Lie symmetry analysis on pricing power options under the Heston dynamic and some fractional financial models

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Abstract

The rise of computational mathematics in financial markets has accelerated the bloom of various financial models. For instance, the Black-Scholes-Merton model, the Vasicek model, the Cox-Ingersoll-Ross model, the Heston model, etc. Each of these models often produces challenging partial differential equations. The Lie symmetry method appears to be a powerful tool to solve these types of equations. In this study, we apply Lie's method to the power options model under the Heston dynamic. The infinitesimal operators, the optimal systems, the invariant solutions, and the conservation laws are presented.

Lie analysis is also an efficient tool to solve the fractional differential equations which involve the differentiation of a function with respect to its independent variable(s) to a non-integer order. Fractional differential equations are well known for their ability to describe the memory effect in various natural phenomena. We apply the Lie symmetry analysis to a time-fractional Black-Scholes-Merton model, as well as an arbitrage-free stock price model. The results of the analysis which include the infinitesimal operators or generators, the optimal systems, and the invariant solutions of the above models are presented. The visual representations of the invariant solutions are provided alongside discussions and comparisons with the solutions from their corresponding non-fractional models.

Declaration

I declare that, *Lie symmetry analysis on pricing power options under the Heston dynamic and some fractional financial models* is my original work. This thesis has not been submitted before for any degree or examination in any other universities. Sources used or quoted have been indicated and acknowledged by complete references.

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Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Background

The publication of Smith's "The Wealth of Nations" [87] in the 18th century marked the birth of modern economics. The three main ideas proposed by Smith: the pursuit of self-interest, division of labour, and freedom of trade, were known as the trinity of individual prerogatives. This blueprint of free markets and free trade had eventually become the hallmarks of modern capitalism today. Since then, modern economics embarked on its long journey toward the utopia imagined by Smith 250 years ago.

Modern economics come a long way, giving birth to many modern economic and financial models that keep inspiring us today. Stokey *et al.* [90] used calculus to model economic phenomena such as supply and demand, market equilibrium, and optimization. Black and Scholes, Merton [12, 61] developed the well-known Black-Scholes-Merton model which introduced a partial differential equation to govern the risk, pricing, and optimization of a portfolio.

Using the Black-Scholes-Merton (BSM) model as the basic ingredient, many financial instruments were introduced to deduce the pricing models for different needs [10, 19, 35, 43, 94]. Most of these financial models derived from the BSM model often ended with partial differential equations. The purpose of this work is to expand on these ideas. We will apply the methods of Lie to help solve the differential equations associated with the pricing of financial derivatives.

Lie symmetry analysis, being a very powerful tool to solve differential equations, is widely used to analyze various financial equations [55, 54]. Ibragimov and Gazizov [28] first introduced the Lie symmetry analysis to the BSM equation. Goard [31] used Lie's method to find new solutions for zero-coupon bonds, assuming a realistic time-dependent, mean-reverting drift form and power-law volatility. Rosa *et al.* [79] used Lie's method to give some solutions and conservation laws for a generalized Fisher Equation with variable coefficients. Kaibe and O'Hara [49] deduced the pricing models for the zero-coupon bond PDE model derived from the functional interest rate model. Tang *et al.* [91] gave the invariant solutions of the Heston model with stock dividends. These are just a few examples of this method.

Recently, the idea of applying the same treatment done to physical science to economics and finance has gained a lot of attention. Econophysics, as the name of this treatment, use the concepts and methods from physics to understand economic and financial systems. Mantegna and Stanley [58] use statistical physics to describe financial systems and present a stochastic model that captures statistical properties in real data, providing a global understanding of economic systems using concepts like stochastic dynamics and scaling, without needing a detailed microscopic description. However, the problem with using science to describe financial behaviour is that it assumes markets are efficient and prices move in a predictable manner. In reality, financial markets are often driven by irrational behaviour and dramatic changes [84].

Fractional calculus, as a branch of calculus, is rising as a new tool to describe the behaviours of the financial market. This old branch of mathematics, which can be traced back to the 17th century, is proven to be useful to describe natural phenomena with memory [6, 34, 64]. A new perspective of finance in describing realistic financial behaviour with memory with fractional calculus is forming. Fallahgoul [25] replaced the geometric Brownian motion in some financial models with the Lèvy process and obtained fractional partial differential equations. Tarasov [92] pointed out that the current "Memory revolution" is filling up the missing pieces of modern economics that are caused by the theory that uses differential and integral operators of integer orders instead of using fractional calculus. Gazizov *et al.* [28, 29, 30] used the Lie symmetry method to analyze various financial models in fractional time frames. Habibi *et al.* [33] studied the time-fractional Fokker-Planck equation using the Lie symmetry method and graphically compared the numerical solutions driven by Chebyshev wavelets's method with the exact solutions. Yue and Shen [97] showed that the fractal dimension bond-pricing formula can better explain price changes in the capital market than the classical bond-pricing models. Chong and O'Hara [17] deduced an invariant solution of a time-fractional Black-Scholes-Merton equation. [1]

1.2 Research purpose

In 1973, Black and Scholes [12], and Merton [61] introduced the Black-Scholes-Merton model to describe the pricing of European options with some assumptions, which include that the stock price process follows the stochastic process that is represented by

$$dS = rSdt + \sigma Sdz,$$

where $S, r,$ and σ are the price, the risk-free interest rate, and the volatility of the underlying stock at time t respectively and z is a standard Brownian motion. This will finally lead to the famous Black-Scholes-Merton (BSM) equation

$$\frac{\partial u}{\partial t} + rS\frac{\partial u}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} = ru,$$

where $u = u(S,t)$ is the price of a derivative. In 2000, Wyss [96] replaced the first derivative of u with respect to time in a transformed BSM equation with a fractional derivative of order $\alpha, 0 < \alpha \leq 1$, and used Green's function to solve the time-fractional BSM equation. In this work, we use Lie symmetry

[1] This part of the research is the original contribution by the author of the thesis to the study of fractional calculus in computational finance.

to analyze the time-fractional version of the BSM equation above, given by

$$\frac{\partial^\alpha u}{\partial t^\alpha} + rS\frac{\partial u}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} = ru.$$

The Lie point symmetries, Lie brackets, and the optimal system of the time-fractional BSM equation above are found as well as the corresponding invariant solutions.

In 2009, Bell and Stelljes [10] described a method of constructing a class of solvable arbitrage-free models, $G(\tilde{S}_t, t)$, for the stock price using the stochastic Bernoulli equation of Stratonovich type

$$d\tilde{S}_t = \mu \tilde{S}_t dt + \sigma \tilde{S}_t^p \circ dw_t,$$

where $1/2 \leq p < 1$, μ and σ are the drift and volatility of the stock with price $\tilde{S}_t$, and w_t is a standard Wiener process. The construction of the model results in a second-degree partial differential equation

$$\frac{\partial G}{\partial t} + \left(rs + \frac{p\sigma^2 s^{2p-1}}{2} \right) \frac{\partial G}{\partial S} + \frac{\sigma^2 s^{2p}}{2} \frac{\partial^2 G}{\partial S^2} - rG = 0,$$

with some conditions. Bell and Stelljes [10] and Sinkala [86] then gave the solutions of the above equation. Our purpose is to extend the work done by them to a time-fractional arbitrage-free stock price model using Lie's approach. We deduce the Lie point symmetries, the infinitesimal generators, the Lie's brackets, and the optimal system. Finally, we propose some invariant solutions to the time-fractional arbitrage-free stock price model.

In 2013, Ibrahim *et al.* [43] introduced a pricing model for power options using Heston dynamics, assuming the asset price is to follow the log-normal process governed by a single Brownian motion. Their model eventually led to the formation of a partial differential equation

$$\frac{\partial u}{\partial t} + \left(r - \frac{1}{2}\beta^2 y\right)\frac{\partial u}{\partial x} + \frac{1}{2}\beta^2 y\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\sigma^2\beta^2 y\frac{\partial^2 u}{\partial y^2}$$
$$+\rho\sigma\beta^2 y\frac{\partial^2 u}{\partial x\partial y} + \kappa(\theta - \beta^2 y)\frac{\partial u}{\partial y} - ru = 0,$$

where u is the value of an option, r is the risk-free rate, β is a constant factor, y is the ratio of the variance to the square of β, $x \equiv lnZ$ where $Z \equiv S_t^\beta$ is an artificial asset with S_t^β a geometric Brownian motion, σ is the volatility of volatility, ρ is the correlation coefficient between the Brownian motions in the model, $\kappa = \kappa^* + \lambda$ for κ^* is the speed of the mean reversion and λ is the volatility risk premium, and $\theta = \frac{\kappa^*\theta^*}{\kappa^*+\lambda}$ with θ^* is the average level of the volatility.

Using Lie's method, we obtain three infinitesimal operators of the above equation. With the operators, we compute the commutators, the adjoint representations, the optimal system, as well as the invariant solutions. Finally, we give the conservation laws.

1.3 Structure of the thesis

This thesis is organized as follows: In Chapter 2, we give a comprehensive review of Lie symmetry analysis and fractional calculus. In Chapter 3, Lie symmetry analysis is applied to a time-fractional Black-Scholes-Merton model. We propose a few invariant solutions as well as the optimal systems for the fractional model. Chapter 4 focuses on a time-fractional arbitrage-free stock price model. The invariant solution obtained is then graphically compared with the non-time-fractional model. The study of power options under the Heston dynamic using Lie's method is listed in Chapter 5, which ended with invariant solutions, optimal systems, and conservation laws. Finally, we conclude our study and include some suggestions for future work in Chapter 6.

Chapter 2

Literature Review

2.1 Differential equations

The history of differential equations comes a long way since the invention of calculus back in the 17th century. It was then becoming a major branch of mathematics. The development of differential equations [4, 68, 81] is a hundreds-years-long bedtime story to tell and it is still being written by scholars around the world nowadays. The application of differential equations has showered almost every field of study.

In the 17th century, when the branch was newly born, scientists found its application in geometry and classical mechanics. After decades of development, many applications were made to astronomy, continuous media, heat theory, optics, electricity, magnetism, etc. Recently, with the introduction of set theory in mathematical analysis and the consequences for theorization, differential equations continue to find their placement in quantum mathematics, dynamical systems, and relativity theory in the 20th century [4].

The story of differential equations unfolded when Leibniz wrote the equation $\int x dx = \frac{1}{2}x^2$ in 1675 [44]. Newton, on the other hand, started to investigate the general methods to integrate the differential equation with the classification of first-order differential equations into three general classes [68]:

$$
\begin{aligned}
&(1) \quad \frac{dy}{dx} = f(x), \\
&(2) \quad \frac{dy}{dx} = f(x, y), \\
&(3) \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u.
\end{aligned}
$$

The first two classes are known as ordinary differential equations (ODE) while the third class, which involves partial derivatives, is known as partial differential equations (PDE). The study of differential equations is focusing on finding the solution to the equations above. Naturally, ODEs are relatively easier to solve compared to PDEs.

The birth of differential equations undoubtedly became one of the biggest things in the 17th century. The most brilliant minds by that time had no hesitation to contribute to the field in succession. Mind of Bernoullis, Riccati, Euler, Lagrange, and Laplace had joined the feast and the result was the birth of numerous well-known differential equations. For instance, the equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n,$$

which is known as the Bernoulli differential equation, where n is a real number not equal to 0 or 1, is named after James Bernoulli in 1695. Leibniz was believed to be the first who offered a solution to the Bernoulli differential equation [74]. The equations are nonlinear with known exact solutions which do not have singular solutions.

Riccati [78], in his discussion about special cases of curves whose radii of curvature were dependent on the corresponding ordinates, introduced the Riccati equation

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2.$$

Daniel Bernoulli [11] later proposed the solution of the Riccati equation.

Euler [24] completed the treatment of the homogeneous linear differential equation with constant coefficients using repeated quadratic factors. With integrating factors and the method of integrating by series, he reduced the order of the differential equations until it was integrable.

In the 18th century, the nature of the series solutions to differential equations was posed [4]. Mathematicians started to view series as tools to approximate solutions of differential equations. At the same time, they were aware that many simple functions could not be integrated by means of elementary functions. This problem of non-elementary integrable functions drove them to compare integration with inverse arithmetical operations [51].

In the late 18th century and early 19th century, Peano [75] and Gramegna [32] posted their findings on the systems of ordinary linear differential equations. Peano applied the method of successive integrations to deal with the homogeneous linear differential equations system

$$\frac{dx_i}{dt} = \alpha_{i1}x_1 + \alpha_{i2}x_2 + \cdots + \alpha_{in}x_n$$

where $i = 1, 2, 3, \ldots, n$ and the coefficients of α_{ij} are functions of t. Gramegna, later in 1910, generalized the work of Peano's to systems of infinite differential equations and to integrodifferential equations by using the method of successive integrations.

Lie Symmetries and Differential Equations

Solving differential equations can be tough and tedious. Different kinds of differential equations, either ordinary or partial, usually require different approaches and techniques. Direct integration, integration by parts, separable equations, integrating factors, Euler's method, homogeneous equations, numerical methods, etc, are among the most frequently used formulae to solve differential equations with different kinds and dimensions.

This chaotic world of differential equations is just like cooking soups that require specific steps and ingredients. To make things worse, most ingredients only work for very limited types of soups. How can one cook soups of an unfamiliar type? Indeed, this was the dilemma of the "cooks" around the middle of the 19th century. An ingredient that fitted in all kinds, if not, most kinds of soups, was desperately sought.

An ingredient for all kinds of soups

Sophus Lie, who lived most of his life in Norway, found the recipe to cook all kinds of soups in the 1880s. He revealed the fact that the most well-known techniques to solve differential equations were based on the invariance of the differential equations under a continuous group of symmetries. The symmetry group, was the ingredient.

This discovery at once unified and extended the available integration techniques. This continuous group of symmetries, now universally known as Lie groups, have reached their influence far beyond all areas of mathematics, both pure and applied, to physics, engineering, finance, economics, quantum mechanics, relativity, continuum mechanics, etc [71].

Sahoo and Saha Ray [80] used Lie symmetry to solve the (3+1) dimensional Yu-Toda-Sasa-Fukuyama equation in physics. Sheftel *et al.* [82] established the relations between the separation of variables and superintegrable systems in quantum mechanics using the structure of the higher order Lie symmetries of the Schrödinger equation in the Euclidean plane. Basquerotto *et al.* [8] presented the application of the Lie symmetries analysis to obtain the solution of a classical nonlinear problem of the dynamics of mechanical systems: the bead on a rotating wire hoop.

In biology, Mechee and Haitham [60] used Lie symmetry to evaluate the uninfected CD_4^+T cells in the human body and gave approximated solutions of the mathematical model of HIV infection. Zheng [98] performed Lie symmetry analysis on a nonlinear Fokker-Planck equation that described cell population growth.

The application of the Lie symmetry does not stop at science and engineering. In finance and economics, Paliathanasis *et al.* [73] performed a classification of the Lie point symmetries for the Black-Scholes-Merton model for European options with stochastic volatility. Gazizov *et al.* [28] completed symmetry analysis of the one-dimensional Black-Scholes-Merton model followed by classification of the two-dimensional Jacobs-Jones model equations. Liu and Wang [56] continued the study on the Black-Scholes-Merton model with dividend yield using the same tool.

In fact, Lie symmetry analysis has become a favourite tool for academics in solving different types of differential equations in their respective fields of study. The reputation of Lie symmetry as an ingredient to cook "almost" all kinds of soups is well established.

A new soup to cook

Fractional calculus, on the other hand, is a newfangled soup to cook. For hundreds of years, academics had been chasing the footprints of fractional calculus since it was first brought up in a letter between Leibniz and L'Hôpital in 1695. In a letter, they debated the possibility to generalize the meaning of integer derivatives to non-integer derivatives, for example a derivative of $n = 1/2$. The discussion ended with the quote from Leibniz "It will lead to a paradox. From this apparent paradox, one day useful consequences will be drawn".

The birth of fractional calculus certainly evoked the interest of researchers around the world. Lacroix [50], being the first to discuss derivatives of non-integer orders, proposed the αth derivative of $y = x^n$ as

$$\frac{d^\alpha y}{dx^\alpha} = \frac{n!}{(n-\alpha)!} x^{n-\alpha}.$$

The above derivative can be used to write the derivative of order 1/2 by replacing α by 1/2

$$\frac{d^{1/2} y}{dx^{1/2}} = \frac{\Gamma(n+1)}{\Gamma(n+1/2)} x^{n-1/2}.$$

Here, $\Gamma(a)$ is an Euler's Gamma function.

The development of fractional calculus has come a long way. Numerous operators and definitions [76] were introduced to suit different conditions and situations. Bernhard Riemann and Joseph Liouville proposed the derivative of a function $u(t, x)$ of independent variables x and t with respect to t of order α, which was later known as the Riemann-Liouville (RL) integral, as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(m-\alpha)} \frac{\partial^m}{\partial t^m} \int_0^t \frac{u(\xi, x)}{(t-\xi)^{\alpha+1-m}} d\xi, \tag{2.1}$$

where $0 < m - 1 < \alpha \leq m, m \in \mathbb{N}$. Caputo then redefined the fractional derivative as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{1}{(t-\xi)^{\alpha+1-m}} \frac{\partial^m u(\xi, x)}{\partial \xi^m} d\xi. \tag{2.2}$$

The application of fractional calculus has been proven to be related to many fields. Metzler and Klafter [64] demonstrated that fractional equations describe anomalous transport processes. Axtell and Bise [6] explored the implications of non-integer order systems in the s-domain of control systems by using fractional calculus and Laplace transformed differintegrals. Henriques *et al.* [34] implemented the algorithms, one of them being the fractional derivative algorithm, of six fractional detectors for colour images and illustrated their performances. Bioengineering and biomedical applications, thermal modeling of engineering systems, wave and diffusion phenomenon, and many more studies continue to reveal the significance of fractional calculus in redefining the classic integer models into brand new non-integer models.

Academics started to apply different methods to seek solutions to fractional differential equations. Wyss [96] proposed a fractional generalization of the Black-Scholes-Merton equation by using Green's function. Li and He [52] used fractional complex transform to convert fractional differential equations into ordinary differential equations. Many more textbooks were published to provide comprehensive reading and study about fractional differential equations [22, 25, 40, 59].

Gazizov *et al.* [29] were among the first to cook this soup with Lie algebras. A prolongation formulae for fractional derivatives was proposed. Huang and Zhdanov [36] later gave an explicit form of the prolongation formulae. Jefferson and Carminati [46] built an algorithmic package `FracSym` using the Maple software to obtain the determining equations.

As a result, research articles exploiting Lie symmetry in solving fractional differential equations grew exponentially recently. Akbulut and Taşcan [3] use Lie symmetry to solve a time-fractional modified Korteweg-de Vries equation. Chen *et al.* [16] extended the KdV equations to a time-fractional generalized equation. Gazizov *et al.* [30] presented the Lie point symmetries and exact solutions of fractional diffusion equations of the orders of 0-2. Jafari *et al.* [45] proposed the Lie point symmetries and exact solutions of a time-fractional Boussinesq equation.

Lie symmetry analysis continues to play its role as a capable tool to solve differential equations, ordinary, partial, or fractional, to this day. In this thesis, Lie symmetry analysis is used to study several time-fractional financial models, as well as power options under the Heston dynamic.

2.2 Lie symmetry analysis

Lie symmetry analysis is an algorithmic procedure that often involves lengthy and tedious calculations. The core of this procedure is the invariance of differential equations. Many books are published to provide substantial reading on this subject [5, 13, 14, 71, 72]. In this section, the fundamental concepts of Lie symmetry analysis are discussed. One may refer to the mentioned books for more comprehensive reading.

2.2.1 Lie group properties, definitions, Lie point symmetries, and Lie algebra

A transformation is the change of the position, size, orientation, or even structure of an object. For instance, a rotation about the center of a unit circle, $x^2 + y^2 = 1$, leaves the circle unchanged. The rotated image, $(\tilde{x}, \tilde{y}) = (cos(\theta + \varepsilon), sin(\theta + \varepsilon))$, with an infinitesimal transformation parameter ε, is a symmetry of the unit circle as it preserves the structure of the original unit circle, $\tilde{x}^2 + \tilde{y}^2 = 1$. In fact, the unit circle has an infinite set of rotational symmetries.

