} \end{aligned} \tag{4}$$

$$\begin{aligned} \text{(C2):}\quad \hat{\phi}(G^{s_1}, G^{s_2}; \mu) \triangleq \sum_{n=0}^{N-1} \phi(\boldsymbol{g}_n^{s_1}, \boldsymbol{g}_{(n+1)_{\bmod N}}^{s_2}; \mu) = 0, \\ \text{for } |\mu| \in \mathcal{I}_2 \text{ and any } s_1, s_2 \in \{0, 1, \ldots, M-1\} \end{aligned} \tag{5}$$

where $G^{s_1}, G^{s_2} \in \mathcal{G}$ with $0 \le s_1, s_2 \le M-1$ and $\mathcal{I} = \{1, 2, \ldots, L-1\}$. When M equals 1, i.e., $s_1 = s_2 = 0$, the E-CZCS becomes CZCS. It can be observed that each G^{s_1} in an E-CZCS is a CZCS. (C1) indicates that the correlation sum $\phi(G^{s_1}, G^{s_2}; \mu)$ exhibits symmetric ZCZs for time shifts over $\mathcal{I}_1$ and $\mathcal{I}_2$. Additionally, (C2) implies that the cross-correlation sum $\hat{\phi}(G^{s_1}, G^{s_2}; \mu)$ possesses a tail-end ZCZ over $\mathcal{I}_2$. The correlation properties of SZCCSs and E-CZCSs are depicted in Fig. 1. It can be seen that E-CZCSs possess an additional cross-correlation property.

We illustrate the relationship between E-CZCSs and relevant sequence sets, including SZCCSs, ZCCSs, and MOCSs, in Fig. 2. Specifically, the E-CZCS incorporates a SZCCS and a ZCCS as specific cases according to condition (C1) in (4). However, the SZCCS and ZCCS do not account for condition (C2) in (5). Moreover, for an (M, N, L, Z)-E-CZCS with $Z \ge L/2$, i.e., $(\mathcal{I}_1 \cup \mathcal{I}_2) \cap \mathcal{I} = \{1, 2, \ldots, L-1\}$, the condition (C1) means that an (M, N, L, Z)-E-CZCS is also an MOCS.

Lemma 1: [28] For an (M, N, L, Z)-E-CZCS $\mathcal{G} = \{G^0, G^1, \ldots, G^{M-1}\}$, there is an upper bound on ZCZ width given as $Z \le \frac{NL}{M} - 1$. For the binary E-CZCS, we have

$$Z \le \frac{NL}{2M}. \tag{6}$$

B. Generalized Boolean Functions

A generalized Boolean function $f(x_1, x_2, \ldots, x_m)$ with m variables is defined as a function mapping from $\mathbb{Z}_2^m = \{(x_1, x_2, \ldots, x_m) | x_1, x_2, \ldots, x_m \in \mathbb{Z}_2\}$ to $\mathbb{Z}_q$. Let $(i_1, i_2, \ldots, i_m)$ be the binary vector of the non-negative integer i such that $i = \sum_{k=1}^{m} i_k 2^{k-1}$. Given a generalized Boolean function f, we can specify a sequence $\boldsymbol{f} = (f_0, f_1, \ldots, f_{2^m-1})$ where $f_i = f(i_1, i_2, \ldots, i_m)$ for $i = 0, 1, \ldots, 2^m - 1$.

Example 1: Suppose that $q = 2$ and $m = 4$. The sequence $\boldsymbol{f}$ of length 16 is as follows:

$$\begin{aligned} \boldsymbol{f} &= (f_0, f_1, \ldots, f_{15}) \\ &= (f(0,0,0,0), f(1,0,0,0), \ldots, f(1,1,1,1)). \end{aligned} \tag{7}$$

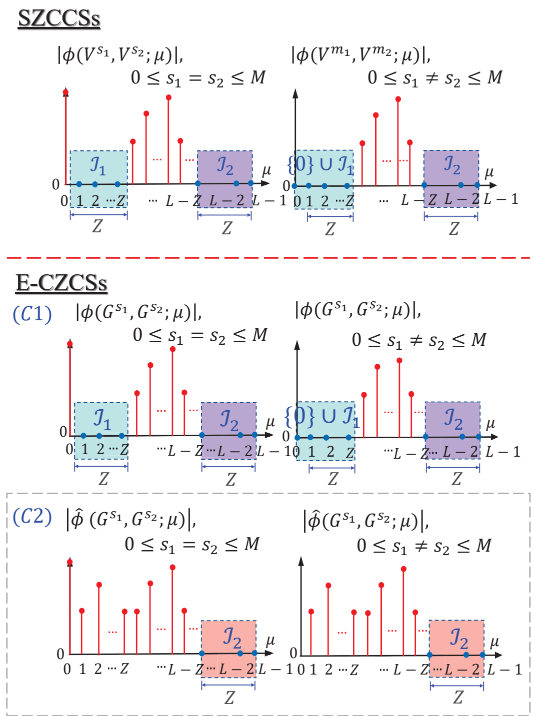


Fig. 1: The correlation properties of SZCCSs and E-CZCSs.

