Knowledge-based structural change*

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Abstract: *How will structural change unfold beyond the rise of services? Motivated by the observed dynamics within the service sector we propose a model of structural change in which productivity is endogenous and output is produced with two intermediate substitutable capital goods. In the productive sector the accumulation of specialized skills leads to an unbounded increase in TFP, as sector becoming asymptotically dominant. We are then able to recover the increasing shares of workers, the increasing real and nominal shares of the output observed in productive service and IT sectors in the US. Interestingly, the economy follows a growth path converging to a particular level of wealth that depends on the initial price of capital and knowledge. As a consequence, countries with the same fundamentals but lower initial wealth will be characterized by lower asymptotic wealth.*

Keywords: *Two-sector model, technological knowledge, constant elasticity of substitution, non-balanced endogenous growth, structural change, Kaldor and Kuznets facts*

Journal of Economic Literature Classification Numbers: C62, E21, E22, E23, O41.

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1 Introduction

The coexistence of structural transformation and sustained balanced growth throughout history stands out as a well-established hallmark of economic development, commonly referred as the Kaldor [48] and Kuznets [51] facts. Recently, attention has pivoted towards the analysis of the dynamic transitions occurring within various sectors, with a special emphasis on the expansive service sector. Notably, this sector now commands a staggering 80% share of the labour force in developed nations, underscoring its pivotal role in shaping contemporary economies.

A compelling trend emerges from the data we present below: certain industries in services (and Information Technology (IT)) exhibit a dual characteristic of heightened productivity alongside an augmented labour share within the services domain. This phenomenon defies the conventional paradigm of structural transformation, which posits that stagnant sectors absorb increasing volumes of factor inputs. Remarkably, these industries reveal the presence of specialized technical expertise. Beyond its intrinsic value, exploring this new patern also bears implications for comprehending the future evolution of the global economy.[1]

What propels the process of structural change? Existing literature delineates two mechanisms capable of instigating substantial economy-wide transformations. On the demand side, when consumer preferences deviate from the Gorman form and relative prices remain constant, the growth in income affects differently the consumption patterns of diverse goods.[2]

On the supply side, disparities in sector-specific productivity growth rates and the extent of substitutability between capital and labour catalyze shifts in relative prices and the reallocation of production factors.[3] Notably, the literature underscores that the absence of substitution either in intermediate or final goods constitutes the primary driving force behind the influx of factors into stagnant sectors (see Ngai and Pissarides [62] and Acemoglu and Guerrieri [4]). Conversely, the work of Ngai and Pissarides [62] reveals that inter-subsector substitution brings the rise of the labour share in the most productive sub-sectors.

In the present paper we conceptualize productivity growth as an outcome of accumulating specialized skills, while also accounting for substitution effects. This contrasts with the literature that predominantly operates under the assumption of exogenous productivity differences.[4] Our dual methodology serves two primary objectives. Firstly, we aim to estab-

[1] In Acemoglu and Restrepo [5], a similar research evaluates the role of automation.

[2] Examples of this literature are Alder et al. [8], Alonso-Carrera and Raurich [10], Boppart [19], Buera and Kaboski [23, 24], Echevarria [34], Foellmi and Zweimueller [35] and Kongsamut et al. [50]

[3] See for instance Acemoglu and Guerrieri [4], Alvarez-Cuadrado et al. [11], Caselli and Coleman [28], Ngai and Pissarides [62], Buera et al. [25] and the survey of Herrendorf *et al.* [40].

[4] See also Hu *et al.* [43] who provide a unified theory with endogenous technology choice in human/knowledge capital accumulation where poverty trap, middle income trap and persistent growth are possible outcomes, or Jaimovich [46] who propose a demand-driven growth theory where process innovations and

lish a connection between productivity differences and the specific occupational categories they relate to. Our model will show that the spillover effect from the concentration of higher technical professions in these industries further reinforces the positive relationship between labour productivity and labour share.

Secondly, we strive to create a framework where economic growth dynamically interacts with structural transformations. It is worth considering the possibility that the reassignment of workers could dampen completly long run growth. For example, with exogenous technolgical progress and substitution in intermediate goods, Acemoglu and Guerrieri [4] find that the progressive sub-sector increases its input share. However, with endogenous growth they find that technological progress is stronger in the sector that is growing less which is the labour intensive sector. On the contrary, our model delivers sustained growth. Importantly, our framework allows for conditional convergence, while most of the literature fails to do so.

We start our analysis by gathering our stylized facts. We partition services into two sectors. We label high-skilled (HS) the sector with a positive relationship between labour share and productivity. This sector is dominated by the presence of Information and Communication industries within services. We obtain four important stylized facts about this sector in addition of the fact that increases in productivity and increases in the labour share are positively correlated; i) The joint labour share of these industries is increasing over time; ii) The relative price of industries in this sector is decreasing; iii) industries in the HS sector require workers with a high level of specialized technical skills and knowledge; and iv) across our set of countries there in no absolute convergence of HS labour shares.

The classification of sectors is slightly different than the ones adopted in the literature. Duernecker *et al.* [32] and Sen [65] use a decomposition based solely on productivity, Duarte and Restuccia [31] distinguish according to positive or negative income elasticities, and Buera and Kabovski [23] and Buera *et al.* [25] use schooling years to proxy skills. Our classification is based on productivities rather than in years of schooling as our mechanism is specifically about skills accumulation. Furthermore, the empirical evidence on the effect of years of schooling on productivity is not conclusive.[5]

We now describe the model in greater detail. We adopt a two intermediate sector model *à la* Acemoglu and Guerrieri [4], where each sector uses capital and labour, only focusing on services economy. We therefore do not account for the goods economy and consider only two service sectors. We think the outcomes of our model do not suffer from this choice, as argued in the model section. In the model, structural transformation relies on relative price effects resulting from differential productivity growth across sectors. Differential growth is endogenous and results from individual choices, similar to specific knowledge accumulation

product innovations are needed for the economy to keep growing in the long run.

[5] Herrendorf et al. [40] adopts a similar partition of sectors.

in Lucas [53]. We believe that skilled intensity of workers can be measured by their knowledge accumulation decisions. Finally, substitutability is attributed to intermediate goods, rather than final goods as per our empirical data, this distinction does not pose a complication (see Sen [65]).

Sector HS employs skilled workers. These agents spend part of their working time to accumulate technological knowledge, they split their individual unit of time between accumulation of individual technological knowledge (research) and work. We assume, as in Lucas [53], that the equation governing knowledge accumulation is linear but we generalise it in Section 8. Sector LS employs unskilled workers which devote all their individual unit of time to work. While the total population is growing at an exogenously given rate, the number of agents working in each sector is endogenously determined over time. The final good, which is consumed, is produced through a CES technology with a constant elasticity of substitution.

The analysis of our model proceeds in several steps. First, we define a workable equilibrium concept. As in Lucas, knowledge accumulation is an externality to the firm, and we consider the planner's solution in which this externality is internalized. We then provide a detailed characterization of the asymptotic non-balanced growth path (NBGP), characterized by different growth rates across sectors, and of the transitional path converging toward this NBGP. A numerical calibrated example provides some further insights into the transition dynamics and its empirical plausibility.

The first finding is that the pattern of labour and capital reallocations along the equilibrium path dramatically depends on whether the two intermediate sectors are complements or substitutes. We show that when inputs are substitutes, technological knowledge ensures that output growth is larger within the HS sector which becomes dominant in the long run. The long run growth rate of the final good sector is determined by the growth rate of the HS sector and there is capital deepening in this sector. As a result, in the long run and along the transition, the real and nominal shares of the HS sector are increasing while the real and nominal shares of the LS sector are decreasing. Similarly, the relative prices of the HS and LS sectors are respectively decreasing and increasing. These conclusions therefore fit with the pattern of labour and capital shares and value added presented in Section 3 below (see also Appendix 11.2). We also consider the case with complementary inputs, but the equilibrium properties are not compatible with the empirical facts previously mentioned. A similar conclusion holds for the case with a unitary elasticity of substitution.

The analysis also reveals the path dependence of economic development. First, we detrend all the variables by their respective endogenous non-balanced growth rates, and prove that there exists a manifold of steady states parameterized by the initial value of the price of knowledge. Each steady state is saddle-point stable and is associated with a set of unique non-balanced growth rates, which depend on the initial value of capital and knowledge.

Thus for each initial condition the economy will follow a particular growth path converging to a particular level of wealth. As a consequence, countries with the same fundamentals but lower initial wealth will be characterized by lower asymptotic wealth. In the case with inputs substitutability, the long run values of knowledge and human capital depend on the initial price of knowledge and long run inequality concerns both stocks. Contrary to Acemoglu and Guerrieri [4] but like Lucas [53], we obtain the existence of non-convergence across countries in a framework with structural change.

We finally provide a numerical illustration in the substitutable case to characterize the transitional dynamics of the main variables. Our aim is to test whether our model is able to replicate qualitatively and possibly quantitatively the US data mentioned above. Our calibration generates increases of the capital and labour shares and of the real and nominal shares in GDP of the HS sector that are qualitatively consistent with the empirical evidence provided in Section 3. Similarly, we obtain decreasing and increasing relative prices of the HS and LS sectors respectively as shown in Section 3. We are also able to replicate quantitatively the variations of the capital share of the HS sector.

The paper is organized as follows. In Section 2 we discuss the position of our paper within the recent literature. In Section 3 we present empirical evidences that support our main theoretical results. Section 4 presents the model. Section 5 characterizes the intertemporal equilibrium. Section 6 shows that our model generates non-balanced growth and structural change consistent with Kaldor facts. Section 7 establishes the existence of a manifold of steady states of the stationarized dynamical system and provides a local stability analysis. Section 8 proves the robustness of our results to the consideration of a non-linear accumulation equation of knowledge. Section 9 contains some conclusions on transitional structural change together with numerical illustration. Section 10 presents conclusions and the Appendix contains all the proofs.

2 Literature review

There is a vast literature on structural change as well as a vast literature on endogenous growth. Here we focus on two specific areas. First, we review the strand that, leveraging the availability of new data, focuses on structural transformation and labour productivity within the service sector. Second, we will look at models of endogenous growth with structural change.

In an insightful paper, Ngai and Pissarides [62] study a general classical multisector model of growth with exogenous differences in TFP growth rates across sectors. Ngai and Pissarides [62] give sufficient conditions on utility[6] and production for the existence of structural change, characterized by sectoral labour reallocation and balanced aggregate

[6]in particular a logarithmic intertemporal utility function

growth. The sectoral employment changes are consistent with the historical trend of structural change,[7] if substitutability between the final goods produced by each sector is low. Conversely, with highly substitutable goods, labour would move from the low stagnant sector to the progressive sector and the assymptotic aggregate growth rate would be the one of the progressive sector.[8]

Duernecker *et al.* [32] model consists of three sectors, a goods producing sector and two service sectors; progressive services and stagnant services. The value added is obtained linearly with labour services via a sector specific total factor productivity. The linear specification implies that sectoral TFP equals labour productivity, but as this is provided by the data, the role of capital accumulation is brought back into the model. They find that progressive services are necessities, stagnant services are luxuries, and the two services are substitutes.

Sen [65] also decompose the service economy into a progressive and a stagnant subsector and show that the definition for high and low-productivity sub-sector is robust and stable for a large majority of developed economies. Differently than Duernecker *et al.* [32], Sen [65] considers the value added of these sub-sectors, and notes that for some services sub-sectors, as business services or wholesale and retail trade, there is no counterparts as final consumption. As in our paper, the author also exhibits substitutability between the two services sub-sectors, which annihilate the Baumol's cost disease effect on productivity growth.

Duarte and Restuccia [31] split services in traditional services characterized by positive income elasticities, and nontraditional services marked by negative income elasticities. They show a substantial reallocation of expenditures in services from traditional ones to nontraditional services as income rises. Their analysis underscores the correlation between relative price decline and income, pinpointing heterogeneity within the services domain as a source of substantial aggregate productivity losses, particularly pronounced in economically disadvantaged nations.

Bárány and Siegel [12] consider a multi-sector growth model populated of agents with heterogeneous skills, where agents optimally select which sector to work in. There is routine labour-augmenting technological changes that generates variations in labour productivity growth. Assuming that produced goods are complements, a change in relative productivities increases labour demand in the relatively slow growing sectors, and wages in these sectors have to increase in order to attract more workers. They find that occupation and sector components are jointly explaining 96.7 percent of occupational labour-augmenting

[7] The decline of agriculture's employment share, the rise and then fall of the manufacturing share, and the rise in the service share (Kuznets [51] and Maddison [60])

[8] See also Mukoyama and Popov [59] who provide a dynamic general equilibrium model with incomplete contracts to examine the interaction between factor accumulation, institutions of contract enforcement, and political-economy frictions. They show that a higher level of institutional capital can enhance industrialization through directly improving production efficiency and indirectly encouraging physical capital accumulation.

technological changes.

Buera and Kabovski [23] and Buera *et al.* [25] consider workers with heterogeneous skill levels, in particular high-skill-intensive versus low-skill-intensive services. These authors focus on the stock of knowledge, based on the number of schooling years of workers obtained away from this sectors. For example, they include real estate in their high skilled sector because it exhibits a share of labour compensation for more educated workers higher than average, but has a lack of TFP growth: techniques to sell houses are barely the same than 40 years ago. Relative prices are increasing in Buera *et al.* [25]. Aditionally, there are rice effect of non-homothetic utility function.

Acemoglu and Guerrieri [4] consider a two-sector growth model with a single consumption good. They focus on the role of differences in factor intensity in an environment with modest differences in technological progress so that capital deepening is the main cause for changes in relative price. While most of the analysis assumes that intermediate goods are complements, they also consider the case of substitutable goods, formalised by Assumption 2, ii). However, they assume that the sector with highest labour intensity has a much larger productivity growth, so that this sector has the largest augmented rate of technological progress. Therefore, the dominant sector has also in this case high labour intensity. Still with substitutable goods, we rather focus on the case in which the sector intensive in capital has also the largest productivity growth, a case excluded from their Assumption 2. Indeed, this allows us to capture well progressive services.

Finally, focusing on automation and artificial intelligence, Acemoglu and Restrepo [5] and Irmen [44] adopt a task-based approach to analyze the particular effect of automation on labour productivity and labour demand. The overall effect is a reallocation of labour from automated tasks to non-automated ones, with lower productivity growth, inducing a decreasing labour income share in the automated industry.[9]

Our decomposition of the service sectors differs to this literature in several ways. In contrast with Duernecker *et al.* [32] our approach distinguish between TFP growth and investment as separate determinants of sectoral labour productivity growth. These authors assume that for the U.S. and along a balanced growth path, aggregate TFP and aggregate capital both grow at the same constant rate. They acknowledge that distinguishing between sectoral TFP growth and capital growth is likely to matter in middle-income and developing countries where capital is scarce.

We contrast with Buera *et al.* [25] as we are looking at the development of new techniques, and new skills, within the industry instead of the actual schooling level of workers. Our definition of skills is based on a measure of technical development within the sector instead

[9]See also Iwaisako and Futagami [45] who investigate how strengthening patent protection affects economic growth in an endogenous growth model where both innovation and capi tal accumulation are the driving forces of economic growth.

of within workers. This theoretical difference is at the origin of our data classification, and explain partially why relative prices in our paper are decreasing while they are increasing in Buera *et al.* [25]. The second explanation lies in the absence of demand-side mechanisms in our paper, which eliminates the price effect of non-homothetic utility function present in Buera *et al.* [25]. Our classification contrasts with Sen [65] and Duernecker *et al.* [32] as we consider both productivity and labour share.

Finally, a few papers combine endogenous growth and structural change. Acemoglu and Guerrieri [4] address the question of whether their results (as described above) are driven by the exogeneity of technological progress. In the endogenous growth version of their model ([3]) they find that technological change tends to offset the non-balanced nature of economic growth. In particular, technological change is stronger in the sector that is growing less which is the labour intensive sector. The dynamic stability of their equilibrium is missing in their analysis.

Bondarev and Greiner [17], Boppart and Weiss [20] and Foellmi and ZweimÃŒller [35] focus on the effect of R&D on the process of structural change. Bondarev and Greiner [17] consider horizontal and vertical innovations to generate endogenous growth with structural change. A representative firm faces a constraint with respect to aggregate R&D spending. Assuming that consumers have preferences for variety, the R&D process generates creative destruction implying that older technologies are continuously replaced by newer ones and new technologies are driven by horizontal innovations and the taste for variety. Interestingly, if spending is insufficient, there may exist two different steady-states of the economy; one with high productivities and less new technologies, and the other with more technologies but lower productivities. If R&D spending is large enough the steady-state becomes unique and all technologies are used.

Boppart and Weiss [20] provide a model of directed technical change in which structural change is driven by both relative price and income effects. There are two final consumption goods, durables and nondurables. Both goods are produced using an identical set of intermediate industries, varying only in their intensities, with which these different industries are used. As (endogenous) innovation takes at industry level, changes in the market sizes induce a shift in industry specific R&D, which finally determines the evolution of final output prices. A unique endogenous balanced growth path is obtained in the long run and endogenous structural change is driven along the transition by income effects. In both papers absolute convergence holds: all countries should converge in the long run toward the same growth rate and level of GDP.

In Foellmi and ZweimÃŒller [35], structural change is generated by hierarchical preferences defined over luxury and necessity goods. The model has monopolistic firms generating R&D-based growth. In this setup, hierarchical preferences generate heterogenous mark-ups across firms and, for a given firm, the change over time of the mark-ups increase the incentives to innovate. Higher growth implies that the market expands more quickly and leads to a faster growth of profits. This latter effect resulting in demand externalities may give rise to multiple equilibria and thus global indeterminacy. In this model, countries may not converge toward the same long run growth rate depending on whether optimistic or pessimistic expectations occur.

Hori *et al.* [42] and Herrendorf and Valentinyi [41] focus on the mechanisms responsible for the reallocation of labour from the goods sectors to the service sectors, and the potential policy implications on the pace of structural transformation. Herrendorf and Valentinyi [41] are motivating the use of endogenous growth, coupled with imperfect competition, as a way to introduce industrial policy, through technological progress, in structural change models. They develop a model of endogenous sector-biased technological change with a unique generalized balanced growth path. In this equilibium there is balanced aggregate growth with more innovation in the services sector but higher productivity growth in the goods economy. In Hori *et al.* [42] the source of endogeneity is twofold: the number of firms in each productive sector and the individual firm productivity, through knowledge spillover effect. By solving the central planner and the decentralized economy problem they obtain labour reallocation from the more productive goods sector to the low-productivity service sector, which is consistent with decreasing prices in the high-productivity sector and balanced aggregate growth. However, in the decentralized economy, structural transformation is faster than for the central planner. Indeed, knowledge spillover creates higher productivity growth for future firms, a phenomenon not taken into account in the decentralized economy, producing an inefficiently faster structural change. Note that in these models the balanced growth path is unique and absolute convergence holds.

In contrast to these five papers, we propose an endogenous growth model where structural change is driven by knowledge accumulation, formalised as in Lucas [53], which affects asymmetricaly two intermediate sectors. We therefore establish a connection between productivity differences and the specific occupational categories they relate to link. We adress the new question whether the reassignment of workers moderate or amplify growth via the endogenous feedback. Finally, the model delivers conditional convergence which is compatible with our empirical facts. Therefore, depending on their initial wealth, countries may converge to the same long run growth path but with different long run wealth.

3 Empirical evidence

In this section we describe our datasets and present five important stylized facts based on our analysis of a subset of economies: i) Increases in productivity and increases in the labour share are positively correlated for two industries (denoted the High-Skilled (HS) sector of services); ii) The joint labour share of these industries is increasing over time; iii) The relative price of industries in the HS sector is decreasing; iv) industries in the HS sector require workers with a high level of specialized technical skills and knowledge; and v) across our set of countries there in no absolute convergence of HS labour shares.

3.1 The data

Our main source is the EUKLEMS/INTANPROD 2023 dataset (Bontadini *et al.* [18]). Additionally, to enhance our results, we have incorporated data from the previous version of EUKLEMS for nine economies: Austria, Germany, Spain, Finland, France, Great Britain, Italy, Japan, and the US. The data spans from 1970 to 2020.

Our analysis employs the NAICS classification to allocate employment, capital, and value-added across different industries. To ensure data completeness, we operate at the 2-digit level, which is represented by letters in our dataset. Notably, we focus solely on the service economy, which leads us to narrow our investigation to the following industries:

G: Wholesale and retail trade; repair of motor vehicles and motorcycles

H: Transportation and storage

I: Accommodation and food service

J: Information and communication

K: Financial and insurance activities

L: Real estate activities

M_N: Professional, scientific, and technical activities and administrative and support service activities

O: Public administration and defense; compulsory social security

P: Education

Q: Human health and social work activities

R: Arts, entertainment, and recreation

S: Other service activities

As our primary focus is to analyze how productivity growth influences the share of employment in these specific sectors, we are particularly interested in three key variables: labour, capital, and Total Factor Productivity (TFP). To measure labour, we use the number of individuals employed in each sector. For capital, we utilize the stock of capital data. As for TFP, we employ the TFP index per hour worked. Since we are integrating data from two versions of EUKLEMS, we encountered the need to align the TFP index to a common base year. The most recent data is based on the year 2015, while the older data relies on the year 2005 as the base year. To ensure consistency, we rescaled the TFP index accordingly. Additionally, we encountered slight differences in the industry classification between the datasets. As a result, we merged subsectors ”M” and ”N” from EUKLEMS/INTANPROD 2023 into the aggregated subsector ”M_N” to match with the older data's classification. For both TFP and the aggregated "M_N" sector, we employed a Torqnvist aggregator.

3.2 Relationship between employment share and productivity growth

In this subsection, our focus is to examine the relationship between productivity growth and the labour share within specific industries. Changes in labour share within an industry could have two possible origins: i) reallocation within the service economy or ii) absorption of labour from the goods economy. Our investigation centers around the former and aims to understand how productivity growth influences this reallocation process. We then define labour share as the share of labour allocated to a particular industry in comparison to the entire service economy. This ensures that the sum of all labour shares in the cited industries remains equal to one, effectively accounting for the reallocation of labour from the goods economy.