In general, a transformation is a symmetry if it preserves the structure of the object, is a diffeomorphism (the inverse of the transformation is smooth), and maps the object to it itself. A symmetry of a differential equation is a transformation that leaves the differential equation invariant. For example, the Riccati equation

$$\frac{dy}{dx} = \frac{y+1}{x} + \frac{y^2}{x^3}$$

has a one-parameter Lie group of inversions as one of its symmetries sounds

$$(\tilde{x}, \tilde{y}) = \Big(\frac{x}{1-\varepsilon x}, \frac{y}{1-\varepsilon x}\Big).$$

One may verify that this transformation finally leads to

$$\frac{d\tilde{y}}{d\tilde{x}} = \frac{\tilde{y}+1}{\tilde{x}} + \frac{\tilde{y}^2}{\tilde{x}^3}.$$

The continuous Lie point symmetries satisfy the properties of a group. We start with some basic definitions of a group.

Definition 2.2.1 *A set G with a law of composition $\phi(a, b)$ is a group if the following properties hold:*

1. *Closure: For $a, b \in G$, $\phi(a, b) \in G$.*

2. *Associative: For a, b and $c \in G$, $\phi(a, \phi(b, c)) = \phi(\phi(a, b), c)$.*

3. *Identity: For any $a \in G$, there exist an identity element $e \in G$ such that $\phi(a, e) = \phi(e, a) = a$.*

4. *Inverse: For any* $a \in G$*, there exist a unique element* $a^{-1} \in G$ *such that* $\phi(a, a^{-1}) = \phi(a^{-1}, a) = e$*.*

Definition 2.2.2 *A group is abelian if* $\phi(a, b) = \phi(b, a)$ *for all the elements in the group.*

Definition 2.2.3 *If a set* $H \subset G$ *forms a group, the set* H *is a subgroup of* G*.*

Definition 2.2.4 *A subgroup* H *of* G *is normal if and only if* $gH = Hg$ *for any* $g \in G$*.*

Definition 2.2.5 *The transformations set*

$$\begin{aligned} \tilde{u} &= U(u, t, x; \varepsilon) \\ \tilde{t} &= T(u, t, x; \varepsilon) \\ \tilde{x} &= \chi(u, t, x; \varepsilon) \end{aligned} \tag{2.3}$$

defined for each (u, t, x) *in a region* D *and parameter* ε *in a set* $S \subset \mathbb{N}$*, with a law of composition* $\phi(\varepsilon, \delta)$*, forms a one-parameter Lie group of transformations on* D*, for* $\varepsilon, \delta \in S$*, if the following hold:*

1. *The transformations are one-to-one and onto* D*.*

2. *The set* S *forms a group* G *with the law of composition* ϕ*.*

3. *For each point in* D *there exists an identity element* $e \in S$ *such that* $\tilde{u} = U(u, t, x, e) = u$*,* $\tilde{t} = T(u, t, x, e) = t$ *and* $\tilde{x} = \chi(u, t, x, e) = x$*.*

4. *For the above transformations set, if* $\tilde{\tilde{x}} = \chi(\tilde{u}, \tilde{t}, \tilde{x}, \delta)$*, then* $\tilde{\tilde{x}} = \chi(u, t, x, \phi(\varepsilon, \delta))$*.*

5. *The parameter* ε *is continuous.* $\varepsilon = 0$ *is taken as the identity element* e*.*

6. *The function* χ*, which is an analytic function of* $\varepsilon \in S$*, is infinitely differentiable with respect to* x *in* D*.*

7. *The above law of composition* $\phi(\varepsilon, \delta)$ *is an analytic function, for* ε *and* $\delta \in S$*.*

Note that if only properties (1-4) are satisfied, the set of transformations is referred to as a group of transformations.

Infinitesimal transformations

Consider the transformations (2.3) where $\varepsilon = 0$ represents the identity element make $u = U(u, t, x; 0)$, $t = T(u, t, x; 0)$ and $x = \chi(u, t, x; 0)$. The Taylor series of the transformations (2.3) about the identity are

$$\begin{aligned} \tilde{u} &= u + \varepsilon\eta(u, t, x) + O(\varepsilon^2), \\ \tilde{t} &= t + \varepsilon\tau(u, t, x) + O(\varepsilon^2), \\ \tilde{x} &= x + \varepsilon\xi(u, t, x) + O(\varepsilon^2), \end{aligned} \tag{2.4}$$

where $\eta(u, t, x)$, $\tau(x, t, u)$ and $\xi(u, t, x)$ are defined by

$$\eta(u, t, x) = \frac{\partial U}{\partial \varepsilon}\Big|_{\varepsilon=0}, \quad \tau(u, t, x) = \frac{\partial T}{\partial \varepsilon}\Big|_{\varepsilon=0}, \quad \xi(u, t, x) = \frac{\partial \chi}{\partial \varepsilon}\Big|_{\varepsilon=0}.$$

When the transformation is infinitesimal, that is when $\varepsilon \approx 0$, the terms of second and higher order in ε are negligible, hence the transformations (2.4) are simplified and known as *infinitesimal transformations*

$$\begin{aligned} \tilde{u} &= u + \varepsilon\eta(u,t,x), \\ \tilde{t} &= t + \varepsilon\tau(u,t,x), \\ \tilde{x} &= x + \varepsilon\xi(u,t,x). \end{aligned} \tag{2.5}$$

The partial differential operator

$$X = \eta(u,t,x)\frac{\partial}{\partial u} + \tau(u,t,x)\frac{\partial}{\partial t} + \xi(u,t,x)\frac{\partial}{\partial x} \tag{2.6}$$

is known as the *infinitesimal generator* of the Lie group. The one-parameter Lie group of transformation (2.3) is equivalent to the infinitesimal generator (2.6). They both can be found by solving the *Lie equations*

$$\frac{d\tilde{u}}{d\varepsilon} = \eta(\tilde{u},\tilde{t},\tilde{x}), \quad \frac{d\tilde{t}}{d\varepsilon} = \tau(\tilde{u},\tilde{t},\tilde{x}), \quad \frac{d\tilde{x}}{d\varepsilon} = \xi(\tilde{u},\tilde{t},\tilde{x}). \tag{2.7}$$

The Lie equations (2.7) should satisfy the initial conditions

$$\tilde{u}\big|_{\varepsilon=0} = u, \quad \tilde{t}\big|_{\varepsilon=0} = t, \quad \tilde{x}\big|_{\varepsilon=0} = x.$$

Lie Point Symmetries

Generally, an infinitely differentiable function, $\phi(u,t,x)$, is an *invariant func-*

tion of the Lie group transformations (2.3) if the function satisfies

$$\phi(\tilde{u}, \tilde{t}, \tilde{x}) = \phi(u, t, x). \tag{2.8}$$

Using the infinitesimal transformations (2.5), the Taylor series (for $\varepsilon \approx 0$) of $\phi(\tilde{u}, \tilde{t}, \tilde{x})$ may be written

$$\begin{aligned} \phi(\tilde{u}, \tilde{t}, \tilde{x}) &= \phi(u + \varepsilon\eta, t + \varepsilon\tau, x + \varepsilon\xi) \\ &= \phi(u, t, x) + \varepsilon\Big(\eta\frac{\partial\phi}{\partial u} + \tau\frac{\partial\phi}{\partial t} + \xi\frac{\partial\phi}{\partial x}\Big) \\ &= (1 + \varepsilon X)\phi(u, t, x), \end{aligned} \tag{2.9}$$

where X is the generator (2.6). Combining equations (2.8) and (2.9) yields $\phi(u, t, x) = (1 + \varepsilon X)\phi(u, t, x)$, which will then lead to $X\phi = 0$:

$$\eta\frac{\partial\phi}{\partial u} + \tau\frac{\partial\phi}{\partial t} + \xi\frac{\partial\phi}{\partial x} = 0. \tag{2.10}$$

Equation (2.10) can be solved using the method of characteristics:

$$\frac{du}{\eta(u, t, x)} = \frac{dt}{\tau(u, t, x)} = \frac{dx}{\xi(u, t, x)}. \tag{2.11}$$

Now, consider a partial differential equation (PDE) with k-th order

$$F(u, t, x, \partial u, \partial^2 u, \ldots, \partial^k u) = 0, \tag{2.12}$$

where u is a dependent variable and (t, x) are two independent variables. The Lie group transformations in equation (2.4) are the *Lie point symmetry group* of the PDE (2.12) if they leave the PDE invariant, that is

$$F(\tilde{u}, \tilde{t}, \tilde{x}, \partial\tilde{u}, \partial^2\tilde{u}, \ldots, \partial^k\tilde{u}) = 0, \tag{2.13}$$

where $\partial^i\tilde{u}$ are known as the extended derivatives of u, or the *prolongations*, are vital in finding the one-parameter Lie groups of point transformations admitted by differential equations in terms of infinitesimal generators. Many books [5, 13, 14, 76, 89] explain in detail how the prolongations are formulated. Here we just give a brief introduction. Suppose we have an infinitesimal generator

$$X = \sum_i \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u}, \tag{2.14}$$

with $u = u(x_1, \ldots, u_n)$ then its kth-prolongations [13] is

$$X^{(k)} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \eta_i^{(1)}\frac{\partial}{\partial u_i} + \cdots + \eta_{i_1\ldots i_k}^{(k)}\frac{\partial}{\partial u_{i_1\ldots i_k}}, \tag{2.15}$$

where $k = 1, 2, \ldots$ and $\eta^{(k)}$ are given by

$$\begin{aligned} \eta_i^{(1)} &= D_i\eta - (D_i\xi)u_j, \\ \eta_{i_1\ldots i_k}^{(k)} &= D_{i_k}\eta_{i_1\ldots i_{k-1}}^{(k-1)} - (D_{i_k}\xi)u_{i_1\ldots i_{k-1}j}, \end{aligned} \tag{2.16}$$

for $i = 1, 2, \ldots, n$ and $i_m = 1, 2, \ldots n$ for $m = 1, 2, \ldots k$ with $k = 2, 3, \ldots$. Here, D_i is the total derivative

$$D_i = \frac{\partial}{\partial x} + u_i\frac{\partial}{\partial u} + u_{ii_1}\frac{\partial}{\partial u_{i_1}} + u_{ii_1i_2}\frac{\partial}{\partial u_{i_1i_2}} + \ldots. \tag{2.17}$$

Note that we use the notation $u_i = \dfrac{\partial u}{\partial x_i}, u_{ij} = \dfrac{\partial^2 u}{\partial x_i \partial x_j}$, etc. The generator in (2.6) is a special case of the generator in (2.14) with one independent variable, says $u = u(\boldsymbol{x})$, and two independent variables $\boldsymbol{x} = (t, x)$. The PDE in (2.12), F, is said to admit a symmetry X if and only if the *infinitesimal criterion for the invariance* of the PDE F hold

$$X^{(k)} F|_{F=0} = 0. \tag{2.18}$$

In fulfilling the criterion (2.18), the equation (2.12) is taken into account while zeroing the kth extension of X on it. The condition (2.18) is commonly leading to a complicated and algorithmic calculation.

Now, let us consider the famous *heat equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{2.19}$$

which has one dependent variable $u = u(t, x)$ and two independents variables (t, x). Let the symmetry (2.6)

$$X = \eta(u, t, x)\frac{\partial}{\partial u} + \tau(u, t, x)\frac{\partial}{\partial t} + \xi(u, t, x)\frac{\partial}{\partial x}$$

be admitted by the heat equation (2.19). To determine the coefficients η, τ and ξ, the second prolongation of X, which is extended as

$$X^{(2)} = \eta\partial_u + \tau\partial_t + \xi\partial_x + \eta^x\partial_{u_x} + \eta^t\partial_{u_t} + \eta^{xx}\partial_{u_{xx}} + \eta^{xt}\partial_{u_{xt}} + \eta^{tt}\partial_{u_{tt}}, \tag{2.20}$$

where $\eta^x \equiv \eta_1^{(1)}, \ldots, \eta^{xt} \equiv \eta_{12}^{(2)}$ and $\partial_u = \dfrac{\partial}{\partial u}, \partial_t = \dfrac{\partial}{\partial t}$, etc, is to act on the equation (2.19) to fit the criterion (2.18), that is

$$X^{(2)}(u_t - u_{xx})|_{u_t - u_{xx}=0} = 0.$$

Here $u_t = \dfrac{\partial u}{\partial t}$ and $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$. This will finally lead to

$$\eta^t = \eta^{xx}. \tag{2.21}$$

The extended coefficients in (2.20) are given [13] as

$$\eta^x = \eta_x + (\eta_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t, \tag{2.22a}$$

$$\eta^t = \eta_t - \xi_t u_x + (\eta_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u u_t^2, \tag{2.22b}$$

$$\begin{aligned} \eta^{xx} = {} & \eta_{xx} + (2\eta_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\eta_{uu} - 2\xi_{xu})u_x^2 \\ & - 2\tau_{xu}u_x u_t - \xi_{uu}u_x^3 - \tau_{uu}u_x^2 u_t + (\eta_u - 2\xi_x)u_{xx} \\ & - 2\tau_x u_{xt} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt}, \end{aligned} \tag{2.22c}$$

$$\begin{aligned} \eta^{tt} = {} & \eta_{tt} - \xi_{tt}u_x + (2\eta_{tu} - \tau_{tt})u_t - 2\xi_{tu}u_x u_t \\ & + (\eta_{uu} - 2\tau_{tu})u_t^2 - \xi_{uu}u_x u_t^2 - \tau_{uu}u_t^3 - 2\xi_t u_{xt} \\ & - 2\xi_u u_t u_{xt} + (\eta_u - 2\tau_t)u_{tt} - \xi_u u_x u_{tt} - 3\tau_u u_t u_{tt}, \end{aligned} \tag{2.22d}$$

$$\begin{aligned} \eta^{xt} = {} & \eta_{xt} + (\eta_{tu} - \xi_{xt})u_x + (\eta_{xu} - \tau_{xt})u_t - \xi_{tu}u_x^2 \\ & + (\eta_{uu} - \xi_{xu} - \tau_{tu})u_x u_t - \tau_{ux}u_t^2 - \xi_{uu}u_x^2 u_t - \tau_{uu}u_x u_t^2 \\ & - \xi_t u_{xx} - \xi_u u_t u_{xx} + (\eta_u - \xi_x - \tau_t)u_{xt} - 2\xi_u u_x u_{xt} \\ & - 2\tau_u u_t u_{xt} - \tau_x u_{tt} - \tau_u u_x u_{tt}. \end{aligned} \tag{2.22e}$$

Substituting η^t and η^{xx} from equations (2.22b) and (2.22c) to equation (2.21), eliminating u_{xx} using the heat equations and finally zeroing the coefficients, will lead to the following solutions:

$$\xi = c_1 + c_4 x + 2c_5 t + 4c_6 xt, \tag{2.23a}$$

$$\tau = c_2 + 2c_4 t + 4c_6 t^2, \tag{2.23b}$$

$$\eta = (c_3 - c_5 x - 2c_6 t - c_6 x^2)u + \alpha(t,x), \tag{2.23c}$$

where c_i are arbitrary constants and $\alpha(t,x)$ is a arbitrary solution of the heat equation. The symmetry generator, hence, is given by

$$\begin{aligned} X =&(c_1 + c_4 x + 2c_5 t + 4c_6 xt)\partial_x + (c_2 + 2c_4 t + 4c_6 t^2)\partial_t \\ &+ ((c_3 - c_5 x - 2c_6 t - c_6 x^2)u + \alpha(t,x))\partial_u. \end{aligned} \tag{2.24}$$

Choosing $c_i = 1$ while zeroing the other $c's$ gives the following generators:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= \frac{\partial}{\partial t}, \\ X_3 &= u\frac{\partial}{\partial u}, \\ X_4 &= x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \\ X_5 &= 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}, \\ X_6 &= 4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2 + 2t)u\frac{\partial}{\partial u}, \\ X_\infty &= \alpha(t,x)\frac{\partial}{\partial u}. \end{aligned} \tag{2.25}$$

The existence of X_∞ shows that the heat equation has infinite-dimensional Lie symmetry algebra.

Lie Algebra

Consider two generators X_1 and X_2, the *commutator*, also known as the *Lie Bracket* of them, $[X_1, X_2]$ is defined as

$$[X_1, X_2] = X_1X_2 - X_2X_1. \tag{2.26}$$

For example, if X_1 and X_2 are two generators defined by

$$X_1 = x\partial_x, \quad X_2 = y\partial_x + x\partial_y,$$

then

$$[X_1, X_2] = x\partial_x(y\partial_x + x\partial_y) - (y\partial_x + x\partial_y)x\partial_x = x\partial_y - y\partial_x.$$

Definition 2.2.6 *A vector space of infinitesimal generators, L, is a Lie algebra if for all $X_i \in L, a, b \in \Re$, the following properties hold:*

1. *Closure:* $[X_1, X_2] \in L$,
2. *Bilinearity:* $[X_1, aX_2 + bX_3] = a[X_1, X_2] + b[X_1, X_3]$,
3. *Anticommutativity:* $[X_1, X_2] = -[X_2, X_1]$,
4. *Jacobi Identity:* $[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0$.

A commutator table is a tabular form to display the structure of a Lie algebra. A n-dimensional Lie algebra produces a $n \times n$ table, with the (i, j)th entry of the table expressing the Lie Bracket $[X_i, X_j]$. In Table (2.1), the commutators of the heat equation are given.

Table 2.1: The commutator table of the infinitesimal generators of the Heat equation (2.19)

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	X_1	$-X_3$	$2X_5$
X_2	0	0	0	$2X_2$	$2X_1$	$4X_4 - 2X_3$
X_3	0	0	0	0	0	0
X_4	$-X_1$	$-2X_2$	0	0	X_5	$2X_6$
X_5	X_3	$-2X_1$	0	$-X_5$	0	0
X_6	$-2X_5$	$2X_3 - 4X_4$	0	$-2X_6$	0	0

2.2.2 Group invariant solutions

Most partial differential equations are challenging to solve due to the existence of multiple independent variables in the equations. Using Lie symmetries, this trouble is lifted as the number of variables in the equations is reduced by one, or more. The reduced differential equations, on some occasions reduced to ordinary differential equations that consist of only one independent variable, are much easier to solve.

Group invariant solutions, also known as *similarity solutions*, are the solutions of differential equations that are characterized by their invariance under some symmetry group of the differential equations.

Definition 2.2.7 *$u = \theta(x)$ is an invariant solution of the PDE resulting from the point symmetry X if and only if*

1. *$u = \theta(x)$ is an invariant surface of the point symmetry X,*

2. *$u = \theta(x)$ is a solution of the PDE.*

The heat equation, which has one dependent variable and two independent variables, has the generators of the form of equation (2.6)

$$X = \eta(u,t,x)\partial_u + \tau(u,t,x)\partial_t + \xi(u,t,x)\partial_x.$$

Solving the characteristic system (2.11), given by

$$\frac{du}{\eta(u,t,x)} = \frac{dt}{\tau(u,t,x)} = \frac{dx}{\xi(u,t,x)},$$

to obtain a general invariant of X will first produce two similarity solutions in the general forms of

$$\theta_1(u,t,x) = I_1 \quad \text{and} \quad \theta_2(u,t,x) = I_2, \tag{2.27}$$

where I_i are constants of integration. X is then found as an arbitrary function of I_1 and I_2.

2.2.3 Optimal systems

In general, it is not practical to list all possible group invariant solutions of a certain differential equation because almost every symmetry group admits an infinite number of subgroups that correspond to a family of group invariant solutions. The need to effectively classify these solutions leads to the construction of an *optimal system*, a small (minimum) set of one-dimensional subalgebras that contains all the one-dimensional subalgebras of the differential equation.