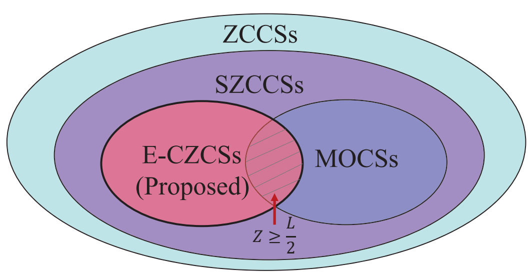


Fig. 2: Relationship between E-CZCSs and relevant sequence sets.

Then, the associated sequences for the generalized Boolean function x_2, x_4, x_2x_4, 1, and $x_2+x_2x_4+1$ are shown below, respectively.

$$\begin{aligned} \boldsymbol{x}_2 &= (0,0,1,1,0,0,1,1,0,0,1,1,0,0,1,1),\\ \boldsymbol{x}_4 &= (0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1),\\ \boldsymbol{x}_2\boldsymbol{x}_4 &= (0,0,0,0,0,0,0,0,0,0,1,1,0,0,1,1),\\ \mathbf{1} &= (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1),\\ \boldsymbol{x}_2+\boldsymbol{x}_2\boldsymbol{x}_4+\mathbf{1} &= (1,1,0,0,1,1,0,0,1,1,1,1,1,1,1,1). \end{aligned} \tag{8}$$

Lemma 2: [29] For positive integers m and k with $k \le m$, the set $\{1,2,\cdots,m\}$ is specifically divided into k nonempty partitions, denoted as $P_1, P_2, \cdots, P_k$. We define π_κ as a bijection mapping from $\{1,2,\cdots,m_\kappa\}$ to P_κ where $m_\kappa = |P_\kappa| \ge 1$ for $\kappa = 1,2,\cdots,k$. Let the generalized Boolean function

$$f = \frac{q}{2}\sum_{\kappa=1}^{k}\sum_{z=1}^{m_\kappa-1} x_{\pi_\kappa(z)}x_{\pi_\kappa(z+1)} + \sum_{z=1}^{m}\nu_z x_z + \nu_0 \tag{9}$$

where q is an even positive integer and ν_z's $\in \mathbb{Z}_q$. Let $(n_1,n_2,\ldots,n_k)$ and $(s_1,s_2,\ldots,s_k)$ be binary vectors of n and s, respectively. For $s = 0,1,\ldots,2^k-1$, we let $V^s = \{\boldsymbol{v}_0^s, \boldsymbol{v}_1^s, \ldots, \boldsymbol{v}_{2^k-1}^s\}$ where

$$\boldsymbol{v}_n^s = \boldsymbol{f} + \frac{q}{2}\sum_{\kappa=1}^{k} n_\kappa \boldsymbol{x}_{\pi_\kappa(1)} + \frac{q}{2}\sum_{\kappa=1}^{k} s_\kappa \boldsymbol{x}_{\pi_\kappa(m_\kappa)} \tag{10}$$

for $n = 0,1,\ldots,2^k-1$. Then, the set $\mathcal{V} = \{V^0, V^1, \ldots, V^{2^k-1}\}$ is a $(2^k, 2^m)$-CCC.

III. Proposed E-CZCSs with Maximum ZCZ

In this section, we will present a theorem to construct E-CZCSs based on generalized Boolean function. The proposed E-CZCSs can have the maximum ZCZ width.

Theorem 1: Given the Boolean function f given by (9). Let $(n_1,n_2,\ldots,n_{k+1})$ and $(s_1,s_2,\ldots,s_k)$ be binary vectors of n and s, respectively. For $s = 0,1,\ldots,2^k-1$, the set $G^s = \{\boldsymbol{g}_0^s, \boldsymbol{g}_1^s, \ldots, \boldsymbol{g}_{2^{k+1}-1}^s\}$ can be constructed as follows:

$$\boldsymbol{g}_n^s = \boldsymbol{f} + \frac{q}{2}\left(\sum_{\kappa=1}^{k} n_\kappa \boldsymbol{x}_{\pi_\kappa(1)} + n_{k+1}n_1\cdot\mathbf{1} + \sum_{\kappa=1}^{k} s_\kappa \boldsymbol{x}_{\pi_\kappa(m_\kappa)}\right) \tag{11}$$

for $n = 0,1,\ldots,2^{k+1}-1$. Then, the set $\mathcal{G} = \{G^0, G^1, \cdots, G^{2^k-1}\}$ is a $(2^k, 2^{k+1}, 2^m, 2^m)$-E-CZCS.