Productivity is measured through the Total Factor Productivity (TFP) data available in our dataset. Additionally, as a control variable, we incorporate the share of capital in each industry. The inclusion of this variable is crucial as capital investments tend to be less volatile than labour and may indicate planned reallocation of labour due to known capital investments.

To establish the relationship between the share of labour in a specific industry and productivity, we employ a relative measure of Total Factor Productivity (TFP) denoted as rTFP. This variable is defined as $rTFP = \frac{TFP^i}{TFP^{services}}$, where i represents a particular industry, and $TFP^{services}$ denotes the aggregated TFP of the entire service economy, achieved using a Torqnvist aggregator. The rationale behind using rTFP is that TFP itself is an index constructed to have a value of 100 for each country and industry in 2005. As such, the absolute value of TFP lacks informative significance. However, by examining relative TFP, we can gauge the performance of a specific sector relative to the aggregated services economy. This relative measure provides more informative insights into the productivity performance of individual industries and their potential impact on labour distribution.

Our dataset is constructed as a three-dimensional panel, with each observation defined by its industry, country, and time information. Given the structure of our data, the most appropriate approach to estimate the productivity effect is to study the following model:

$$LS_{i,j,t} = \beta_0 + \beta_1 rTFP_{i,j,t} + \beta_2 KS_{i,j,t} + \lambda_i + \mu_j + \nu_t + \varepsilon_{i,j,t} \tag{1}$$

where $LS_{i,j,t}$ represent the labour share of industry i in country j in year t and KS represents the capital share. Additionally, λ_i, μ_j and ν_t are the associated fixed effects and ε denotes the residuals. However, our main focus is on understanding the value of the estimator β_1, which represents the effect of relative TFP on the labour share, but for each industry separatly. Therefore, we will restructure equation (1) to obtain a two-dimensional panel with country-time indexes for each industry. The revised model is as follows:

$$LS^i_{j,t} = \beta^i_0 + \beta^i_1 rTFP^i_{j,t} + \beta^i_2 KS^i_{j,t} + \mu^i_j + \nu^i_t + \varepsilon^i_{j,t} \tag{2}$$

where i stills denotes the industry level data but now we have one equation for each industry. This formulation will allow us to get an industry specific estimator for the effect of relative total factor productivity (TFP) on labour distribution. We are particularly interested in the values of the individual estimators, denoted as β^i_1, which quantify this effect for each industry, and to obtain this estimator we are running a two-dimensional panel model on a country, year basis. Essentially, we execute the same model twelve times, once for each industry, and compile the results of relative TFP on labour share into a single graph, as presented in Figure 1. The comprehensive estimation details are provided in Table 3 in the appendix.

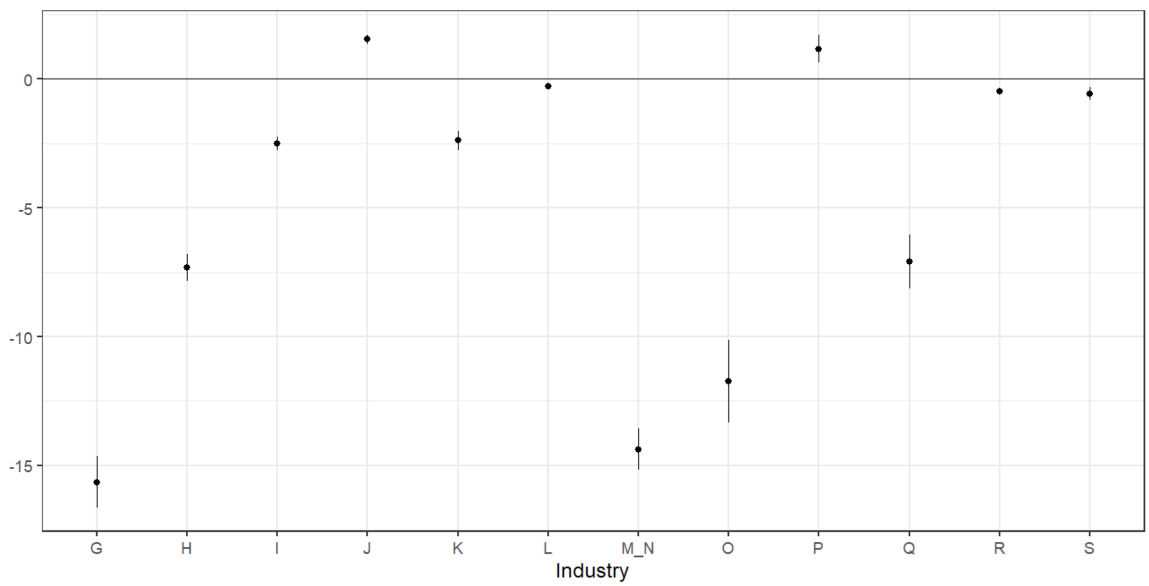


Figure 1: The relationship between productivity and labour share

In this figure, each point represents the value of the estimator, and the bar indicates the 95% confidence interval. Our model reveals a noteworthy finding: a positive and significant relationship exists for two industries, Information and communication (J) and Education (P) within the services economy. Specifically, as the relative productivity of these industries increases, the labour share of the considered industry also rises. These results stand in contrast to what is typically observed in the goods/services structural change literature, where lower relative productivity in the service sector usually leads to an increasing share of labour. This effect is often attributed to the complementarity between goods and services (Acemoglu and Guerrieri [4] or Buera and Kaboski [24] among many others).

At this stage, we conjecture that the positivity of these estimators stems from the existence of a substitution effect between the two sectors of the services economy. The sectors are labeled as HS (High-Skilled Sector) for industries with estimators larger than 0, indicating a positive effect of labour productivity, and LS (Low-Skilled Sector) for industries with a negative estimator. The labour force migrates to the HS sector, which is also the sector with the fastest productivity growth**.** Note that the numenclature is motivated by the occupations in these sectors, as reveled by the relevant empirical subsection below

3.3 Evolution of labour share and prices in the service sectors

In this section we aim to further investigate the presence of a substitution effect between the HS and LS sectors. The classification relies on the relationship between labour productivity on worker distribution, but we aim to further investigate the presence of a substitution effect between these two service sub-sectors. To ascertain the existence of a substitution effect between HS and LS, we examine the evolution of the labour share and relative prices of the HS sector over time and during the development process in a subset of economies.

To observe the evolution of labour and prices in the HS sector, our analysis will focus on nine economies: Austria, Germany, Spain, Finland, France, Great Britain, Italy, Japan, and the US. These countries were chosen based on their data availability, spanning back to 1970, providing us with a comprehensive 50-year observation period to study the mechanisms of structural change. By merging two EUKLEMS datasets, we gain access to valuable historical data, enabling a thorough examination of the long-term phenomenon of structural change in three different regions of the world: Europe,[10] the USA, and Japan.

As described earlier, calculating the labour share of the HS sector involves summing the values of industries J and P to obtain the overall share of this sector. However, for the price analysis, we employ a Tornqvist aggregator to aggregate the prices. This allows us to compare the aggregated price of the HS sector with that of the entire service economy. By doing so, we can discern whether the relative prices of HS have been increasing or decreasing over time. The graphical representation of the evolution of labour and relative prices is provided in the figure 2 below.

As evident from the data, the labour share and relative prices in the HS sector are moving in opposite directions for most of the countries under study, except for Austria, where the labour share shows fluctuations but experiences a slight overall increase. Across the eight countries (excluding Austria), the increase in the labour share of the HS sub-sector is accompanied by a decrease in its relative price. This trend has remained stable over the entire 50-year period, with a small atypical episode due to the early 2000's internet crisis.

[10]It is worth noting that even within Europe, while Italy, Germany, Finland, and Great Britain are part of the same trade agreements, there are still notable differences in their industries and cultures, which can potentially influence the outcomes of our analysis. These nuances and distinctions between the countries under study should be considered while interpreting the results.

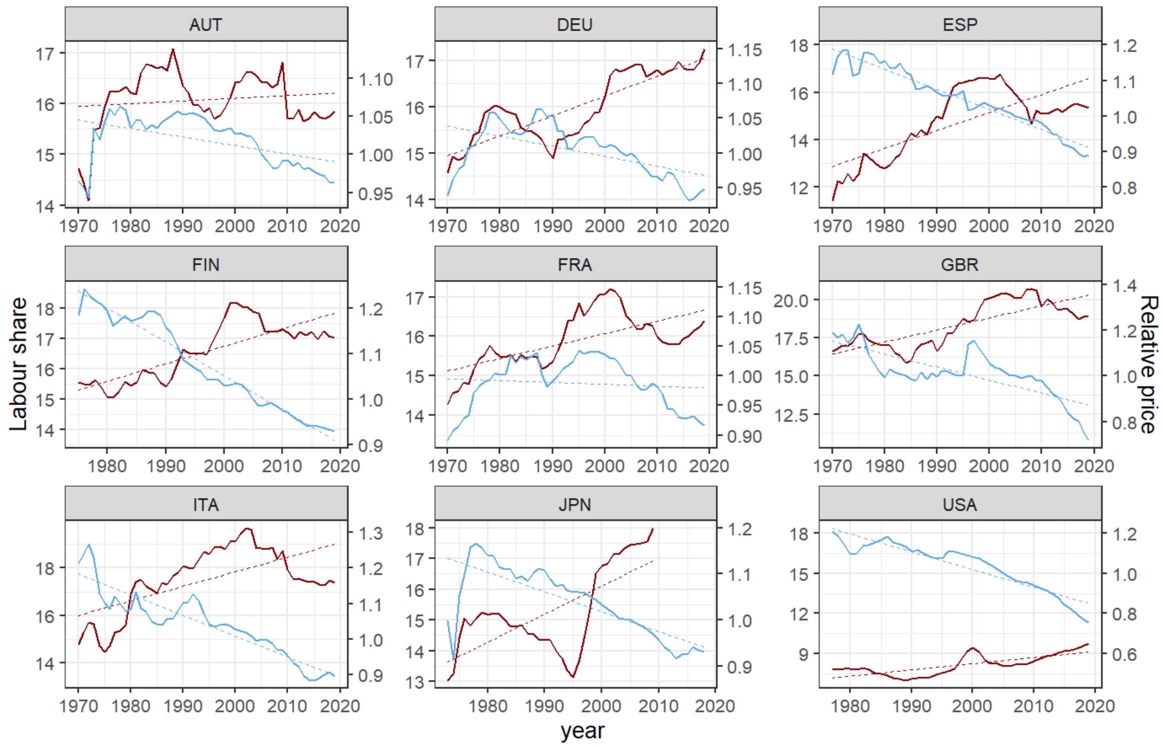


Figure 2: Labour share (red line) and relative price (blue line) of the HS sub-sector

The early 2000's internet crisis, commonly referred to as the dot-com bubble, had a noticeable effect on the labour share of HS in several countries. Notably, France, Italy, the USA, Spain, and Finland experienced a drop in the labour share of HS mainly around 2003-2005. This decline can be attributed to the fact that the HS sector comprises the information and communication industry (ICT), which was particularly affected by the burst of the internet bubble. As a result, the labour share in the HS sector witnessed a temporary decline during this period.

It is important to note that despite the setback during the dot-com bubble, the general trend of increasing labour share and decreasing relative prices in the HS sector persisted for the other countries and the overall 50-year period. This suggests that the structural change mechanism driving these trends is robust and not solely dependent on short-term economic fluctuations.

The fact that the labour share of the HS sector increased while its relative price decreased over time, strongly suggest the presence of a substitution effect between HS and LS and provide a compelling evidence to support our hypothesis.

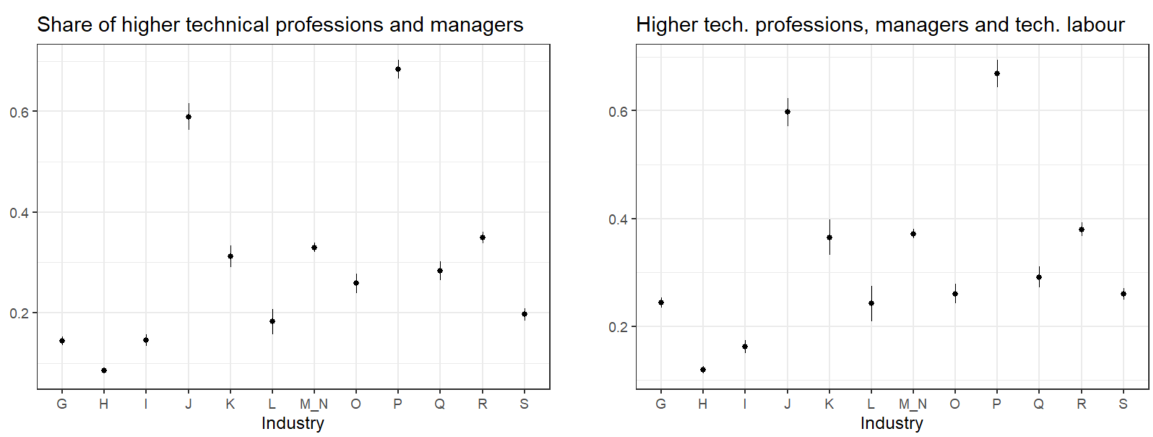
3.4 Occupations in the service sectors

To investigate the reasons behind the higher labour productivity in the HS sector, we turn to Eurostat's data on occupations, specifically utilizing the lfsa_eisn2 dataset from the LFS survey series. These data offer valuable insights into the share of higher technical professions, which we consider as an explanatory variable for the observed higher labour productivity in the HS sector. Our hypothesis is that the presence of knowledge and specialized skills within the workforce is a significant factor contributing to this effect.

Eurostat's data on occupations possess a crucial advantage over other variables, such as education or wages, as they provide a more nuanced perspective. Over the covered time period, there has been a considerable increase in educational attainment, making the educational level alone less informative as a determinant of labour productivity. By focusing on occupations, we can better capture the specific technical skills and knowledge required in the workforce, which are likely to have a more direct impact on productivity.

In Figure 3, we present the results of our analysis, focusing on the share of higher technical professions and managers (panel a) along with technical labour (panel b) within various services industries. We maintain the same coverage as in our previous results for consistency.

The figure highlights the distribution of higher technical professions and technical labour across different services industries. This visual representation allows us to discern any patterns or correlations between the prevalence of specialized occupations and the higher labour productivity observed in the HS sector. However, it is important to note that the data only focuses on European countries, and further research could expand the analysis to include a broader geographical scope for more comprehensive conclusions.



(a) (b)

Figure 3: Share of higher occupations in total labour

The analysis of Eurostat's data reveals a significant finding - the two industries that make up the HS sector also have the largest share of higher technical professions compared to any other industry. This notable difference of more than 20 percentage points emphasizes the crucial role of technical professions in driving the observed positive relationships between labour productivity and labour share. It appears that the high level of specialized technical skills and knowledge is at the core of the mechanisms underlying the growth process in the HS sector.

3.5 Relative convergence in the HS sector

We now examin whether a cross-country convergence exists within the labour shares of the HS sector. In figure 4 we plot the relationship between the labour share in the HS sector (x axis) and the relative log GDP per capita.

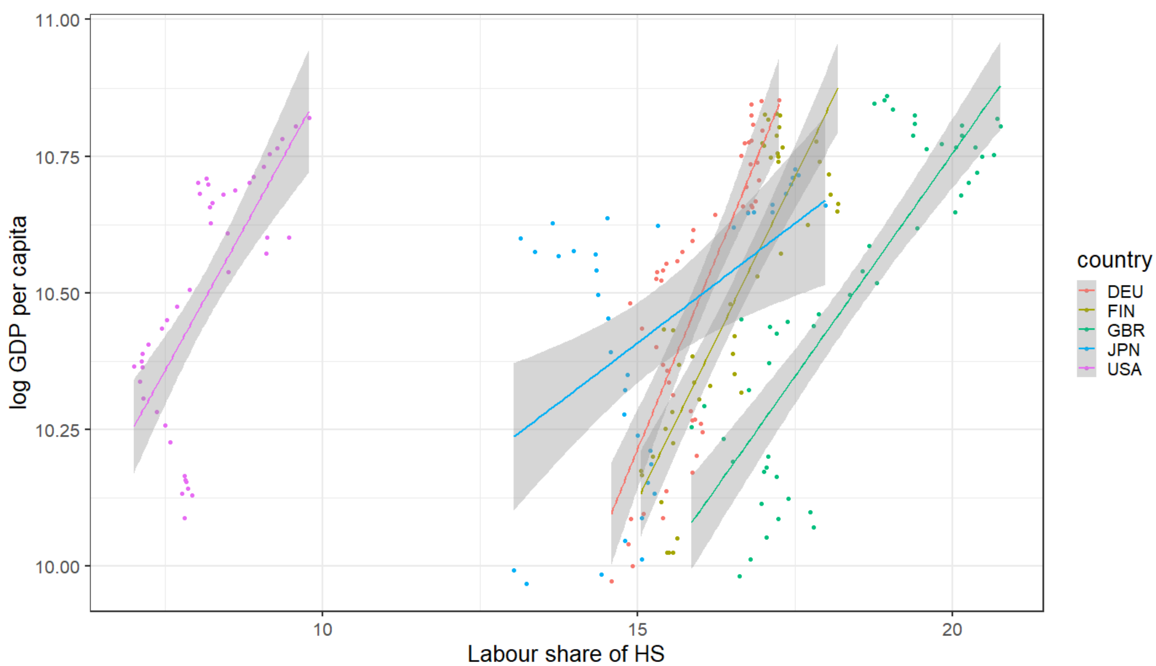


Figure 4: The relationship between labour share in HS and Log GDP per capita

The illustration features a selection of five countries, chosen for the sake of clarity. However, the comprehensive graph encompassing the entire dataset is available in the appendix, revealing a comparable pattern. Germany and Finland exhibit closely aligned yet consistently parallel trends, rather than displaying a converging trajectory. The trends for Great-Britain and the US are far from the others but still almost parallel, surely not converging. It is worth mentioning that Japan stands as an exception due to a prolonged phase of sluggish growth. Examining the historical trends of these nations, it becomes evident that absolute convergence is not observed. Instead, a pattern of conditional convergence emerges.

The extensive body of growth literature has generated numerous mechanisms that underpin conditional convergence, some of which could conceivably apply to the realm of the service sector, and more specifically the HS sector. Within this paper, we introduce an alternative mechanism anchored in the concept of knowledge accumulation, primarily facilitated by highly skilled professionals within the HS sector. We contend that this particular mechanism, coupled with certain constraints governing the movement of these skilled workers across international borders, offers valuable insights into the differences observed across countries in the process of structural transformation within the service sector.

3.6 Introducing an endogenous growth mechanism

From the empirical evidence outlined above, we can discern several crucial findings. Firstly, there exists a HS service sector that displays a positive correlation between labour share and productivity. This sector is characterized by an increasing labour share and a decreasing relative price compared to the broader services economy.

Secondly, the industries comprising the HS sector exhibit a significantly higher proportion of technical labour, with a difference of at least 20 percentage points. Consequently, this sector boasts a greater concentration of skilled labour and a more substantial accumulation of knowledge.

Lastly, when examining the level of development achieved based on the labour share in the HS sector, we observe parallel trends across the last five decades, which do not align with an absolute convergence scheme among countries. Given this consistent pattern across the five countries under consideration, it leads us to advocate in favor of conditional convergence.

All these factors collectively lead us to consider the possibility of introducing an endogenous growth mechanism to elucidate the behavior of the HS sector. Notably, Hori *et al.* [42] and Herrendorf and Valentinyi [41] have already integrated endogenous growth mechanisms into the structural change literature, as elaborated in section 2. However, their focus was primarily on accounting for industrial policy, knowledge spillovers, and size effects.

In our context, we posit that endogenous growth can serve as a valuable framework for explaining conditional convergence with knowledge accumulation. As previously mentioned, our sector delineation is based on the interplay between productivity and labour share, resulting in a categorization distinct from the conventional "progressive" versus "stagnant" classification found in the literature.

Since the seminal work of Lucas [53], we know that conditional convergence is a feature of endogenous growth models. Moreover, given that our classification appears to stems from the accumulation of skills, employing an exogenous mechanism seems at odd with the broader literature on technical knowledge and skill accumulation.

For these reasons, we believe that employing an endogenous growth model will be instrumental in capturing the dynamics of the HS sector within each country and, importantly, in replicating the observed phenomenon of conditional convergence in our empirical data.

In the forthcoming sections, we will delve into the details of the endogenous structural change model, with a series a series of robustness checks, aligning it closely with the empirical findings detailed earlier

4 The model

In this section we propose a two-sector model that captures the stylized facts from the previous section. That is: i) There is a sector (noted HS) where increases in productivity and increases in labour share occur synchronously; ii) The relative price of industries in the HS sector is decreasing; iii) industries in the HS sector require workers with a high level of specialized technical skills and knowledge; and iv) across countries there in no absolute convergence of HS labour shares.

In this paper, our focus is exclusively on the services economy. As elucidated in section 2, the existing literature extensively delves into the driving forces behind structural changes from agriculture and manufacturing to the services sector. However, it's noteworthy that services have consistently represented a substantial portion, exceeding 60%, of all developed economies since the 1970s, and this figure has risen to approximately 80% in recent years. Even within the last quarter-century, the share of services has only experienced a modest 5% increase, accompanied by minimal labour reallocation to from manufacturing and agriculture to services.

For the sake of tracatability, we have chosen to construct a two-sector model exclusively centered on services. From the existing literature with three sectors (e.g., Duernecker *et al.* [32] and Sen [65]) we can hint that, at least qualitatively, the dynamics within the service sector is not affected by the existence of the goods sector. In our model, the effect of omitting the goods economy is mitigeted by the fact that the realocation of labour from manufacturing and agriculture to services can be assimilated to population growth. Indeed, within our model, population can expand at a rate denoted as n. Given that our population solely comprises workers, n can originate from population growth itself or from the reallocation of workers from the goods economy. Independently, workers are will eventually be allocated to HS and LS endogenously.