In this thesis, the direct method used in Olver [71] is adopted. First, the adjoint representations for each X_i of the basis symmetry group are used to determine the adjoint representation as follows:

$$\mathrm{Ad}(\exp(\varepsilon X_i))X_j = X_j - \varepsilon[X_i, X_j] + \frac{1}{2}\varepsilon^2[X_i,[X_i,X_j]] - \dots, \tag{2.28}$$

where $[X_i, X_j]$ is the commutator of the generators X_i and X_j. The adjoint representations are then used to identify the optimal system by simplifying a given arbitrary element of the Lie algebra

$$X = \sum_i a_i X_i, \tag{2.29}$$

where a_i are constants. Chou and Li [18] gave detailed work to obtain the optimal system of the heat equation.

2.2.4 Law of conservation

Neother's theorem [69] shows that there is a connection between conservation laws for variational problems with the symmetries of differential equations or Euler-Lagrange equations particularly. Ibragimov [41, 42] later redefined the conservation theorem by eliminating the need for the existence of a Lagrangian. It is proved that the adjoint equation inherits all symmetries of the original equation. As a result, we are now free to associate a conservation law with any Lie group to find conservation laws for differential equations without the need to classify its Lagrangians. Ibragimov [42] provided that any Lie point, Lie-Bäcklund or non-local symmetry

$$X = \xi^i(x, u, u_{(1)}, \dots)\frac{\partial}{\partial x^i} + \eta(x, u, u_{(1)}, \dots)\frac{\partial}{\partial u} \tag{2.30}$$

of equation

$$F(u, t, x, \partial u, \partial^2 u, \cdots, \partial^k u) = 0 \tag{2.31}$$

provides a conservation law $D_i(C^i) = 0$ for the simultaneous system for equation (2.31) and its adjoint equation with a new dependent variable v

$$F^*(u, v, t, x, y, \partial u, \partial v, \partial^2 u, \partial^2 v, \dots, \partial^k u, \partial^k v) = \frac{\delta(vF)}{\delta u} = 0 \tag{2.32}$$

where

$$\frac{\delta}{\delta u} = \frac{\delta}{\delta u} + \sum_{s=1}^{\infty}(-1)^s D_{i_1} \dots D_{i_s}\frac{\partial}{\partial u_{i_1 \dots i_s}}. \tag{2.33}$$

The conserved vector is given by

$$\begin{aligned} C^i =& \xi^i \mathcal{L} + W\left[\frac{\partial \mathcal{L}}{\partial u_i} - D_j\left(\frac{\partial \mathcal{L}}{\partial u_{ij}}\right) + D_j D_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}}\right) - \dots\right] \\ &+ D_j W\left[\frac{\partial \mathcal{L}}{\partial u_{ij}} - D_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}}\right) + \dots\right] + D_j D_k W\left[\frac{\partial \mathcal{L}}{\partial u_{ijk}} - \dots\right] + \dots \end{aligned} \tag{2.34}$$

where W and $\mathcal{L}$ are defined as

$$W = \eta - \xi^j u_j, \quad \mathcal{L} = vF. \tag{2.35}$$

For more details about conservation laws, one may refer to the above-recommended references.

2.3 Fractional derivatives and integrals

The theory of fractional derivatives was mainly developed as pure theoretical knowledge since the first discussion between Leibniz and L'Hôpital. Liouville, Grünwald, Letnikov, and Riemann were among the pioneers who gave different definitions to the fractional derivatives. Advanced modern sciences and technologies have slowly revealed the roles played by fractional derivatives in various fields, especially in describing the memory and hereditary properties of materials and natural processes, which have been neglected in classical integer-order models. The study of fractional calculus always involves some special functions, including the functions discussed in 2.3.1.

2.3.1 Special functions

2.3.1.1 Gamma function

The Euler's gamma function $\Gamma(z)$, which gives the generalization of the factorial of any real number, is no doubt one of the most important basic functions in describing fractional calculus. Here we give the definition and some basic properties of the gamma function.

Definition 2.3.1 *The gamma function* $\Gamma(z)$*, for* $z \in \mathbb{C}$*, is defined by the integral*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt. \tag{2.36}$$

The gamma function satisfies the functional equation

$$\Gamma(z+1) = z\Gamma(z), \tag{2.37}$$

which if $z = 1, 2, 3 \ldots$, leads to

$$\Gamma(z+1) = z!. \tag{2.38}$$

2.3.1.2 Beta function

Definition 2.3.2 *The beta function* $B(x, y)$*, for* $x, y \in \mathbb{R}^+$*, is defined as*

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt. \tag{2.39}$$

The beta function can be represented in gamma functions:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{2.40}$$

We will use beta function in the later discussion.

2.3.1.3 Mittag-Leffler function

Introduced by Mittag-Leffler [65, 66, 67], the Mittag-Leffler function which is an exponential function, e^z, with its one-parameter general definition is given by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{2.41}$$

The two-parameter Mittag-Leffler function, on the other hand, was introduced by Agarwal [2]:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha > 0, \quad \beta > 0). \tag{2.42}$$

Equations (2.38) and (2.42) give

$$\begin{aligned}
E_{1,1} &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z, \\
E_{1,2} &= \frac{1}{z}(e^z - 1), \\
E_{1,3} &= \frac{1}{z^2}(e^z - 1 - z).
\end{aligned}$$

2.3.2 Riemann-Liouville fractional derivatives

Fractional derivatives (or integrals) are the derivatives (or integrals) of an arbitrary real order α, namely,

$$D_t^{\alpha} f(t) \quad \text{or} \quad D_t^{-\alpha} f(t).$$

A couple of definitions of fractional derivatives were proposed differently by various mathematicians to suit different conditions.

Among all, the most famous are the Riemann-Liouville fractional derivatives (2.1) and the Caputo's fractional derivative (2.2). Each of these definitions has its own advantages and disadvantages. The Riemann-Liouville always leads to initial conditions which is the result of the limit values of the definition itself at the origin of time, $t = 0$. The solutions of the initial conditions, which can be solved mathematically, have unknown physical interpretations to this day. On the hand, Caputo's fractional derivatives have the same initial conditions in the fractional form as its integer-order differential equations. There exist different links and connections between these fractional derivatives. For more details, one may refer to Podlubny [76].

In this thesis, the focus is given to the Riemann-Liouville fractional derivative (2.1). The Riemann-Liouville fractional derivative with one independent variable is defined as:

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{\partial}{\partial t}\right)^m \int_0^t (t-\tau)^{m-\alpha-1} f(\tau) d\tau, \tag{2.43}$$

where $0 < m-1 < \alpha \leq m, m \in \mathbb{N}$. On the other hand, a fractional integral is defined as:

$$D_t^{-\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau) d\tau, \tag{2.44}$$

where $0 < \beta < 1$. Consider the fractional integral of a simple function t^μ:

$$\begin{aligned}
D_t^{-\beta} t^\mu &= \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \tau^\mu d\tau \\
&= \frac{1}{\Gamma(\beta)} \int_0^t (1-\tau/t)^{\beta-1} t^{\beta-1} \tau^\mu d\tau \\
&= \frac{1}{\Gamma(\beta)} \int_0^t (1-u)^{\beta-1} t^{\beta-1} (ut)^\mu t du, \quad \text{(letting } u = \tau/t\text{)} \\
&= \frac{1}{\Gamma(\beta)} t^{\beta+\mu} \int_0^1 (1-u)^{\beta-1} u^\mu du \\
&= \frac{1}{\Gamma(\beta)} t^{\beta+\mu} B(\mu+1, \beta) \\
&= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\beta+1)} t^{\beta+\mu}, \quad \text{(from equation (2.40))}.
\end{aligned}$$

The above example gives the fractional integral of a simple function, t^μ:

$$D_t^{-\beta} t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\beta+1)} t^{\beta+\mu}. \tag{2.45}$$

Now, letting $\alpha = n - \beta$ and n be the smallest integer that is greater than α, the fractional derivative of $f(t)$ of order α is given by

$$D^\alpha f(t) = D^n [D^{-\beta} f(t)]. \tag{2.46}$$

Using the example above, the fractional derivative of the function t^{μ} gives

$$\begin{aligned}
D^{\alpha}t^{\mu} &= D^{1}[D^{-(1-\alpha)}t^{\mu}], \quad \text{(here, } n = 1 \text{ and } \beta = 1-\alpha\text{)} \\
&= D^{1}\Big(\frac{\Gamma(\mu+1)}{\Gamma((\mu+1-\alpha)+1)}t^{\mu+1-\alpha}\Big), \quad \text{(from equation (2.45))} \\
&= (\mu+1-\alpha)\frac{\Gamma(\mu+1)}{(\mu+1-\alpha)\Gamma(\mu+1-\alpha)}t^{\mu-\alpha}, \quad \text{(from equation(2.37))} \\
&= \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\alpha)}t^{\mu-\alpha}.
\end{aligned}$$

Hence, we have the fractional derivative rule for power

$$D^{\alpha}t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\alpha)}t^{\mu-\alpha}, \quad \mu \geq 0, \quad 0 < \alpha < 1. \tag{2.47}$$

Putting $\mu = 0$ and $\alpha = 1/2$ in equation (2.47) reveals the fact that the half-derivative of a *constant* in Riemann-Liouville approach is a non-zero value, which sometimes is considered as one of the shortcomings of the Riemann-Liouville definition:

$$D^{1/2}t^{0} = \frac{\Gamma(1)}{\Gamma(1/2)}t^{-1/2} = \frac{1}{\sqrt{\pi t}}.$$

These two examples demonstrate the complexity of the direct fractional derivatives and integrals, even the simple power functions. In many cases, different approaches or transformations are applied to simplify the tedious calculation.

2.3.2.1 Laplace transforms

The Laplace transforms play a very important role in simplifying the fractional derivatives. We will use them to solve some basic fractional differential equations later. Here we briefly discuss the basic definitions and properties of the Laplace transform.

Definition 2.3.3 *The Laplace transform of the function $f(t)$ (or the original) is a function $F(s)$, for $s \in \mathbb{C}$, defined by*

$$F(s) = L\{f(t); s\} = \int_0^\infty e^{-st} f(t) dt. \tag{2.48}$$

There exist positive constants M and T such that $e^{-\alpha t}|f(t)| \leq M$ for all $t > T$, for the function $f(t)$ must grow slower than an exponential function when $t \to \infty$.

Definition 2.3.4 *The inverse Laplace transform to restore the original $f(t)$ is defined by*

$$f(t) = L^{-1}\{F(s); t\} = \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds, \quad c = \Re(s). \tag{2.49}$$

The Laplace transform is linear, that is, for an arbitrary constant a,

$$L(f(t) + g(t)) = F(s) + G(s) \quad \text{and} \quad L(af(t)) = aF(s). \tag{2.50}$$

The product of the Laplace transforms of two functions $f(t)$ and $g(t)$, if exist, is the Laplace transform of the convolution of them, $f(t) * g(t)$. That is

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau, \tag{2.51}$$

gives

$$L\{f(t) * g(t); s\} = F(s)G(s). \tag{2.52}$$

Equations (2.51) and (2.52) are crucial to evaluate the Laplace transform of the Riemann-Liouville fractional integral. Recall the fractional integral of the function $f(t)$ of order β, equation (2.44),

$$D_t^{-\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau)d\tau,$$

which the integral in fact is a convolution of two functions:

$$\int_0^t (t-\tau)^{\beta-1} f(\tau)d\tau = t^{\beta-1} * f(t). \tag{2.53}$$

Using equation (2.52), the Laplace transform of the Riemann-Liouville fractional integral can be written as

$$\begin{aligned} L\{D_t^{-\beta} f(t); s\} &= \frac{1}{\Gamma(\beta)} \Big(L\{t^{\beta-1}; s\} \cdot L\{f(t); s\} \Big) \\ &= \frac{1}{\Gamma(\beta)} \Big(\Gamma(\beta) s^{-\beta} \cdot F(s) \Big) \\ &= s^{-\beta} F(s). \end{aligned} \tag{2.54}$$

The Laplace transform of the derivative of an integer order n of a function

$f(t)$ is given by

$$L\{f^{(n)}(t); s\} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0). \tag{2.55}$$

To extend the Laplace transform to the Riemann-Liouville fractional derivative, we write the fractional derivative in the form of

$$D_t^\alpha f(t) = g^{(n)}(t), \quad \text{and} \tag{2.56}$$

$$g(t) = D^{-(n-\alpha)} f(t), \tag{2.57}$$

for $n - 1 \leq \alpha < n$. Using equation (2.55) on equation (2.56) gives

$$L\{D_t^\alpha f(t); s\} = s^n G(s) - \sum_{k=0}^{n-1} s^k g^{(n-k-1)}(0), \tag{2.58}$$

with $G(s) = s^{-(n-\alpha)} F(s)$ from equation (2.54) and $g^{(n-k-1)}(0) = D^{\alpha-k-1} f(0)$ from equation (2.56). Hence, equation (2.58) is finally written as

$$L\{D_t^\alpha f(t); s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k [D^{\alpha-k-1} f(0)], \tag{2.59}$$

for $n-1 \leq \alpha < n$. Here we give some useful Laplace transforms in Table (2.2) [83].

$f(t)$	$F(s)$
$t^{\alpha-1}E_{\alpha,\alpha}(rt^{\alpha})$	$\dfrac{1}{s^{\alpha}-r}$
$E_{\alpha}(-rt^{\alpha})$	$\dfrac{s^{\alpha-1}}{s^{\alpha}+r}$
$1-E_{\alpha}(-rt^{\alpha})$	$\dfrac{r}{s(s^{\alpha}+r)}$
$t^{\beta-1}E_{\alpha,\beta}(rt^{\alpha})$	$\dfrac{s^{\alpha-\beta}}{s^{\alpha}-r}$

Table 2.2: Laplace transforms

2.3.2.2 The Cauchy problem

The Cauchy problem is an integro-differential equation that sounds

$$D^{\alpha}f(t)+\lambda D^{-\beta}f(t)=h(t), \tag{2.60}$$

for $\lambda,\alpha,\beta\in\mathbb{C}$ with $\Re(\alpha)>0,\Re(\beta)>0$ and $h(t)$ is an arbitrary integrable function. The Cauchy problem comes with the condition

$$D^{\alpha-k-1}f(0)=a_k, k=0,1,\ldots,[\Re(\alpha)], \tag{2.61}$$

where $[\Re(\alpha)]$ is the integer part of $\Re(\alpha)$. Taking Laplace transform on equation (2.60), we have

$$
\begin{aligned}
&L\{D^{\alpha}f(t) + \lambda D^{-\beta}; s\} = L\{h(t); s\} \\
&s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{k}a_k + \lambda s^{-\beta}F(s) = H(s) \\
&F(s)(s^{\alpha} + \lambda s^{-\beta}) = \sum_{k=0}^{n-1} s^{k}a_k + H(s) \\
&F(s) = \sum_{k=0}^{n-1} a_k \cdot \frac{s^{k+\beta}}{s^{\alpha+\beta} + \lambda} + H(s) \cdot \frac{s^{\beta}}{s^{\alpha+\beta} + \lambda}
\end{aligned}
$$

Note that from Table (2.2),

$$L^{-1}\Big\{\frac{s^{k+\beta}}{s^{\alpha+\beta} + \lambda}; s\Big\} = t^{\alpha-k-1}E_{\alpha+\beta,\alpha-k}(-\lambda t^{\alpha+\beta}),$$

and

$$L^{-1}\Big\{\frac{s^{\beta}}{s^{\alpha+\beta} + \lambda}; s\Big\} = t^{\alpha-1}E_{\alpha+\beta,\alpha}(-\lambda t^{\alpha+\beta}).$$

Hence, we have

$$
\begin{aligned}
L^{-1}\{F(s); s\} &= L^{-1}\left\{\sum_{k=0}^{n-1} a_k \cdot \frac{s^{k+\beta}}{s^{\alpha+\beta} + \lambda} + H(s) \cdot \frac{s^{\beta}}{s^{\alpha+\beta} + \lambda}; s\right\} \\
&= \sum_{k=0}^{n-1} a_k t^{\alpha-k-1}E_{\alpha+\beta,\alpha-k}(-\lambda t^{\alpha+\beta}) \\
&+ \int_0^t (t-\tau)^{\alpha-1}E_{\alpha+\beta,\alpha}(-\lambda(t-\tau)^{\alpha+\beta})h(\tau)d\tau.
\end{aligned} \tag{2.62}
$$

That is the solution of the Cauchy problem (2.60) is in the form of equation (2.62). By taking $\alpha = 1/2, \beta = 0$ and $h(t) = 0$, the Cauchy problem sounds

$$D^{\frac{1}{2}} f(t) + \lambda f(t) = 0, \tag{2.63}$$

for the fractional integral of $f(t)$ is invariant and no integral is involved [59]. With the condition $D^{-\frac{1}{2}} f(t)|_{t=0} = c$, the solution of equation (2.63), by applying solution (2.62), is given by

$$f(t) = ct^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}(-\lambda t^{\frac{1}{2}}).$$

2.3.3 Lie symmetry analysis of fractional derivative

Solving fractional differential equations in a conventional way can be very challenging. The example in equation (2.63) has shown that for the simplest form of a fractional differential equation, a massive transformation is actually taking place behind the scenes. The calculation will only get uglier when a fractional differential equation gets more complicated.

With more and more attention paid to the fractional differential equations in the past decades due to their impressive applications in natural sciences, the secret ingredient to cooking all kinds of soup finally joined the feast in 2007 when Gazizov *et al.* [29] presented an approach to use the Lie symmetry analysis to solve fractional differential equations (FDEs).

Consider a time-fractional differential equation of the form

$$\frac{\partial^\alpha}{\partial t^\alpha} u(t,x) = F(u,t,x,u_x,u_{xx}), \quad (0 < \alpha < 1). \tag{2.64}$$

Similar to the Lie symmetry analysis of the integer order differential equations, the FDE (2.64) is to admit a symmetry X if and only if the infinitesimal criterion

$$X^{(\alpha,2)}\Delta|_{\Delta=0} = 0, \quad \Delta = \frac{\partial^\alpha u}{\partial t^\alpha} - F, \tag{2.65}$$

is fulfilled, where the prolongation of $X^{(\alpha,2)}$ is extended (similar to equation(2.20)) as

$$X^{(\alpha,2)} = X + \eta^{\alpha,t}\partial_{\partial_t^\alpha u} + \eta^x \partial_{u_x} + \eta^{xx}\partial_{u_{xx}}, \tag{2.66}$$

with η^x and η^{xx} are as given in equations (2.22a) and (2.22c), and

$$\begin{aligned} \eta^{\alpha,t} =& D_t^\alpha(\eta) + \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) + D_t^\alpha(u D_t(\tau)) \\ & - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u). \end{aligned} \tag{2.67}$$

Equation (2.67) is not convenient to use for the lengthy total derivatives in it. To write the equation (2.67) in an explicit form [36], we start with the Leibniz rule for fractional differentiation:

$$D_t^\alpha(f(t)g(t)) = \sum_{n=0}^{\infty} \binom{\alpha}{n} f^{(n)}(t) D_t^{\alpha-n} g(t), \tag{2.68}$$

where

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1-n)}.$$

Leibniz rule gives

$$\xi D_t^{\alpha}(u_x) - D_t^{\alpha}(\xi u_x) = -\sum_{n=1}^{\infty} \binom{\alpha}{n} \xi^{(n)}(t) D_t^{\alpha-n}(u_x), \tag{2.69}$$

and

$$\begin{aligned} D_t^{\alpha}(uD_t(\tau)) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u) = &-\alpha D_t(\tau) D_t^{\alpha} u \\ &- \sum_{n=1}^{\infty} \binom{\alpha}{n+1} D_t^{n+1}(\tau) D_t^{\alpha-n}(u). \end{aligned} \tag{2.70}$$

With the generalized chain rule [70] of the form

$$\frac{d^{\alpha} f(g(t))}{dt^{\alpha}} = \sum_{n=0}^{\infty} \frac{U_n}{n!} \frac{d^n f(g(t))}{d(g(t))^n}, \quad U_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} g^k(t) \partial_t^{\alpha}(g^{(n-k)}(t)),$$

and the Leibniz rule (2.68), we obtain

$$D_t^{\alpha}(\eta) = \partial_t^{\alpha}\eta + \eta_u \partial_t^{\alpha} u - u \partial_t^{\alpha} \eta_u + \sum_{n=1}^{\infty} \binom{\alpha}{n} \partial_t^n \eta_u \partial_t^{\alpha-n} u + \zeta, \tag{2.71}$$

where

$$\zeta = \sum_{n=2}^{\infty} \sum_{m=2}^{n} \sum_{k=2}^{m} \binom{\alpha}{n} \binom{n}{m} \frac{t^{n-\alpha} U_k}{k!\Gamma(n+1-\alpha)} \frac{\partial^{n-m+k}\eta}{\partial t^{n-m} \partial u^k}.$$

When η is a function in the form $\eta = f(t,x)u$, its the second derivative of η (and above) with respect to u gives $\zeta = 0$. Finally, combining equations

(2.69), (2.70) and (2.71), we have the explicit form of $\eta^{\alpha,t}$

$$\eta^{\alpha,t} = \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\partial_t^n \eta_u - \binom{\alpha}{n+1}D_t^{n+1}(\tau)\right]\partial_t^{\alpha-n}u$$
$$- \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)\partial_t^{\alpha-n}(u_x) + \partial_t^\alpha \eta + (\eta_u - \alpha D_t(\tau))\partial_t^\alpha u - u\partial_t^\alpha \eta_u + \zeta. \tag{2.72}$$

The explicit form of $\eta^{\alpha,t}$ reveals the fact that since α is a fraction, the first two terms of equation (2.72) with fractional integrals $\partial_t^{\alpha-n}u$ and $\partial_t^{\alpha-n}(u_x)$ will not coincide with any of the other differential quantities that present, we actually "earn" the following conditions

$$\binom{\alpha}{n}\partial_t^n \eta_u - \binom{\alpha}{n+1}D_t^{n+1}(\tau) = 0 \quad \text{and} \quad \binom{\alpha}{n}D_t^n(\xi) = 0,$$

for $n = 1, 2, \ldots$, which in most cases (at least in ours), can be simplified to

$$\eta_{ut} - \frac{1}{2}\tau_{tt} = 0, \tag{2.73a}$$

$$\xi_t = 0. \tag{2.73b}$$

However, it is recommended to expand the conditions by at least two terms to avoid unnecessary loss. Jefferson and Carminati [46] in 2013 presented an algorithmic package `FracSym` to solve the Lie equations of fractional differential equations. In our study, the package is used to verify the solutions of the determining equations found. Henceforth, the process to obtain optimal systems and invariant solutions of fractional differential equations are similar to the process we discussed in sections (2.2.2) and (2.2.3) earlier.