Proof: To prove that the set $\mathcal{G}$ is a $(2^k, 2^{k+1}, 2^m, 2^m)$-E-CZCS, it is sufficient to prove the (C1) and (C2) in (4) and (5), respectively. First, according to *Lemma 2*, we know that $\{\{\boldsymbol{g}_0^s, \boldsymbol{g}_1^s, \ldots, \boldsymbol{g}_{2^k-1}^s\} | s \in \{0,1,\ldots,2^k-1\}\}$ is a $(2^k, 2^m)$-CCC. Then, $\{\{\boldsymbol{g}_{2^k}^s, \boldsymbol{g}_{2^k+1}^s, \ldots, \boldsymbol{g}_{2^{k+1}-1}^s\} | s \in \{0,1,\ldots,2^k-1\}\}$ is also a $(2^k, 2^m)$-CCC. For $G^{s_1}, G^{s_2} \in \mathcal{G}$ with $0 \le s_1, s_2 \le 2^k-1$, we have

$$\begin{aligned} &\phi(G^{s_1}, G^{s_2}; \mu)\\ &= \sum_{n=0}^{2^k-1}\phi(\boldsymbol{g}_n^{s_1}, \boldsymbol{g}_n^{s_2}; \mu) + \sum_{n=2^k}^{2^{k+1}-1}\phi(\boldsymbol{g}_n^{s_1}, \boldsymbol{g}_n^{s_2}; \mu) = 0, \end{aligned}$$

for $|\mu| \ne 0$, $s_1 = s_2$, and for all $|\mu|$, $s_1 \ne s_2$. Therefore, (4) holds. Second, we want to prove that, for $0 \le |\mu| \le 2^m-1$,

$$\begin{aligned} \hat{\phi}(G^{s_1}, G^{s_2}; \mu) &= \sum_{n=0}^{N-1}\phi(\boldsymbol{g}_n^{s_1}, \boldsymbol{g}_{(n+1)_{\bmod N}}^{s_2}; \mu)\\ &= \phi(\boldsymbol{g}_0^{s_1}, \boldsymbol{g}_1^{s_2}; \mu) + \phi(\boldsymbol{g}_1^{s_1}, \boldsymbol{g}_2^{s_2}; \mu) + \cdots + \phi(\boldsymbol{g}_{N-1}^{s_1}, \boldsymbol{g}_0^{s_2}; \mu)\\ &= \sum_{i=0}^{L-1-\mu}\Big(\xi^{g_{0,i+\mu}^{s_1} - g_{1,i}^{s_2}} + \xi^{g_{1,i+\mu}^{s_1} - g_{2,i}^{s_2}} + \cdots\\ &\qquad + \xi^{g_{N-2,i+\mu}^{s_1} - g_{N-1,i}^{s_2}} + \xi^{g_{N-1,i+\mu}^{s_1} - g_{0,i}^{s_2}}\Big)\\ &= \sum_{i=0}^{L-1-\mu}\sum_{\eta=0}^{\frac{N}{2}-1}\Big(\xi^{g_{\eta,j}^{s_1} - g_{(\eta+1),i}^{s_2}}\\ &\qquad + \xi^{g_{(\eta+2^{k+1}),j}^{s_1} - g_{(\eta+2^{k+1}+1)_{\bmod N},i}^{s_2}}\Big) \end{aligned}$$

where $N = 2^{k+1}$ and $j = i + u$. Therefore, we can obtain

$$\begin{aligned} &g^{s_1}_{\eta,j} - g^{s_2}_{(\eta+1),i} - g^{s_1}_{(\eta+2^{k+1}),j} + g^{s_2}_{(\eta+2^{k+1}+1)_{\bmod N},i} \\ &\qquad = \begin{cases} \frac{q}{2}, \text{ for } \eta = 0, 2, \ldots; \\ -\frac{q}{2}, \text{ for } \eta = 1, 3, \ldots \end{cases} \end{aligned}$$

implying $\xi^{g^{s_1}_{\eta,j} - g^{s_2}_{(\eta+1),i}} + \xi^{g^{s_1}_{(\eta+2^{k+1}),j} - g^{s_2}_{(\eta+2^{k+1}+1)_{\bmod N},i}} = 0$. Hence, (5) also holds. Combining the above two parts, the set $\mathcal{G}$ is a $(2^k, 2^{k+1}, 2^m, 2^m)$-E-CZCS. ■

Remark 1: By considering $q = 2$ in *Theorem 1*, binary $(2^k, 2^{k+1}, 2^m, 2^m)$-E-CZCS can be constructed and the ZCZ width satisfies the binary upper bound

$$2^m = \frac{2^m \cdot 2^{k+1}}{2 \cdot 2^k}.$$

It implies that the maximum ZCZ width of the $(2^k, 2^{k+1}, 2^m, 2^m)$-E-CZCS is equal to its sequence length. This $(2^k, 2^{k+1}, 2^m, 2^m)$-E-CZCS cannot be constructed by [28] and has not been reported in the literature.

Remark 2: The $(2^k, 2^{k+1}, 2^m, 2^m)$-E-CZCS from Theorem 1 are also MOCSs. Furthermore, the proposed E-CZCSs can exhibit the unique cross-correlation property (C2) in (5).