Focusing on a two-sector model has also an implication regarding LS. Indeed, sector LS has obviously residual heterogeneity in industries characteristics. However, in the LS sector productivity is negatively associated to labour shares, a sign of complementarity (Ngai and Pissarides [62]). We could then decompose the LS sector into two-subsectors with complementary outputs. We don't expect this addition to modify the role of substitution in the rise of the HS sector. Furthermore, The resulting three-sector endogenous growth model would become analytically untractable.

In line with the empirical evidence, we make the assumption that the two sectors produce substitutable inputs. While the model attributes these inputs to intermediate goods, rather than final goods as per our empirical data, this distinction doesn't pose a complication. The literature convincingly underscores the blurred boundary between intermediate and final goods (see Sen *[65]).* Addressing point (iii) above, we adopt a knowledge accumulation framework akin to that outlined by Lucas [53], which aptly captures the trajectory of progress through the accrual of skill in the workforce. This choice in modeling is resonant with the inherent nature of skill development and knowledge enhancement within industries. We consider an economy in which at each time t there is a continuum $[0, N(t)]$ of infinitely-lived agents characterized by homogeneous preferences. We assume a standard formulation for the utility function which is compatible with endogenous growth such that

$$u(c_i(t)) = \int_0^{+\infty} \frac{c_i(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt \tag{3}$$

where $c_i(t)$ is consumption of agents of type $i \in [0, N(t)]$ at time t, $\theta > 0$ is the inverse of the elasticity of intertemporal substitution in consumption and $\rho > 0$ is the discount factor. We assume that the total population grows at the constant exponential rate $n \in [0, \rho)$, so that $N(t) = e^{nt}N(0)$. Agents are heterogeneous only with respect to the sector in which they work. As explained below, this difference arises from different decisions on the allocation of labour time.

Our economy consists of two sectors, with three factors of production, capital, $K(t)$, labour, $L(t)$, and individual technological knowledge, $a(t)$. In line with Lucas [53], we interpret $a(t)$ as the outcome of an individual's choice. Final output, $Y(t)$, is an aggregate of the output of two intermediate sectors, $Y_H(t)$ a "knowledge-intensive sector" or "high-skilled" (HS) sector (e.g. sector H), and $Y_L(t)$ a "non knowledge-intensive" or "low-skilled" (LS) sector (e.g. sector L). The numbers of workers $N_H(t)$ and $N_L(t)$ in these two sectors, together with their respective growth rates $\dot{N}_H(t)/N_H(t) = n_H(t)$ and $\dot{N}_L(t)/N_L(t) = n_L(t)$, are endogenously determined at the equilibrium.

Agents working in the HS sector devote a fraction $u(t) \in (0, 1)$ of their unit of time to production. Total labour in this sector is $L_H(t) = u(t)N_H(t)$. The rest of time $1 - u(t)$ is devoted to the accumulation of individual technological knowledge $a(t)$. Here, $L_H(t)$ denotes the total number of hours worked in the HS sector and differs from the number $N_H(t)$ of workers. Similarly to how Lucas [53] treats newborns, in our model each agent entering the HS sector at any time t_0 acquires the available knowledge $a(t_0)$. To simplify the computations throughout the paper, we assume a linear formulation:[11]

[11] We show in Section 8 that all our results are robust to the consideration of a non-linear accumulation equation of knowledge formulated as a Cobb-Douglas production function mixing knowledge and physical capital. We thank an anonymous referee for having suggested to check for such a robustness.

$$\dot{a}(t) = z[1 - u(t)]a(t) - \eta a(t) \tag{4}$$

with $z > 0$ and $\eta > 0$ the depreciation rate of knowledge.[12] By contrast, agents working in the LS sector will spend all their unit of time working so that total labour in this sector is $L_L(t) = N_L(t)$. Contrary to the HS sector, in this sector the number of hours worked is identical to the number of workers, since the total individual available working time is normalized to one.

The final good is produced through a CES technology such that

$$Y(t) = \left(\gamma Y_H(t)^{\frac{\epsilon-1}{\epsilon}} + (1-\gamma)Y_L(t)^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}} \tag{5}$$

with $\gamma \in (0,1)$ and ϵ the constant elasticity of substitution. The value of ϵ is important as it determines whether the inputs are substitutable ($\epsilon > 1$) or complementary ($\epsilon < 1$). Sector H produces the HS intermediate good using capital, labour and knowledge through the following Cobb-Douglas technology:

$$Y_H(t) = [L_H(t)a(t)]^{\alpha} K_H(t)^{1-\alpha} \tag{6}$$

with $\alpha \in (0,1)$. The product $L_H(t)a(t) = u(t)N_H(t)a(t)$ then represents total efficient labour.[13] Note that, considering capital and labour (hours worked) as inputs, the total factor productivity (TFP) is given by $a(t)^{\alpha}$ and increases endogenously through equation (4).[14] Sector L produces the LS intermediate good using only capital and labour through the following Cobb-Douglas technology:

$$Y_L(t) = L_L(t)^{\beta} K_L(t)^{1-\beta} \tag{7}$$

with $L_L(t) = N_L(t)$ and $\beta \in (0,1)$. Contrary to the HS sector, the LS sector has a constant TFP.[15] While we do not impose any restriction on the capital intensity difference $(\alpha - \beta)$ across sectors, we assume $\alpha, \beta > 1/2$ to match standard empirical estimates of labour shares.

Denoting total capital by $K(t)$ and total labour by $L(t)$, capital and labour market clearing require at each date $K(t) \geq K_H(t) + K_L(t)$ and $L(t) \geq L_H(t) + L_L(t) = N_H(t)u(t) + N_L(t)$. The capital accumulation equation is standard

$$\dot{K}(t) = Y(t) - \delta K(t) - N_H(t)c_H(t) - N_L(t)c_L(t) \tag{8}$$

where $\delta > 0$ is the depreciation rate of capital and c_i the consumption of any agent working in sector $i = H, L$.

[12] As in the formulation of Lucas [53] with human capital, our equation of knowledge accumulation is at the individual level and does not deliver the scale effect (e.g. Romer [64] or Aghion and Howitt [7] and Jones [47]).

[13] As in Lucas [53], we define each worker's output according to $(u(t)a(t))^{\alpha} k(t)^{1-\alpha}$ where $k(t) = K_H(t)/N_H(t)$.

[14] See Hartley et al. [38] where a formulation similar to ours is used to study the optimal transition from fossil fuels to renewable en classical growth economy with endogenous technological progress in energy production. See also Adao et al. [6].

[15] Again, this assumption is introduced to simplify the computations. We could also assume that the LS sector is characterized by an exogenous growth rate of TFP without altering our results. This claim is proved in Remark 1 below.

5 Planner solution and intertemporal equilibrium

In the model, as in Lucas [53], the welfare theorems hold and the intertemporal competitive equilibrium can be found via the planner's problem.[16] Assume that the planner has a Benthamite objective function and consider the following intertemporal optimization problem

$$
\begin{aligned}
\max_{\{c_i(t),K_i(t),L_i(t)\}_{i=H,L},u(t),a(t)} \quad & \int_0^{+\infty} \left(N_H(t)\frac{c_H(t)^{1-\theta}}{1-\theta} + N_L(t)\frac{c_L(t)^{1-\theta}}{1-\theta} \right) e^{-\rho t} dt \\
s.t. \quad & (4), (5), (6), (7), (8) \text{ and} \\
& K(t) \geq K_H(t) + K_L(t) \\
& L(t) \geq L_H(t) + L_L(t) = N_H(t)u(t) + N_L(t) \\
& K(0), a(0), N(0) \text{ given}
\end{aligned} \tag{9}
$$

From now on, let us denote the value of increments in aggregate capital by the "price" P, the price of the HS good by P_H, the price of the LS good by P_L and the price of knowledge by Q. Solving the first order conditions of this optimization problem allows to state the following Proposition:

Proposition 1. *All agents, no matter which sector they work in, have the same labour and capital income at the equilibrium and have the same intertemporal profile of consumption as given by $c_H(t) = c_L(t) = P(t)^{-1/\theta}$ for any $t \geq 0$. Aggregate consumption is thus $C(t) = N(t)P(t)^{-\frac{1}{\theta}}$. Moreover, for any given initial conditions $(K(0), a(0))$, and considering the rental rate of capital*

$$
R(t) \quad = \quad (1-\alpha)\gamma \left(\tfrac{Y}{Y_H}\right)^{\frac{1}{\epsilon}} \tfrac{Y_H}{K_H} = (1-\beta)(1-\gamma)\left(\tfrac{Y}{Y_L}\right)^{\frac{1}{\epsilon}} \tfrac{Y_L}{K_L}, \tag{10}
$$

any path $\{K(t), a(t), P(t), Q(t)\}_{t\geq 0}$ that satisfies the following system of differential equations

$$
\frac{\dot{P}}{P} = -\left[R - \delta - \rho\right], \quad \tfrac{\dot{Q}}{Q} = -(z - \eta - \rho) \tag{11}
$$

$$
\frac{\dot{K}}{K} = \frac{Y}{K} - \delta - \frac{NP^{-\frac{1}{\theta}}}{K}, \quad \tfrac{\dot{a}}{a} = z(1-u) - \eta \tag{12}
$$

together with the transversality conditions[17]

$$
\lim_{t\to+\infty} P(t)K(t)e^{-\rho t} = 0 \text{ and } \lim_{t\to+\infty} Q(t)a(t)e^{-\rho t} = 0 \tag{13}
$$

is an optimal solution of problem (9) and therefore an intertemporal equilibrium.[18]

So far we have interpreted knowledge accumulation along the lines of Lucas [53] (see footnote 2). Another interpretation of the model is as follows. Knowledge now becomes an aggregated variable, a public good used by the firms in the high-tech sector. Since there is no private incentive to invest in $a(t)$, the regulator has to step in. The government manages

[16] The economic interpretation of competitive equilibrium is as in Lucas [53].

[17] See Michel [57] and Kamihigashi [49] for some proof of the necessity of the transversality condition.

[18] See Boucekkine [21, 22] for closed-form solutions of Lucas-type models.

the production of $a(t)$ by imposing lump-sum taxes on consumers and use the revenue to hire workers from the high-tech sector. As the two sectors (low- and high-tech) are perfectly competitive, the wage per hour needs to be identical across sectors. Note that the tax revenue used to buy efforts from consumers working in the high-tech industry might involve a time dependent tax system.

6 Non-balanced growth paths

We now provide the existence result and characterize the non-balanced growth paths (NBGP) along which the variables P, Q, K, a grow at constant but potentially different rates. Note that this implies that C, Y, C_i, K_i and Y_i also grow at constant rates. We introduce the following notations valid along the NBGP, which highlights the fact that the growth rates of the various variables can differ:

$$g_C = \frac{\dot{C}}{C}, g_Y = \frac{\dot{Y}}{Y}, g_K = \frac{\dot{K}}{K}, g_{Y_i} = \frac{\dot{Y}_i}{Y_i}, g_{K_i} = \frac{\dot{K}_i}{K_i}, g_{c_i} = \frac{\dot{c}_i}{c_i}, g_a = \frac{\dot{a}}{a}$$

and

$$g_P = \frac{\dot{P}}{P}, g_Q = \frac{\dot{Q}}{Q}$$

for $i = H, L$. Along an NBGP we have $K(t) = k(t)e^{g_K t}$, $a(t) = x(t)e^{g_a t}$, $Q(t) = q(t)e^{g_Q t}$ and $P(t) = p(t)e^{g_P t}$. Note that NBGP can differ in their levels but still have the same 8-tuple in $\mathbb{R}^8$ of growth rates.

As the final good is produced via a CES function, the dynamics strongly depends on the value of the elasticity of substitution ϵ between the two intermediary sectors. When inputs are substitute, i.e., $\epsilon > 1$, as technological progress only occurs in the HS sector, it is optimal to reallocate capital and labour inputs to sector H. Thus, eventually the HS sector becomes dominant. On the contrary, when sectors are complement, i.e. $\epsilon < 1$, both inputs are necessary to produce efficiently the final good. In this case, in addition to the advantage of the HS sector due to technological progress, complementarity between inputs provides incentive to increase the production and use of the LS sector. The total effect is therefore a priori not determined.

6.1 The case of substitutable inputs

We consider first the case of substitutable inputs, $\epsilon > 1$, which will turn out to be the one delivering outcomes in line with the recent data on services we described in the introduction. We introduce the following restrictions which guarantee positiveness of growth rates and interiority of the share u of time devoted to work by agents in the HS sector:

Assumption 1. *$\epsilon > 1$, $z > \eta + \rho - n$ and*

$$\theta > \max\left\{1 - \frac{\rho-n}{z-\eta}, \frac{\beta(\epsilon-1)(z-\eta-\rho+n)}{n}\right\}$$

From the first-order conditions (29)-(36) and the differential equations (11)-(12) we derive:

Theorem 1. *Under Assumption 1, the set of non-balanced growth paths (NBGP) is non-empty and is characterised by the same 8-tuple of growth rates. The growth rates are such that* $g_Y = g_{Y_H} = g_K = g_{K_H} = g_C = n + g_{c_H} = n + g_{c_L} = n - \frac{g_P}{\theta} = n + g_a, g_Y = g_{Y_H} > g_{Y_L}, g_K = g_{K_H} > g_{K_L}$ *and* $n_H > n_L$*. Moreover, the time devoted to production in the HS sector and the rental rate of capital are both constant.*[19]

Proof. See Appendix 11.3.

We can derive a number of important implications from Theorem 1. First, we obtain that $g_Y = g_{Y_H} > g_{Y_L}$, $g_K = g_{K_H} > g_{K_L}$ and $n_H > n_L$. Sector H exhibits a higher growth rate than sector L and capital and labour are allocated more intensively to the HS sector, which becomes dominant in the economy. Consequently, in the long run there is capital deepening, as the capital-labour ratio of both sectors is increasing and the share of skilled-labour is increasing while the share of unskilled-labour is decreasing. This pattern of structural change is observed in the data, as shown by Figure 2 in Section 3.

Clearly, as $g_K = g_Y = g_{K_H} = g_{Y_H} = n + g_a$, technical knowledge, a, is the engine of growth. It is worth noting, however, that although the HS sector is asymptotically dominant in output, the amount of inputs used is not vanishing and workforce does not shrink. Finally, the price of the final good decreases, $g_P < 0$. In fact, if we decompose this price effect using equations (10) and (11), we see that this drop is mainly due to the decreasing price of the intermediate good H.

Remark 1. ***Introducing TFP growth into the LS sector***

In order to get structural change, it has been necessary to consider asymmetric intermediary sectors in terms of productivity growth. We have then considered some HS sector where TFP increases because of knowledge accumulation and some LS sector where TFP is constant. Such an assumption may appear as quite extreme and one may wonder what are the robustness of our results if the LS sector is also characterized by some TFP growth. One easy way to proceed is to consider some exogenous TFP growth as in the standard structural change literature (see e.g. Acemoglu and Guerrieri [4]). Let us then consider that the production function of sector L is given by

$$Y_L(t) = A_L(t)L_L(t)^{\beta}K_L(t)^{1-\beta} \tag{14}$$

where the TFP $A_L(t)$ *is such that* $\dot{A}_L(t)/A_L(t) = g_{A_L} \geq 0$*, with* g_{A_L} *exogenously given. Adapting the proof of Theorem 1 shows that the same results hold with the only exception of the growth rates*

[19] As shown through equation (63) in Appendix 11.3, the growth rate of knowledge g_a is not proportional to the size of population and thus there is no scale effect.

associated to sector L which become

$$\begin{aligned} g_{Y_L} &= \epsilon g_{A_L} + n + (1-\beta\epsilon) g_a \\ g_{K_L} &= (\epsilon - 1) g_{A_L} + n + (1 + \beta - \beta\epsilon) g_a \\ n_L &= \tfrac{1+\beta(\epsilon-1)}{\beta} g_{A_L} + n - (\epsilon-1)\beta g_a \end{aligned}$$

Considering that the dominant sector is naturally the HS one, it seems reasonable to assume that the exogenous TFP growth rate g_{A_L} is sufficiently small to ensure $g_{Y_H} > g_{Y_L}$ and $n_H > n_L$. This is obtained under the assumption

$$g_a > \tfrac{1+\beta(\epsilon-1)}{\beta^2(\epsilon-1)} g_{A_L}$$

We can also easily prove that all our results presented below in Corollaries 1 and 2, Theorems 3 and 4 still hold.

We obtain the following characterisation of the NBGP which includes the Kaldor facts:

Corollary 1. *Under Assumption 1, the non-balanced growth path has the following properties:*

1. *There is capital deepening, i.e., the ratio K/L is increasing.*
2. *The growth rate of real GDP, or Y, is constant.*
3. *The capital-output ratio K/Y is constant.*
4. *The nominal share of capital income in GDP is constant and equal to $s_K = 1 - \alpha$.*
5. *The real interest rate R is constant.*
6. *The relative prices P_H/P and P_L/P are respectively decreasing and increasing. As a result, P_L/P_H is increasing.*
7. *The real and nominal shares in GDP of the HS sector and of the LS sector, as defined by Y_H/Y, $P_H Y_H/PY$, and Y_L/Y, $P_L Y_L/PY$, are respectively increasing and decreasing.*

Proof. See Appendix 11.4.

The mechanism at work in the economy can be described as follows. The accumulation of individual knowledge in the HS sector is used as labour-augmenting technological progress. Knowledge accumulation thus leads to an unbounded increase in TFP in the HS sector and to capital deepening. The relative price of the HS good decreases because of knowledge accumulation and TFP growth, while the relative price of the LS good increases. The change in relative prices is associated with changes in the demand for both intermediate goods by the final sector. This mechanism endogenously determines the growth rates of the two sectors such that the growth rate of real GDP remains constant and determined by the growth rate of the HS sector. The consequences of this are that the relative real and nominal shares of the HS sector with respect to the LS sector, Y_H/Y_L and $P_H Y_H/P_L Y_L$, are both increasing, and the real and nominal shares of the dominated LS sector are both decreasing. Finally, endogenous technological progress makes production of the final good more and more efficient, so that its price decreases.

These results are in line with documented facts of recent structural change in the services which shows that both real and nominal shares of HS sector have grown (see Figure 7 in Section 9.1). Moreover, the relative number of hours (and number of workers) in LS service sectors have fallen (see Figure 2). Our model is also able to explain the decreasing relative price of the HS sector in most countries (see Figure 2). Finally, our results are compatible with the fact that the relative share of modern market services (here the HS sector) has increased significantly with respect to the other sectors (here the LS sector) (see again Figure 7). Although the mechanism at work is quite different, we get similar conclusions than Buera et al. [25] who derive along the exogenous growth process a systematic shift in the composition of value added to sectors that are intensive in high-skilled labour. (See Figure 7 in Section 9.1).

Note that Acemoglu et al. [1] also supports input substitution. However, their results are obtained in a framework that focuses on the role of the environment and directed technical change and assume two intermediary sectors characterized by dirty and clean technologies. In our framework, we conjecture that substitution between inputs will be favored by the development of digital goods and artificial intelligence.

6.2 The case of complementary inputs

We now consider the case of complementary inputs, i.e., $\epsilon < 1$. We introduce the following restrictions which guarantee positiveness of growth rates and interiority of the share u of time devoted to work by agents in the HS sector:

Assumption 2. *$\epsilon < 1$ and $z \in \left\{\eta + \rho - n, \eta + \rho + \frac{n[\epsilon + \alpha(1-\epsilon)}{1+\alpha(1-\epsilon)}\right\}$*

We obtain the following theorem:

Theorem 2. *Under Assumption 2, the set of NBGR is non-empty and is characterised by the same 8-tuple of growth rates. The growth rates are constant across this set and given by: $g_K = g_Y = g_{K_L} = g_{Y_L} = n_L = n$, $g_{K_H} = n_H < g_{K_L}$, $n_H < n_L$ and $g_P = g_C = 0$. Moreover, the time devoted to production in the HS sector and the rental rate of capital are both constant.*

Proof. See Appendix 11.5.

The mechanism at work in this case can be described as follows. When $\epsilon < 1$ there exist two opposite forces: complementarity of inputs and endogenous technical change. The accumulation of individual knowledge in the HS sector is used as labour-augmenting technological progress. Knowledge accumulation thus leads to an unbounded increase in TFP in the HS sector. But as the LS intermediate sector output is also needed in the production of the final good because of the complementarity assumption, the productivity gap has to be compensated by a stronger reallocation of capital and labour in that sector, $g_{K_H} < g_{K_L}$ and $n_H < n_L$. Importantly, there is no capital deepening in any sector, i.e., the ratio K/L is constant in the long run. In addition, the share of skilled-labour is decreasing while the share of unskilled-labour is increasing. These results here are clearly not in line with the US data.

The relative price of the HS good still decreases because of TFP growth, while the relative price of the LS good increases. The striking result here is that these two effects offset each other so that the price of the final good P and thus consumption are constant. The change in relative prices is however still associated with changes in the demand for both intermediate goods by the final sector. This mixed mechanism endogenously determines the growth rates of the two sectors such that the growth rate of real GDP remains constant and now determined by the growth rate of sector L, but the HS sector remains dominant in the long run. It is also worth noting that the productivity lag effect is more than compensated by the technical change effect as the HS sector grows faster, i.e. $g_{Y_H} > g_{Y_L}$, which implies that the dominant sector is again the HS sector.

The consequences of this are that the relative real and nominal shares of sector L with respect to the HS sector, Y_L/Y_H and $P_L Y_L / P_H Y_H$, are both decreasing, and the real and nominal shares of the HS sector are respectively increasing and decreasing. Again, these properties are not compatible with the well documented facts of the structural change literature mentioned in Section 3.