Chapter 3

Time-fractional Black-Scholes-Merton model

3.1 Preliminaries of financial mathematics

The financial market, like most traditional markets, is a place where all kinds of products are traded. Instead of selling dairy products, the merchants are selling financial products or contracts, which are commonly known as *derivatives* nowadays. The derivatives market is big, much bigger than the stock market when measured in terms of underlying assets. By hedging or speculation, derivatives are able to transfer the risks in the economy from one entity to another.

The value of a derivative, as a financial instrument, depends on the values of the underlying assets. Currently the most frequently traded derivatives are *forward contracts*, *futures contracts*, *options*, *bonds*, and *swaps*.

Forward contracts

The forward contract is an agreement to buy or sell an asset at a certain future for a certain price. One of the contract holders agrees to buy the underlying asset on a certain date in the future (the long position) while the other contract holder agrees to sell that asset on the same date for the same price (the short position).

This is very common and popular in foreign exchange markets to hedge foreign currency risk. If the delivery price (the price set in advance in the contract) and the spot price (the current market price) of the asset are K and S_T respectively at maturity, in general, the payoff from a long position in a forward contract is

$$S_T - K$$

while the payoff of a short position is

$$K - S_T.$$

The payoff from the contract is the trader's total gain or loss from the contract since it costs nothing to enter the contract. The good side (and also the bad) of forward contracts is the gain (or the loss) is theoretically indefinite.

Futures contracts

Similar to a forward contract, a futures contract entangle two parties in agreeing to trade an asset at a certain time in the future for a certain price. The futures contracts are normally traded on an exchange, between a wide range of commodities such as gold, old, sugar, etc, and financial assets.

Options

A *call option* gives the holder the right, but not the obligation, to buy the underlying asset by a certain date for a certain price. A *put option*, on the other hand, gives the right to the holder to sell the underlying asset by a certain date for a certain price. *European options* can only be exercised on the expiration date while *American options* can be exercised at any time up to the expiration date. Unlike futures and forwards contracts, option contracts come with prices and the payoff of an option depends only on the price at maturity.

If the price of an asset at maturity, S_T, exceeds the strike price (or exercise price), K, that is when $S_T > K$, a call option should always be exercised and gain a payoff of $S_T - K$. The opposite of this situation, if $S_T \leq K$, the holder of a call option should not exercise the option because the asset can be bought with a cost less than or equal to the exercise price, K.

There are four types of option positions with their respective payoffs:

1. A long position (i.e., has bought the option) in a call option, payoff = $\max(S_T - K, 0)$.

2. A long position in a put option, payoff = $\max(K - S_T, 0)$.

3. A short position (i.e., has sold the option) in a call option, payoff = $\min(K - S_T, 0)$.

4. A short position in a put option, payoff = $\min(S_T - K, 0)$.

Figure 3.1 illustrates these payoffs.

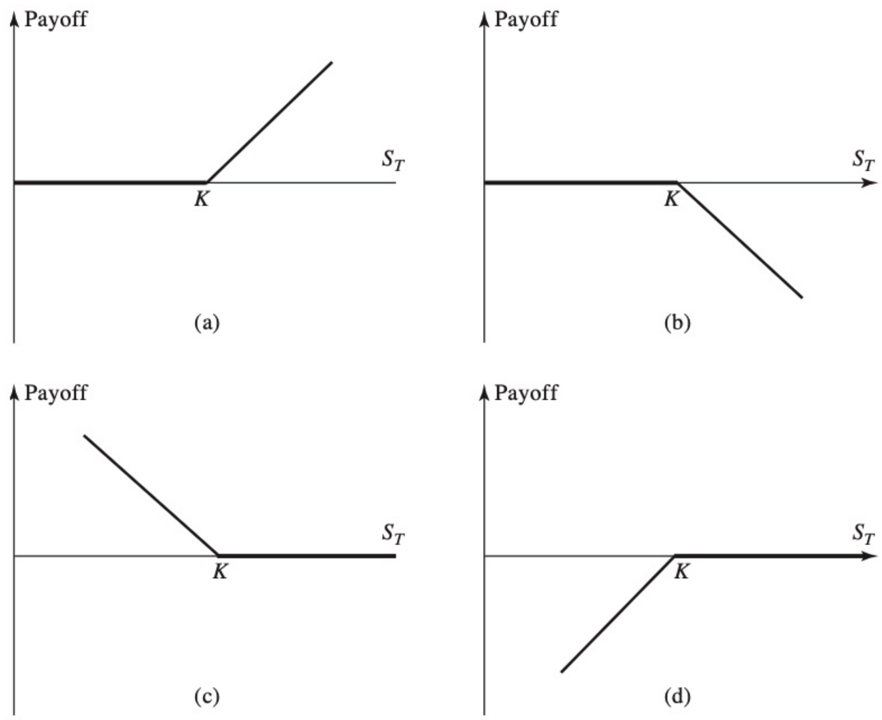


Figure 3.1: Payoffs from positions in European options: (a) long call; (b) short call; (c) long put; (d) short put. Strike price = K; price of the asset at maturity = S_T.

Bonds and Swaps

Bonds and swaps are not our study focus. We briefly introduce them here for fundamental financial knowledge. A bond is a contract written by a big party to raise capital where the up-front premium is regarded as a loan to the bond writer. Upon maturity, the bond writer is to reclaim the bond from the holders with an amount agreed upon by both parties. A swap is a contract between two parties to agree on exchanging assets, says cash flows or stocks, in the future.

One important thing when dealing with financial derivatives is pricing them. Determining a fair price of a derivative always starts with the construction of a model to simulate the movement of the asset. In this context, the *geometric Brownian motion* is commonly used to simulate the asset price in many financial models. The stock price, S, is assumed to be governed by the *stochastic differential equation*

$$dS = rSdt + \sigma SdB_t, \tag{3.1}$$

where r is the expected rate of return of the asset, σ is the volatility and B_t is a standard Brownian motion. A further type of stochastic process can be defined as

$$dx = a(x,t)dt + b(x,t)dz, \tag{3.2}$$

where dz is a Wiener process, is known as an *Itô process*. This process is Markov since the change in x at time t depends only on the value of x at time t, not on its history.

Itô lemma

Suppose a variable x follows the Itô process (3.2), then it has a drift rate of a and a variance rate of b^2. Itô's lemma shows that a function $G(x, t)$ follows the process

$$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b \cdot dz, \tag{3.3}$$

also follows an Itô process. The drift rate and variance rate of the process $G(x, t)$ are

$$\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2$$

and

$$\left(\frac{\partial G}{\partial x} b \right)^2$$

respectively. It is well recognized that equation (3.1) is a reasonable model of stock price movements [37]. On top of this, the Itô lemma (3.3) can be extended to the fact that the process followed by a function $G(S, t)$ is

$$dG = \left(\frac{\partial G}{\partial S} rS + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S \cdot dz. \tag{3.4}$$

The functions S and G are affected by the same underlying source of uncertainty, dz. Equation (3.4) is proven to be the key in the derivation of the famous Black-Scholes-Merton results.

3.2 The Black-Scholes-Merton equation

The Black-Scholes-Merton model is undoubtedly one of the most important models in finance. Established and presented by Fischer Black and Myron Scholes [12], and Robert Merton [61] in 1973, the model hugely inspired the way that traders price and hedge derivatives. The discovery was later given the highest recognition from the Nobel Prize committee in 1995 for its magnificent impact on the economy.

The Black-Scholes-Merton (BSM) model introduces a linear parabolic partial differential equation that gives theoretical values for the European call and the European put options. This model is derived based on the following assumptions:

1. The stock price follows the process developed in equation (3.1).
2. The full use of the proceeds is allowed and the short selling of the securities is not penalized.
3. The transactions cost nothing, nor taxes. The assets are perfectly divisible.
4. The derivative pays no dividends during its life circle.
5. The risk-free arbitrage opportunities do not exist.

6. Trading in the market is time-continuous.

7. The risk-free interest rate, r, is deterministic and constant over time.

To derive the Black-Scholes-Merton equation, recall the assumption made earlier that the stock price process follows the stochastic process (3.1) that sounds

$$dS = rSdt + \sigma Sdz.$$

Suppose that $u(S,t)$ is the price of a call option (or other derivatives) at time t when the price of the underlying stock is S. Equation (3.4) gives

$$du = \left(\frac{\partial u}{\partial S} rS + \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial u}{\partial S}\sigma S \cdot dz. \tag{3.5}$$

Define a new portfolio that holds a short one derivative and long an amount $\partial u/\partial S$ of share as:

$$\Pi = \frac{\partial u}{\partial S} S - u, \tag{3.6}$$

where Π is the value of the portfolio.

The changes of the value of portfolio (3.6) in time, dt, is

$$d\Pi = \frac{\partial u}{\partial S} dS - du. \tag{3.7}$$

Combining equations (3.1) and (3.5) into equation (3.7), we have

$$d\Pi = \left(-\frac{\partial u}{\partial t} - \frac{1}{2}\frac{\partial^2 u}{\partial S^2}\sigma^2 S^2\right)dt. \tag{3.8}$$

The absence of the Wiener process, dz, in equation (3.8) indicates that the portfolio is riskless during the time dt and earns the same rate of returns as other riskless securities due to the absence of arbitrage opportunity (in assumption 5). Hence,

$$d\Pi = r\Pi dt. \tag{3.9}$$

Substituting equations (3.6) and (3.8) into equation (3.9) yields

$$\left(\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial S^2}\sigma^2 S^2\right)dt = r\left(u - \frac{\partial u}{\partial S}S\right)dt$$

which finally lead to the Black-Scholes-Merton equation

$$\frac{\partial u}{\partial t} + rS\frac{\partial u}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 u}{\partial S^2} = ru. \tag{3.10}$$

The Black-Scholes-Merton equation has many solutions corresponding with different boundary conditions: for a European call option, the condition sounds

$$u = \max(S - K, 0) \quad \text{when } t = T,$$

and for a European put option,

$$u = \max(K - S, 0) \quad \text{when } t = T,$$

with K as the exercise price at maturity. Among the solutions obtained from equation (3.10), the most famous solutions are the Black-Scholes-Merton formulas for the prices of European call (c) and put (p) options:

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2) \tag{3.11}$$

and

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1), \tag{3.12}$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}},$$
$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}.$$

Here, S_0 is the initial stock price, K is the strike price, r is the riskless rate of interest, σ is the stock price volatility, and T is the time to maturity of the option. The function $N(\cdot)$ is the *cumulative probability distribution function* of a variable with a standard normal distribution, $\phi(0,1)$. Equations (3.11) and (3.12) are related with a very important relationship which is known as the *put-call parity*:

$$c + ke^{-rT} = p + S_0. \tag{3.13}$$

3.2.1 Lie point symmetries admitted by the Black-Scholes-Merton equation

Gazizov and Ibragimov [28] in 1998 presented the Lie symmetry analysis on the Black-Scholes-Merton equation and showed that the equation is included in Lie's classification of linear second-order partial differential equation with two independent variables. The Black-Scholes-Merton equation was then transformed into the heat equation. Finally, the exact solutions were presented.

The infinite-dimensional vector space of the infinitesimal symmetries of equation (3.10) spanned by the following operators:

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \\
X_2 &= S\frac{\partial}{\partial S}, \\
X_3 &= 2t\frac{\partial}{\partial t} + (\ln S + Dt)S\frac{\partial}{\partial S} + 2rtu\frac{\partial}{\partial u}, \\
X_4 &= \sigma^2 tS\frac{\partial}{\partial S} + (\ln S - Dt)u\frac{\partial}{\partial u}, \\
X_5 &= 2\sigma^2 t^2\frac{\partial}{\partial t} + 2\sigma^2 tS\ln S\frac{\partial}{\partial S} + ((\ln S - Dt)^2 + 2\sigma^2 rt^2 - \sigma^2 t)u\frac{\partial}{\partial u}, \\
X_6 &= u\frac{\partial}{\partial u}, \\
X_\infty &= \phi(t, S)\frac{\partial}{\partial u},
\end{aligned}
\tag{3.14}
$$

where $D = r - \sigma^2/2$ and $\phi(t, S)$ is any solutions of equation (3.10).

3.2.2 Solutions from the admitted Lie point symmetries

Equation (3.10) is a partial differential equation with many solutions. In fact, each generator of (3.14), as well as a linear combination of the generators, admits an invariant solution of (3.10). For instance, consider the one-parameter subgroup with the generator

$$
X = X_1 + X_2 + X_6 = \frac{\partial}{\partial t} + S\frac{\partial}{\partial S} + u\frac{\partial}{\partial u}, \tag{3.15}
$$

which has the associated characteristics equations as

$$\frac{dt}{1} = \frac{dS}{S} = \frac{du}{u}.$$

The solution of the above characteristics equations gives the two independent invariants $t - \ln S$ and u/S. Hence, the invariants of the combined generator (3.15) take the form

$$u = SG(t - \ln S),$$

which will finally reduce equation (3.10) to a second order ordinary differential equation

$$\frac{1}{2}\sigma^2 G''(z) + AG'(z) = 0,$$

where $z = t - \ln S$, $A = r + \sigma^2/2 - 1$ and $G'(z) = dG/dz$. The solution of the second-order ordinary differential equation can be easily found

$$G(z) = \frac{k_1\sigma^2 \exp(2Az/\sigma^2)}{2A} + k_2$$

where k_1 and k_2 are arbitrary constants. Hence, the solution of equation (3.10) corresponding to the generator (3.15) is obtained:

$$u(S,t) = S\left(\frac{k_1\sigma^2 \exp(2A(t - \ln S)/\sigma^2)}{2A} + k_2\right).$$

Apparently constructing the invariant solutions for all possible linear combinations of the operators (3.14) is impractical. The *optimal system*, which is a small representative of the set of the symmetries that is possible to generate any other solution via a simple transformation, is constructed. For simplicity, equation (3.10) is written as

$$u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru = 0, \tag{3.16}$$

where S is replaced by x, u_t, u_x and u_{xx} are partial derivatives of u respect to t and x respectively.

3.3 Time-fractional Black-Scholes-Merton equation

The success of the Black-Scholes-Merton equation in providing the values of options undoubtedly put it in position as one of the most important discoveries in financial mathematics in the past few decades despite the fact that the equation was established under strict assumptions. To weaken those restrictions, various improved models have been proposed. Merton proposed the stochastic interest model [62] and the jump-diffusion model [63], Hull and White [38] proposed a stochastic volatility model, Davis *et al.* [21] and Barles and Soner [7] proposed models with transactions costs, etc.

Recently, numerous fractional Black-Scholes-Merton models were proposed as the result of the discovery of the fractal structure of the financial market. Wang [95] presented a European call option pricing formula of a discrete-time option pricing by the fractional Black-Scholes-Merton model with transaction costs by the mean of the self-financing delta-hedging argument. Liang *et al.* [53] established a bi-fractional Black-Merton-Scholes model with the Hurst exponent H in [1/2,1] and proposed the explicit option pricing formulas for the model.

In 2000, Wyss [96] presented a solution of a time-fractional Black-Scholes-Merton equation by using Green's function. His proof was solely done on the basis of mathematical content. Cartea and Castillo-Negrete [15] showed that the prices of the financial derivatives of CGMY, KoBoL, and FMLS models, which follow a jump process or Lévy process, satisfy a fractional partial differential equation. Jumarie [47, 48] derived two new families of fractional Black-Scholes-Merton equations in coarse-grained space and time by modeling a fractional stochastic differential equation as a fractional dynamics driven by white noise. Jumarie's works were done on the *modified* Riemann-Liouville fractional derivative, in which the effects of the non-zero initial value of the respective function are removed. Song and Wang [88] combined Jumarie's time-fractional Black-Scholes-Merton equation with the terminal and boundary conditions satisfied by the standard put options and gave the numerical solutions to the model. Ravi Kanth and Aruna [77] presented the fractional differential transform method and modified fractional differential method for the solution of the time-fractional Black-Scholes-Merton European option pricing equation.

3.3.1 Lie point symmetries admitted by a time-fractional Black-Scholes-Merton equation

In this section, the Lie symmetries analysis is used on a time-fractional Black-Scholes-Merton equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru = 0, \tag{3.17}$$

where $\dfrac{\partial^\alpha u}{\partial t^\alpha}$ is the partial derivative of the function $u(t,x)$ respect to t of order α, defined as the Riemann-Liouville fractional derivative

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(m-\alpha)}\left(\frac{\partial}{\partial t}\right)^m \int_0^t \frac{u(\xi,x)}{(t-\xi)^{\alpha+1-m}}d\xi, \tag{3.18}$$

where $0 < m-1 < \alpha \leq m, m \in \mathbb{N}$, σ and r are two different scalars that represent volatility and interest rate respectively. The invariance condition of equation (3.17) that satisfies the infinitesimal criterion of the invariance (2.65), that is

$$X^{(\alpha,2)}\left(\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru\right) = 0,$$

is given by

$$\eta^{\alpha,t} + \frac{1}{2}\sigma^2 x^2 \eta^{xx} + \sigma^2 x u_{xx}\xi + rx\eta^x + ru_x\xi - r\eta = 0, \tag{3.19}$$

where the infinitesimals η, ξ and τ are functions of variables (u,t,x) that define the infinitesimal generator (2.6).