Example 2: Let $m = 4$, $k = 2$, and $q = 2$. The set $\{1, 2, 3, 4\}$ is divided into two nonempty partitions $P_1 = \{2, 3\}$ and $P_2 = \{1, 4\}$ with $m_1 = 2$ and $m_2 = 2$. We also let $\pi_1 = (3, 2)$ and $\pi_2 = (1, 4)$ indicating $\pi_1(1) = 3$, $\pi_1(2) = 2$, $\pi_2(1) = 1$, and $\pi_2(2) = 4$. Then, the Boolean function f in (9) can be expressed as $f = x_3x_2 + x_1x_4 + x_2 + x_4$ by letting $\nu_0 = 0$, $\nu_1 = 0$, $\nu_2 = 1$, $\nu_3 = 0$, and $\nu_4 = 1$. According to *Theorem 1*, the set $\mathcal{G} = \{G^s = \{\boldsymbol{g}^s_0, \boldsymbol{g}^s_1, \ldots, \boldsymbol{g}^s_7\} | s \in \{0, 1, 2, 3\}\}$ is a binary $(4, 8, 16, 16)$-E-CZCS of which $\boldsymbol{g}^s_n = \boldsymbol{f} + n_1\boldsymbol{x}_3 + n_2\boldsymbol{x}_1 + n_3n_1\boldsymbol{1} + s_1\boldsymbol{x}_2 + s_2\boldsymbol{x}_4$. In Fig. 3-a, we can see that the correlation sum $\phi(G^{s_1}, G^{s_2}; \mu) = 0$ for all μ when $s_1 \neq s_2$ and $\phi(G^{s_1}, G^{s_2}; \mu) = 0$ for $\mu \neq 0$ when $s_1 = s_2$. Also, the correlation sum $\hat{\phi}(G^{s_1}, G^{s_2}; \mu) = 0$ for all μ and $s_1, s_2 = 0, 1, 2, 3$ as shown in Fig. 3-b. Therefore, the set $\mathcal{G} = \{G^0, G^1, G^2, G^3\}$ is indeed a $(4, 8, 16, 16)$-E-CZCS.

In Table I, the comparison of our proposed construction with the existing constructions of E-CZCSs in [28] is outlined. Our proposed E-CZCSs differ from existing works in that their maximum ZCZ is equal to their sequence length.

IV. Proposed GSM Training Framework: Design and Simulation

In this section, we will introduce the training design based on the proposed E-CZCSs for the broadband GSM system. Then, we evaluate its channel estimation performances over frequency-selective fading channels.

A. Training Framework

The reader is referred to [28] for a detailed formulation of training framework employing E-CZCSs for broadband GSM system. For the $(4, 8, L, L)$-E-CZCS $\mathcal{G} = \{G^s = \{\boldsymbol{g}^s_0, \boldsymbol{g}^s_1, \ldots, \boldsymbol{g}^s_7\} | s \in \{0, 1, 2, 3\}\}$ from *Theorem 1*, we can generate a GSM training matrix as illustrated in (12) where $\mathbf{0}$ denotes the $1 \times L$ all-zero vector.

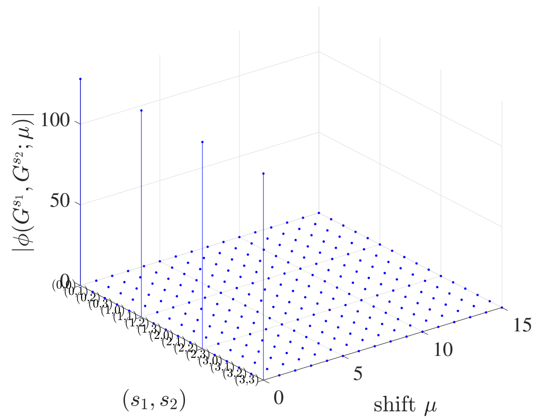
B. Simulation Result

In this subsection, we evaluate the GSM channel estimation performances of the training framework based on the proposed E-CZCS by comparing it with various sequence sets including ZCCS, CCC, SZCCS, Zadoff-Chu sequences, and binary random sequences. The setup is as follows. We set $N_t = 8$ TAs, $N_a = 4$ RF chains, and one receive antenna. The $(\lambda + 1)$-paths are separated by integer symbol durations as $h[t] = \sum_{l=0}^{\lambda} h_l \delta[t - lT]$ where h_l's are complex Gaussian random variables with zero mean and $E\left(|h_l|^2\right) = 1/(\lambda + 1)$ for all l. We use the binary $(4, 8, 16, 16)$-E-CZCS from *Example 2* to generate the GSM training matrix $\boldsymbol{\Psi}$ as depicted in (12). For the SZCCS and the ZCCS, their corresponding training matrix $\boldsymbol{\Psi}'$ is expressed as