6.3 The case of a unitary elasticity

For reference, we also consider the limit case $\epsilon = 1$ where the production function of the final good sector is Cobb-Douglas such that

$$Y(t) = Y_H(t)^{\gamma} Y_L(t)^{1-\gamma} \tag{15}$$

with $\gamma \in (0, 1)$, and the two intermediary sectors have an elasticity of substitution equal to one. Perfect substitution between the two intermediate inputs implies that the only driving force determining the long run growth properties is the knowledge accumulation in the HS sector. As a consequence, the asymptotic dominant sector is still the HS one but compared to Theorems 1 and 2, there is a discontinuity since the growth rate of the final sector is now determined by a convex combinaison of the growth rates of the two intermediate sectors. It can be shown indeed that there exists a unique set of NBGR such that

$$g_Y = g_K = g_{K_L} = g_{K_H} = g_C = n - \frac{g_P}{\theta} = \gamma g_{Y_H} + (1-\gamma) g_{Y_L},\ n_H = n_L = n,$$
$$g_{Y_H} = \frac{\alpha[\gamma+(1-\gamma)\beta](z-\eta+n-\rho)}{\theta\gamma\alpha+(1-\gamma)\beta} + n,\ g_{Y_L} = \frac{\gamma\alpha(1-\beta)(z-\eta+n-\rho)}{\theta\gamma\alpha+(1-\gamma)\beta} + n,$$
$$g_a = \frac{[\gamma\alpha+(1-\gamma)\beta](z-\eta+n-\rho)}{\theta\gamma\alpha+(1-\gamma)\beta}$$

with $g_{Y_H} > g_Y > g_{Y_L}$. In this configuration, the ratios K_H/K_L, L_H/L_L and N_H/N_L are constant along the NBGP and along the transition implying that the HS labour share and the share of capital in the HS sector are constant. Moreover, the nominal value added of the high-skilled sector is also constant. We disregard this case as the characteristics of the NBGP are not in line with the empirical evidence.

7 Local dynamics

The asymptotic properties of the NBGP described in the previous section only provide a long-run necessary condition. Yet interesting and empirically relevant properties also occur along the transition path. Note that in the rest of the paper we will focus on the case with substitution, i.e., $\epsilon > 1$. Indeed, given the analysis in the previous Section only this case generates a growth pattern compatible with the empirical evidence. However, we will mention some results for the case $\epsilon \leq 1$ for completeness.

We now aim to generate structural change along the transition path. We can reformulate the dynamical system given by equations (11)-(12) using the normalization of variables introduced by Caballe and Santos [26]. The stationarized NBGP is obtained by "removing" the NBGR trend from the variables, namely: $k(t) = K(t)e^{-g_K t}$, $x(t) = a(t)e^{-g_a t}$, $q(t) = Q(t)e^{-g_Q t}$ and $p(t) = P(t)e^{-g_P t}$, for all $t \geq 0$, with $k(t)$, $x(t)$, $q(t)$ and $p(t)$ the stationarized values for $K(t)$, $a(t)$, $Q(t)$ and $P(t)$. As the price of knowledge Q is characterized by a constant growth rate g_Q, the solution of the corresponding equation in (11) is given by $Q(t) = Q(0)e^{-g_Q t}$ and its stationarized value is constant with $\dot{q}(t) = 0$. We then get $q(t) = q(0) = q_0$ for all $t \geq 0$. Recall that as the population is growing at the exponential rate n, we have $N(t) = e^{nt}N(0)$ with $N(0) = N_0$ given.

Substituting these stationarized variables into (11)-(12), we obtain an equivalent stationarized system of differential equations that characterizes the equilibrium path. Of course, the expression of this dynamical system slightly differs depending on the value of the elasticity of substitution ϵ. As assumed above we focus on the case with $\epsilon > 1$.

Lemma 1. *Let N_0 be given and suppose Assumption 1 holds. Along a stationarized equilibrium path and for any given q_0, knowledge x, capital k and its price p are solutions of the following dynamical system*

$$
\begin{aligned}
\frac{\dot{p}}{p} &= -\left[(1-\alpha)\gamma\psi^{\frac{1}{\epsilon}}x^{\alpha}\left(\frac{L_H(k,x,p,q_0,N_0)}{K_H(k,x,p,q_0,N_0)}\right)^{\alpha} + g_P - \rho - \delta\right] \\
\frac{\dot{k}}{k} &= \psi x^{\alpha}\left(\frac{L_H(k,x,p,q_0,N_0)}{K_H(k,x,p,q_0,N_0)}\right)^{\alpha}\frac{K_H(k,x,p,q_0,N_0)}{k} - \delta - g_K - \frac{N_0 p^{-\frac{1}{\theta}}}{k} \\
\frac{\dot{x}}{x} &= z\Big(1-u(k,x,p,q_0,N_0)\Big) - g_a - \eta
\end{aligned} \tag{16}
$$

with $\psi = \gamma^{\frac{\epsilon}{\epsilon-1}}\left(1 + (\frac{1-\alpha}{1-\beta})\frac{k-K_H(k,x,p,q_0,N_0)}{K_H(k,x,p,q_0,N_0)}\right)^{\frac{\epsilon}{\epsilon-1}}$.

Proof. See Appendix 11.6.[20]

Note also that the transversality conditions (13) become

$$\lim_{t\to+\infty} p(t)k(t)e^{-[\rho-g_K-g_P]t} = 0 \text{ and } \lim_{t\to+\infty} x(t)e^{-(\rho-g_a-g_Q)t} = 0$$

Using the expressions of the growth rates g_K, g_a, g_Q and g_P given in Theorem 1, we easily derive under Assumption 1 that $zu^* = z - g_a - \eta = \rho - g_K - g_P = \rho - g_a - g_Q > 0$.

Considering the stationarized dynamical systems given in Lemma 1, we can now focus on proving the existence of a steady state solution, i.e., $\dot{p}/p = \dot{k}/k = \dot{x}/x = 0$, and $\dot{q}(t) = 0$ which obviously corresponds to the NBGR exhibited in Theorems 1.

Theorem 3. *Let N_0 be given and suppose Assumption 1 holds. The projection of the set of NBGP on the subspace (k, x, p, q) is a one-dimensional manifold, noted $\mathcal{M} \subset \mathcal{R}^4$, parameterized by q_0. Furthermore, for any given $q_0 > 0$, there exists a unique steady state $(k^*(q_0), x^*(q_0), p^*(q_0))$, solution of the dynamical system (16). Moreover, $k^{*'}(q_0) < 0$, $x^{*'}(q_0) < 0$ and $p^{*'}(q_0) > 0$.*

Proof. See Appendix 11.7.

Theorem 3 proves that when $\epsilon > 1$, there exists a one-dimensional manifold of steady states for the capital stock k, technological knowledge x and the price of capital p parameterized by the constant price of knowledge q_0. Importantly, the asymptotic amount of time devoted to production in the HS sector u^* and the asymptotic rental rate of capital R^*, as given in Theorem 1, do not depend on q_0. Note that an analogous result to Theorem 3 for the case $\epsilon < 1$ can be obtained and shows that there still exists a manifold of steady state but this is degenerate as only technological knowledge x depends on q_0 while the capital stock k and the price p do not.

The existence of a manifold of steady states in levels is fairly standard in endogenous growth models (see for instance Lucas [53]) where the asymptotic equilibrium of the economy depends on some initial conditions. We will show that there exists a set $\mathcal{K}$ containing the set $\mathcal{M}$ such that for initial values of physical capital $k(0)$ and technological knowl-

[20] Analogous results for the case with complementarity ($\epsilon < 1$) or the case with a unitary elasticity of substitution ($\epsilon = 1$) can be obtained and are available upon request.

edge $x(0)$ in $\mathcal{K}$, a value of q_0 is "automatically" selected in order for the economy to leap onto the optimal path (i.e., the stable manifold) and then converge to the particular steady state $(k^*(q_0), x^*(q_0), p^*(q_0))$ situated on the manifold $\mathcal{M}$. Note that for any given pair $(k(0), x(0)) \in \mathcal{K}$ there exists a unique value of the price of knowledge q_0 compatible with the equilibrium conditions.

To prove such a result, we need to study the local stability properties of the steady state. Linearizing the dynamical systems (16) around the steady state $(k^*(q_0), x^*(q_0), p^*(q_0))$ for a given $q_0 > 0$, the local stability property of $(k^*(q_0), x^*(q_0), p^*(q_0))$ is appraised through the characteristic roots of the associated Jacobian matrix. As shown by Martinez-Garcia [56] (see also Bond, Wang and Yip [16] and Xie [70]), since we have two state variables, k and x, and two forward variables, p and q, with q being constant, the standard saddle-point stability occurs if there exists a one-dimensional stable manifold, i.e. if only one characteristic root is negative.

Lemma 2. *Let Assumption 1 hold. Then for any given $q_0 > 0$, the steady state $(k^*(q_0), x^*(q_0), p^*(q_0))$ is saddle-point stable.*

Proof. See Appendix 11.8.[21]

In the dynamical system (16), the predetermined variables are the capital stock and the level of individual knowledge. For given initial conditions $K(0) = k(0) = k_0$ and $a(0) = x(0) = x_0$, we generically cannot find a value of q_0 such that $(k_0, x_0) = (k^*(q_0), x^*(q_0))$ and the economy is not in the set of NBGP from the initial date. In other words, non-trivial transitional dynamics generically occurs starting from $(k_0, x_0) \neq (k^*, x^*)$. The initial values of the forward variables $p(0) = p_0$ and $q(0) = q_0$ are chosen such that the one-dimensional stable optimal path converging toward an NBGP is selected. As the stable manifold is one-dimensional, for any given $K(0) = k(0) = k_0$ and $x(0) = x_0$ there exists a unique pair (q_0, p_0) compatible with an equilibrium path. This property therefore defines a conditional convergence depending on the value of the initial price of knowledge q_0.

The arguments supporting the truth of the previous statement are as follows. The dynamical system has two state and two forward looking variables. The steady state is then a 4-uple (k, x, p, q). There is a one-dimensional manifold of steady states. Each of these is saddle-path stable and q is constant on any equilibrium path. For each of these q there is a one-dimensional stable manifold leading to the NBGP. When q spans the feasible set, this describes a stable planar manifold. Generically there is a two-dimensional plane in space of dimension four with a given $(x(0), k(0))$. The intersection of the two planes in dimension four is a set of dimension zero, a point. So for a given $(x(0), k(0))$ there is a unique $(p(0), q(0))$.

[21] The same conclusion holds for the complementary case ($\epsilon < 1$), i.e., Assumption 2, or the case with a unitary elasticity of substitution ($\epsilon = 1$).

A striking property is that, although the steady state values $(k^*(q_0), x^*(q_0), p^*(q_0))$ depend on the selected q_0, the eigenvalues do not. It follows that the rate of convergence along any transitional path is the same, regardless of the initial conditions and thus of the asymptotic value of the steady state.

Building on Theorem 3, it can be shown (see Appendix 11.7) that $k^*(x^*)$ is a linear function of x^*. The conditional convergence property can be illustrated by the following Figure.

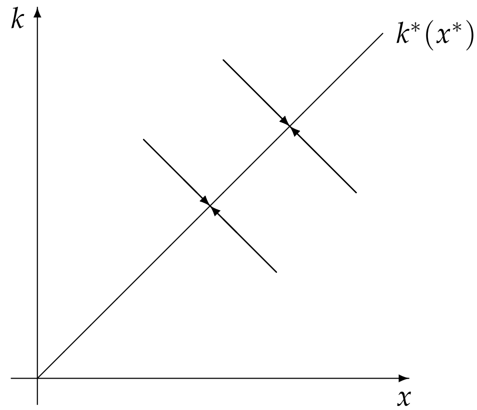


Figure 5: Manifold of steady states when $\epsilon > 1$.

The economy is characterized by two initial conditions, $k(0) = k_0$ and $x(0) = x_0$, while $p(0)$ and $q(0)$ are pinned down by the equilibrium conditions. All pairs (k^*, x^*) satisfying $k^* = k^*(x^*)$ correspond to a common asymptotic NBGP but different optimal paths along the transition according to the initial condition (k_0, x_0). For a given (k_0, x_0), the optimal path will converge toward an asymptotic position located on the curve $k^*(x^*)$ which depends on the initial position (k_0, x_0) that defines the admissible initial values p_0 and q_0. Arrows in Figure 5 illustrate some possible trajectories. It can be shown indeed that $k(t)$ and $x(t)$ evolve in opposite directions. Such a property is easy to explain. As k and x are stationnarized values of K and a, if we consider initial values k_0 and x_0 that are above the curve $k^*(x^*)$, the transitory growth rate of $a(t)$ will be lower than the long run growth rate g_a and will increase progressively to g_a while $x(t)$ converges to the long run value x^*. On the contrary, the transitory growth rate of $K(t)$ will be initially higher than its long run value and decrease progressively converging toward g_K while $k(t)$ converges to the long run value k^*. Opposite results are of course obtained if we consider initial values k_0 and x_0 that are below the curve $k^*(x^*)$.

Remark: In the case of complementarity with $\epsilon < 1$, k^* is independent from x^* and the manifold is horizontal in the (x, k) space. The intuition on this degenerate manifold can easily be explained as follows: as sector L is dominant in the long run, it determines the level of capital independently from the level of technical knowledge. There is therefore no synergy in the long run between knowledge and capital accumulation. In the case of a unitary elasticity of substitution $\epsilon = 1$, we get a similar manifold as in Figure 5 but which is a non-linear increasing concave function.

The following Theorem summarizes the results.

Theorem 4. *Let Assumption 1 hold. There exists a set $\mathcal{K}$ containing the set $\mathcal{M} \subset \mathbb{R}^4$ such that for initial values of physical capital $k(0)$ and technological knowledge $x(0)$ in $\mathcal{K}$, the economy converges to the steady state $(k^*(q_0), x^*(q_0), p^*(q_0))$ situated on the manifold $\mathcal{M}$, where the price of knowledge q_0 and p_0 are uniquely determined and the optimal path converging to the NBGP is unique.*

Proof. See Appendix 11.9.

The existence of a manifold of steady states, while standard in the endogenous growth literature, is a fundamental difference with respect to exogenous growth models where the technological progress is exogenous, the steady state is unique and countries with the same fundamentals but different initial endowments of physical capital will all converge toward the same asymptotic level of wealth and the same asymptotic growth rate. By contrast, in the present model, while all countries with the same fundamentals are characterized by the same growth rate, they will follow different optimal paths along the transition and be asymptotically characterized by different wealth levels.[22] Importantly, the long run heterogeneity of wealth will concern both physical capital and knowledge.[23]

8 Robustness with a non linear accumulation of knowledge

We have assumed through equation (4) a linear accumulation of knowledge. Such a formulation may appear as highly specific and one may wonder whether our main conclusions are robust to a more general formulation. A fundamental property to ensure the existence of endogenous growth is to preserve a linear homogeneity of the accumulated factor that drives the whole process. We may then generalize equation (4) through the consideration of a production function of knowledge based both on the stock of knowledge and the stock of physical capital through a linear homogeneous Cobb-Douglas formulation. Since we focus on an individual accumulation of knowledge, we need therefore to consider an individual stock of capital entering the production function of knowledge. Let us assume that

$$\dot{a}(t) = Y_a(t) \equiv z\left[(1-u(t))a(t)\right]^{\phi}\left(\frac{K_a(t)}{N_H(t)}\right)^{1-\phi} = z\left[(1-u(t))a(t)\right]^{\phi}\left(\frac{u(t)K_a(t)}{L_H(t)}\right)^{1-\phi} \tag{17}$$

[22]See also Marsiglio and Tolotti [55] who consider an endogenous growth model with heterogeneous firms where innovation and social interactions may lead to the existence of multiple BGP equilibria.

[23]Note that when $\epsilon < 1$, countries will converge to the same capital stock but will have heterogeneous levels of knowledge, which is less plausible from an empirical perspective.

with the new capital stock constraint $K(t) \geq K_H(t) + K_a(t) + K_L(t)$.[24] Adjusting accordingly the intertemporal problem (9), we easily show that Proposition 1 still holds but with the following modifications for the dynamic equations of the stock of knowledge and its price

$$\begin{aligned} \frac{\dot{Q}}{Q} &= -z(1-u)^{\phi}\left(\frac{uK_a}{aL_H}\right)^{1-\phi}\frac{1-u(1-\phi)}{1-u} \\ \frac{\dot{a}}{a} &= z(1-u)^{\phi}\left(\frac{uK_a}{aL_H}\right)^{1-\phi} \end{aligned} \tag{18}$$

Obviously, contrary to the linear formulation, the growth rate of the price of knowledge is no longer constant along the transition.

In the case of substitutable inputs, $\epsilon > 1$, proceeding as in the proof of Theorem 1, let us define the shares of capital and labour allocated to the HS sector and the share of capital allocated for the accumulation of knowledge as

$$\kappa_K(t) \equiv \frac{K_H(t)}{K(t)}, \qquad \kappa_a(t) \equiv \frac{K_a(t)}{K(t)}, \qquad \lambda(t) \equiv \frac{L_H(t)}{L(t)} \tag{19}$$

such that $1 - \kappa_H - \kappa_a(t)(t) \equiv \frac{K_L(t)}{K(t)}$ and $1 - \lambda(t) \equiv \frac{L_L(t)}{L(t)}$. As shown in a proof available upon request, we obtain:

$$\kappa_a(t) = \frac{\alpha}{1-\alpha}\frac{(1-\phi)(1-u(t))}{\phi u(t) + (1-\phi)(1-u(t))}\kappa_K(t) \tag{20}$$

$$\kappa(t) = \left[1 + \left(\frac{1-\beta}{1-\alpha}\right)\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{Y_L(t)}{Y_H(t)}\right)^{\frac{\epsilon-1}{\epsilon}} + \frac{\alpha}{1-\alpha}\frac{(1-\phi)(1-u(t))}{\phi u(t) + (1-\phi)(1-u(t))}\right]^{-1} \tag{21}$$

and

$$\lambda(t) = \left[1 + \frac{\beta(1-\alpha)}{\alpha(1-\beta)}\frac{1-\kappa_H(t)-\kappa_a(t)}{\kappa_H(t)}\right]^{-1} \tag{22}$$

and we also derive that

$$Y(t) = \psi(t)Y_H(t) \tag{23}$$

with

$$\psi(t) = \gamma^{\frac{\epsilon}{\epsilon-1}}\left(1 + \frac{1-\alpha}{1-\beta}\frac{1-\kappa_H(t)-\kappa_a(t)}{\kappa_H(t)}\right)^{\frac{\epsilon}{\epsilon-1}} \tag{24}$$

We can then compute the following differential equation for $\kappa_H(t)$:

$$\frac{\dot{\kappa}_H}{\kappa_H} = G(\kappa) \equiv \frac{(\epsilon-1)\beta g_a\left[1-\kappa_H\left[1+\frac{\alpha}{1-\alpha}\frac{(1-\phi)(1-u)}{\phi u+(1-\phi)(1-u)}\right]\right]}{\epsilon + \frac{(\beta-\alpha)(1-\alpha)\left[1-\kappa_H\left[1+\frac{\alpha}{1-\alpha}\frac{(1-\phi)(1-u)}{\phi u+(1-\phi)(1-u)}\right]\right]}{\alpha\left[1-\alpha-\kappa_H\left[\beta-\frac{\alpha\phi u}{\phi u+(1-\phi)(1-u)}\right]\right]}} \tag{25}$$

Assuming that $\beta > \alpha$, we have $G'(\kappa) < 0$ with $G(0) > 0$ and

$$G\left(\left[1 + \frac{\alpha}{1-\alpha}\frac{(1-\phi)(1-u)}{\phi u+(1-\phi)(1-u)}\right]^{-1}\right) = 0$$

This implies that equation (25) has a unique solution such that

[24] To simplify the formulation, we assume here that the depreciation rate of knowledge η is equal to zero.

$$\lim_{t\to\infty}\kappa_H(t)=\kappa_H^*=\frac{1}{1+\frac{\alpha}{1-\alpha}\frac{(1-\phi)(1-u)}{\phi u+(1-\phi)(1-u)}} \text{ and } \lim_{t\to\infty}\kappa_a(t)=\kappa_a^*=\frac{\frac{\alpha}{1-\alpha}\frac{(1-\phi)(1-u)}{\phi u+(1-\phi)(1-u)}}{1+\frac{\alpha}{1-\alpha}\frac{(1-\phi)(1-u)}{\phi u+(1-\phi)(1-u)}} \tag{26}$$

Using these asymptotic values into λ and ψ gives the following results: $\lim_{t\to\infty}\lambda(t)=\lambda^*=1$ and $\lim_{t\to\infty}\psi(t)=\psi^*=\gamma^{\frac{\epsilon}{\epsilon-1}}$. We can then derive the NBGR such that $n_H=n$, $n_L=n+(1-\epsilon)\beta g_a$, $g_P=-\theta g_a$, $g_Q=g_P+n$, $g_Y=k_K=g_{Y_H}=g_{K_H}=g_{K_a}=g_a+n$, $g_{Y_L}=n+(1-\beta\epsilon)g_a$ and $g_{K_L}=n+(1+\beta-\beta\epsilon)g_a$ with

$$g_a=\frac{(1-u)(\rho-n)}{1-u(1-\phi)-\theta(1-u)}=g_{Y_a}$$

We need finally to compute the stationary value of u. From equation (17) we derive that u is a solution of $F(u)=H(u)$ with

$$F(u)\equiv\frac{\rho-n}{1-u(1-\phi)-\theta(1-u)} \text{ and } H(u)\equiv z\left(\frac{\alpha(1-\phi)u}{(1-\alpha)\phi u+(1-\phi)(1-u)}\right)^{1-\phi} \tag{27}$$

Assuming $\rho>n$, $\theta<1$ and

$$z>\frac{\rho-n}{\phi}\left(\frac{(1-\alpha)\phi}{\alpha(1-\phi)}\right)^{1-\phi}$$

we easily prove that there exists a unique solution $u^*\in(0,1)$. The NBGRs are then finally computed plugging the value u^* into the expression of g_a. We then conclude that our main results on the characteristics of the NBGRs and NBGPs as derived in Theorem 1 and Lemma 1 are robust to the consideration of a non linear accumulation equation of knowledge. It is also possible to prove that the transitional structural change properties as provided in Corollary 2 in the next Section also hold.