Substituting the derivatives η^x, η^{xx}, as defined in (2.22a, 2.22c), and $\eta^{\alpha,t}$ from (2.72) into equation (3.19) and equating the coefficients of u_t, u_x, u_{xx}, u_{xt}, $u_t u_{xx}$, $u_x u_{xx}$, u_x^2, $\partial_t^{\alpha-n} u$ and $\partial_t^{\alpha-n} u_x$ to zero gives the following system of determining equations:

$$\frac{1}{2}\sigma^2 x^2 \tau_{xx} + rx\tau_x = 0, \tag{3.20a}$$

$$\sigma^2 x^2 \eta_{xu} - \frac{1}{2}\sigma^2 x^2 \xi_{xx} + r\alpha x\tau_t - rx\xi_x + r\xi = 0, \tag{3.20b}$$

$$\frac{1}{2}\alpha x\tau_t - x\xi_x + \xi = 0, \tag{3.20c}$$

$$\tau_x = \tau_u = \xi_u = \xi_t = 0, \tag{3.20d}$$

$$\frac{1}{2}\sigma^2 x^2 \eta_{uu} - \sigma^2 x^2 \xi_{xu} - rx\xi_u = 0, \tag{3.20e}$$

$$\text{for } n = 1, 2, 3, \cdots, \quad \binom{\alpha}{n}\partial_t^n \eta_u - \binom{\alpha}{n+1} D_t^{n+1}(\tau) = 0. \tag{3.20f}$$

Notice that equations (3.20d), which suggest

$$\xi = \xi(x) \quad \text{and} \quad \tau = \tau(t), \tag{3.21}$$

show that equation (3.20a) is trivial and equation (3.20e) is equivalent to $\eta_{uu} = 0$, that is

$$\eta = uA(t, x) + B(t, x). \tag{3.22}$$

Since $\xi = \xi(x)$, differentiating equation (3.20c) with respect to t gives $\tau_{tt} = 0$, or $\tau = c_1 t + c_i$, for c_1 and c_i being two arbitrary constants. The transformations of variables should retain the structure of the Riemann-Liouville fractional derivative operator, that is $\tau(u,t,x)|_{t=0} = 0$ Hence, the function $\tau(t)$ is found

$$\tau(t) = c_1 t. \tag{3.23}$$

Substituting equation (3.23) to equation (3.20c) yields

$$\frac{1}{2}c_1\alpha x - x\xi_x + \xi = 0,$$

which gives the solution of the function $\xi(x)$

$$\xi(x) = \frac{1}{2}c_1\alpha x\ln x + c_2 x, \tag{3.24}$$

Combining equation (3.23) with the free condition (2.73a) discussed in the previous section (or with equation (3.20f)) gives $\eta_{ut} = 0$, which subsequently suggests the function $A(t,x)$ in equation (3.22) being $A(x)$. Substituting $\eta = uA(x) + B(t,x)$ and equations (3.24, 3.23) into equation (3.20b) gives

$$A(x) = c_1\alpha M\ln x + c_3, \tag{3.25}$$

where $M = \dfrac{1}{4} - \dfrac{1}{2}\dfrac{r}{\sigma^2}$ and c_3 is an arbitrary constant. Hence, the function $\eta(u,t,x)$ is solved:

$$\eta(u,t,x) = (c_1\alpha M\ln x + c_3)u + B(t,x), \tag{3.26}$$

where $B(t,x)$ is any solution of equation (3.17).

Hence, the symmetry group of the fractional Black-Scholes-Merton equation is spanned by the vector fields

$$\begin{aligned}
X_1 &= t\frac{\partial}{\partial t} + \frac{1}{2}\alpha x \ln x \frac{\partial}{\partial x} + (\alpha M \ln x)u\frac{\partial}{\partial u}, \\
X_2 &= x\frac{\partial}{\partial x}, \\
X_3 &= u\frac{\partial}{\partial u}, \\
X_\infty &= B(x,t)\frac{\partial}{\partial u}.
\end{aligned} \tag{3.27}$$

The one-parameter Lie group of transformation (2.3) of the operators (3.27) is obtained by solving Lie equations (2.7):

$$\begin{aligned}
&X_1 : \bar{u} = ux^{2M(e^{\alpha\varepsilon/2}-1)}, && \bar{t} = e^{\varepsilon}t, && \bar{x} = x^{e^{\alpha\varepsilon/2}} \\
&X_2 : \bar{u} = u, && \bar{t} = t, && \bar{x} = xe^{\varepsilon} \\
&X_3 : \bar{u} = ue^{\varepsilon}, && \bar{t} = t, && \bar{x} = x
\end{aligned} \tag{3.28}$$

3.3.2 Optimal system of time-fractional Black-Scholes-Merton equation

To minimize the group of invariant solutions generated by the infinitesimal generators (3.27), we construct the optimal system admitted by the generators (3.27). First, we compute the commutators or the Lie brackets as defined in (2.26):

$$[X_i, X_j] = X_iX_j - X_jX_i.$$

For instance, $[X_1, X_2]$ is calculated as:

$$
\begin{aligned}
[X_1, X_2] &= \left(t\frac{\partial}{\partial t} + \frac{1}{2}\alpha x \ln x \frac{\partial}{\partial x} + (\alpha M \ln x) u \frac{\partial}{\partial u}\right) \cdot x\frac{\partial}{\partial x} \\
&\quad - x\frac{\partial}{\partial x} \cdot \left(t\frac{\partial}{\partial t} + \frac{1}{2}\alpha x \ln x \frac{\partial}{\partial x} + (\alpha M \ln x) u \frac{\partial}{\partial u}\right) \\
&= \frac{1}{2}\alpha x \ln x \frac{\partial}{\partial x} - x\left(\frac{1}{2}\alpha \ln x + \frac{\alpha}{2}\right)\frac{\partial}{\partial x} - x\frac{\alpha M}{x} u \frac{\partial}{\partial u} \\
&= -\frac{\alpha}{2} x \frac{\partial}{\partial x} - \alpha M u \frac{\partial}{\partial u} \\
&= -\frac{\alpha}{2} X_2 - \alpha M X_3
\end{aligned}
$$

using the anticommutativity property of a commutator,

$$
[X_2, X_1] = \frac{\alpha}{2} X_2 + \alpha M X_3.
$$

Similarly, we obtained the Lie brackets $[X_1, X_3] = [X_2, X_3] = 0$.We listed all the commutators generated by the operators (3.27) in Table 3.1. The classical BSM equation emits more generators and hence produces more complicated Lie brackets [85].

Table 3.1: Lie bracket of the admitted system algebra for the time-fractional Black-Scholes-Merton model.

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	$-\frac{\alpha}{2} X_2 - \alpha M X_3$	0
X_2	$\frac{\alpha}{2} X_2 + \alpha M X_3$	0	0
X_3	0	0	0

Using the commutators Table 3.1, the adjoint representations for each X_i of the basis symmetries group are determined as defined in equation (2.28).

First, we determined $Ad(\exp(\varepsilon X_1))X_2$:

$$
\begin{aligned}
Ad(\exp(\varepsilon X_1))X_2 &= X_2 - \varepsilon[X_1, X_2] + \frac{\varepsilon^2}{2}[X_1,[X_1,X_2]] \\
&- \frac{\varepsilon^3}{6}[X_1,[X_1,[X_1,X_2]]] + \cdots \\
&= X_2 - \varepsilon\big(- \frac{\alpha}{2}X_2 - \alpha M X_3\big) + \frac{\varepsilon^2}{2}[X_1, -\frac{\alpha}{2}X_2 - \alpha M X_3] \\
&- \frac{\varepsilon^3}{6}[X_1,[X_1, -\frac{\alpha}{2}X_2 - \alpha M X_3]] + \cdots \\
&= X_2 + \varepsilon\big(\frac{\alpha}{2}X_2 + \alpha M X_3\big) - \frac{\alpha\varepsilon^2}{4}[X_1, X_2] \\
&+ \frac{\varepsilon^3}{6}[X_1, \frac{\alpha}{2}[X_1,X_2]] + \cdots \\
&= X_2 + \varepsilon\big(\frac{\alpha}{2}X_2 + \alpha M X_3\big) + \frac{\alpha\varepsilon^2}{4}\big(\frac{\alpha}{2}X_2 + \alpha M X_3\big) \\
&- \frac{\varepsilon^3}{6}[X_1, \frac{\alpha}{2}\big(\frac{\alpha}{2}X_2 + \alpha M X_3\big)] + \cdots \\
&= X_2 + \varepsilon\big(\frac{\alpha}{2}X_2 + \alpha M X_3\big) + \frac{\alpha\varepsilon^2}{4}\big(\frac{\alpha}{2}X_2 + \alpha M X_3\big) \\
&+ \frac{\varepsilon^3}{6}\frac{\alpha^2}{4}\big(\frac{\alpha}{2}X_2 + \alpha M X_3\big) + \cdots \\
&= \left(1 + \frac{\alpha\varepsilon}{2} + \frac{\alpha^2\varepsilon^2}{8} + \frac{\alpha^3\varepsilon^3}{48} + \cdots\right)X_2 \\
&+ \left(\alpha\varepsilon + \frac{\alpha^2\varepsilon^2}{4} + \frac{\alpha^3\varepsilon^3}{24} + \cdots\right)MX_3 \\
&= \exp\big\{\frac{\alpha\varepsilon}{2}\big\}X_2 + 2\big(\exp\big\{\frac{\alpha\varepsilon}{2}\big\} - 1\big)MX_3
\end{aligned}
$$

The adjoint representations of other X_i's are computed similarly and listed in Table 3.2.

To determine the optimal system, we need to simplify as many of the

Table 3.2: Adjoint representation of the admitted system algebra for the time-fractional Black-Scholes-Merton model.

$[X_i, X_j]$	X_1	X_2	X_3
X_1	X_1	$\exp\{\frac{\alpha\varepsilon}{2}\}X_2 + 2\big(\exp\{\frac{\alpha\varepsilon}{2}\} - 1\big)MX_3$	X_3
X_2	$X_1 - \frac{\varepsilon\alpha}{2}X_2 - \varepsilon\alpha MX_3$	X_2	X_3
X_3	X_1	X_2	X_3

coefficients a_i as possible of the nonzero vector

$$X = a_1X_1 + a_2X_2 + a_3X_3.$$

First, suppose $a_1 \neq 0$ and assume $a_1 = 1$. Referring to Table 3.2, the coefficient of X_2 will disappear if we act X with $Ad\big(\exp\big(\frac{2a_2}{\alpha}\big)X_2\big)$:

$$\begin{aligned}
X' &= Ad\big(\exp\big(\frac{2a_2}{\alpha}\big)X_2\big)(X_1 + a_2X_2 + a_3X_3) \\
&= X_1 - \frac{2a_2}{\alpha}\frac{\alpha}{2}X_2 - \frac{2a_2}{\alpha}\alpha MX_3 + a_2X_2 + a_3X_3 \\
&= X_1 + (a_3 - 2a_2M)X_3 \\
&= X_1 + \rho X_3 \quad (\text{let } a_3 - 2a_2M = \rho)
\end{aligned}$$

No further simplification is possible on such an X'. This means every one-dimensional subalgebra generated by X with $a_1 \neq 0$ is equivalent to the subalgebra spanned by $X_1 + \rho X_3$.

Now consider the one-dimensional subalgebras that are spanned by X with

$a_1 = 0$. Similarly, we suppose $a_2 \neq 0$ and scale it to 1, that is

$$X'' = X_2 + a_3 X_3.$$

The coefficient of X_3 will disappear if we act on X'' with a specific adjoint representation of X_1:

$$\begin{aligned}
X''' &= Ad\Bigg(\exp\left\{\frac{2}{\alpha}\ln\left(1-\frac{a_3}{2M}\right)\right\}X_1\Bigg)(X_2+a_3X_3) \\
&= \exp\left\{\frac{\frac{2}{\alpha}\ln\left(1-\frac{a_3}{2M}\right)\alpha}{2}\right\}X_2 + \Bigg(a_3 + 2\Bigg(\exp\left\{\frac{\frac{2}{\alpha}\ln\left(1-\frac{a_3}{2M}\right)\alpha}{2}\right\} - 1\Bigg)M\Bigg)X_3 \\
&= \left(1-\frac{a_3}{2M}\right)X_2 \\
&= \tilde{a}_2 X_2
\end{aligned}$$

Hence, any subalgebra spanned by X with $a_1 = 0, a_2 \neq 0$ is equivalent to the subalgebra spanned by X_2. Lastly, the remaining cases $a_1 = a_2 = 0$ and $a_3 \neq 0$ are similarly seen to be equivalent to X_3. Hence, the set of one-dimensional optimal systems for the time-fractional Black-Scholes-Merton model is

$$\big\{X_1 + \rho X_3, X_2, X_3\big\}.$$

3.3.3 Group invariant solutions of time-fractional Black-Scholes-Merton equation

To obtain the group invariant solutions, the method of characteristics (2.11) is applied to the infinitesimal generators (or the optimal system). The function $u = u(t,x)$ is an invariant solution of a fractional differential equation corresponding to its infinitesimal operator if and only if it fulfills the invariant surface condition

$$\tau(u,t,x)u_t + \xi(u,t,x)u_x = \eta(u,t,x).$$

Suppose ξ and τ are not both zero, then the above invariant surface condition can be solved by the method of characteristics:

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}.$$

3.3.3.1 Invariant solution generated by X_2

Consider the infinitesimal operator $X_2 = x\frac{\partial}{\partial x}$ we obtained earlier. The characteristic equations

$$\frac{dt}{0} = \frac{dx}{x} = \frac{du}{0}$$

give the similarity variables t and u. It is more convenient to write it as $u = F(t)$. Inserting it into the equation (3.17) yields the following fractional ordinary differential equation

$$\frac{\partial^\alpha}{\partial t^\alpha}F(t) = rF(t).$$

Similar to the Cauchy problem (2.63), the solution to the above equation is

$$u(t,x) = F(t) = k_1 t^{\alpha-1} E_{\alpha,\alpha}(rt^\alpha), \tag{3.29}$$

where $k_1 = D^{-(1-\alpha)}F(0)$ and $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$ is a Mittag-Leffler function.

3.3.3.2 Invariant solution generated by $X_2 + \rho X_3$

The generator $X_2 + \rho X_3$, where ρ is an arbitrary constant, gives

$$X_2 + \rho X_3 = x\frac{\partial}{\partial x} + \rho u\frac{\partial}{\partial u}$$

and leads to the characteristic equations

$$\frac{dt}{0} = \frac{dx}{x} = \frac{du}{\rho u}.$$

Solving the above equations gives the similarity variables t and $ux^{-\rho}$ which is more conveniently written as $u = x^\rho G(t)$. Substitute this into equation (3.17) will lead to the following fractional ordinary differential equation

$$\frac{\partial^\alpha}{\partial t^\alpha}G(t) + \left((\rho - 1)\left(\frac{1}{2}\sigma^2\rho + r\right)\right)G(t) = 0.$$

Solving the above fractional equation for $G(t)$ eventually give the solution of equation (3.17) as

$$u(t, x) = k_2 x^\rho t^{\alpha-1} E_{\alpha,\alpha}\left((1 - \rho)\left(\frac{1}{2}\sigma^2\rho + r\right)t^\alpha\right), \tag{3.30}$$

where $k_2 = D^{-(1-\alpha)}G(0)$. The invariant solution provided by the generator $X_1 + \rho X_3$ as listed in the optimal systems is not found for the calculation involved is unmanageable. Note that $X_4 = u\dfrac{\partial}{\partial u}$ does not provide any invariants.

3.4 Conclusion

Partial differential equations always generate multiple solutions. So do the corresponding fractional differential equations. Equation (3.29) we suggested above is a one-variable function in t. This solution is true from the mathematical perspective. However, the lack of variable x in the solution is somehow less appealing from the financial point of view. Here we illustrate the newly obtained equations (3.29) and (3.30) in figures 3.2 and 3.3. The parameters chosen for illustration are as follows

- risk-free interest rate $r = 0.1$,
- volatility $\sigma = 0.2$,
- constant $k_1 = k_2 = 1$,
- constant $\rho = 1.5$.

Figure 3.2 illustrates the solution (3.29) using the parameters with three different values of $\alpha = 0.5$, 0.8, and 0.9 against time. Figure 3.3 shows the surfaces of equation (3.30).

The solutions of fractional differential equations, in general, differ significantly from each other when the values of α are small. However, with the

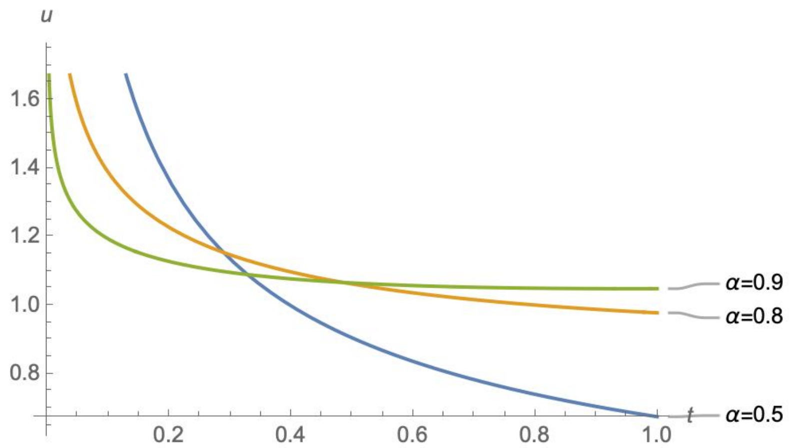


Figure 3.2: Equation (3.29) with $\alpha = 0.5, 0.8$, and 0.9, against time using $r = 0.1, \sigma = 0.2$.

increase of α, they appear to be emerging to each other. In Figure 3.2, the graphs of equation (3.29) have a big difference between the value of $\alpha = 0.5$ and $\alpha = 0.8$. The gap between the graphs shrinks when α ranges from 0.8 to 0.9. Furthermore, figure 3.2 shows an obvious discrepancy between graphs when t is short (less than 0.4).

Figure 3.3 illustrate equation (3.30) with the same values of α which are 0.5, 0.8, and 0.9. Figure 3.3 shows that the value of the option increases slowly when x increases. Similar to figure 3.2, the values of the option differ significantly when the value of time is small. The values of the option slowly decline when time increases.

The cause of the discrepancy between the solutions of fractional differential equations and non-fractional differential equations is uncertain. Tarasova and Tarasov [93] pointed out that fractional derivatives describe the memory effect

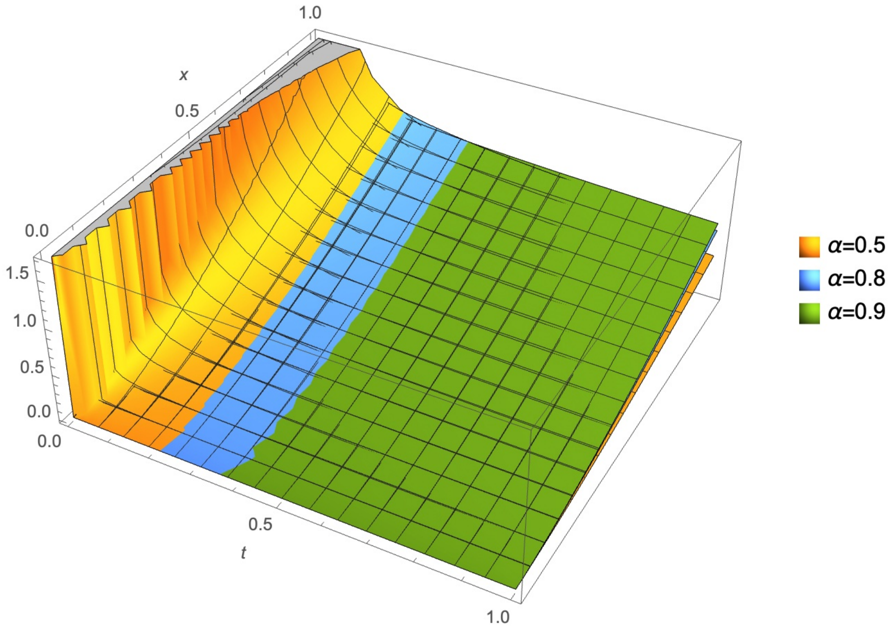


Figure 3.3: Equation (3.30) with $\alpha = 0.5, 0.8,$ and 0.9, against time using $r = 0.1, \sigma = 0.2, \rho = 1.5$.

in financial and economic processes.