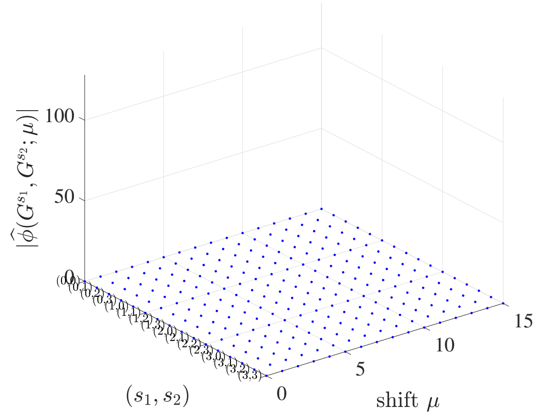
$$\boldsymbol{\Psi}' = \begin{bmatrix} \boldsymbol{v}^0_0 & \mathbf{0} & \boldsymbol{v}^0_1 & \mathbf{0} \\ \boldsymbol{v}^1_0 & \mathbf{0} & \boldsymbol{v}^1_1 & \mathbf{0} \\ \boldsymbol{v}^2_0 & \mathbf{0} & \boldsymbol{v}^2_1 & \mathbf{0} \\ \boldsymbol{v}^3_0 & \mathbf{0} & \boldsymbol{v}^3_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{v}^0_0 & \mathbf{0} & \boldsymbol{v}^0_1 \\ \mathbf{0} & \boldsymbol{v}^1_0 & \mathbf{0} & \boldsymbol{v}^1_1 \\ \mathbf{0} & \boldsymbol{v}^2_0 & \mathbf{0} & \boldsymbol{v}^2_1 \\ \mathbf{0} & \boldsymbol{v}^3_0 & \mathbf{0} & \boldsymbol{v}^3_1 \end{bmatrix}_{8 \times 256} \tag{13}$$

where $\{\{\boldsymbol{v}^0_0, \boldsymbol{v}^0_1\}, \{\boldsymbol{v}^1_0, \boldsymbol{v}^1_1\}, \{\boldsymbol{v}^2_0, \boldsymbol{v}^2_1\}, \{\boldsymbol{v}^3_0, \boldsymbol{v}^3_1\}\}$ represents the $(4, 2, 64, 32)$-ZCCS and the first four sequence sets of the $(8, 2, 64, 15)$-SZCCS from [26], respectively. In the case of binary random sequences, the elements of $\boldsymbol{v}^0_0$, $\boldsymbol{v}^0_1$, $\boldsymbol{v}^1_0$, $\boldsymbol{v}^1_1$, $\boldsymbol{v}^2_0$, $\boldsymbol{v}^2_1$, $\boldsymbol{v}^3_0$, $\boldsymbol{v}^3_1$ in $\boldsymbol{\Psi}'$ are randomly generated from the alphabet set of $\{-1, +1\}$. When using Zadoff-Chu sequences for the training matrix $\boldsymbol{\Psi}'$, the sequences represented by $\boldsymbol{v}^0_0$, $\boldsymbol{v}^0_1$, $\boldsymbol{v}^1_0$, $\boldsymbol{v}^1_1$, $\boldsymbol{v}^2_0$, $\boldsymbol{v}^2_1$, $\boldsymbol{v}^3_0$, $\boldsymbol{v}^3_1$ are determined by eight distinct Zadoff-Chu sequences, each of length 64, with low cross-correlation. For the $(4, 4, 32)$-CCC $\mathcal{V} = \{V^s = \{\boldsymbol{v}^s_0, \boldsymbol{v}^s_1, \boldsymbol{v}^s_2, \boldsymbol{v}^s_3\} | s \in$