We need finally to prove that even with a non linear accumulation equation of knowledge, the one-dimensional manifold of steady states as proved in Theorem 3 still exists. We need therefore to stationarize the NBGP as in Section 7. Considering $k(t)=K(t)e^{-g_K t}$, $x(t)=a(t)e^{-g_a t}$, $q(t)=Q(t)e^{-g_Q t}$, $p(t)=P(t)e^{-g_P t}$ and $\ell(t)=L(t)e^{-nt}$, for all $t\geq 0$, with $k(t)$, $x(t)$, $q(t)$, $p(t)$ and $\ell(t)$ the stationarized values for $K(t)$, $a(t)$, $Q(t)$, $P(t)$ and $L(t)$, and that $N(t)=e^{nt}N(0)$ with $N(0)=N_0$ given, we can derive the following stationarized dynamical system:

$$\begin{aligned}
\frac{\dot{k}}{k} &= \psi x^\alpha\lambda^\alpha\kappa_H^{1-\alpha}\left(\frac{\ell}{k}\right)^\alpha-\delta-g_K-\frac{N_0p^{-1/\theta}}{k}\\
\frac{\dot{x}}{x} &= z(1-u)^\phi u^{1-\phi}x^{\phi-1}\lambda^{\phi-1}\kappa_a^{1-\phi}\left(\frac{k}{\ell}\right)^{1-\phi}-g_a\\
\frac{\dot{p}}{p} &= -\left[(1-\alpha)\gamma\psi^{1/\epsilon}x^\alpha\lambda^\alpha\kappa_H^{-\alpha}\left(\frac{\ell}{k}\right)^\alpha+g_P-\delta-\rho\right]\\
\frac{\dot{q}}{q} &= -\left[z(1-u)^\phi u^{1-\phi}x^{\phi-1}\lambda^{\phi-1}\kappa_a^{1-\phi}\left(\frac{k}{\ell}\right)^{1-\phi}+g_Q-\rho\right]
\end{aligned}$$

The steady state of this dynamical system is obtained considering that $u=u^*$, $\lambda=\lambda^*=1$ and $\ell=\ell^*=N_0u^*$. Considering equation (26), we first find that

$$\begin{aligned} k &= \left(\frac{(1-\alpha)\gamma^{\frac{\epsilon}{\epsilon-1}}}{\delta+\rho-g_P}\right)^{1/\alpha} \frac{\ell^*}{\kappa_H^*} x \equiv \mathcal{Z}_1 x \\ p &= \left[\frac{(1-\alpha)N_0}{\left[(\delta+\rho-g_P)\kappa_H^*-(1-\alpha)(\delta+g_K)\right]\mathcal{Z}_1}\right]^{\theta} x^{-\theta} \equiv \mathcal{Z}_2 x^{-\theta} \end{aligned} \tag{28}$$

From the first order conditions of the Hamiltonian maximization program we can get

$$p\alpha\gamma\phi^{1/\epsilon}x^\alpha\lambda^\alpha\kappa_H^{1-\alpha}\ell^\alpha k^{1-\alpha} = qz(1-u)^\phi u^{1-\phi}x^\phi\lambda^{\phi-1}\kappa_a^{1-\phi}\ell^{\phi-1}k^{1-\phi}\left(\frac{\phi u+(1-\phi)(1-u)}{1-u}\right)$$

Using $u = u^*$, $\lambda = \lambda^* = 1$ and $\ell = \ell^* = N_0 u^*$ with (28) and solving this equation with respect to x we find

$$x^* = \left(\frac{\alpha\gamma\psi^{1/\epsilon}\mathcal{Z}_2\mathcal{Z}_1^{\phi-\alpha}\ell^{*1-\phi+\alpha}\kappa_H^{\phi-\alpha}}{q_0 z\left(\frac{\alpha}{1-\alpha}\frac{(1-\phi)u^*}{\phi u^*+(1-\phi)(1-u^*)}\right)^{1-\phi}[\phi u^*+(1-\phi)(1-u^*)]}\right)^{1/\theta} \equiv x^*(q_0)$$

and thus

$$\begin{aligned} k^* &= \mathcal{Z}_1\left(\frac{\alpha\gamma\psi^{1/\epsilon}\mathcal{Z}_2\mathcal{Z}_1^{\phi-\alpha}\ell^{*1-\phi+\alpha}\kappa_H^{\phi-\alpha}}{q_0 z\left(\frac{\alpha}{1-\alpha}\frac{(1-\phi)u^*}{\phi u^*+(1-\phi)(1-u^*)}\right)^{1-\phi}[\phi u^*+(1-\phi)(1-u^*)]}\right)^{1/\theta} \equiv k^*(q_0) \\ p^* &= \mathcal{Z}_2\frac{q_0 z\left(\frac{\alpha}{1-\alpha}\frac{(1-\phi)u^*}{\phi u^*+(1-\phi)(1-u^*)}\right)^{1-\phi}[\phi u^*+(1-\phi)(1-u^*)]}{\alpha\gamma\psi^{1/\epsilon}\mathcal{Z}_2\mathcal{Z}_1^{\phi-\alpha}\ell^{*1-\phi+\alpha}\kappa_H^{\phi-\alpha}} \equiv p^*(q_0) \end{aligned}$$

Therefore, for any given $q_0 > 0$, there exists a unique steady state $(k^*(q_0), x^*(q_0), p^*(q_0))$ with $k^{*'}(q_0) < 0$, $x^{*'}(q_0) < 0$ and $p^{*'}(q_0) > 0$. We have then proved that all our main results, in particular the existence of a manifold of steady states, are not peculiar to the linear formulation of the accumulation equation of knowledge and are robust to a non-linear specification.[25]

9 Transitional structural change

Having shown the existence of paths converging to the NBGP, we focus here on the properties of these paths. When inputs are substitutable, for any given pair (k_0, x_0) the optimal path will converge toward an asymptotic position located on the manifold $\mathcal{M}$. From the property that the manifold is parameterized by q_0 and that $k^{*'}(q_0) < 0$, $x^{*'}(q_0) < 0$ in Theorem 3, we know that the set of NBGP is such that greater value of capital along the NBGP is associated with greater value of knowledge (also seen in Fig. 5). It is also likely that an economy with initially low levels of physical capital and technological knowledge will remain permanently behind an initially better-endowed economy, as suggested by Fig. 5. In other words, q_0 is decreasing when $k(0)$ and $a(0)$ increase.

Theorems 1, 3 and 4, and Corollary 1 allows us to give a characterisation of the transition

[25]Saddle-point stability could also be proved but at the cost of cumbersome computations which are beyond the goal of this paper.

path toward the non-balanced growth path.

Corollary 2. *Under Assumption 1, the transition toward the non-balanced growth path is characterized by the following properties:*

1. *The shares of capital and labour in the HS sector, $\kappa = K_H/K$ and $\lambda = L_H/L$, are increasing.*
2. *The real and nominal shares in GDP of the HS sector, respectively Y_H/Y and $P_H Y_H/PY$, are increasing.*
3. *The nominal share of capital income in GDP, $s_K = RK/Y$, is increasing (decreasing) if and only if $\beta > (<) \alpha$.*
4. *The relative price of the HS sector P_H/P is decreasing while the relative price of the LS sector P_L/P is increasing.*

Proof. See Appendix 11.10.

The results 1, 2 and 3 in Corollary 2 are in line with the empirical evidence collected in Sections 3 and 9.1. First, along the transition path, the share of workers in the HS sector increases while converging toward its stationary value, and the labour force in the LS sector decreases. As illustrated in Section 3, the share of workers in service sectors in most countries over the period 1970-2020 (see Figure 2) follows this pattern. Other patterns reinforce this fact. The number of hours and workers in the LS sectors relative to those in HS service sectors have fallen and both real and nominal shares of LS sectors have also fallen. Moreover, the relative share of modern market services plus durable goods (here the HS sector) has increased dramatically with respect to the other sectors (here the LS sector). Our results are thus also fully in line with these empirical facts (see Figure 7 in Section 9.1).

Results 4 is also well in line with the dynamics of relative prices highlighted in Section 3 (see Figure 2). Over the period $1970 - 2020$, the relative prices of services of the progressive sector and of the stagnant sector are indeed respectively increasing and decreasing as derived from our model. Our results are clearly explained by the productivity growth differential across the two intermediate sectors. Indeed, as the stagnant sector has a relatively lower growth rate than the HS sector, its relative price obviously has to increase over time.

9.1 Transitional dynamics: an illustrative calibration

We now provide an illustrative calibration to investigate whether the equilibrium dynamics generated by our model is consistent with the patterns shown by the data described in Section 3. The first step is to obtain a plausible NBGR. Our model is characterized by 10 parameters, namely $\epsilon, \delta, \rho, \theta, \gamma, \alpha, \beta, n, z$ and η. Following Barro and Sala-i-Martin [13], we adopt the standard values for the annual depreciation rate $\delta = 0.05$, the annual discount rate

$\rho = 0.02$ and the long-run annual interest rate $R^* = 0.08$. We use the EUKLEMS database to evaluate the annual population growth rate over the period 1970-2020 and we find $n = 0.01$. Concerning the sectoral labour shares, we focus on the US data and we use[26] the National Income and Product Accounts (NIPA) between 1948 and 2005 where industries are classified according to the North American Industrial Classification System at the 22-industry level. We classify industries according to the requirement of technological knowledge by the workers. That is, we consider an industry to be HS if workers exhibit a higher growth of compensation per capita than average. The Table 2 provided in Appendix 11.11 shows the average capital share of each industry together with the sector classification. This classification allows us to compute average shares of capital for two "aggregate sectors" in which $\alpha = 0.62$ and $\beta = 0.64$.[27] Recent contributions by Mulligan [60], Vissing-Jorgensen and Attanasio [69] and Gruber [37] provide robust estimates of the elasticity of intertemporal substitution in consumption between 1 and 2.3. We consider here an intermediate value of 1.5 leading therefore to $\theta = 10/15$. As we do not have any empirical evidence to calibrate the values of the parameters z and η characterizing the accumulation of individual knowledge, we adjust these values to match the endogenous annual output growth rate g_Y. The total output growth between 1970 and 2020 in the EUKLEMS/INTANPROD 2021 database is 2.5%, leading to $g_Y = 0.025$. The corresponding values of z and η are then $z = 0.11$ and $\eta = 0.09$.

As in Acemoglu and Guerrieri [4] we compute the capital share of the high-skilled, $\kappa(t)$, considering equation (55) in Appendix 11.2 for five distinct countries, namely the United States, Germany, Japan, Finland and the United Kingdom. Each κ then allow us to derive the labour share of the high-skilled sector, $\lambda(t)$ as given by (56) in Appendix 11.2. In figure 6, we show how the model perform compare to the data for each country on the period 1970-2019. We then compute the nominal (value added) share in GDP, $P_H(t)Y_H(t)/P(t)Y(t)$, and the real (value added) share in GDP of the HS sector, $Y_H(t)/Y(t)$ as given by (83) (in Appendix 11.10), the graphs are shown in Figure 7. The model allows for a numerical characterization of the dynamics of these quantities along the transition for $\epsilon = 1.8$ (reported in Figure 7). Importantly the predictions are not very sensitive to the choice of ϵ within the range $(1.5, 2)$.

We observe that for the 5 selected countries, the model fits quite well with the data and shows a similar trend along the years, even if each starting point is different. This feature is due to the endogenous growth mechanism and the multiplicity of equilibrium described in the empirical evidence. Our model is able to reproduce the dynamics of some European countries by simply changing the starting point, validating the endogenous mechanism at stake in our model, and the multiplicity of trends.

[26] As Acemoglu and Guerrieri [4]

[27] These two numbers are obtained as the weighted average of the shares mentioned in Table 1 in Appendix 10.1 according to the relative size of each sector.

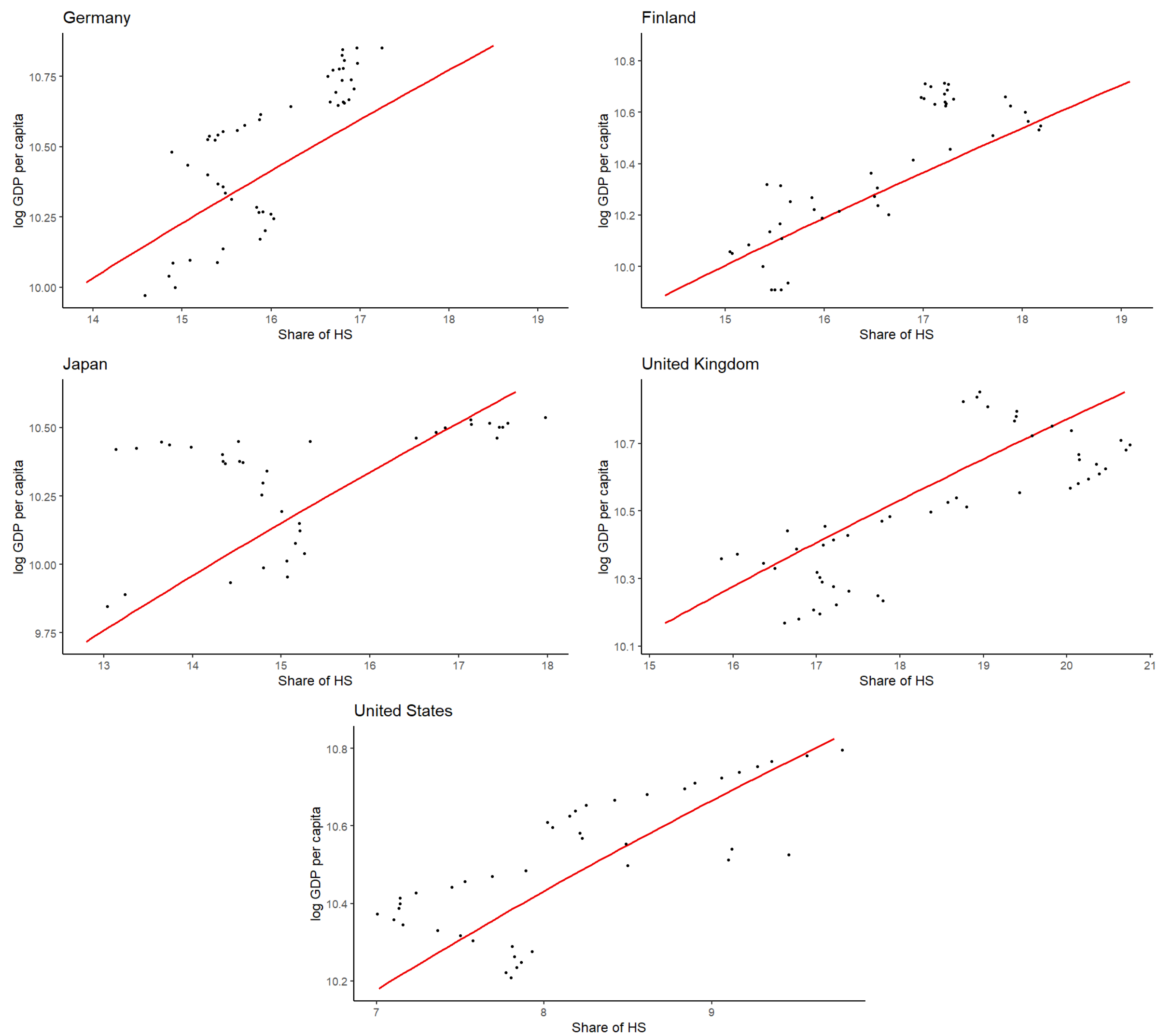


Figure 6: labour shares transitional dynamics, for 5 countries in the model (red line) and the data (black dots)

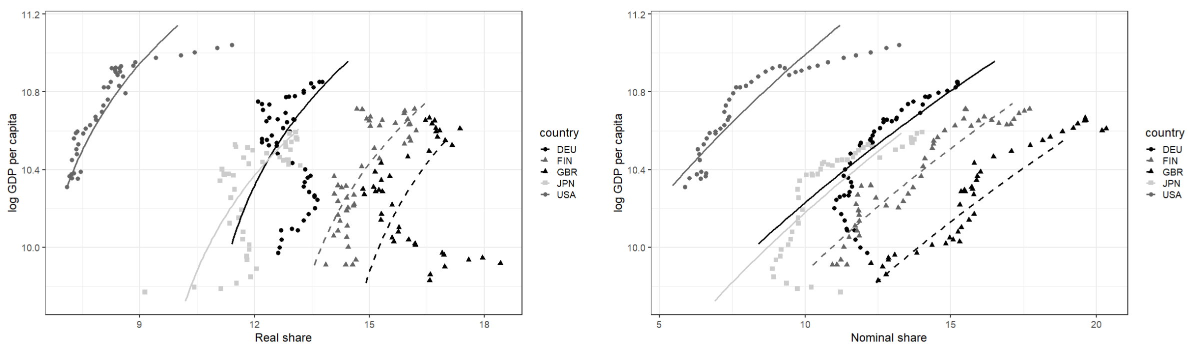


Figure 7: log GDP per capita according to real and nominal value added in the HS sector, in the data (dots) and in the model (plain line)

A number of features are worth noting. First, the dynamics of the labour shares and of the real and nominal shares of value added in GDP of the HS sector match well the data. We are able to reproduce this positive correlation between labour accumulation in the HS sector and log GDP per capita. Secondly, the endogenous growth mechanism allows us to retrieve the multiplicity of trend observed in the data. In figure 7 we clearly observe the parallel trends for our 5 countries and the non-convergence of them in the model and in the data. According to the endowment level in HS labour, each country will reach a different level of development for the same share of labour/value added in the HS sector. Even if there are still some perturbations in the data, due to the internet bubble and the 2008 financial crisis (or the reunification for Germany), the match is consistent over the 50 years of observations allowed by the data. We believe introducing endogenous growth in structural change models is a promising venue to study heterogeneity of economies and industries.

Finally, the following Figure contain the dynamics of the relative prices of the HS sector. As mentioned in the empirical part of Section 3, prices are given as an index, which does not allow us to make level comparisons between countries. However, as observed in Section 3 for numerous countries, the relative price of the HS sector is decreasing in the model. Due to the nature of the data we are not able to quantitatively compare the model and the data, but at least qualitatively they are both going in the same direction.

As proved in Corollary 2 and in line with the data, we find that the relative price of the HS sector P_H/P is decreasing and reproduce the price trend observed in Austria. Moreover, when we compare to Figure 2 we see that the model provides on average a quite good match to the decreasing trend of price for most countries.

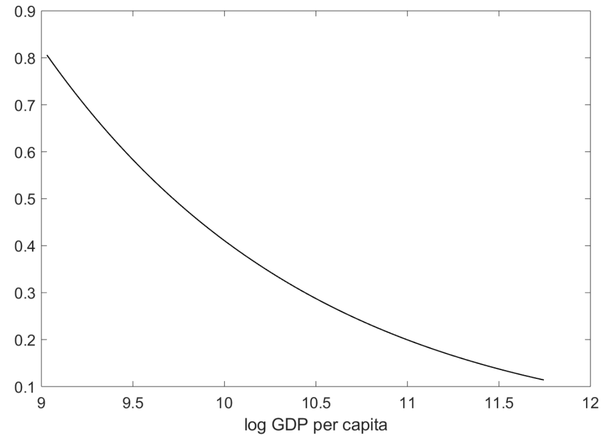


Figure 8: Transitional dynamics of the relative prices

10 Conclusion

Motivated by the recent dynamics of highly productive sectors, we propose a two-sector model of non-balanced endogenous growth. The final good is produced through a CES technology using two intermediate sectors. The progressive sector is HS as in Lucas [53] and the accumulation of knowledge leads to an unbounded increase in TFP. Along the NBGP capital and labour reallocate across sectors. When intermediary capital goods are substitutable, the real and nominal shares of the HS sector are increasing while the real and nominal shares of the LS sector are decreasing. Interestingly, the shares of capital and labour in the HS sector depend on the initial value of capital and knowledge. As a consequence, countries with the same fundamentals but lower initial wealth will be characterized by lower asymptotic wealth. We finally provided a numerical illustration to characterize the transitional dynamics of the main variables.

In this paper we explore the dynamics of an economy in which productivity is endogenous and intermediate sectors are all substitutes in the production of a final output. The next step is to extend the economy to include also complementary intermediates. We expect that the general dynamics will depend on the details of the model as there are obvious conflicting tendencies. A realistic model would also include demand side effect generated by non-homotheticities in preferences, in the spirit of Comin et al. [29].

A more satisfactory theory of structural change would require deeper microeconomic foundations. A promising approach is to follow recent developments able to endogenize the linkages in an input output economy.[28] Indeed, the way innovation pushes to new or better substitutes or complements depends on these details. Similarly, the role of automation on

[28]See for instance Acemoglu and Azar [2], Carvalho and Tahbaz-Salehi [27], Ghiglino [36], Miranda-Pinto [58], Oberfield [63].

structural change through its impact on labour productivity could be precisely analyzed.[29]

11 Appendix

11.1 Additional empirics

In this appendix we will provide some additional empirical facts to complement section 3. We will start by providing descriptive statistics about EUKLEMS data and then the complete estimation related to the relationship between employment share and productivity growth (reported in Section 3.2 and Fig. 1). We will also provide some further facts about capital dynamics and value added, omitted from the core document.

11.1.1 Descriptive statistics

We first provide some descriptive statistics of the EUKLEMS data for the 9 selected countries providing the relevance of each industry and their relative TFP. In table 1 we provide the mean share of each industry as well as its growth rate, while in table 2 we provide information about relative TFP and its growth rate. From these descriptive statistics it appears that

Industry	Mean	Std err	Min	Max	Growth	G std err
G	19.53	0.25	14.39	26.01	-0.96	0.13
H	7.54	0.19	3.87	11.81	-0.31	0.16
I	6.83	0.21	4.19	11.02	-0.13	0.30
J	4.29	0.07	3.10	5.92	1.00	0.20
K	3.65	0.08	2.22	5.28	-1.35	0.23
L	1.42	0.03	0.99	2.09	0.56	0.25
M_N	14.79	0.41	5.42	20.39	1.63	0.20
O	10.03	0.16	7.11	12.30	-0.74	0.20
P	8.63	0.18	2.92	10.39	-0.07	0.14
Q	16.79	0.34	9.71	22.83	0.67	0.12
R	2.35	0.04	1.61	3.41	0.33	0.27
S	3.87	0.06	2.90	5.09	0.33	0.18

Table 1: Employment share and its growth rate in each industry

G, *wholesale and retail trade*, M_N, *professional, scientific and technical activities and administrative and support services activities*, as well as Q, *Human Health and social work activities* represent the biggest share in the service economy, and only G has a decreasing share. M_N is also the sector with the most vivid growth followed by J, *Information and communication*. L, *real estate*

[29] See Acemoglu and Restrepo [5].

activities, is the industry with the lowest share even if it is increasing by half a percentage point per year in mean. The sign of the correlation between employment and TFP growth is reported in the next table.