Figures 3.2 and 3.3 show that the memory effect on the time-fractional Black-Scholes-Merton equation is prominent for smaller t, that is when the option just gets started. When time passes by, the effect fades and the graphs merge. We believe the memory effect on the time-fractional Black-Scholes-Merton equation is somehow connected to the variable t and the constant α.

In this chapter, Lie symmetry analysis is used on a time-fractional Black-Scholes-Merton equation. The time-fractional Black-Scholes-Merton equation admits three point symmetries and an infinite dimensional sub-algebra X_∞. With the commutators and adjoint representations obtained, the set of one-dimensional optimal systems for the time-fractional Black-Scholes-Merton model is found. Two exact invariant solutions of the time-fractional Black-Scholes-Merton equation are proposed and presented in the form of graphs.

Chapter 4

Time-fractional arbitrage-free stock price model

4.1 Arbitrage-free stock price model

The Black-Scholes-Merton model is based on the assumption that the stock price follows an Itô process described by the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dw_t$$

where μ representing the drift of the stock, σ is the volatility of the stock, and w_t is a standard Wiener process. In fact, the value of S_t at any time t is expected to be

$$E^Q[S_t] = S_0 e^{rt},$$

where Q is the risk-neutral measure. In 1976, Cox and Ross [20] proposed an alternative to the above model, that is

$$dS_t = \mu S_t dt + \sigma\sqrt{S_t}dw_t.$$

The above equation has no closed-form solution. Hence, the analogue of the Black-Scholes-Merton formula for the Cox-Ross model can not be found. Stock price models depend heavily on the arbitrage-free condition, which is not realistic. Bell and Stelljes [10], in 2009, proposed a method to construct a class of solvable arbitrage-free models for stock prices by following the stochastic Bernoulli equation of Stratonovich type

$$d\tilde{S}_t = \mu\tilde{S}_t dt + \sigma\tilde{S}_t^p \circ dw_t, \tag{4.1}$$

where $1/2 \leq p \leq 1$. The solution to the above model is an integro-differential equation

$$\tilde{S}_t = e^{rt}\left\{(1-p)\sigma\int_0^t e^{r(p-1)u}dw_u + \tilde{S}_0^{1-p}\right\}^{1/(1-p)}, \tag{4.2}$$

which generally contrasts with the arbitrage-free condition. To overcome this shortage, Bell and Stelljes constructed a function Z so that the process $S_t \equiv Z(\tilde{S}_t, t)$ is arbitrage-free. This yields a second-order partial differential equation for Z that is similar to the classical Black-Scholes-Merton equation

$$Z_t + \left(rs + \frac{p\sigma^2 s^{2p-1}}{2}\right)Z_s + \frac{\sigma^2 s^{2p}}{2}Z_{ss} - rZ = 0, \tag{4.3}$$

with the condition that there exists n such that for every future time $T > 0$,

$$\sup_{0 \le t \le T} |Z_s(s,t)| \le C|s|^n, \quad \forall s \in \mathbb{R}.$$

The solutions of equation (4.3) for $p = 1$ and $p = 1/2$ were suggested at the end of the paper [10]. When $p = 1$, equation (4.3) emerges to the classical Black-Scholes-Merton model and admit a solution $Z(s,t) = s^{-\sigma^2 t/2}$. Similarly, when $p = 1/2$, the equation (4.3) emerges to the Cox-Ross model and admits a solution $Z(s,t) = s + \sigma^2/4r$.

Sinkala [86], in 2016, extended this result for all values of $p, 1/2 \le p \le 1$, for which the equation is tractable by using Lie symmetry analysis. Replacing the variables x and u in place of s and Z, we have

$$\frac{\partial u}{\partial t} + \left(rx + \frac{p\sigma^2 x^{2p-1}}{2} \right) \frac{\partial u}{\partial x} + \frac{\sigma^2 x^{2p}}{2} \frac{\partial^2 u}{\partial x^2} - ru = 0. \tag{4.4}$$

Equation (4.4) admits a rich symmetry group. Sinkala showed that for $p \neq 1$, the basis of the infinite-dimensional vector space of infinitesimal symmetries of equation (4.4) are as follows:

$$
\begin{aligned}
X_1 &= e^{Dt}x^{1-p}\left(\sigma^2 x^{2p-1}\frac{\partial}{\partial x} - 2ru\frac{\partial}{\partial u}\right),\\
X_2 &= e^{-Dt}x^{p}\frac{\partial}{\partial x},\\
X_3 &= e^{2Dt}\left[rx\frac{\partial}{\partial x} - \frac{\partial}{\partial t} + \left(D - r - \frac{2r^2x^{2(1-p)}}{\sigma^2}u\frac{\partial}{\partial u}\right)\right],\\
X_4 &= e^{-2Dt}\left(rx\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + ru\frac{\partial}{\partial u}\right),\\
X_5 &= \frac{\partial}{\partial t} + \left(r - \frac{D}{2}\right)u\frac{\partial}{\partial u},\\
X_6 &= u\frac{\partial}{\partial u},\\
X_\phi &= \phi(t,x)\frac{\partial}{\partial u},
\end{aligned}
\tag{4.5}
$$

where $D = (p-1)r$ and $\phi(t,x)$ is any solution of (4.4). The corresponding invariant solutions generated by (4.5) are

$$X_1: \quad u(t,x) = \kappa_1 e^{r\left[(2-p)t - \frac{x^{2(1-p)}}{(1-p)\sigma^2}\right]}, \tag{4.6a}$$

$$X_2: \quad u(t,x) = \kappa_1 \exp\left\{rt - \frac{e^{(p-1)rt}x^{1-p}}{p-1} - \frac{e^{2(p-1)rt}\sigma^2}{4(p-1)r}\right\}, \tag{4.6b}$$

$$X_3: \quad u(t,x) = \exp\left\{r(2(1-p)t) - \frac{x^{2(1-p)}}{(1-p)\sigma^2}\right\}\left(\kappa_1 e^{prt} + \kappa_2 e^{rt}x^{1-p}\right), \tag{4.6c}$$

$$X_4: \quad u(t,x) = \kappa_1 e^{rt} + \kappa_2 e^{prt}x^{1-p}. \tag{4.6d}$$

Note that X_5 in equation (4.5) leads to a reduced second-order ordinary differential equation which Sinkala did not suggest a solution to it.

For $p = 1$, equation (4.4) reduces to

$$\frac{\partial u}{\partial t} + \left(r + \frac{\sigma^2}{2}\right) x \frac{\partial u}{\partial x} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 u}{\partial x^2} - ru = 0, \tag{4.7}$$

which admits infinitesimal symmetries that is spanned by the following operators:

$$\begin{aligned}
X_1 &= x\sigma^2 \frac{\partial}{\partial x} - ru \frac{\partial}{\partial u}, \\
X_2 &= tx\sigma^2 \frac{\partial}{\partial x} + (\ln x - rt) u \frac{\partial}{\partial u}, \\
X_3 &= \frac{\partial}{\partial t} + Nru \frac{\partial}{\partial u}, \\
X_4 &= x \ln x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} + \left(2Nrt - \frac{1}{2} - \frac{r}{\sigma^2} \ln x\right) u \frac{\partial}{\partial u}, \\
X_5 &= tx \ln x \frac{\partial}{\partial x} + t^2 \frac{\partial}{\partial t} + \left[Nrt^2 + \frac{1}{2\sigma^2} (\ln x)^2 - t\left(\frac{1}{2} + \frac{r}{\sigma^2} \ln x\right)\right] u \frac{\partial}{\partial u}, \\
X_6 &= u \frac{\partial}{\partial u}, \qquad X_\phi = \phi(t,x) \frac{\partial}{\partial u},
\end{aligned} \tag{4.8}$$

where $N = 1 + \frac{r}{2\sigma^2}$ and $\phi(t, x)$ is any solution of equation (4.7). The respective invariant solutions generated by (4.8) are

$$X_1 : \quad u(t,x) = \kappa_1 e^{Nrt} x^{-\frac{r}{\sigma^2}}, \tag{4.9a}$$

$$X_2 : \quad u(t,x) = \kappa_1 e^{Nrt} x^{\frac{\ln x}{2t\sigma^2} - \frac{r}{\sigma^2}}, \tag{4.9b}$$

$$X_3 : \quad u(t,x) = e^{Nrt} x^{-\frac{r}{\sigma^2}} (\kappa_1 + \kappa_2 \ln x), \tag{4.9c}$$

$$X_5 : \quad u(t,x) = e^{Nrt} x^{\frac{-2rt + \ln x}{2t\sigma^2}} \left(\frac{\kappa_1}{\sqrt{t}} + \frac{\kappa_2 \ln x}{t^{3/2}}\right). \tag{4.9d}$$

The invariant solution generated by the generator X_4 was not suggested.

Sinkala's work complemented the work done by Bell and Stelljes who proposed a method for constructing explicitly solvable arbitrage-free models for the stock price and reported the challenge to finding such solutions for a general parameter p, which Bell *et al.* had only found two simple solutions for $p = 1$ and $p = 1/2$.

In this study, we extend the arbitrage-free model (4.3) to a time-fractional arbitrage-free stock price model using Lie symmetry analysis. Recall the second order partial differential equation (4.4), the time-fractional version of equation (4.4) becomes

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \left(rx + \frac{p\sigma^2 x^{2p-1}}{2}\right)\frac{\partial u}{\partial x} + \frac{\sigma^2 x^{2p}}{2}\frac{\partial^2 u}{\partial x^2} - ru = 0, \tag{4.10}$$

where $0 < \alpha < 1, p \neq 1$, and $\dfrac{\partial^\alpha u}{\partial t^\alpha}$ is the Riemann-Liouville fractional derivative of u of order α with respect to t . When $p = 1$, equation (4.10) becomes

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \left(r + \frac{\sigma^2}{2}\right)x\frac{\partial u}{\partial x} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 u}{\partial x^2} - ru = 0. \tag{4.11}$$

4.2 Lie symmetries of a time-fractional arbitrage-free stock price model

First, we study the time-fractional arbitrage-free stock price model with $p \neq 1$. Recall the time fractional arbitrage-free stock price model given in equation (4.10)

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \left(rx + \frac{p\sigma^2 x^{2p-1}}{2} \right) \frac{\partial u}{\partial x} + \frac{\sigma^2 x^{2p}}{2} \frac{\partial^2 u}{\partial x^2} - ru = 0,$$

where $0 < \alpha < 1, p \neq 1$, and $\dfrac{\partial^\alpha u}{\partial t^\alpha}$ is defined as Riemann-Liouville fractional derivative of u of order α with respect to t. The invariance condition of equation (4.10) is

$$\begin{aligned}\eta^{\alpha,t} + \left(rx + \frac{1}{2} p\sigma^2 x^{2p-1} \right) \eta^x + \left(r\xi + \frac{1}{2}(2p-1) p\sigma^2 x^{2p-2}\xi \right) u_x \\ + \frac{1}{2}\sigma^2 x^{2p} \eta^{xx} + p\sigma^2 x^{2p-1} \xi u_{xx} - r\eta = 0.\end{aligned} \tag{4.12}$$

The prolongations η^x, η^{xx}, and $\eta^{\alpha,t}$, which are listed in equations (2.22a, 2.22c, 2.72), are substituted into equation (4.12). The coefficients of u_t, u_x, u_{xx}, $u_{xt}, u_t u_{xx}, u_x u_{xx}, u_x^2$, $\partial_t^{\alpha-n}$ and $\partial_t^{\alpha-n}$ are then set to zero to obtain the following non-trivial system of determining equations:

$$\eta_{uu} = \xi_t = \xi_u = \tau_x = \tau_u = 0, \tag{4.13a}$$

$$-\tau_t\alpha - 2\xi x^{-1}p + 2\xi_x = 0, \tag{4.13b}$$

$$\tau_{tt}(1-\alpha) + 2\eta_{ut} = 0, \tag{4.13c}$$

$$(\alpha\tau_t - \xi_x)\left(rx + \frac{1}{2}p\sigma^2 x^{2p-1}\right) +$$
$$\left(r + \frac{(2p-1)p}{2}\sigma^2 x^{2p-2}\right)\xi + \frac{\sigma^2 x^{2p}}{2}(2\eta_{xu} - \xi_{xx}) = 0. \tag{4.13d}$$

From equation (4.13a), we know

$$\eta(t,x) = uA(t,x) + B(t,x), \quad \xi = \xi(x), \quad \text{and} \quad \tau = \tau(t).$$

Differentiating equation (4.13b) with respect to t will eliminate ξ and ξ_x because $\xi = \xi(x)$. This will lead to $\tau_{tt} = 0$ and hence,

$$\tau = c_1 t. \tag{4.14}$$

Here, the constant of integration is dropped to preserve the structure of the Riemann-Liouville fractional derivative. Equation (4.14) then lends to the solution of ξ:

$$\xi = c_1\frac{\alpha}{2(1-p)}x + c_2 x^p. \tag{4.15}$$

Finally, substitute τ, ξ and their derivatives into equation (4.13d) to obtain

$$\eta = \left(c_1\frac{N\alpha}{2(p-1)}x^{2-2p} - c_2 M x^{1-p} + c_3\right)u + B(t,x). \tag{4.16}$$

Here c_1, c_2, c_3 are arbitrary constant, $N = r/\sigma^2$ and $B(t,x)$ is any solution to equation (4.10). The arbitrary constants in η, ξ, and τ make up an infinite dimensional Lie algebra of symmetries as below

$$\begin{aligned}
X_1 &= t\frac{\partial}{\partial t} + \frac{\alpha}{2(1-p)}x\frac{\partial}{\partial x} + \left(\frac{\alpha N}{2(p-1)}x^{2-2p}\right)u\frac{\partial}{\partial u},\\
X_2 &= x^p\frac{\partial}{\partial x} - Nx^{1-p}u\frac{\partial}{\partial u},\\
X_3 &= u\frac{\partial}{\partial u},\\
X_\infty &= B(t,x)\frac{\partial}{\partial u}.
\end{aligned} \tag{4.17}$$

For the case when $p = 1$, equation (4.11)

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \left(r + \frac{\sigma^2}{2}\right)x\frac{\partial u}{\partial x} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 u}{\partial x^2} - ru = 0$$

gives the following invariance condition

$$\eta^{\alpha,t} + \left(r + \frac{\sigma^2}{2}\right)x\eta^x + \left(r\xi + \frac{\sigma^2\xi}{2}\right)u_x + \frac{\sigma^2 x^2}{2}\eta^{xx} + \sigma^2 x\xi_{xx} - r\eta = 0. \tag{4.18}$$

Similar to the above case, the following system of determining equations is obtained:

$$\eta_{uu} = \xi_t = \xi_u = \tau_x = \tau_u = 0, \tag{4.19a}$$

$$-\alpha x \tau_t + 2x\xi_x - 2\xi = 0, \tag{4.19b}$$

$$\tau_{tt}(1-\alpha) + 2\eta_{ut} = 0, \tag{4.19c}$$

$$\begin{aligned} &\left(r + \frac{\sigma^2}{2}\right)\xi + \left(r + \frac{\sigma^2}{2}\right)\alpha x \tau_t \\ -&\left(r + \frac{\sigma^2}{2}\right) x\xi_x + \sigma^2 x^2 \eta_{ux} - \frac{1}{2}\sigma^2 x^2 \xi_{xx} = 0. \end{aligned} \tag{4.19d}$$

Equation (4.19a) provides a similar condition of the function $\tau(t), \xi(x)$, and $\eta(t, x)$ as in the previous case. Differentiating equation (4.19b) with respect to t gives us

$$\tau = c_1 t. \tag{4.20}$$

Equation (4.20) then leads to the solution of ξ by using equation (4.19c):

$$\xi = c_1 \frac{1}{2} \alpha x \ln x + c_2 x. \tag{4.21}$$

Finally, equation (4.19d) gives

$$\eta = \left(-c_1 \frac{\alpha}{2} N \ln x + c_3\right) u + C(t, x), \tag{4.22}$$

where $N = r/\sigma^2$ and $C(t, x)$ is any solution to equation (4.11). The admitted infinite dimensional vector space of infinitesimal symmetries is spanned by the following operators:

$$
\begin{aligned}
X_1 &= t\frac{\partial}{\partial t} + \frac{\alpha}{2}x\ln x\frac{\partial}{\partial x} - \left(\frac{\alpha}{2}N\ln x\right)u\frac{\partial}{\partial u},\\
X_2 &= x\frac{\partial}{\partial x},\\
X_3 &= u\frac{\partial}{\partial u},\\
X_\infty &= C(t,x)\frac{\partial}{\partial u}.
\end{aligned}
\tag{4.23}
$$

The infinitesimal generators in equations (4.17) and (4.23) give the one-parameter groups of symmetries by solving Lie equations (2.7). The results corresponding to the infinitesimal generators (4.17) and (4.23) are presented in (4.24) and (4.25) respectively.

$$
\begin{aligned}
&X_1 : \bar{u} = ue^{\left\{\frac{Nx^{2-2p}}{p-1}\left(e^{\frac{\alpha\varepsilon}{2}}-1\right)\right\}}, && \bar{t} = e^{\varepsilon}t, && \bar{x} = e^{\frac{\alpha\varepsilon}{2(1-p)}}x\\
&X_2 : \bar{u} = ue^{\left\{-N\left(\frac{(1-p)\varepsilon^2}{2}+x^{1-p}\varepsilon\right)\right\}}, && \bar{t} = t, && \bar{x} = [(1-p)\varepsilon + x^{1-p}]^{\frac{1}{1-p}}\\
&X_3 : \bar{u} = ue^{\varepsilon}, && \bar{t} = t, && \bar{x} = x\\
&X_\infty : \bar{u} = u + B(t,x)\varepsilon, && \bar{t} = t, && \bar{x} = x
\end{aligned}
\tag{4.24}
$$

$$
\begin{aligned}
&X_1 : \bar{u} = ux^{N\left(1-e^{\frac{\alpha\varepsilon}{2}}\right)}, && \bar{t} = e^{\varepsilon}t, && \bar{x} = x^{e^{\frac{\alpha\varepsilon}{2}}}\\
&X_2 : \bar{u} = u, && \bar{t} = t, && \bar{x} = e^{\varepsilon}x\\
&X_3 : \bar{u} = ue^{\varepsilon}, && \bar{t} = t, && \bar{x} = x\\
&X_\infty : \bar{u} = u + B(t,x)\varepsilon, && \bar{t} = t, && \bar{x} = x
\end{aligned}
\tag{4.25}
$$

4.3 Optimal systems and group invariant solutions

Any linear combination of the generators found in the previous section may construct different group invariant solutions. To minimize the set of reductions that are not equivalent by any transformation, we construct the optimal systems [71] for each of the cases when $p \neq 1$ and $p = 1$ respectively by using infinitesimal generators (4.17) and (4.23). First, the commutators of the admitted symmetries are constructed.