$$\boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \\ \boldsymbol{x}_4 \\ \boldsymbol{x}_5 \\ \boldsymbol{x}_6 \\ \boldsymbol{x}_7 \\ \boldsymbol{x}_8 \end{bmatrix} = \begin{bmatrix} \boldsymbol{g}^0_0 & \mathbf{0} & \boldsymbol{g}^0_1 & \mathbf{0} & \boldsymbol{g}^0_2 & \mathbf{0} & \boldsymbol{g}^0_3 & \mathbf{0} & \boldsymbol{g}^0_4 & \mathbf{0} & \boldsymbol{g}^0_5 & \mathbf{0} & \boldsymbol{g}^0_6 & \mathbf{0} & \boldsymbol{g}^0_7 & \mathbf{0} \\ \boldsymbol{g}^1_0 & \mathbf{0} & \boldsymbol{g}^1_1 & \mathbf{0} & \boldsymbol{g}^1_2 & \mathbf{0} & \boldsymbol{g}^1_3 & \mathbf{0} & \boldsymbol{g}^1_4 & \mathbf{0} & \boldsymbol{g}^1_5 & \mathbf{0} & \boldsymbol{g}^1_6 & \mathbf{0} & \boldsymbol{g}^1_7 & \mathbf{0} \\ \boldsymbol{g}^2_0 & \mathbf{0} & \boldsymbol{g}^2_1 & \mathbf{0} & \boldsymbol{g}^2_2 & \mathbf{0} & \boldsymbol{g}^2_3 & \mathbf{0} & \boldsymbol{g}^2_4 & \mathbf{0} & \boldsymbol{g}^2_5 & \mathbf{0} & \boldsymbol{g}^2_6 & \mathbf{0} & \boldsymbol{g}^2_7 & \mathbf{0} \\ \boldsymbol{g}^3_0 & \mathbf{0} & \boldsymbol{g}^3_1 & \mathbf{0} & \boldsymbol{g}^3_2 & \mathbf{0} & \boldsymbol{g}^3_3 & \mathbf{0} & \boldsymbol{g}^3_4 & \mathbf{0} & \boldsymbol{g}^3_5 & \mathbf{0} & \boldsymbol{g}^3_6 & \mathbf{0} & \boldsymbol{g}^3_7 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{g}^0_0 & \mathbf{0} & \boldsymbol{g}^0_1 & \mathbf{0} & \boldsymbol{g}^0_2 & \mathbf{0} & \boldsymbol{g}^0_3 & \mathbf{0} & \boldsymbol{g}^0_4 & \mathbf{0} & \boldsymbol{g}^0_5 & \mathbf{0} & \boldsymbol{g}^0_6 & \mathbf{0} & \boldsymbol{g}^0_7 \\ \mathbf{0} & \boldsymbol{g}^1_0 & \mathbf{0} & \boldsymbol{g}^1_1 & \mathbf{0} & \boldsymbol{g}^1_2 & \mathbf{0} & \boldsymbol{g}^1_3 & \mathbf{0} & \boldsymbol{g}^1_4 & \mathbf{0} & \boldsymbol{g}^1_5 & \mathbf{0} & \boldsymbol{g}^1_6 & \mathbf{0} & \boldsymbol{g}^1_7 \\ \mathbf{0} & \boldsymbol{g}^2_0 & \mathbf{0} & \boldsymbol{g}^2_1 & \mathbf{0} & \boldsymbol{g}^2_2 & \mathbf{0} & \boldsymbol{g}^2_3 & \mathbf{0} & \boldsymbol{g}^2_4 & \mathbf{0} & \boldsymbol{g}^2_5 & \mathbf{0} & \boldsymbol{g}^2_6 & \mathbf{0} & \boldsymbol{g}^2_7 \\ \mathbf{0} & \boldsymbol{g}^3_0 & \mathbf{0} & \boldsymbol{g}^3_1 & \mathbf{0} & \boldsymbol{g}^3_2 & \mathbf{0} & \boldsymbol{g}^3_3 & \mathbf{0} & \boldsymbol{g}^3_4 & \mathbf{0} & \boldsymbol{g}^3_5 & \mathbf{0} & \boldsymbol{g}^3_6 & \mathbf{0} & \boldsymbol{g}^3_7 \end{bmatrix}_{8 \times 16L} \tag{12}$$



(a) The correlation sum $\phi(G^{s_1}, G^{s_2}; \mu)$.



(b) The correlation sum $\hat{\phi}(G^{s_1}, G^{s_2}; \mu)$.

Fig. 3: The correlation properties of the $(4, 8, 16, 16)$-E-CZCS in Example 2.

TABLE I: The Comparison with Different Constructions of (M, N, L, Z)-E-CZCSs

Ref.	Set Size	Flock Size	Length	ZCZ width	Z/L	Based on
[28, Th. 2]	M	N	$2L$	Z	$\frac{Z}{2L}$	$(M, N, L, Z+1)$-ZCCS
	M	N	$2L$	L	$\frac{1}{2}$	(M, N, L)-MOCS
	M	M	$2L$	L	$\frac{1}{2}$	(M, L)-CCC
[28, Th. 3]	2^k	2^v	2^m	$2^{m-k+v-1}$	2^{v-k-1} $(v \leq k)$	Generalized Boolean functions
Theorem 1	2^k	2^{k+1}	2^m	2^m	1	Generalized Boolean functions

$\{0, 1, 2, 3\}\}$ from [29], the training matrix $\mathbf{\Psi}''$ is expressed as

$$\mathbf{\Psi}'' = \left[\begin{array}{cc:cc:cc:cc} \boldsymbol{v}_0^0 & \mathbf{0} & \boldsymbol{v}_1^0 & \mathbf{0} & \boldsymbol{v}_2^0 & \mathbf{0} & \boldsymbol{v}_3^0 & \mathbf{0} \\ \boldsymbol{v}_0^1 & \mathbf{0} & \boldsymbol{v}_1^1 & \mathbf{0} & \boldsymbol{v}_2^1 & \mathbf{0} & \boldsymbol{v}_3^1 & \mathbf{0} \\ \boldsymbol{v}_0^2 & \mathbf{0} & \boldsymbol{v}_1^2 & \mathbf{0} & \boldsymbol{v}_2^2 & \mathbf{0} & \boldsymbol{v}_3^2 & \mathbf{0} \\ \boldsymbol{v}_0^3 & \mathbf{0} & \boldsymbol{v}_1^3 & \mathbf{0} & \boldsymbol{v}_2^3 & \mathbf{0} & \boldsymbol{v}_3^3 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{v}_0^0 & \mathbf{0} & \boldsymbol{v}_1^0 & \mathbf{0} & \boldsymbol{v}_2^0 & \mathbf{0} & \boldsymbol{v}_3^0 \\ \mathbf{0} & \boldsymbol{v}_0^1 & \mathbf{0} & \boldsymbol{v}_1^1 & \mathbf{0} & \boldsymbol{v}_2^1 & \mathbf{0} & \boldsymbol{v}_3^1 \\ \mathbf{0} & \boldsymbol{v}_0^2 & \mathbf{0} & \boldsymbol{v}_1^2 & \mathbf{0} & \boldsymbol{v}_2^2 & \mathbf{0} & \boldsymbol{v}_3^2 \\ \mathbf{0} & \boldsymbol{v}_0^3 & \mathbf{0} & \boldsymbol{v}_1^3 & \mathbf{0} & \boldsymbol{v}_2^3 & \mathbf{0} & \boldsymbol{v}_3^3 \end{array}\right]_{8 \times 256} \tag{14}$$