Industry	Mean	Std err	Min	Max	Growth	G std err
G	1.01	0.01	0.69	1.44	1.26	0.25
H	0.94	0.01	0.70	1.08	-0.42	0.33
I	0.92	0.01	0.69	1.06	-0.74	0.45
J	1.10	0.03	0.54	2.30	2.03	0.31
K	0.97	0.02	0.58	1.47	0.56	0.47
L	1.07	0.01	0.91	1.40	-0.21	0.18
M_N	1.03	0.02	0.85	1.54	-0.26	0.19
O	1.08	0.01	0.98	1.22	0.10	0.16
P	0.99	0.01	0.73	1.43	0.017	0.019
Q	1.01	0.02	0.71	1.37	-0.94	0.17
R	0.99	0.01	0.66	1.35	-1.14	0.45
S	0.99	0.01	0.78	1.42	-0.83	0.25

Table 2: Relative TFP and its growth rate in each industry

From this table we observe that among the industries with positive relative TFP growth, G, J, K and O, only J exhibits both a positive growth for employment and for TFP.

11.1.2 The relationship between labour share and TFP : a complete estimation

We now report the complete estimation related to the relationship between employment share and relative TFP (Section 3.2 and Fig. 1). We used the panel data model described in equation (2) on each of the 12 industries in the services sector. Table 3 shows the results. As capital is less volatile than labour, it is important to take into account that capital movements often precede labour movements. From this estimation we obtain that capital and labour are always positively correlated, except in the case of industry Q, human health services, as the effect is far from being significant. In this table we obtain the effect of capital share on labour income share to correct for the TFP effect. Once we correct for capital, we obtain that J and P are the only industries with a strctly positive relationship between relative TFP and labour share, even when we take into account capital movements, country and yearly fixed effects. As explained in section 3, we believe these movements are induced because of the high share of technical labour within those industries.

Dependent variable : Labour share

Industry	G	H	I	J	K	L	M_N	O	P	Q	R	S
Relative TFP	-15.64^{***}	-7.31^{***}	-2.50^{***}	1.54^{***}	-2.38^{***}	-0.30^{***}	-14.38^{***}	-11.73^{***}	1.17^{***}	-7.09^{***}	-0.49^{***}	-0.57^{***}
	(0.50)	(0.26)	(0.14)	(0.09)	(0.19)	(0.06)	(0.41)	(0.82)	(0.27)	(0.53)	(0.04)	(0.13)
Capital Share	0.13^{***}	0.10^{***}	0.04^{***}	0.03^{**}	0.00	0.02^{***}	0.15^{***}	0.55^{***}	0.04^{*}	-0.02	0.01	0.02
	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.00)	(0.03)	(0.05)	(0.02)	(0.04)	(0.02)	(0.02)
Country FE	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES
Year FE	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES
R^2	0.70	0.68	0.46	0.41	0.27	0.17	0.74	0.43	0.06	0.29	0.32	0.05
N	443	443	443	413	443	443	439	418	443	443	413	413

The results are obtained using a dynamic panel estimation and the within group estimator. It therefore take into account country and year specification.

$^{***}p < 0.01$; $^{**}p < 0.05$; $^{*}p < 0.1$

Table 3: The relationship between labour share and relative TFP

11.1.3 Capital share

In section 3 we have seen how labour share in the HS sector is evolving compared to prices, but we have been agnostic on capital share. In figure 9 we show how the capital share in the HS sector evolves over the last 50 years for our 9 countries.

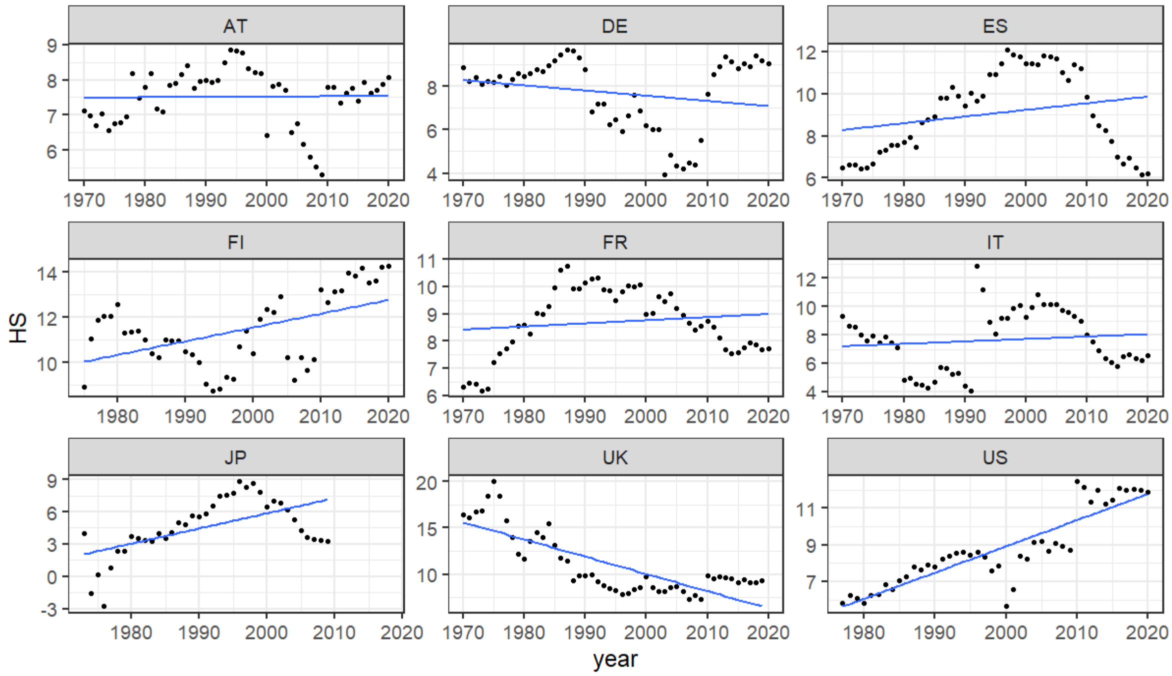


Figure 9: Capital share in the HS sector

We observe that capital fluctuates more than labour and exhibits some unusual patterns : Austria in the 2000, Germany after reunification or Italy in the 90's. However, except for the UK, the capital share in the HS sector is increasing or stable for those countries, with some fluctuations according to the period.

11.2 Proof of Proposition 1

The Hamiltonian in current value associated to the optimization problem (9) is (we omit subscript for t to simplify notations):

$$\begin{aligned}\mathbb{H} = {} & N_H \frac{c_H^{1-\theta}}{1-\theta} + N_L \frac{c_L^{1-\theta}}{1-\theta} + P_H\left[(L_H a)^\alpha K_H^{1-\alpha} - Y_H\right] + P_L\left[L_L^\beta K_L^{1-\beta} - Y_L\right] \\ & + P\left[\left(\gamma Y_H^{\frac{\epsilon-1}{\epsilon}} + (1-\gamma) Y_L^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}} - \delta K - N_H c_H - N_L c_L\right] + Q\Big[z(1-u) - \eta\Big] a \\ & + \mu_K\left[K - K_H - K_L\right] + \mu_L\left[L - L_H - L_L\right]\end{aligned}$$

with $L_H = uN_H$ and μ_K and μ_L the Lagrange multipliers associated with the capital and labour market clearing conditions. The first-order conditions with respect to the control variables $c_H, c_L, u, L_H, L_L, K_H, K_L, Y_H$ and Y_L give:

$$
\begin{aligned}
c_i^{-\theta} &= P \text{ for any } i = H, L && (29)\\
P_H \alpha \frac{Y_H}{L_H a} N_H &= Qz && (30)\\
P_H &= P\gamma \left(\frac{Y}{Y_H}\right)^{\frac{1}{\epsilon}} && (31)\\
P_L &= P(1-\gamma)\left(\frac{Y}{Y_L}\right)^{\frac{1}{\epsilon}} && (32)\\
\mu_K = P_H(1-\alpha)\frac{Y_H}{K_H} &= P_L(1-\beta)\frac{Y_L}{K_L} && (33)\\
\mu_L = P_H \alpha \frac{Y_H}{L_H} &= P_L \beta \frac{Y_L}{L_L} && (34)
\end{aligned}
$$

Substituting (31) and (32) into (33) and (34) gives

$$
\begin{aligned}
(1-\alpha)\gamma \left(\frac{Y}{Y_H}\right)^{\frac{1}{\epsilon}} \frac{Y_H}{K_H} &= (1-\beta)(1-\gamma)\left(\frac{Y}{Y_L}\right)^{\frac{1}{\epsilon}} \frac{Y_L}{K_L} && (35)\\
\alpha\gamma \left(\frac{Y}{Y_H}\right)^{\frac{1}{\epsilon}} \frac{Y_H}{L_H} &= \beta(1-\gamma)\left(\frac{Y}{Y_L}\right)^{\frac{1}{\epsilon}} \frac{Y_L}{L_L} && (36)
\end{aligned}
$$

In the HS sector, the rental rate of capital r_H and the individual wage rate w_H are given by

$$r_H = (1-\alpha)\tfrac{Y_H}{K_H}, \quad w_H = \alpha \tfrac{Y_H}{N_H u a} \tag{37}$$

while in the LS sector, r_L and w_L are given by

$$r_L = (1-\beta)\tfrac{Y_L}{K_L}, \quad w_L = \beta \tfrac{Y_L}{N_L} \tag{38}$$

Note first that substituting (31) and (32) into (37) and (38) allows (36) to be written as follows

$$w_H(t)a(t)P_H(t) = w_L(t)P_L(t)$$

which gives the equality between nominal wages per hour devoted to production in the relevant sector. Similarly (35) is equivalent to

$$r_H(t)P_H(t) = r_L(t)P_L(t) \tag{39}$$

which gives the equality between the capital return in the two sectors. These two properties imply that despite the fact that agents work in different sectors, they all consume the same amount since $c_H(t) = c_L(t) = P^{-1/\theta}$ for any $t \geq 0$.

The equilibrium rental rate of capital can then be defined as

$$
\begin{aligned}
R(t) &= (1-\alpha)\gamma\left(\tfrac{Y}{Y_H}\right)^{\frac{1}{\epsilon}}\tfrac{Y_H}{K_H} = (1-\beta)(1-\gamma)\left(\tfrac{Y}{Y_L}\right)^{\frac{1}{\epsilon}}\tfrac{Y_L}{K_L} \\
&= \tfrac{P_K r_K}{P} = \tfrac{P_L r_L}{P}
\end{aligned}
\tag{40}
$$

From the Hamiltonian, we also derive the optimality conditions that provide differential equations for the prices P and Q of aggregate capital and knowledge:

$$
\dot{P} = \rho P - \frac{\partial \mathbb{H}}{\partial K} = \rho P - \frac{\partial \mathbb{H}}{\partial K} = P(\rho+\delta) - P_H(1-\alpha)\frac{Y_H}{K_H} \tag{41}
$$

$$
\dot{Q} = \rho Q - \frac{\partial \mathbb{H}}{\partial a} = -Q(z-\eta-\rho) + Qzu - P_H\alpha\frac{Y_H}{a} \tag{42}
$$

Substituting equation (31) into (41) and using (40) then gives

$$
\dot{P} = -P\left[R-\delta-\rho\right] \tag{43}
$$

Note now that using $L_H = uN_H$, equation (30) becomes $P_H\alpha Y_H/(ua) = Qz$. Substituting this expression into (42) gives

$$
\dot{Q} = -Q(z-\eta-\rho) \tag{44}
$$

Moreover, since $c_H(t) = c_L(t) = P^{-\frac{1}{\theta}}$, we get aggregate consumption as

$$
C = C_H + C_L = N_H c_H + N_L c_L = NP^{-\frac{1}{\theta}}
$$

We finally obtain the following differential equations for prices and stocks as given by (11)-(12). The result follows.

□

11.3 Proof of Theorem 1

From (12) we immediately get $g_a = z(1-u) - \eta$. Differentiating equation (29) gives using (11)

$$
\begin{aligned}
g_{c_H} = g_{c_L} = -\tfrac{1}{\theta}g_P &= \tfrac{1}{\theta}\left[(1-\alpha)\gamma\left(\tfrac{Y}{Y_H}\right)^{\frac{1}{\epsilon}}\tfrac{Y_H}{K_H} - \delta - \rho\right] \\
&= \tfrac{1}{\theta}\left[(1-\beta)(1-\gamma)\left(\tfrac{Y}{Y_L}\right)^{\frac{1}{\epsilon}}\tfrac{Y_L}{K_L} - \delta - \rho\right]
\end{aligned}
\tag{45}
$$

It follows that $g_P = -\theta g_{c_H} = -\theta g_{c_L}$. Since g_P is constant along a NBGP, we get

$$
g_{K_H} = \tfrac{\epsilon-1}{\epsilon}g_{Y_H} + \tfrac{1}{\epsilon}g_Y \tag{46}
$$

and

$$
g_{K_L} = \tfrac{\epsilon-1}{\epsilon}g_{Y_L} + \tfrac{1}{\epsilon}g_Y \tag{47}
$$

The capital accumulation equation (8) can be written as

$$
g_K = \tfrac{Y}{K} - \delta - \tfrac{NP^{-1/\theta}}{K}
$$

Differentiating this expression using the fact that along a NBGP $\dot{g}_K = 0$ yields

$$
g_Y = n - g_P/\theta \tag{48}
$$

Since aggregate consumption is given by $C = NP^{-1/\theta}$ we conclude that $g_C = g_Y$. Differentiating (6) gives:

$$g_{Y_H} = \alpha g_a + \alpha n_H + (1-\alpha) g_{K_H} \tag{49}$$

Combining (30) and (31) and differentiating gives:

$$g_P + \tfrac{\epsilon-1}{\epsilon} g_{Y_H} + \tfrac{1}{\epsilon} g_Y - g_a = g_Q = -(z - \eta - \rho) \tag{50}$$

Now differentiating equations (35) and (36) yields:

$$\begin{aligned} \tfrac{\epsilon-1}{\epsilon} g_{Y_H} - g_{K_H} &= \tfrac{\epsilon-1}{\epsilon} g_{Y_L} - g_{K_L} \\ \tfrac{\epsilon-1}{\epsilon} g_{Y_H} - n_H &= \tfrac{\epsilon-1}{\epsilon} g_{Y_L} - n_L \end{aligned} \tag{51}$$

The differentiation of (7) gives

$$g_{Y_L} = \beta n_L + (1-\beta) g_{K_L} \tag{52}$$

and using (51) we get

$$g_{Y_L} = (1-\beta)\epsilon g_{K_H} + \beta \epsilon n_H + (1-\epsilon) g_{Y_H} \tag{53}$$

Equations (46)-(53) are not enough to determine explicitly the values of the growth rates. We need also to determine the relationship between the growth rate of K and K_H, L and L_H, Y and Y_H. We thus use the same methodology as in Acemoglu and Guerrieri [4].

Considering the first order conditions (30)-(34) allows to define the following maximized value of current output given the physical capital stock $K(t)$ and the stock of knowledge $a(t)$ at time t as

$$\Phi(K(t), a(t), t) = \max_{K_H(t), K_L(t), L_H(t), L_L(t), u(t)} \left(\gamma Y_H^{\frac{\epsilon-1}{\epsilon}} + (1-\gamma) Y_L^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}}$$

Let us define the shares of capital and labour allocated to the HS sector (sector H) as

$$\kappa(t) \equiv \tfrac{K_H(t)}{K(t)}, \qquad \lambda(t) \equiv \tfrac{L_H(t)}{L(t)} \tag{54}$$

We also have $1 - \kappa(t) \equiv \frac{K_L(t)}{K(t)}$ and $1 - \lambda(t) \equiv \frac{L_L(t)}{L(t)}$. And combining this statement with the equations (35) and (36) we obtain:

$$\kappa(t) = \left[1 + \left(\frac{1-\beta}{1-\alpha} \right) \left(\frac{1-\gamma}{\gamma} \right) \left(\frac{Y_L(t)}{Y_H(t)} \right)^{\frac{\epsilon-1}{\epsilon}} \right]^{-1} \tag{55}$$

and

$$\lambda(t) = \left[1 + \left(\frac{1-\alpha}{1-\beta} \right) \left(\frac{\beta}{\alpha} \right) \left(\frac{1-\kappa(t)}{\kappa(t)} \right) \right]^{-1} \tag{56}$$

Using equation (5), we can write the maximized value of current ouput $\Phi(K(t), a(t), t)$ as follows:

$$Y(t) = \Phi(K(t), a(t), t) = \psi(t) a(t)^{\alpha} \lambda(t)^{\alpha} \kappa(t)^{1-\alpha} L(t)^{\alpha} K(t)^{1-\alpha} \tag{57}$$

with

$$\psi(t) = \gamma^{\frac{\epsilon}{\epsilon-1}} \left(1 + \tfrac{1-\alpha}{1-\beta} \tfrac{1-\kappa(t)}{\kappa(t)} \right)^{\frac{\epsilon}{\epsilon-1}} \tag{58}$$

Considering that $c_H = c_L = c$, we can rewrite the Hamiltonian in maximized value:

$$\mathbb{H}(c, K, a, P) = N\frac{c^{1-\theta}}{1-\theta} + P\Big[\Phi(K(t), a(t), t) - \delta K - Nc\Big]$$

and the first order conditions with respect to consumption gives:

$$\frac{\dot{c}}{c} = \frac{1}{\theta}\Big[(1-\alpha)\gamma\psi(t)^{\frac{1}{\epsilon}}a^{\alpha}\lambda^{\alpha}\kappa^{-\alpha}L^{\alpha}K^{-\alpha} - \delta - \rho\Big] = -\frac{1}{\theta}\frac{\dot{P}}{P} \tag{59}$$

Since along the NBGP $\dot{g_c} = 0$, we get:

$$\frac{1}{\epsilon}\frac{\dot{\psi}(t)}{\psi(t)} + \alpha g_a + \alpha\frac{\dot{\lambda}(t)}{\lambda(t)} - \alpha\frac{\dot{\kappa}(t)}{\kappa(t)} + \alpha n - \alpha g_K = 0$$

Differentiating equations (56) and (58) gives

$$\begin{aligned}
\frac{\dot{\psi}(t)}{\psi(t)} &= \frac{\dot{\kappa}(t)}{\kappa(t)}\left(\frac{\epsilon(1-\alpha)}{[\Delta\kappa+(1-\alpha)](1-\epsilon)}\right)\\
\frac{\dot{\lambda}(t)}{\lambda(t)} &= \frac{\dot{\kappa}(t)}{\kappa(t)}\left(\frac{(1-\beta)\alpha}{(1-\alpha)\beta} + \frac{1-\kappa}{\kappa}\right)^{-1}\\
\frac{\dot{K}(t)}{K(t)} &= \frac{\dot{\kappa}(t)}{\kappa(t)}\left[\frac{(1-\alpha)\epsilon}{\alpha(1-\epsilon)[\Delta\kappa+1-\alpha]} + \left(\frac{(1-\beta)\alpha}{(1-\alpha)\beta} + \frac{1-\kappa}{\kappa}\right)^{-1}\right] + \Big(z(1-u) - \eta + n\Big)
\end{aligned}$$

where $\Delta = \alpha - \beta$. Now differentiating (55) and substituting with what we have just found allow to write the law of motion of κ as follows:

$$\frac{\dot{\kappa}(t)}{\kappa(t)} = G(\kappa(t))\beta g_a \tag{60}$$

with

$$G(\kappa(t)) = \frac{(1-\kappa)(\epsilon-1)}{\epsilon+(\epsilon-1)\Big(\Delta KS(1-\kappa)-\delta\kappa-(1-\alpha)-\lambda S(\alpha-\lambda\Delta)\Big(\frac{(1-\alpha)\beta+\Delta\kappa}{(1-\alpha)\beta}\Big)\Big)} \tag{61}$$

where

$$KS = \frac{(1-\alpha)\epsilon}{\alpha(1-\epsilon)[\Delta\kappa+1-\alpha]} + \left(\frac{(1-\beta)\alpha}{(1-\alpha)\beta} + \frac{1-\kappa}{\kappa}\right)^{-1}, \quad \lambda S = \left(\frac{(1-\beta)\alpha}{(1-\alpha)\beta} + \frac{1-\kappa}{\kappa}\right)^{-1} \tag{62}$$

It is then easy to check under $\alpha, \beta > 1/2$ that $G(0) > 0$, $G(1) = 0$ and $G'(\kappa) < 0$ for any κ. This implies that equation (60) has a unique solution such that $\lim_{t\to\infty}\kappa(t) = \kappa^* = 1$. Using this asymptotic value into λ and ψ give the following results: $\lim_{t\to\infty}\lambda(t) = \lambda^* = 1$ and $\lim_{t\to\infty}\psi(t) = \psi^* = \gamma^{\frac{\epsilon}{\epsilon-1}}$.