Recall the commutators of operators X_i and X_j as defined in (2.26), $[X_i, X_j] = X_iX_j - X_jX_i$. For instance,

$$
\begin{aligned}
[X_1, X_2] =& X_1X_2 - X_2X_1 \\
=& \left(t\frac{\partial}{\partial t} + \frac{\alpha}{2(1-p)}x\frac{\partial}{\partial x} + \frac{\alpha N x^{2-2p}}{2(p-1)}u\frac{\partial}{\partial u}\right)\left(x^p\frac{\partial}{\partial x} - Nx^{1-p}u\frac{\partial}{\partial u}\right) \\
&- \left(x^p\frac{\partial}{\partial x} - Nx^{1-p}u\frac{\partial}{\partial u}\right)\left(t\frac{\partial}{\partial t} + \frac{\alpha}{2(1-p)}x\frac{\partial}{\partial x} + \frac{\alpha N x^{2-2p}}{2(p-1)}u\frac{\partial}{\partial u}\right) \\
=& \frac{\alpha}{2(1-p)}x\left[px^{p-1}\frac{\partial}{\partial x} - N(1-p)x^{-p}u\frac{\partial}{\partial u}\right] - \frac{\alpha N x^{2-2p}}{2(p-1)}u\left[Nx^{1-p}\frac{\partial}{\partial u}\right] \\
&- x^p\left[\frac{\alpha}{2(1-p)}\frac{\partial}{\partial x} - \alpha N x^{1-2p}u\frac{\partial}{\partial u}\right] + Nx^{1-p}u\left[\frac{\alpha N x^{2-2p}}{2(p-1)}\frac{\partial}{\partial u}\right] \\
=& \frac{\alpha}{2(1-p)}px^p\frac{\partial}{\partial x} - \frac{\alpha}{2}Nx^{1-p}u\frac{\partial}{\partial u} - \frac{\alpha N^2 x^{3(1-p)}}{2(p-1)}u\frac{\partial}{\partial u} - \frac{\alpha}{2(1-p)}x^p\frac{\partial}{\partial x} \\
&+ \alpha N x^{1-p}u\frac{\partial}{\partial u} + \frac{\alpha N^2 x^{3(1-p)}}{2(p-1)}u\frac{\partial}{\partial u} \\
=& -\frac{\alpha}{2}x^p\frac{\partial}{\partial x} + \frac{\alpha}{2}Nx^{1-p}u\frac{\partial}{\partial u} = -\frac{\alpha}{2}\left(x^p\frac{\partial}{\partial x} - Nx^{1-p}u\frac{\partial}{\partial u}\right) = -\frac{\alpha}{2}X_2
\end{aligned}
$$

Using the anticommutativity property of a commutator,

$$[X_2, X_1] = \frac{\alpha}{2} X_2.$$

Similarly, we obtained the Lie brackets $[X_1, X_3] = [X_2, X_3] = 0$. The Lie brackets (or commutators) admitted when $p \neq 1$ and $p = 1$ are listed in Table 4.1 and Table 4.2 respectively.

Table 4.1: Lie bracket of the admitted system algebra for the case $p \neq 1$.

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	$-\frac{\alpha}{2}X_2$	0
X_2	$\frac{\alpha}{2}X_2$	0	0
X_3	0	0	0

Table 4.2: Lie bracket of the admitted system algebra for the case $p = 1$.

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	$-\frac{\alpha}{2}X_2 + \frac{N\alpha}{2}X_3$	0
X_2	$\frac{\alpha}{2}X_2 - \frac{N\alpha}{2}X_3$	0	0
X_3	0	0	0

Next, we need to construct the adjoint representation by using the formula

$$\mathrm{Ad}(\exp(\varepsilon X_i))X_j = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(\mathrm{ad}X_i)^n X_j = X_j - \varepsilon[X_i, X_j] + \frac{\varepsilon^2}{2!}[X_i, [X_i, X_j]] - \cdots .$$

To determine $Ad(\exp(\varepsilon X_1))X_2$,

$$\begin{aligned}
Ad(\exp(\varepsilon X_1))X_2 =& X_2 - \varepsilon[X_1, X_2] + \frac{\varepsilon^2}{2}[X_1,[X_1,X_2]] + \cdots \\
=& X_2 - \varepsilon\left[-\frac{\alpha}{2}X_2\right] + \frac{\varepsilon^2}{2}[X_1,[-\frac{\alpha}{2}X_2]] + \cdots \\
=& X_2 - \varepsilon\left[-\frac{\alpha}{2}X_2\right] - \frac{\alpha\varepsilon^2}{4}[X_1, X_2] + \cdots \\
=& X_2 + \frac{\alpha\varepsilon}{2}X_2 + \frac{\alpha^2\varepsilon^2}{8}X_2 + \cdots \\
=& \left(1 + \frac{\alpha\varepsilon}{2} + \frac{\alpha^2\varepsilon^2}{8} + \cdots\right)X_2 \\
=& \exp\left\{\frac{\alpha\varepsilon}{2}\right\}X_2
\end{aligned}$$

The adjoint representation of other X_i's for the case when $p \neq 1$ and $p = 1$ are computed similarly and listed in Table 4.3 and Table 4.4.

Table 4.3: Adjoint representation of subalgebra for the case $p \neq 1$.

Ad	X_1	X_2	X_3
X_1	X_1	$\exp(\frac{\varepsilon\alpha}{2})X_2$	X_3
X_2	$X_1 - \frac{\varepsilon\alpha}{2}X_2$	X_2	X_3
X_3	X_1	X_2	X_3

Table 4.4: Adjoint representation of subalgebra for the case $p = 1$.

Ad	X_1	X_2	X_3
X_1	X_1	$\exp(\frac{\varepsilon\alpha}{2})X_2 + N(1 - \exp(\frac{\varepsilon\alpha}{2}))X_3$	X_3
X_2	$X_1 - \frac{\varepsilon\alpha}{2}X_2 + N(\frac{\varepsilon\alpha}{2})X_3$	X_2	X_3
X_3	X_1	X_2	X_3

For the case when $p \neq 1$, consider the linear combination of the symmetry generators:

$$X = a_1X_1 + a_2X_2 + a_3X_3.$$

Suppose first that $a_1 \neq 0$, then we can rescale it to 1. If we act on such an X by $\exp(\frac{2a_2}{\alpha})X_2$, we can make the coefficient of X_2 disappears:

$$X' = \mathrm{Ad}\left(\exp\left(\frac{2a_2}{\alpha}X_2\right)\right)X = X_1 + a_3X_3.$$

No more simplification is possible. Thus the one-dimensional subalgebra spanned by X with $a_1 \neq 0$ is equivalent to the one spanned by $X_1 + \rho X_3$, where ρ is an arbitrary constant. The remaining one-dimensional subalgebras are spanned by the vectors of the above form with $a_1 = 0$. Suppose $a_2 \neq 0$ and setting $a_2 = 1$, acting on X by the group generated by X_1:

$$X'' = \mathrm{Ad}(\exp(\varepsilon X_1))X = \exp\left(\frac{\varepsilon\alpha}{2}\right)X_2 + a_3X_3.$$

This is a scalar multiple of $X''' = X_2 + a_3\exp\left(-\frac{\varepsilon\alpha}{2}\right)X_3$ which depends on the sign of a_3. We can make the coefficient of X_3 either $+1$, -1, or 0. In other words, any one-dimensional subalgebra spanned by X with $a_1 = 0, a_2 \neq 0$ is equivalent to one spanned by either $X_2 + X_3, X_2 - X_3$ or X_2. Finally, setting $a_1 = a_2 = 0$ and $a_3 = 1$, we have only X_3. Hence the set of one-dimensional optimal systems for the case when $p \neq 1$ is

$$\Big\{X_1 + \rho X_3, X_2 + X_3, X_2 - X_3, X_2, X_3\Big\}.$$

Similarly, one can check that the optimal system for the case when $p = 1$ is

$$\Big\{X_1 + \rho X_3, X_2 + \rho X_3, X_3\Big\}.$$

4.3.1 Group invariant solution of time-fractional arbitrage-free stock price model

As discussed in Chapter 2, the method of characteristics (2.11) is applied to the infinitesimal generators (or the optimal system) to obtain the group invariant solutions. Finding the group invariant solutions of fractional differential equations is equivalent to finding the invariant solutions corresponding to its infinitesimal operator while keeping the invariant surface condition

$$\tau(u,t,x)u_t + \xi(u,t,x)u_x = \eta(u,t,x)$$

satisfied. The results from solving the characteristic equations are then combined and substituted in the original FDE. This will reduce the FDE to an "easier" differential equation which will lead to the completion of the whole process.

Invariant solution for the case $p = 1$

For the case when $p = 1$, let's consider the generator

$$X_2 + \rho X_3 = x\frac{\partial}{\partial x} + \rho u\frac{\partial}{\partial u}.$$

The characteristic equations of the generators are

$$\frac{dx}{x} = \frac{du}{\rho u}, \tag{4.26}$$

which give the similarity variables t and $ux^{-\rho}$. The solution of equation (4.11) has the form $u = x^{\rho}H(t)$, where $H(t)$ is a function in t, with its derivatives can be written as $u_x = \rho x^{p-1}H(t)$ and $u_{xx} = \rho(\rho-1)x^{\rho-2}H(t)$. Substitute the derivatives into equation (4.11) gives the equation

$$\partial_t^{\alpha}H(t) = -\left(\frac{1}{2}\rho^2\sigma^2 + r(\rho-1)\right)H(t).$$

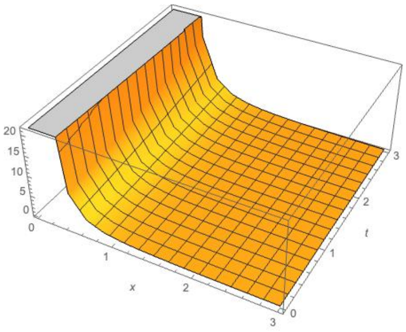
Finally, the invariant solution of equation (4.11) is obtained as

$$u = \lambda_1 x^{\rho} t^{\alpha-1} E_{\alpha,\alpha}\left(-\left(\frac{1}{2}\rho^2\sigma^2 + r(\rho-1)\right)t^{\alpha}\right) \tag{4.27}$$

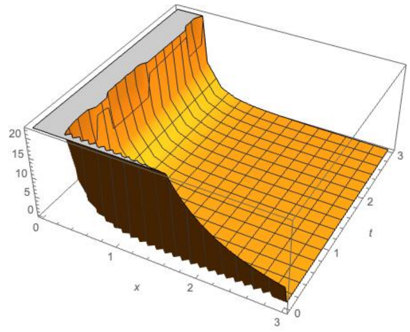
where $\lambda_1 = D^{-(1-\alpha)}H(0)$.

The invariant solutions for the case when $p \neq 1$ are not found. Equation (4.10) fails to reduce to a solvable fractional differential equation after the simplification. Similarly, the invariant solution for the case when $p = 1$ corresponds to the generator $X_1 + \rho X_3$ is left unsolved. The generator $u\frac{\partial}{\partial u}$ has no invariant solution.

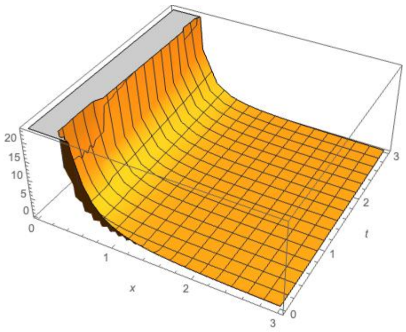
4.4 Conclusion



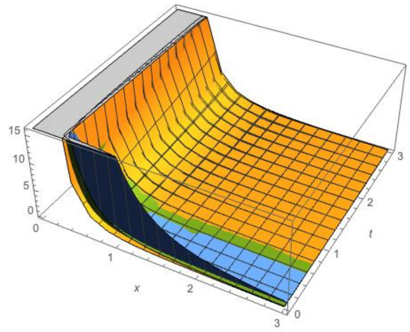
(a) Equation (4.28)



(b) Equation (4.27),$\alpha = 0.7$



(c) Equation (4.27),$\alpha = 0.9$



(d) Comparison of equation (4.28) and equation (4.27) with different α

Figure 4.1: $p = 1$

Sinkala [86], in his work on the integer version of the model, gave the respective invariant solutions of the model as

$$u = c_1 x^{-N} \exp\left\{ \left(1 + \frac{N}{2}\right) rt \right\} \tag{4.28}$$

for the case when $p = 1$, where c_1 is an arbitrary constant and $N = r/\sigma^2$. Having both the solutions of the model in fractional and non-fractional versions allows us to examine the differences and similarities of the models in detail. Generally, most researches show that when the value of α approaches 1, the fractional version tends to be similar to the integer version. Using $\lambda_1 = \kappa_1 = c_1 = 1, r = 0.1, \sigma = 0.2, \rho = -2.5$ and different values of α, the comparison of equation (4.27) and (4.28) are shown in Figures 4.1. Figure 4.1(d) shows the combined comparison of figures 4.1(a), 4.1(b) and 4.1(c).

The figures show that when the values of α get closer to 1, the respective surfaces get closer to Sinkala's solutions. The disparity of the solutions of integer differential equations (Figure 4.1(a)) and the solutions of fractional differential equations (Figure 4.1(b) and Figure 4.1(c)) is due to the presence of memory in financial agents. The concept of derivative of non-integer order is widely used to describe the processes with memory in natural sciences and in financial processes recently [57, 93]. The memory allows that at repeated changes the agents can react to these changes in a different way than they did before. Our figures show that when the order of the fractional derivative equations, α, is low (far from 1) and when the time is short, the disparity is obvious. The effect of memory is noticeable. When the value of α increases and approaches 1, the disparity seems to be offset.

The effect of fractional calculus in computational finance is undoubtedly significant. Before an interpretation of the rule of fractional calculus in computational finance is commonly acknowledged, looking into the differences and similarities of the existing models and their fractional versions is indispensable. The figures above show that the differences between a time-fractional model and an integer model are noticeably huge when the value of α is away from

1 and when the time is short. When the time fractional model unfolds, as the values of α and time increase, it convergences to the corresponding integer model.

Chapter 5

Power Options under the Heston dynamic

5.1 Introduction

The Black-Scholes-Merton model introduced by Black and Scholes [12], and Merton [61] undoubtedly is the most celebrated model for option pricing since its introduction in 1973. However, the assumptions made by this model, which include constant volatility, make this model too idealistic. In reality, most financial assets consider stochastic volatility. The extension of the Black-Scholes-Merton model to stochastic volatility triggered the blooming of various stochastic volatility models. Among these models, Heston's [35] is the most recognized one with stochastic interest rates being introduced in pricing a European call option on an asset with stochastic volatility. This model allows an arbitrary correlation between volatility and spot-asset returns.

Ibrahim *et al.* [43] later evaluated the power option prices based on the Heston dynamics, assuming the asset price is to follow the log-normal process governed by a single Brownian motion. A second Brownian motion drives the volatility process. The two processes, the asset price process and the volatility process, are correlated by a constant correlation coefficient. Assuming the market is complete [23], a partial differential equation representing the portfolio is derived.

In a probability space (Ω, F, Q) on two Brownian motions $W_{1,t}$ and $W_{2,t}$ for $t > 0$ where $F_t, 0 \leq t \leq T$, is the filtration caused by the Brownian motions with T being the maturity and Q is a risk-neutral probability, the asset price S_t is governed by the model

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{V_t} S_t dW_{1,t} \\ dV_t &= \kappa^*(\theta^* - V_t)dt + \sigma\sqrt{V_t} dW_{2,t} \\ \langle dW_{1,t}, dW_{2,t} \rangle &= \rho dt, \end{aligned} \tag{5.1}$$

which is a geometric Brownian motion with volatility that follows a stochastic process where the variance V_t follows a square-root mean reverting process [20], r is the risk-free rate, κ^* is the mean reversion speed, θ^* is the average level of volatility, σ is the volatility of volatility, and ρ is the correlation coefficient between the two Brownian motions. Ibrahim *et al.* [43] proposed a model to adapt the stock price for a power option as

$$\begin{aligned} dS_t^\beta &= \left(\beta r + \frac{1}{2}V_t\beta(\beta-1)\right)S_t^\beta dt + \beta\sqrt{V_t}S_t^\beta dW_{1,t} \\ dV_t &= \kappa^*(\theta^* - \beta^2 V_t)dt + \sigma\beta\sqrt{V_t}dW_{2,t}, \end{aligned} \tag{5.2}$$

where β is a positive constant. Inspired by Gatheral [27] , Ibrahim *et al.* derived a partial differential equation based on equation (5.2):

$$\begin{aligned} \frac{\partial u}{\partial t} + \left(r - \frac{1}{2}\beta^2 y\right)\frac{\partial u}{\partial x} + \frac{1}{2}\beta^2 y\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\sigma^2\beta^2 y\frac{\partial^2 u}{\partial y^2} \\ +\rho\sigma\beta^2 y\frac{\partial^2 u}{\partial x\partial y} + \kappa(\theta - \beta^2 y)\frac{\partial u}{\partial y} - ru = 0, \end{aligned} \tag{5.3}$$

where $u = u(t,x,y)$ is the value of the underlying power option with $x = \ln S_t^\beta$ and $y = V_t$ for simplicity. Here $\kappa = \kappa^* + \lambda$ and $\theta = \dfrac{\kappa^*\theta^*}{\kappa^* + \lambda}$ are to eliminate λy, the volatility risk premium. In this study, Lie symmetry analysis is carried out on equation (5.3). The infinitesimal generators, the symmetry groups, the optimal systems, the invariant solutions, and the conservation laws emitted by equation (5.3) are presented.

5.2 Lie symmetry analysis of a (2+1) - dimensional partial differential equation

The Lie symmetry analysis of a (2+1)-dimensional partial differential equation is similar to the Lie symmetry analysis discussed in section 2.2, with more tedious and complicated terms. Consider a (2+1) partial differential equation of second order of the form

$$F(t, x, y, u_t, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0, \tag{5.4}$$

where $u_t = \dfrac{\partial u}{\partial t}, u_x = \dfrac{\partial u}{\partial x}, \ldots, u_{yy} = \dfrac{\partial^2 u}{\partial y^2}$. The transformations of the points u, t, x and y in the forms of

$$\begin{aligned}
\bar{u} &= u + \varepsilon\eta(u, t, x, y) + O(\varepsilon^2), \\
\bar{t} &= t + \varepsilon\tau(u, t, x, y) + O(\varepsilon^2), \\
\bar{x} &= x + \varepsilon\xi(u, t, x, y) + O(\varepsilon^2), \\
\bar{y} &= y + \varepsilon\varphi(u, t, x, y) + O(\varepsilon^2),
\end{aligned} \tag{5.5}$$

with

$$\eta = \frac{d\bar{x}}{d\varepsilon}\Big|_{\varepsilon=0}, \tau = \frac{d\bar{t}}{d\varepsilon}\Big|_{\varepsilon=0}, \xi = \frac{d\bar{x}}{d\varepsilon}\Big|_{\varepsilon=0}, \varphi = \frac{d\bar{y}}{d\varepsilon}\Big|_{\varepsilon=0},$$

and ε being the infinitesimal parameter, are the symmetry transformations of equation (5.4) if the transformation points (5.5) satisfy equation (5.4). The collection of all such possible symmetry transformations, says the set G, exhibits the characteristics of a continuous group which include the existence of identity element, inverse and the composition of any two elements in the same group.