Fig. 4 demonstrates the channel estimation MSE performances with different numbers of multi-paths at $E_b/N_0 = 16$ dB. We can observe that our GSM training matrix based on the proposed $(4, 8, 16, 16)$-E-CZCS outperforms, whilst achieving the MSE lower bound when the number of multi-paths is not larger than the ZCZ width. For the SZCCS-based, ZCCS-based, and CCC-based GSM training matrices, they suffer from worse channel estimation performances because their corresponding sequence sets ignore the condition (C2) in (5), leading to nonzero IAI.

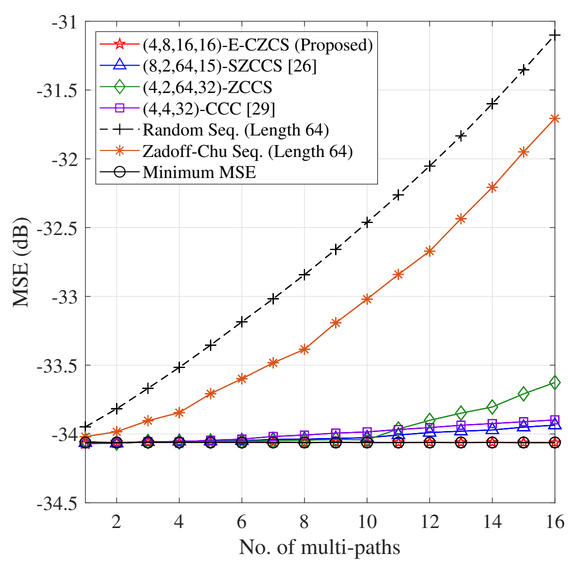


Fig. 4: The comparison of MSE for GSM training based on different sequences with 8 TAs.

V. Conclusion

In this paper, we have presented a new construction of E-CZCSs with maximum ZCZ (equal to their sequence length) based on generalized Boolean functions. The proposed GSM training framework achieves the minimum channel estimation MSE over frequency-selective channels when the number of multi-paths is not larger than the ZCZ width. A potential topic for future research is to construct E-CZCSs with flexible lengths based on generalized Boolean functions.

References

[1] P. Spasojevic and C. N. Georghiades, "Complementary sequences for ISI channel estimation," *IEEE Trans. Inf. Theory*, vol. 47, no. 3, pp. 1145–1152, Mar. 2001.

[2] J. M. Groenewald and B. T. Maharaj, "MIMO channel synchronization using Golay complementary pairs," in *Proc. AFRICON 2007*, Windhoek, South Africa, Sep. 2007, pp. 1–5.

[3] R. D. J. van Nee, "OFDM codes for peak-to-average power reduction and error correction," in *Proc. IEEE Global Telecommun. Conf.*, London, U.K., Nov. 1996, pp. 740–744.

[4] J. A. Davis and J. Jedwab, "Peak-to-mean power control in OFDM, Golay complementary sequences, and Reed-Müller codes," *IEEE Trans. Inf. Theory*, vol. 45, no. 7, pp. 2397–2417, Nov. 1999.

[5] K. G. Paterson, "Generalized Reed-Müller codes and power control in OFDM modulation," *IEEE Trans. Inf. Theory*, vol. 46, no. 1, pp. 104–120, Jan. 2000.

[6] C.-Y. Chen, "Complementary sets of non-power-of-two length for peak-to-average power ratio reduction in OFDM," *IEEE Trans. Inf. Theory*, vol. 62, no. 12, pp. 7538–7545, Dec. 2016.

[7] S.-M. Tseng and M. R. Bell, "Asynchronous multicarrier DS-CDMA using mutually orthogonal complementary sets of sequences," *IEEE Trans. Commun.*, vol. 48, pp. 53–59, Jan. 2000.

[8] H.-H. Chen, J.-F. Yeh, and N. Suehiro, "A multicarrier CDMA architecture based on orthogonal complete complementary codes for new generations of wideband wireless communications," *IEEE Commun. Mag.*, vol. 39, no. 10, pp. 126–134, Oct. 2001.

[9] Z. Liu, Y. L. Guan, and H.-H. Chen, "Fractional-delay-resilient receiver design for interference-free MC-CDMA communications based on complete complementary codes," *IEEE Trans. Wireless Commun.*, vol. 14, no. 3, pp. 1226–1236, Mar. 2015.