We conclude therefore that $g_K = g_{K_H}$ and $n = n_H$, and using the maximized value of the output $\Phi(K(t), a(t), t)$ as is given by (57), we obtain $Y(t) = \psi^* Y_H(t)$, which gives $g_{Y_H} = g_Y$. Now we can replace all these equalities into the equations (46)-(53) to obtain the explicit values of the growth rates. From (46), (48) and (49), we have $g_K = g_Y$, $g_P = \theta(n - g_K)$ and $g_K = g_a + n$. Using (50) we then get $g_P = g_Q - n = -(z - \eta - \rho + n)$ and thus

$$g_a = \frac{z-\eta-\rho+n}{\theta} \text{ and } g_K = n + \frac{z-\eta-\rho+n}{\theta} \tag{63}$$

From (51) and (53) we derive

$$g_{Y_L} = n + (1-\beta\epsilon)g_a,\ g_{K_L} = n + (1+\beta-\beta\epsilon)g_a \text{ and } n_L = n + (1-\epsilon)\beta g_a$$

Using $g_a = z(1-u) - \eta$ we finally obtain:

$$u^* = \frac{1}{z}\left[z - \eta - \frac{1}{\theta}(n - \rho + z - \eta)\right] \tag{64}$$

and from (11)

$$R^* = \delta + \rho - g_P = \delta + \rho - \theta(n - g_K)$$

$\square$

11.4 Proof of Corollary 1

Let us consider the first-order conditions (30)-(36) and Theorem 1.

1. We derive from Theorem 1 that the growth rate of K/L is equal to

$$g_{K/L} = g_K - n = g_a > 0$$

2. This result is obvious from Theorem 1.
3. Along the NBGP the capital-output ratio is given by

$$g_{K/Y} = g_K - g_Y = 0$$

4. The share of capital income in GDP is given by

$$s_K = \frac{RK}{Y} = (1-\alpha)\gamma\left(\frac{Y}{Y_H}\right)^{\frac{1-\epsilon}{\epsilon}}\frac{1}{\kappa}$$

Along the NBGP with $\kappa = 1$ and $Y = \psi^* Y_H$ we then get

$$s_K = (1-\alpha)$$

5. This result is obvious from Theorem 1.
6. From (5), (31) and (32) we derive that

$$\begin{aligned} \frac{P_H}{P} &= \gamma\left(\frac{Y}{Y_H}\right)^{\frac{1}{\epsilon}} = \gamma\left(\gamma + (1-\gamma)\left(\frac{Y_L}{Y_H}\right)^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{1}{\epsilon-1}} \\ \frac{P_L}{P} &= (1-\gamma)\left(\frac{Y}{Y_L}\right)^{\frac{1}{\epsilon}} = (1-\gamma)\left(\gamma\left(\frac{Y_H}{Y_L}\right)^{\frac{\epsilon-1}{\epsilon}} + (1-\gamma)\right)^{\frac{1}{\epsilon-1}} \end{aligned} \tag{65}$$

We derive from these expressions

$$\begin{aligned} \frac{\frac{d}{dt}(P_H/P)}{P_H/P} &= g_{P_H/P} = \frac{1}{\epsilon}\frac{(1-\gamma)\left(\frac{Y_L}{Y_H}\right)^{\frac{\epsilon-1}{\epsilon}}}{\gamma + (1-\gamma)\left(\frac{Y_L}{Y_H}\right)^{\frac{\epsilon-1}{\epsilon}}}(g_{Y_L} - g_{Y_H}) \\ \frac{\frac{d}{dt}(P_L/P)}{P_L/P} &= g_{P_L/P} = \frac{1}{\epsilon}\frac{\gamma\left(\frac{Y_H}{Y_L}\right)^{\frac{\epsilon-1}{\epsilon}}}{\gamma\left(\frac{Y_H}{Y_L}\right)^{\frac{\epsilon-1}{\epsilon}} + (1-\gamma)}(g_{Y_H} - g_{Y_L}) \end{aligned} \tag{66}$$

Since $g_{Y_H} = g_Y > g_{Y_L}$, we conclude that $g_{P_H/P} < 0$, $g_{P_L/P} > 0$ and thus $g_{P_L/P_H} > 0$.

7. Using the expressions of Y/Y_H and Y/Y_L in (65), we derive

$$
\begin{aligned}
\frac{\frac{d}{dt}(Y_H/Y)}{Y_H/Y} &= g_{Y_H/Y} = \frac{(1-\gamma)\left(\frac{Y_L}{Y_H}\right)^{\frac{\epsilon-1}{\epsilon}}}{\gamma+(1-\gamma)\left(\frac{Y_L}{Y_H}\right)^{\frac{\epsilon-1}{\epsilon}}}(g_{Y_H}-g_{Y_L}) > 0 \\
\frac{\frac{d}{dt}(Y_L/Y)}{YL/Y} &= g_{Y_L/Y} = \frac{\gamma\left(\frac{Y_H}{Y_L}\right)^{\frac{\epsilon-1}{\epsilon}}}{\gamma\left(\frac{Y_H}{Y_L}\right)^{\frac{\epsilon-1}{\epsilon}}+(1-\gamma)}(g_{Y_L}-g_{Y_H}) < 0
\end{aligned}
\tag{67}
$$

It follows that the real shares of the HS and LS sectors in GDP, Y_H/Y and Y_L/Y, are respectively increasing and decreasing. Using (65), we also derive

$$
\begin{aligned}
\frac{P_H Y_H}{PY} &= \gamma\left(\gamma+(1-\gamma)\left(\frac{Y_L}{Y_H}\right)^{\frac{\epsilon-1}{\epsilon}}\right)^{-1} \\
\frac{P_L Y_L}{PY} &= (1-\gamma)\left(\gamma\left(\frac{Y_H}{Y_L}\right)^{\frac{\epsilon-1}{\epsilon}}+(1-\gamma)\right)^{-1}
\end{aligned}
\tag{68}
$$

and thus

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{P_H Y_H}{PY}\right) &= \frac{\epsilon-1}{\epsilon}\frac{P_H Y_H}{PY}\frac{(1-\gamma)\left(\frac{Y_L}{Y_H}\right)^{\frac{\epsilon-1}{\epsilon}}}{\gamma+(1-\gamma)\left(\frac{Y_L}{Y_H}\right)^{\frac{\epsilon-1}{\epsilon}}}(g_{Y_H}-g_{Y_L}) \\
\frac{d}{dt}\left(\frac{P_L Y_L}{PY}\right) &= \frac{\epsilon-1}{\epsilon}\frac{P_L Y_L}{PY}\frac{\gamma\left(\frac{Y_H}{Y_L}\right)^{\frac{\epsilon-1}{\epsilon}}}{\gamma\left(\frac{Y_H}{Y_L}\right)^{\frac{\epsilon-1}{\epsilon}}+(1-\gamma)}(g_{Y_L}-g_{Y_H})
\end{aligned}
\tag{69}
$$

Since $\epsilon > 1$, we conclude that the nominal shares of the HS and LS sectors, as defined by $P_H Y_H/PY$ and $P_L Y_L/PY$, are respectively increasing and decreasing.

□

11.5 Proof of Theorem 2

Basically, we use the same type of methodology to find the NBGP when inputs are complementary. We define however in a different way the output of the final good as follows

$$
Y(t) = \phi(t)(1-\lambda)^{\beta}(1-\kappa)^{1-\beta}K^{1-\beta}L^{\beta} \tag{70}
$$

with

$$
\phi = (1-\gamma)^{\frac{\epsilon}{\epsilon-1}}\left(1+\left(\frac{1-\beta}{1-\alpha}\right)\left(\frac{\kappa}{1-\kappa}\right)\right)^{\frac{\epsilon}{\epsilon-1}}
$$

To simplify notations let us define $v = 1-\kappa$ and $w = 1-\lambda$.

Using again the hamiltonian in maximized value, and the fact that along the NBGP $\dot{g_C} = 0$, we obtain:

$$
\frac{\dot\phi(t)}{\phi(t)} + \beta\frac{\dot w(t)}{w(t)} - \beta\frac{\dot v(t)}{v(t)} + \beta\frac{\dot L(t)}{L(t)} - \beta\frac{\dot K(t)}{K(t)} = 0
$$

We need to transform equations (20) and (21) to express them in terms of v and w, then we differentiate these transformed equations and also the definition of ϕ. We modify the formulations and we express them in terms of $\frac{\dot v}{v}$. Once we have these 3 differential equations we use the previous equality to express $\frac{\dot K}{K}$ in terms of $\frac{\dot v}{v}$. We have:

$$\begin{aligned}
\frac{\dot{v}(t)}{v(t)} &= -\frac{\epsilon-1}{\epsilon}(1-v)(g_{Y_H}-g_{Y_L}) \\
\frac{\dot{\phi}(t)}{\phi(t)} &= \frac{\dot{v}(t)}{v(t)}\left(\frac{\epsilon(1-\beta)}{[\Delta v+(1-\alpha)](1-\epsilon)}\right) \equiv D_1\frac{\dot{v}(t)}{v(t)} \\
\frac{\dot{w}(t)}{w(t)} &= \frac{\dot{v}(t)}{v(t)}\left(\frac{(1-\alpha)\beta}{(1-\beta)\alpha}+\frac{1-v}{v}\right)^{-1} \equiv D_2\frac{\dot{v}(t)}{v(t)} \\
\frac{\dot{K}(t)}{K(t)} &= \frac{1}{\beta}\frac{\dot{\phi}(t)}{\phi(t)}+\frac{\dot{w}(t)}{w(t)}-\frac{\dot{v}(t)}{v(t)}+n
\end{aligned}$$

The last step is to prove that $\frac{\dot{v}(t)}{v}=0$ when $v=1$ implying that $\kappa=0$ and $\lambda=0$. To this end, we compute $(g_{Y_H}-g_{Y_L})$ and we obtain the following equation:

$$\frac{\dot{v}(t)}{v(t)}=\frac{(1-\epsilon)\alpha g_a(1-v)}{\epsilon+(1-\epsilon)\left(D_2\alpha\frac{1-v}{1-w}+1-\alpha+\frac{\Delta}{\beta}(1-v)D_1\right)} \equiv G(v)$$

Under $\alpha,\beta>1/2$, $G(.)$ is a positive and either decreasing or hump-shaped function in v, with $G(0)>0$, $G'(1)<0$ and $G(1)=0$. As v is defined over a compact subset $[0,1]$, we conclude that $v=1$ is the steady state value of the model, which implies that $\kappa^*=0$, $\lambda^*=0$ and $\phi^*=(1-\gamma)^{\frac{\epsilon}{\epsilon-1}}$. We conclude therefore that $g_K=g_{K_L}$ and $n=n_L$, and using equation (70) we obtain $Y(t)=\phi^*Y_L(t)$, which gives $g_{Y_L}=g_Y$. Now we can replace all these equalities into the equations (46)-(53) to obtain the explicit values of the growth rates. From (52) and (47), (48), we have $g_K=g_Y$, $g_K=n$ and $g_P=0$. From (51) and (49) we get $g_{K_H}=n_H$, $g_{Y_H}=\epsilon\alpha g_a+n$ and $n_H=n+(\epsilon-1)g_a$. Using (50) we get $n-g_Q=g_a[1+\alpha(1-\epsilon)]$ and thus

$$g_a=\frac{z-\eta-\rho+n}{1+\alpha(1-\epsilon)} \text{ and } n_H=\frac{n[\epsilon+\alpha(1-\epsilon)]-(1-\epsilon)(z-\eta-\rho)}{1+\alpha(1-\epsilon)}$$

Using $g_a=z(1-u)-\eta$ we finally obtain:

$$u^*=\frac{1}{z}\left[z-\eta-\frac{z-\eta-\rho+n}{1+\alpha(1-\epsilon)}\right]$$

and from (11)

$$R^*=\delta+\rho-g_P=\delta+\rho$$

□

11.6 Proof of Lemma 1

Let us consider the stationarized values for $K(t)$, $a(t)$ and $P(t)$ as defined by $k(t)=K(t)e^{-g_Kt}$, $x(t)=a(t)e^{-g_at}$ and $p(t)=P(t)e^{-g_Pt}$, for all $t\geq 0$. Recall also that as population is growing at the exponential rate n, we have $N(t)=e^{nt}N(0)$ with $N(0)=N_0$ given. Let Assumption 1 hold and let us substitute the maximized output $Y(t)=\Phi(K(t),a(t),t)$ as given by (57) into equations (23) and (26). We obtain

$$
\begin{aligned}
\frac{\dot{P}}{P} &= -\left[(1-\alpha)\gamma\psi^{\frac{1}{\epsilon}}x^{\alpha}\left(\frac{L_H}{K_H}\right)^{\alpha} - \rho - \delta\right] \\
\frac{\dot{K}}{K} &= \psi a^{\alpha}\left(\frac{L_H}{K_H}\right)^{\alpha}\frac{K_H}{K} - \delta - \frac{NP^{-\frac{1}{\theta}}}{K}
\end{aligned}
$$

We need to prove that $L_H = L_H(k, a, p, q_0, N_0)$, $K_H = K_H(k, a, p, q_0, N_0)$ and $u = u(k, a, p, q_0, N_0)$, and to compute the derivatives of these functions. From (14) and (15) we derive:

$$
\begin{aligned}
(14) \quad &\Leftrightarrow \quad \frac{\gamma(1-\alpha)}{(1-\gamma)(1-\beta)}\frac{[(L_H a)^{\alpha}K_H^{1-\alpha}]^{\frac{\epsilon-1}{\epsilon}}}{K_H} = \frac{[(L-L_H)^{\beta}(K-K_H)^{1-\beta}]^{\frac{\epsilon-1}{\epsilon}}}{K-K_H} \\
(15) \quad &\Leftrightarrow \quad \frac{\gamma\alpha}{(1-\gamma)\beta}\frac{[(L_H a)^{\alpha}K_H^{1-\alpha}]^{\frac{\epsilon-1}{\epsilon}}}{L_H} = \frac{[(L-L_H)^{\beta}(K-K_H)^{1-\beta}]^{\frac{\epsilon-1}{\epsilon}}}{L-L_H} \\
\frac{(14)}{(15)} \quad &\Leftrightarrow \quad \frac{B_H}{B_L}L_H(K-K_H) = K_H(L-L_H)
\end{aligned}
$$

with $B_H = \frac{\gamma(1-\alpha)}{(1-\gamma)(1-\beta)}$ and $B_L = \frac{\gamma\alpha}{(1-\gamma)\beta}$. Let us then denote

$$
\begin{aligned}
\omega_H &= \frac{B_H}{B_L}L_H(K-K_H) - K_H(L-L_H) = 0 \\
\omega_L &= B_H\frac{[(L_H a)^{\alpha}K_H^{1-\alpha}]^{\frac{\epsilon-1}{\epsilon}}}{K_H} - \frac{[(L-L_H)^{\beta}(K-K_H)^{1-\beta}]^{\frac{\epsilon-1}{\epsilon}}}{K-K_H} = 0
\end{aligned}
$$

If the matrix

$$
J_1 = \begin{pmatrix} \frac{\partial\omega_H}{\partial K_H} & \frac{\partial\omega_H}{\partial L_H} \\ \frac{\partial\omega_L}{\partial K_H} & \frac{\partial\omega_L}{\partial L_H} \end{pmatrix}
$$

is non-singular, there exists locally unique functions $K_H = \tilde{K}_H(K, a, L)$ and $L_H = \tilde{L}_H(K, a, L)$, with:

$$
\begin{pmatrix} \frac{\partial\tilde{K}_H}{\partial K} & \frac{\partial\tilde{K}_H}{\partial a} & \frac{\partial\tilde{K}_H}{\partial L} \\ \frac{\partial\tilde{L}_H}{\partial K} & \frac{\partial\tilde{L}_H}{\partial a} & \frac{\partial\tilde{L}_H}{\partial L} \end{pmatrix} = J_1^{-1}\begin{pmatrix} \frac{\partial\omega_H}{\partial K} & \frac{\partial\omega_H}{\partial a} & \frac{\partial\omega_H}{\partial L} \\ \frac{\partial\omega_L}{\partial K} & \frac{\partial\omega_L}{\partial a} & \frac{\partial\omega_L}{\partial L} \end{pmatrix}
$$

Tedious but straightforward computations then give

$$
\begin{aligned}
\frac{\partial\tilde{K}_H}{\partial K} &= \frac{\tilde{K}_H}{\Upsilon}\left[(\alpha-\beta)(\epsilon-1)(L-L_H) - L\right] \\
\frac{\partial\tilde{K}_H}{\partial a} &= -\frac{1}{\Upsilon}\alpha(\epsilon-1)(K-\tilde{K}_H)\tilde{K}_H L\frac{1}{a} \\
\frac{\partial\tilde{K}_H}{\partial L} &= -\frac{\tilde{K}_H}{\Upsilon}(\epsilon-1)(\alpha-\beta)(K-\tilde{K}_H) \\
\frac{\partial\tilde{L}_H}{\partial K} &= \frac{1}{\Upsilon}\frac{B_H\tilde{L}_H}{B_L\tilde{K}_H}\tilde{L}_H(\alpha-\beta)(\epsilon-1)(K-\tilde{K}_H) \\
\frac{\partial\tilde{L}_H}{\partial a} &= -\frac{1}{\Upsilon}\frac{B_H\tilde{L}_H}{B_L\tilde{K}_H}\alpha(\epsilon-1)K\tilde{L}_H(K-\tilde{K}_H)\frac{1}{a} \\
\frac{\partial\tilde{L}_H}{\partial L} &= \frac{\tilde{L}_H}{\Upsilon}\left[(\alpha-\beta)(\epsilon-1)(K-\tilde{K}_H) - K\right]
\end{aligned}
$$

with $\Upsilon = \Upsilon(L, K, L_H, K_H) = \left[(\epsilon-1)(\alpha-\beta)(\frac{K_H}{K} - \frac{L_H}{L}) - 1\right]LK$.

We need now to prove that $K_H = K_H(K, a, P, Q, N)$ and $L_H = L_H(K, a, P, Q, N)$. We have shown that $K_H = \tilde{K}_H(K, a, L)$ and $L_H = \tilde{L}_H(K, a, L)$. We use market clearing conditions to assess $L = N - (1-u)N_H$ in order to obtain $K_H = \tilde{L}_H(K, a, N - (1-u)N_H)$ and $L_H = \tilde{L}_H(K, a, N - (1-u)N_H)$. The first step is to prove that $N_H = N_H(K, a, u, N)$, and thus $u = u(K, a, P, Q, N)$. Let us denote $H \equiv uN_H - \tilde{L}_H(K, a, N - (1-u)N_H) = 0$. Assuming $\partial H/\partial N_H \neq 0$ implies that here exists a locally unique function $N_H = \tilde{N}_1(K, a, u, N)$ such that

$$\frac{\partial \tilde{N}_1}{\partial K} = \frac{\frac{\partial \tilde{L}_H}{\partial K}}{u+(1-u)\frac{\partial \tilde{L}_H}{\partial L}}, \quad \frac{\partial \tilde{N}_1}{\partial a} = \frac{\frac{\partial \tilde{L}_H}{\partial a}}{u+(1-u)\frac{\partial \tilde{L}_H}{\partial L}} \quad \frac{\partial \tilde{N}_1}{\partial N} = \frac{\frac{\partial \tilde{L}_H}{\partial L}}{u+(1-u)\frac{\partial \tilde{L}_H}{\partial L}}, \quad \frac{\partial \tilde{N}_1}{\partial u} = -\frac{\tilde{N}_1(1-\frac{\partial \tilde{L}_H}{\partial L})}{u+(1-u)\frac{\partial \tilde{L}_H}{\partial L}}$$

Let us consider now equation (30) such that:

$$G(K, a, u, P, Q, N) = \frac{\alpha\gamma}{zau} PY(K, a, u, N)^{\frac{1}{\epsilon}} Y_H(K, a, u, N)^{\frac{\epsilon-1}{\epsilon}} - Q = 0$$

Assuming $\partial G/\partial u \neq 0$, there exists a locally unique function $u = u(K, a, P, Q, N)$ such that:

$$\frac{\partial u}{\partial K} = -\frac{\frac{\partial G}{\partial K}}{\frac{\partial G}{\partial u}}; \quad \frac{\partial u}{\partial a} = -\frac{\frac{\partial G}{\partial a}}{\frac{\partial G}{\partial u}}; \quad \frac{\partial u}{\partial P} = -\frac{\frac{\partial G}{\partial P}}{\frac{\partial G}{\partial u}}; \quad \frac{\partial u}{\partial Q} = -\frac{\frac{\partial G}{\partial Q}}{\frac{\partial G}{\partial u}}; \quad \frac{\partial u}{\partial N} = -\frac{\frac{\partial G}{\partial N}}{\frac{\partial G}{\partial u}}$$

with:

$$\begin{aligned}
\frac{\partial G}{\partial K} &= Q\left[\frac{1}{\epsilon}\frac{1}{Y}\frac{\partial Y}{\partial K} + \frac{1-\epsilon}{\epsilon}\frac{1}{Y_H}\frac{\partial Y_H}{\partial K}\right], \; \frac{\partial G}{\partial a} = Q\left[-\frac{1}{a} + \frac{1}{\epsilon}\frac{1}{Y}\frac{\partial Y}{\partial a} + \frac{1-\epsilon}{\epsilon}\frac{1}{Y_H}\frac{\partial Y_H}{\partial a}\right], \; \frac{\partial G}{\partial P} = \frac{Q}{P}, \\
\frac{\partial G}{\partial u} &= Q\left[\frac{1}{\epsilon}\frac{1}{Y}\frac{\partial Y}{\partial u} + \frac{1-\epsilon}{\epsilon}\frac{1}{Y_H}\frac{\partial Y_H}{\partial u} - \frac{1}{u}\right], \; \frac{\partial G}{\partial Q} = -1, \; \frac{\partial G}{\partial N} = Q\left[\frac{1}{\epsilon}\frac{1}{Y}\frac{\partial Y}{\partial N} + \frac{1-\epsilon}{\epsilon}\frac{1}{Y_H}\frac{\partial Y_H}{\partial N}\right]
\end{aligned}$$

Let us then denote

$$\begin{aligned}
& K_H = \tilde{K}_H(K, a, N - (1 - u(K, a, P, Q, N)\tilde{N}_1(K, a, u(K, a, P, Q, N))) \\
\Leftrightarrow \quad & K_H = K_H(K, a, P, Q, N) \\
& L_H = \tilde{L}_H(K, a, N - (1 - u(K, a, P, Q, N)\tilde{N}_1(K, a, u(K, a, P, Q, N))) \\
\Leftrightarrow \quad & L_H = L_H(K, a, P, Q, N)
\end{aligned} \tag{71}$$