[10] C.-C. Tseng and C. L. Liu, "Complementary sets of sequences," *IEEE Trans. Inf. Theory*, vol. IT-18, no. 5, pp. 644–652, Sep. 1972.

[11] N. Suehiro and M. Hatori, "N-shift cross-orthogonal sequences," *IEEE Trans. Inf. Theory*, vol. 34, no. 1, pp. 143–146, Jan. 1988.

[12] M. J. E. Golay, "Complementary series," *IRE Trans. Inf. Theory*, vol. 7, no. 2, pp. 82–87, Apr. 1961.

[13] P. Fan, W. Yuan, and Y. Tu, "Z-complementary binary sequences," *IEEE Signal Process. Lett.*, vol. 14, no. 8, pp. 509–512, Aug. 2007.

[14] Z. Liu, P. Yang, Y. L. Guan, and P. Xiao, "Cross Z-complementary pairs for optimal training in spatial modulation over frequency selective channels," *IEEE Trans. Signal Process.*, vol. 68, pp. 1529–1543, Feb. 2020.

[15] C. Fan, D. Zhang, and A. R. Adhikary, "New sets of binary cross Z-complementary sequence pairs," *IEEE Commun. Lett.*, vol. 24, no. 8, pp. 1616–1620, Aug. 2020.

[16] A. R. Adhikary, Z. Zhou, Y. Yang, and P. Fan, "Constructions of cross Z-complementary pairs with new lengths," *IEEE Trans. Signal Process.*, vol. 68, pp. 4700–4712, 2020.

[17] Z.-M. Huang, C.-Y. Pai, and C.-Y. Chen, "Binary cross Z-complementary pairs with flexible lengths from Boolean functions," *IEEE Commun. Lett.*, vol. 25, no. 4, pp. 1057–1061, Apr. 2021.

[18] M. Yang, S. Tian, N. Li, and A. R. Adhikary, "New sets of quadriphase cross Z-complementary pairs for preamble design in spatial modulation," *IEEE Signal Process. Lett.*, vol. 28, pp. 1240–1244, 2021.

[19] F. Zeng, X. He, Z. Zhang, and L. Yan, "Quadriphase cross Z-complementary pairs for pilot sequence design in spatial modulation systems," *IEEE Signal Process. Lett.*, vol. 29, pp. 508–512, 2022.

[20] S. Das, A. Banerjee, and Z. Liu, "New family of cross Z-complementary sequences with large ZCZ width," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Espoo, Finland, Jun 2022, pp. 522–527.

[21] H. Zhang, C. Fan, Y. Yang, and S. Mesnager, "New binary cross Z-complementary pairs with large CZC ratio," *IEEE Trans. Inf. Theory*, vol. 69, no. 2, pp. 1328–1336, Feb. 2023.

[22] Z.-M. Huang, C.-Y. Pai, and C.-Y. Chen, "Cross Z-complementary sets for training design in spatial modulation," *IEEE Trans. Commun.*, vol. 70, no. 8, pp. 5030–5045, Aug. 2022.

[23] R. Y. Mesleh, H. Haas, S. Sinanovic, C. W. Ahn, and S. Yun, "Spatial modulation," *IEEE Trans. Veh. Technol.*, vol. 57, no. 4, pp. 2228–2241, Jul. 2008.

[24] A. Younis, N. Serafimovski, R. Mesleh, and H. Haas, "Generalised spatial modulation," in *Proc. Conf. Rec. Forty 4th Asilomar Conf. Signals, Syst. Comput.*, Pacific Grove, CA, USA, Nov. 2010, pp. 1498–1502.

[25] J. Fu, C. Hou, W. Xiang, L. Yan, and Y. Hou, "Generalised spatial modulation with multiple active transmit antennas," in *Proc. IEEE Globecom Workshops*, Miami, FL, USA, Dec. 2010, pp. 839–844.

[26] Y. Zhou, Z. Zhou, Z. Liu, Y. Yang, P. Yang, and P. Fan, "Symmetrical Z-complementary code sets for optimal training in generalized spatial modulation," *Signal Processing*, vol. 208, Jul. 2023, Art. no. 108990.

[27] Z.-M. Huang, C. Hu, Z. Liu, and C.-Y. Chen, "On the mates of cross Z-complementary pairs for training sequence design in generalized spatial modulation," *IEEE Access*, vol. 11, pp. 145 231–145 237, Dec. 2023.

[28] Z.-M. Huang, C.-Y. Pai, Z. Liu, and C.-Y. Chen, "Enhanced cross Z-complementary set and its application in generalized spatial modulation," *arXiv:2311.18390v2*, Dec. 2023, [Online]. Available: https://arxiv.org/pdf/2311.18390.pdf.

[29] C.-Y. Chen, C.-H. Wang, and C.-C. Chao, "Complete complementary codes and generalized Reed-Muller codes," *IEEE Commun. Lett.*, vol. 12, pp. 849–851, Nov. 2008.