Tedious but straightforward computations allow therefore to express all the derivatives we need. For example, considering the derivatives with respect to K we get:

$$\begin{aligned}
\frac{\partial Y}{\partial K} &= \left(\gamma \frac{\partial Y_H}{\partial K} Y_H^{-\frac{1}{\epsilon}} + (1-\gamma)\frac{\partial Y_L}{\partial K} Y_L^{-\frac{1}{\epsilon}}\right) Y^{\frac{1}{\epsilon}} \\
\frac{\partial Y_H}{\partial K} &= Y_H\left(\alpha \frac{1}{L_H}\frac{\partial L_H}{\partial K} + (1-\alpha)\frac{1}{K_H}\frac{\partial K_H}{\partial K}\right) \\
\frac{\partial Y_L}{\partial K} &= -Y_L\left(\beta \frac{1}{L-L_H}\frac{\partial L_H}{\partial K} + (1-\beta)\frac{1}{K-K_H}\frac{\partial K_H}{\partial K}\right) \\
\frac{\partial L_H}{\partial K} &= \frac{\partial \tilde{L}_H}{\partial K}\left(\frac{u}{u+(1-u)\frac{\partial \tilde{L}L_H}{\partial L}}\right) \\
\frac{K_H}{\partial K} &= \frac{\partial \tilde{K}_H}{\partial K} - (1-u)\frac{\partial \tilde{K}_H}{\partial L}\frac{\frac{\partial \tilde{L}_H}{\partial K}}{u+(1-u)\frac{\partial \tilde{L}_H}{\partial L}}
\end{aligned}$$

We do similar computations to obtain all the derivatives K_H and L_H with respect to K, a, P, Q and N. Recalling that N_H as follow: $N_H = \tilde{N}_1(K, a, u(K, a, P, Q, N), N) = N_H(K, a, P, Q, N)$ we finally derive:

$$
\begin{aligned}
\frac{\partial N_H}{\partial K} &= \frac{\partial \tilde{N}_1}{\partial K} + \frac{\partial \tilde{N}_1}{\partial u}\frac{\partial u}{\partial K}, \; \frac{\partial N_H}{\partial A} = \frac{\partial \tilde{N}_1}{\partial A} + \frac{\partial \tilde{N}_1}{\partial u}\frac{\partial u}{\partial A}, \; \frac{\partial N_H}{\partial P} = \frac{\partial \tilde{N}_1}{\partial u}\frac{\partial u}{\partial P}, \\
\frac{\partial N_H}{\partial Q} &= \frac{\partial \tilde{N}_1}{\partial u}\frac{\partial u}{\partial Q}, \; \frac{\partial N_H}{\partial N} = \frac{\partial \tilde{N}_1}{\partial N} + \frac{\partial \tilde{N}_1}{\partial u}\frac{\partial u}{\partial N}
\end{aligned}
$$

Using these results into (71) we obtain:

$$
\frac{\partial K_H}{\partial K} = \frac{\partial \tilde{K}_H}{\partial K} + \frac{\partial \tilde{K}_H}{\partial L}\left[\frac{\partial u}{\partial K}N_H - (1-u)\frac{\partial N_H}{\partial K}\right], \quad \frac{\partial K_H}{\partial A} = \frac{\partial \tilde{K}_H}{\partial A} + \frac{\partial \tilde{K}_H}{\partial L}\left[\frac{\partial u}{\partial A}N_H - (1-u)\frac{\partial N_H}{\partial A}\right]
$$

$$
\frac{\partial K_H}{\partial P} = \frac{\partial \tilde{K}_H}{\partial L}\left[\frac{\partial u}{\partial P}N_H - (1-u)\frac{\partial N_H}{\partial P}\right], \quad \frac{\partial K_H}{\partial Q} = \frac{\partial \tilde{K}_H}{\partial L}\left[\frac{\partial u}{\partial Q}\tilde{N}_H - (1-u)\frac{\partial N_H}{\partial Q}\right]
$$

$$
\frac{\partial K_H}{\partial N} = \frac{\partial \tilde{K}_H}{\partial N} + \frac{\partial \tilde{K}_H}{\partial L}\left[\frac{\partial u}{\partial N}N_H - (1-u)\frac{\partial N_H}{\partial N}\right], \quad \frac{\partial L_H}{\partial K} = \frac{\partial \tilde{L}_H}{\partial K} + \frac{\partial \tilde{L}_H}{\partial L}\left[\frac{\partial u}{\partial K}N_H - (1-u)\frac{\partial N_H}{\partial K}\right]
$$

$$
\frac{\partial L_H}{\partial A} = \frac{\partial \tilde{L}_H}{\partial A} + \frac{\partial \tilde{L}_H}{\partial L}\left[\frac{\partial u}{\partial A}N_H - (1-u)\frac{\partial N_H}{\partial A}\right], \quad \frac{\partial L_H}{\partial P} = \frac{\partial \tilde{L}_H}{\partial L}\left[\frac{\partial u}{\partial P}N_H - (1-u)\frac{\partial N_H}{\partial P}\right]
$$

$$
\frac{\partial L_H}{\partial Q} = \frac{\partial \tilde{L}_H}{\partial L}\left[\frac{\partial u}{\partial Q}N_H - (1-u)\frac{\partial N_H}{\partial Q}\right], \quad \frac{\partial L_H}{\partial N} = \frac{\partial \tilde{L}_H}{\partial N} + \frac{\partial \tilde{L}_H}{\partial L}\left[\frac{\partial u}{\partial N}N_H - (1-u)\frac{\partial N_H}{\partial N}\right]
$$

We have proved therefore that K_H and L_H are functions of K, a, P, Q and N, and we have all the derivatives of our main variables. Using the property that a homogeneous of degree 1 CES function generates input demand functions which are homogeneous of degree 0, we state finally:

$$
\begin{aligned}
K_H(K, a, P, Q, N) &= K_H(k, x, p, q_0, N_0) \\
L_H(K, a, P, Q, N) &= L_H(k, x, p, q_0, N_0) \\
u(K, a, P, Q, N) &= u(k, x, p, q_0, N_0)
\end{aligned}
$$

The stationarized dynamical system (16) is then easily derived. □

11.7 Proof of Theorem 3

Consider the stationarized dynamical system as given by (16). Using (57) we can rewrite it as follows

$$
\begin{aligned}
\frac{\dot{p}}{p} &= -\left[(1-\alpha)\gamma\psi^{\frac{1}{\epsilon}}x^{\alpha}\lambda^{\alpha}\kappa^{1-\alpha}l^{\alpha}k^{-\alpha} + g_P - \rho - \delta\right] \\
\frac{\dot{k}}{k} &= \psi x^{\alpha}\lambda^{\alpha}\kappa^{1-\alpha}l^{\alpha}k^{-\alpha} - \delta - g_K - \frac{N_0 p^{-\frac{1}{\theta}}}{k} \\
\frac{\dot{x}}{x} &= z(1-u) - g_a - \eta
\end{aligned}
\tag{72}
$$

A steady-state is therefore a solution of the following system

$$
(1-\alpha)\gamma\psi^{\frac{1}{\epsilon}}x^{\alpha}\lambda^{\alpha}\kappa^{-\alpha}l^{\alpha}k^{-\alpha} = \rho + \delta - g_P \tag{73}
$$

$$
\psi x^{\alpha}\lambda^{\alpha}\kappa^{1-\alpha}l^{\alpha}k^{-\alpha} = \delta + g_K + \frac{N_0 p^{-\frac{1}{\theta}}}{k} \tag{74}
$$

$$
z(1-u) = g_a + \eta \tag{75}
$$

From (75) we get $u = (z - g_a - \eta)/z \equiv u^*$ as given by (64). Since at steady state we have $\lambda^* = 1$ we derive $L_H = L$ and $N_H = N$. Recalling that $L = N - (1-u)N_H$, we get $l^* = u^* N_0$ with N_0 the initial value of $N(t)$. Taking the ratio of (74) on (73) gives after simplification

$$g_K + \delta + \tfrac{N_0 p^{-\frac{1}{\theta}}}{k} = \tfrac{\delta+\rho-g_P}{(1-\alpha)\gamma}\kappa\psi^{\frac{\epsilon-1}{\epsilon}} \tag{76}$$

Using the fact that $\kappa^* = 1$ and $\psi^* = \gamma^{\frac{\epsilon}{\epsilon-1}}$, substituting (76) into (74) and solving for k gives

$$k = \left(\tfrac{(1-\alpha)\gamma^{\frac{\epsilon}{\epsilon-1}}}{\rho+\delta-g_P}\right)^{\frac{1}{\alpha}} l^* x \equiv \mathcal{Z}_1 x \tag{77}$$

Solving (76) with respect to p using (77) gives

$$p = \left[\tfrac{(1-\alpha)N_0}{[\delta+\rho-g_P-(1-\alpha)(\delta+g_K)]\mathcal{Z}_1}\right]^{\theta} x^{-\theta} \equiv \mathcal{Z}_2 x^{-\theta} \tag{78}$$

Consider finally equation (30) which can be written at the steady state as

$$p\gamma\psi^{\frac{1}{\epsilon}}\alpha l^{\alpha} x^{\alpha} l^{\alpha} k^{1-\alpha} = q_0 z u x \tag{79}$$

Substituting (77) and (78) into (79) and solving for x finally gives

$$x^* = \left(\tfrac{\mathcal{Z}_2 \mathcal{Z}_1^{1-\alpha}\gamma^{\frac{\epsilon}{\epsilon-1}}\alpha l^{*\alpha}}{q_0 z u^*}\right)^{\frac{1}{\theta}} \equiv x^*(q_0) \tag{80}$$

Therefore, substituting (80) into (78) and (77), we find

$$\begin{aligned} k^* &= \mathcal{Z}_1\left(\tfrac{\mathcal{Z}_2 \mathcal{Z}_1^{1-\alpha}\gamma^{\frac{\epsilon}{\epsilon-1}}\alpha l^{*\alpha}}{q_0 z u^*}\right)^{\frac{1}{\theta}} \equiv k^*(q_0) \\ p^* &= \mathcal{Z}_2 \tfrac{q_0 z u^*}{\mathcal{Z}_1^{1-\alpha}\gamma^{\frac{\epsilon}{\epsilon-1}}\alpha l^{*\alpha}} \equiv p^*(q_0) \end{aligned} \tag{81}$$

We conclude that for any given $q_0 > 0$, there exists a unique steady state $(k^*(q_0), x^*(q_0), p^*(q_0))$ with $k^{*'}(q_0) < 0$, $x^{*'}(q_0) < 0$ and $p^{*'}(q_0) > 0$.

$\square$

11.8 Proof of Lemma 2

Under Assumption 1, let us consider the dynamical system

$$\begin{aligned} \dot{p} &= -p\left\{(1-\alpha)\gamma\psi^{\frac{1}{\epsilon}}x^{\alpha}\left(\frac{L_H(k,x,p,q_0,N_0)}{K_H(k,x,p,q_0,N_0}\right)^{\alpha} + g_P - \rho - \delta\right\} \equiv \mathcal{F}(p,k,x) \\ \dot{k} &= k\left\{\psi x^{\alpha}\left(\frac{L_H(k,x,p,q_0,N_0)}{K_H(k,x,p,q_0,N_0}\right)^{\alpha}\frac{K_H(k,x,p,q_0,N_0)}{k} - \delta - g_K - \frac{N_0 p^{-\frac{1}{\theta}}}{k}\right\} \equiv \mathcal{G}(p,k,x) \\ \dot{x} &= x\left\{z\Big(1 - u(k,x,p,q_0,N_0)\Big) - g_a - \eta\right\} \equiv \mathcal{H}(p,k,x) \end{aligned}$$

The linearization around the steady state yields the following Jacobian matrix:

$$\mathcal{J} = \begin{pmatrix} \mathcal{F}_1(p^*,k^*,x^*) & \mathcal{F}_2(p^*,k^*,x^*) & \mathcal{F}_3(p^*,k^*,x^*) \\ \mathcal{G}_1(p^*,k^*,x^*) & \mathcal{G}_2(p^*,k^*,x^*) & \mathcal{G}_3(p^*,k^*,x^*) \\ \mathcal{H}_1(p^*,k^*,x^*) & \mathcal{H}_2(p^*,k^*,x^*) & \mathcal{H}_3(p^*,k^*,x^*) \end{pmatrix}$$

with

$$\begin{aligned}
\mathcal{F}_1(p^*,k^*,x^*) &= -(\rho+\delta-g_P)\tfrac{\alpha}{1-\alpha} < 0,\ \mathcal{F}_2(p^*,k^*,x^*) = 0 \\
\mathcal{F}_3(p^*,k^*,x^*) &= (\rho+\delta-g_P)\tfrac{\alpha}{1-\alpha}\tfrac{p^*}{x^*} > 0 \\
\mathcal{G}_1(p^*,k^*,x^*) &= \tfrac{k^*}{p^*}\left[(\rho+\delta-g_P)\tfrac{\alpha}{(1-\alpha)^2} + \tfrac{1}{\theta}N_0\tfrac{p^{*\frac{-1}{\theta}}}{k^*}\right] > 0 \\
\mathcal{G}_2(p^*,k^*,x^*) &= (\rho+\delta-g_P)\tfrac{1}{1-\alpha} - \delta - g_K > 0 \\
\mathcal{G}_3(p^*,k^*,x^*) &= (\rho+\delta-g_P)\tfrac{1}{(1-\alpha)^2}\tfrac{k^*}{x^*} > 0 \\
\mathcal{H}_1(p^*,k^*,x^*) &= -\tfrac{x^*}{p^*}\tfrac{zu^*}{1-\alpha} < 0,\ \mathcal{H}_2(p^*,k^*,x^*) = -\tfrac{x^*}{k^*}zu^* < 0 \\
\mathcal{H}_3(p^*,k^*,x^*) &= z(1-u^*) - g_a - \eta + \tfrac{zu^*}{1-\alpha} = \tfrac{zu^*}{1-\alpha} > 0
\end{aligned}$$

We then derive the characteristic polynomial

$$\mathcal{Q}(\lambda) = \lambda^3 - \lambda^2\mathcal{T} + \lambda\mathcal{S} - \mathcal{D} \tag{82}$$

with

$$\begin{aligned}
\mathcal{T} &= \mathcal{F}_1(p^*,k^*,x^*) + \mathcal{G}_2(p^*,k^*,x^*) + \mathcal{H}_3(p^*,k^*,x^*) \\
\mathcal{S} &= \mathcal{F}_1(p^*,k^*,x^*)\mathcal{G}_2(p^*,k^*,x^*) + \mathcal{G}_2(p^*,k^*,x^*)\mathcal{H}_3(p^*,k^*,x^*) - \mathcal{H}_2(p^*,k^*,x^*)\mathcal{G}_3(p^*,k^*,x^*) \\
&+ \mathcal{F}_1(p^*,k^*,x^*)\mathcal{H}_3(p^*,k^*,x^*) - \mathcal{H}_1(p^*,k^*,x^*)\mathcal{F}_3(p^*,k^*,x^*) \\
\mathcal{D} &= \mathcal{F}_1(p^*,k^*,x^*)\left[\mathcal{G}_2(p^*,k^*,x^*)\mathcal{H}_3(p^*,k^*,x^*) - \mathcal{G}_3(p^*,k^*,x^*)\mathcal{H}_2(p^*,k^*,x^*)\right] \\
&+ \mathcal{F}_3(p^*,k^*,x^*)\left[\mathcal{G}_1(p^*,k^*,x^*)\mathcal{H}_2(p^*,k^*,x^*) - \mathcal{G}_2(p^*,k^*,x^*)\mathcal{H}_1(p^*,k^*,x^*)\right]
\end{aligned}$$

Consider first the expression of $\mathcal{T}$. We get

$$\mathcal{T} = \rho - g_p - g_K + \tfrac{u^*z}{1-\alpha} = u^*z + \tfrac{u^*z}{1-\alpha} > 0$$

which does not depend on q_0. Consider now the expression of $\mathcal{D}$. We get

$$\begin{aligned}
\mathcal{G}_2\mathcal{H}_3 - \mathcal{G}_3\mathcal{H}_2 &= \tfrac{u^*z}{(1-\alpha)^2}\left[(\rho+\delta-g_P)(1+\alpha) + (1-\alpha)zu^*\right] > 0 \\
\mathcal{G}_1\mathcal{H}_2 - \mathcal{G}_2\mathcal{H}_1 &= \tfrac{x^*}{p^*}u^*z\left[\tfrac{\rho-g_P-g_K}{1-\alpha} - \tfrac{1}{\theta}N_0\tfrac{p^{*\frac{-1}{\theta}}}{k^*}\right]
\end{aligned}$$

and thus

$$\mathcal{D} = -(\rho+\delta-g_P)\tfrac{\alpha zu^*}{1-\alpha}\left[\tfrac{1}{\theta}N_0\tfrac{p^{*\frac{-1}{\theta}}}{k^*} + (\rho+\delta-g_P)\tfrac{1+\alpha}{(1-\alpha)^2}\right] < 0$$

which does not depend on q_0 either. Finally, straightforward computations also show that $\mathcal{S}$ does not depend on q_0 either. We conclude that the eigenvalues do not depend on the value of q_0 and nor on the value of the steady state $(p^*(q_0), k^*(q_0), x^*(q_0))$. Therefore, since $\mathcal{T} > 0$

and $\mathcal{D} < 0$, we conclude that there exists a unique negative eigenvalue and thus that any given steady state $(p^*(q_0), k^*(q_0), x^*(q_0))$ on the manifold is a saddle-point.

□

11.9 Proof of Theorem 4

In the case $\epsilon > 1$, we have shown in the proof of Lemma 2 that the eigenvalues of the linearized dynamical system do not depend on the value of q_0 and nor, therefore, on the value of the steady state $(p^*(q_0), k^*(q_0), x^*(q_0))$. Therefore, since $\mathcal{T} > 0$ and $\mathcal{D} < 0$, we conclude that for any given steady state $(p^*(q_0), k^*(q_0), x^*(q_0))$ on the manifold, the local stability properties are the same. For any given $q_0 > 0$, there exists a unique characteristic root with negative real part and the steady state $(p^*(q_0), k^*(q_0), x^*(q_0))$ is saddle-point stable. Therefore, for any given $q_0 > 0$, there exists a unique $p_0 > 0$ such that the unique converging path is on the stable manifold of dimension one. Along this converging path all the variables are bounded and the transversality conditions are satisfied. Therefore this converging path is the unique optimal solution.

□

11.10 Proof of Corollary 2

Let Assumption 1 holds.

i) From Theorems 3 and 4, and Corollary 1, we know that along the NBGP, $\kappa = \lambda = 1$, and that the unique steady state is saddle-point stable. As $\kappa(0)$ and $\lambda(0)$ are necessarily less than 1, the result follows.

ii) From (5), (31) and (55) we get

$$\frac{Y_H}{Y} = \gamma^{\frac{\epsilon}{1-\epsilon}} \left(\frac{(1-\beta)\kappa}{(1-\beta)\kappa + (1-\alpha)(1-\kappa)} \right)^{\frac{\epsilon}{\epsilon-1}}, \quad \frac{P_H Y_H}{PY} = \frac{(1-\beta)\kappa}{(1-\beta)\kappa + (1-\alpha)(1-\kappa)} \tag{83}$$

and thus

$$\frac{\partial Y_H/Y}{\partial \kappa} = \frac{\epsilon}{\epsilon-1} \frac{Y_H}{Y} \frac{(1-\alpha)}{\kappa[(1-\beta)\kappa + (1-\alpha)(1-\kappa)]} > 0, \quad \frac{\partial P_H Y_H/PY}{\partial \kappa} = \frac{(1-\beta)(1-\alpha)}{[(1-\beta)\kappa + (1-\alpha)(1-\kappa)]^2} > 0$$

The result follows from i).

iii) From (10) and (83) we get

$$s_K = \frac{RK}{Y} = \frac{(1-\beta)(1-\alpha)}{(1-\beta)\kappa + (1-\alpha)(1-\kappa)} \text{ and thus } \frac{\partial s_K}{\partial \kappa} = \frac{(1-\beta)(1-\alpha)(\beta-\alpha)}{[(1-\beta)\kappa + (1-\alpha)(1-\kappa)]^2} \tag{84}$$

The result follows.

iv) From (65) and (83) we derive

$$\frac{P_H}{P} = \gamma^{\frac{\epsilon}{\epsilon-1}} \left[1 + \frac{(1-\kappa)(1-\alpha)}{\kappa(1-\beta)} \right]^{\frac{1}{\epsilon-1}} \text{ and } \frac{P_L}{P} = (1-\gamma)^{\frac{\epsilon}{\epsilon-1}} \left[1 + \frac{\kappa(1-\beta)}{(1-\kappa)(1-\alpha)} \right]^{\frac{1}{\epsilon-1}} \tag{85}$$

It follows therefore

$$\frac{\partial P_H/P}{\partial \kappa} < 0 \text{ and } \frac{\partial P_L/P}{\partial \kappa} > 0 \tag{86}$$

The result follows.

□

11.11 Data used for the computation of the capital shares

We followed Acemoglu and Guerrieri [4]. We use the National Income and Product Accounts (NIPA) between 1948 and 2005 where industries are classified according to the North American Industrial Classification System at the 22-industry level, and classify industries according to the requirement of technological knowledge by the workers. That is, we consider an industry to be HS if workers exhibit a higher growth of compensation per capita than average. The following Table shows the average capital share of each industry together with the sector classification.

Industry	Sector	Capital share
Educational services	H	0.10
Information	H	0.53
Management of companies and enterprises	L	0.20
Health care and social assistance	L	0.22
Administrative and waste management services	L	0.28
Other services, except government	L	0.33
Professional, scientific and technical services	L	0.34
Transportation and warehousing	L	0.35
Accommodation and food services	L	0.36
Retail trade	L	0.42
Arts, entertainment and recreation	L	0.42
Finance and insurance	L	0.45
Wholesale trade	L	0.46
Utilities	L	0.77

Table 4: Industry capital shares

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