The infinitesimal operators of the group G are presented as

$$X = \eta(u, t, x, y)\frac{\partial}{\partial u} + \tau(u, t, x, y)\frac{\partial}{\partial t} + \xi(u, t, x, y)\frac{\partial}{\partial x} + \varphi(u, t, x, y)\frac{\partial}{\partial y}. \tag{5.6}$$

which when prolonged to derivatives by adding all terms of $\eta^J \partial_{u_J}$ up to the necessary order give

$$X^{(2)} = X + \eta^t \partial_{u_t} + \eta^x \partial_{u_x} + \eta^y \partial_{u_y} + \eta^{xx} \partial_{u_{xx}} + \eta^{yy} \partial_{u_{yy}} + \eta^{xy} \partial_{u_{xy}}. \tag{5.7}$$

In accordance with [13] η^J is expressed as

$$\begin{aligned}
\eta^x &= \eta_x + (\eta_u - \xi_x)u_x - u_t\tau_x - u_x u_t \tau_u - u_x^2 \xi_u - u_y \varphi_x - u_x u_y \varphi_u, &(5.8a)\\
\eta^t &= \eta_t + u_t\eta_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_x u_t \xi_u - u_y\varphi_t - u_t u_y \varphi_u, &(5.8b)\\
\eta^y &= \eta_y + u_y\eta_u - u_t\tau_y - u_t u_y \tau_u - u_x\xi_y - u_x u_y \xi_u - u_y\varphi_y - u_y^2\varphi_u, &(5.8c)\\
\eta^{xx} &= \eta_{xx} + u_x(2\eta_{xu} - \xi_{xx}) + u_x^2(\eta_{uu} - 2\xi_{xu}) + u_{xx}(\eta_u - 2\xi_x) - u_t\tau_{xx} \\
&\quad - u_x u_t 2\tau_{xu} - u_{xt}2\tau_x - u_x^2 u_t \tau_{uu} - u_t u_{xx}\tau_u - u_x u_{xt} 2\tau_u - u_x^3\xi_{uu} \\
&\quad - u_x u_{xx} 3\xi_u - u_y\varphi_{xx} - u_x u_y 2\varphi_{xu} - u_{xy}2\varphi_x - u_x^2 u_y \varphi_{uu} \\
&\quad - u_y u_{xx}\varphi_u - u_x u_{xy} 2\varphi_u, &(5.8d)\\
\eta^{xy} &= \eta_{xy} + u_x(\eta_{yu} - \xi_{xy}) + u_y\eta_{xu} + u_x u_y(\eta_{uu} - \xi_{xu} - \varphi_{yu}) \\
&\quad + u_{xy}(\eta_u - \xi_x - \varphi_y) - u_t\tau_{xy} - u_x u_t \tau_{yu} - u_{xt}\tau_y - u_t u_y \tau_{xu} \\
&\quad - u_x u_t u_y \tau_{uu} - u_t u_{xy}\tau_u - u_y u_{xt}\tau_u - u_{ty}\tau_x - u_x u_{ty}\tau_u - u_x^2\xi_{yu} \\
&\quad - u_{xx}\xi_y - u_x^2 u_y \xi_{uu} - u_y u_{xx}\xi_u - u_x u_{xy} 2\xi_u - u_y u_{xy} 2\varphi_u - u_y\varphi_{xy} \\
&\quad - u_y^2\varphi_{xu} - u_x u_y^2 \varphi_{uu} - u_{yy}\varphi_x - u_x u_{yy}\varphi_u, &(5.8e)\\
\eta^{yy} &= \eta_{yy} + u_y(2\eta_{yu} - \varphi_{yy}) + u_y^2(\eta_{uu} - 2\varphi_{yu}) + u_{yy}(\eta_u - 2\varphi_y) - u_t\tau_{yy} \\
&\quad - u_t u_y 2\tau_{yu} - u_{ty}2\tau_y - u_t u_y^2 \tau_{uu} - u_t u_{yy}\tau_u - u_y u_{ty} 2\tau_u - u_x\xi_{yy} \\
&\quad - u_x u_y 2\xi_{yu} - u_{xy}2\xi_y - u_x u_y^2 \xi_{uu} - u_x u_{yy}\xi_u - u_y u_{xy} 2\xi_u - u_y^3\varphi_{uu} \\
&\quad - u_y u_{yy} 3\varphi u. &(5.8f)
\end{aligned}$$

The determining equations of equation (5.3) can be found from the invariance condition

$$X^{(2)}|_{\text{equation(5.3)}} = 0.$$

The calculation to solve the invariance condition is lengthy and convoluted if it is done manually. We use the software package MathLie [9] to simplify the work. The result from the calculation gives the following infinitesimal operators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad \text{and} \quad X_3 = u\frac{\partial}{\partial u}. \tag{5.9}$$

For simplicity, the infinite generator X_∞ is dropped. The corresponding transformations of the generators (5.9) are translation along the x-axis

$$\bar{t} = t, \quad \bar{x} = x + \varepsilon, \quad \bar{y} = y, \quad \bar{u} = u,$$

translation along the t-axis

$$\bar{t} = t + \varepsilon, \quad \bar{x} = x, \quad \bar{y} = y, \quad \bar{u} = u,$$

and scaling along the u-axis

$$\bar{t} = t, \quad \bar{x} = x, \quad \bar{y} = y, \quad \bar{u} = ue^{\varepsilon}.$$

5.3 Commutators, adjoint representations and optimal system

An optimal system of generators is a set consisting of exactly one generator from each class of associated symmetry generators to obtain the invariant solutions. This classification is essential for the equity of the generators caused by the characteristics of the symmetry group. Olver [71] provided the commutator or the Lie bracket (2.26) as

$$[X_i, X_j] = X_i X_j - X_j X_i,$$

which is skew-symmetric, $[X_i, X_j] = -[X_j, X_i]$, and $[X_i, X_i] = 0$. Using the operators (5.9),

$$[X_1, X_2] = X_1 X_2 - X_2 X_1 = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\right) - \frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\right) = 0.$$

Similarly, we found that $[X_i, X_j] = 0$ for all i and j, as shown in Table 5.1.

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	0
X_2	0	0	0
X_3	0	0	0

Table 5.1: Commutators table

Hence, using the adjoint representations (2.28)

$$Ad(\exp(\varepsilon X_i))X_j = X_j - \varepsilon[X_i, X_j] + \frac{\varepsilon^2}{2!}[X_i, [X_i, X_j]] - \ldots,$$

the adjoint representations of the corresponding commutators are listed in Table 5.2.

The optimal system is constructed by simplifying the coefficients of the general infinitesimal generator

$$X = aX_1 + bX_2 + cX_3$$

Adj	X_1	X_2	X_3
X_1	X_1	X_2	X_3
X_2	X_1	X_2	X_3
X_3	X_1	X_2	X_3

Table 5.2: Adjoint representation table

as much as possible using the adjoint representation table. First, assuming the constant $c \neq 0, c = 1$, the generator now is in the form of $X = aX_1 + bX_2 + X_3$. There is no further simplification that can be done base on Table 5.2. Assuming $c = 0, b \neq 0, b = 1$, we have $X = aX_1 + X_2$ which can be further simplified no more. Finally, letting $b = c = 0, a = 1$, we have $X = X_1$. Hence, the optimal system is presented as

$$\{X_1, aX_1 + X_2, aX_1 + bX_2 + X_3\}. \tag{5.10}$$

For details work to obtain commutators, adjoint representations, and the optimal system, one may refer to [71].

5.4 Invariant solutions

In this section, the similarity reductions are used to obtain the invariant solutions of the optimal system found in section 5.3. The invariant solutions can be found by solving the invariant surface condition

$$\eta = \tau u_t + \xi u_x + \varphi u_y \tag{5.11}$$

through the characteristic method

$$\frac{du}{\eta} = \frac{dt}{\tau} = \frac{dx}{\xi} = \frac{dy}{\varphi}. \tag{5.12}$$

Recall the infinitesimal operators (5.9) obtained in section 5.2:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad \text{and} \quad X_3 = u\frac{\partial}{\partial u}.$$

Let's consider the possible cases from the optimal system in section 5.3.

5.4.1 Case 1: X_1

The generator $X_1 = \dfrac{\partial}{\partial x}$ makes the characteristic equation of $dx = 0$. Introducing the similarity variables $t = \tilde{t}$ and $y = \tilde{y}$, the solution of equation (5.3) is written as $u = F(\tilde{t}, \tilde{y})$, which gives the derivatives

$$u_t = F_{\tilde{t}}, \quad u_x = u_{xx} = u_{xy} = 0, \quad u_y = F_{\tilde{y}}, \quad u_{yy} = F_{\tilde{y}\tilde{y}}. \tag{5.13}$$

Equations (5.13) reduce equation (5.3) to

$$F_{\tilde{t}} + \frac{1}{2}\sigma^2\beta^2\tilde{y}F_{\tilde{y}\tilde{y}} + [\kappa(\theta - \beta^2\tilde{y})]F_{\tilde{y}} - rF = 0. \tag{5.14}$$

The reduced equation (5.14) is still complicated. To reduce it to a simpler differential equation, we run the package MathLie on it one more time, resulting in the infinitesimal generators below:

$$
\begin{aligned}
\Gamma_1 &= e^{\beta^2\kappa\tilde{t}}\tilde{y}\frac{\partial}{\partial\tilde{y}} + \frac{e^{\beta^2\kappa\tilde{t}}}{\beta^2\kappa}\frac{\partial}{\partial\tilde{t}} + e^{\beta^2\kappa\tilde{t}}\left(\frac{2\beta^2\kappa^2\tilde{y} - 2\theta\kappa^2 + r\sigma^2}{\beta^2\kappa\sigma^2}\right)F\frac{\partial}{\partial F},\\
\Gamma_2 &= e^{-\beta^2\kappa\tilde{t}}\tilde{y}\frac{\partial}{\partial\tilde{y}} - \frac{e^{-\beta^2\kappa\tilde{t}}}{\beta^2\kappa}\frac{\partial}{\partial\tilde{t}} - \frac{e^{-\beta^2\kappa\tilde{t}}r}{\beta^2\kappa}F\frac{\partial}{\partial F},\\
\Gamma_3 &= \frac{\partial}{\partial\tilde{t}},\\
\Gamma_4 &= F\frac{\partial}{\partial F}.
\end{aligned}
\tag{5.15}
$$

To serve the purpose to simplify equation (5.14), we consider the combined generator

$$\Gamma_3 + \Gamma_4 = \frac{\partial}{\partial\tilde{t}} + F\frac{\partial}{\partial F}$$

which gives the characteristics equation

$$\frac{d\tilde{t}}{1} = \frac{dF}{F}. \tag{5.16}$$

Equation (5.16) gives the similarity variables $\tilde{y} = \hat{y}$ and $F(\tilde{t}, \tilde{y}) = e^{\tilde{t}}G(\hat{y})$, which derivatives are given by

$$F_{\tilde{t}} = e^{\tilde{t}}G(\hat{y}), \quad F_{\tilde{y}} = e^{\tilde{t}}G'(\hat{y}), \quad F_{\tilde{y}\tilde{y}} = e^{\tilde{t}}G''(\hat{y}). \tag{5.17}$$

Substitute equations (5.17) into equation (5.14) will reduce it to a second degree differential equation

$$\frac{1}{2}\sigma^2\beta^2\hat{y}G''(\hat{y}) + [\kappa(\theta - \beta^2\hat{y})]G'(\hat{y}) + (1-r)G(\hat{y}) = 0, \tag{5.18}$$

which has a solution

$$G(\hat{y}) = c_1 \hat{y}^{\mathcal{A}} U\left(\mathcal{B}, 1+\mathcal{A}, \mathcal{C}\hat{y}\right) + c_2 \hat{y}^{\mathcal{A}} L^{\mathcal{A}}_{-\mathcal{B}}(\mathcal{C}\hat{y}), \tag{5.19}$$

where c_i are arbitrary constants, $U(a, b, x)$ is confluent hypergeometric function of second kind, $L^{\alpha}_{n}(x)$ is the associated Laguerre polynomial L which are described in more depth in reference [1], $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are constants defined as

$$\mathcal{A} = 1 - \frac{\mathcal{C}\theta}{\beta^2}, \quad \mathcal{B} = \mathcal{A} + \frac{r-1}{\beta^2 \kappa}, \quad \mathcal{C} = \frac{2\kappa}{\sigma^2}.$$

Equation (5.19) gives the solution of equation (5.14) as

$$F(\tilde{t}, \tilde{y}) = e^{\tilde{t}} \tilde{y}^{\mathcal{A}} [c_1 U\left(\mathcal{B}, 1+\mathcal{A}, \mathcal{C}\tilde{y}\right) + c_2 L^{\mathcal{A}}_{-\mathcal{B}}(\mathcal{C}\tilde{y})]$$

which finally provides the solution of equation (5.3) as

$$u_1(t, y) = e^{t} y^{\mathcal{A}} [c_1 U\left(\mathcal{B}, 1+\mathcal{A}, \mathcal{C}y\right) + c_2 L^{\mathcal{A}}_{-\mathcal{B}}(\mathcal{C}y)]. \tag{5.20}$$

5.4.2 Case 2: $aX_1 + X_2$

Now, consider the generator $aX_1 + X_2 = a\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial t}$. The characteristics equation of the generator

$$\frac{dx}{a} = \frac{dt}{1}$$

introduces the similarity variables $x - at = \tilde{w}$ and $y = \tilde{y}$. The solution of equation (5.3) can be written as $u = F(\tilde{y}, \tilde{w})$ with

$$\begin{aligned} &u_t = -aF_{\tilde{w}}, \quad u_x = F_{\tilde{w}}, \quad u_{xx} = F_{\tilde{w}\tilde{w}}, \quad u_y = F_{\tilde{y}}, \\ &u_{yy} = F_{\tilde{y}\tilde{y}}, \quad u_{xy} = F_{\tilde{w}\tilde{y}}. \end{aligned} \tag{5.21}$$

Equations (5.21) reduces equation (5.3) to

$$\begin{aligned} \left(r - a - \frac{1}{2}\beta^2\tilde{y}\right) F_{\tilde{w}} + \frac{1}{2}\beta^2\tilde{y}F_{\tilde{w}\tilde{w}} + \frac{1}{2}\sigma^2\beta^2\tilde{y}F_{\tilde{y}\tilde{y}} + \rho\sigma\beta^2\tilde{y}F_{\tilde{w}\tilde{y}} \\ +[\kappa(\theta - \beta^2\tilde{y})]F_{\tilde{y}} - rF = 0. \end{aligned} \tag{5.22}$$

Similar to the case before, equation (5.22) is further analyzed using Lie symmetry and the generators below are obtained

$$\Gamma_1 = \frac{\partial}{\partial \tilde{w}} \quad \text{and} \quad \Gamma_2 = F\frac{\partial}{\partial F}. \tag{5.23}$$

Combining the generators (5.23), $\Gamma_1 + \Gamma_2 = \dfrac{\partial}{\partial \tilde{w}} + F\dfrac{\partial}{\partial F}$ introduces the similarity variables $\tilde{y} = \hat{y}$ and $F(\tilde{y}, \tilde{w}) = e^{\tilde{w}}G(\hat{y})$. Hence, we have

$$F_{\tilde{w}} = F_{\tilde{w}\tilde{w}} = e^{\tilde{w}}G(\hat{y}), \quad F_{\tilde{y}} = F_{\tilde{y}\tilde{w}} = e^{\tilde{w}}G'(\hat{y}), \quad F_{\tilde{y}\tilde{y}} = e^{\tilde{w}}G''(\hat{y}). \tag{5.24}$$

Equations (5.24) reduce equation (5.22) to a second degree differential equation

$$\frac{1}{2}\sigma^2\beta^2\hat{y}G''(\hat{y}) + [\kappa\theta + (\rho\sigma - \kappa)\beta^2\hat{y}]G'(\hat{y}) - aG(\hat{y}) = 0. \tag{5.25}$$

The solution of equation (5.25) will finally reveal the solution of equation (5.3) as

$$u_2(t,x,y) = e^{x-at}y^{\mathcal{A}}[c_3U(\mathcal{D}, 1+\mathcal{A}, \mathcal{E}y) + c_4L^{\mathcal{A}}_{-\mathcal{D}}(\mathcal{E}y)], \tag{5.26}$$

where $\mathcal{D}$ and $\mathcal{E}$ are constants defined as

$$\mathcal{D} = \mathcal{A} - \frac{a}{\beta^2(\rho\sigma - \kappa)}, \mathcal{E} = -\frac{2(\rho\sigma-\kappa)}{\sigma^2}.$$

5.4.3 Case 3: $aX_1 + bX_2 + X_3$

The generator $aX_1 + bX_2 + X_3 = a\dfrac{\partial}{\partial x} + b\dfrac{\partial}{\partial t} + u\dfrac{\partial}{\partial u}$, with the characteristic equations

$$\frac{dx}{a} = \frac{dt}{b} = \frac{du}{u},$$

provides the similarity variables $bx - at = \tilde{w}$ and $y = \tilde{y}$. Rewriting the solution of equation (5.3) as $u = e^{t/b}F(\tilde{w}, \tilde{y})$ gives the derivatives of u as

$$\begin{aligned} &u_t = e^{t/b}\left(\frac{1}{b}F - aF_{\tilde{w}}\right), \quad u_x = be^{t/b}F_{\tilde{w}}, \quad u_{xx} = b^2e^{t/b}F_{\tilde{w}\tilde{w}}, \\ &u_y = e^{t/b}F_{\tilde{y}}, \quad u_{yy} = e^{t/b}F_{\tilde{y}\tilde{y}}, \quad u_{xy} = be^{t/b}F_{\tilde{w}\tilde{y}}. \end{aligned} \tag{5.27}$$

Using equations (5.27), equation (5.3) is reduced to

$$\begin{aligned} \left(rb - a - \frac{1}{2}b\beta^2\tilde{y}\right)F_{\tilde{w}} + \frac{1}{2}b^2\beta^2\tilde{y}F_{\tilde{w}\tilde{w}} + \frac{1}{2}\sigma^2\beta^2\tilde{y}F_{\tilde{y}\tilde{y}} + b\rho\sigma\beta^2\tilde{y}F_{\tilde{w}\tilde{y}} \\ +[\kappa(\theta - \beta^2\tilde{y})]F_{\tilde{y}} + \left(\frac{1}{b} - r\right)F = 0. \end{aligned} \tag{5.28}$$

Equation (5.28) gives the infinitesimal generators

$$\Gamma_1 = \frac{\partial}{\partial \tilde{w}} \quad \text{and} \quad \Gamma_2 = F\frac{\partial}{\partial F}. \tag{5.29}$$

The combined generator $\Gamma_1+\Gamma_2 = \dfrac{\partial}{\partial \tilde{w}}+F\dfrac{\partial}{\partial F}$ with the characteristics equation

$$\frac{d\tilde{w}}{1} = \frac{dF}{F}$$

introduces the similarity variables $\tilde{y} = \hat{y}$ and $F(\tilde{w}, \tilde{y}) = e^{\tilde{w}}G(\hat{y})$, which derivatives are given as

$$F_{\tilde{w}} = F_{\tilde{w}\tilde{w}} = e^{\tilde{w}}G(\hat{y}), \quad F_{\hat{y}} = F_{\hat{y}\tilde{w}} = e^{\tilde{w}}G'(\hat{y}), \quad F_{\hat{y}\hat{y}} = e^{\tilde{w}}G''(\hat{y}). \tag{5.30}$$

Equations (5.30) reduce equation (5.3) to a second degree differential equation

$$\begin{aligned} &\frac{1}{2}\sigma^2\beta^2\hat{y}G''(\hat{y}) + [\kappa(\theta - \beta^2\hat{y}) + b\rho\sigma\beta^2\hat{y}]G'(\hat{y}) \\ &+ \left(\frac{1}{b} - a + r(b-1) + \frac{1}{2}b\beta^2(b-1)\hat{y}\right) G(\hat{y}) = 0. \end{aligned} \tag{5.31}$$

Similar to the previous cases, the solution of equation (5.3) is then obtained as

$$u_3(t, x, y) = e^{\mathcal{G}y+bx+(1/b-a)t}y^{\mathcal{A}}[c_5U(\mathcal{H}, 1+\mathcal{A}, \mathcal{I}y) + c_6L^{\mathcal{A}}_{-\mathcal{H}}(\mathcal{I}y)], \tag{5.32}$$

with

$$
\begin{aligned}
\mathcal{F} &= \sqrt{\kappa^2 - 2b\kappa\rho\sigma + b(b(\rho^2 - 1) + 1)\sigma^2}, \\
\mathcal{G} &= -\frac{\mathcal{F} - \kappa + b\rho\sigma}{\sigma^2}, \\
\mathcal{H} &= \frac{\sigma(\theta\kappa\rho - r\sigma)b^2 + (\sigma^2(\mathcal{F}\beta^2 + a + r) - \theta\kappa(\kappa + \mathcal{F}))b - \sigma^2}{b\beta^2\sigma^2\mathcal{F}}, \\
\mathcal{I} &= \frac{2\mathcal{F}}{\sigma^2},
\end{aligned}
$$

are constants.

5.5 Conservation Laws

A connection between the symmetries of differential equations and conservation laws exists if the equations are founded from the variational principle, according to Noether [69]. Ibragimov [39] defined an adjoint equation for a non-linear differential equation and constructed a Lagrangian for equations with the adjoint equation. Conservation laws are useful to check the accuracy of numerical solutions [26]. One may refer to the above-mentioned references for more details.

Any Lie point or Lie-Bäcklund of a differential equation (5.4) provides a conservation law $D_i(C^i) = 0$ for the system of differential equations comprising equation (5.4). The adjoint equation is given by

$$
F^*(t, x, y, u_x, v_x, \ldots, u_{yy}, v_{yy}) = \frac{\delta(vF)}{\delta u},
$$

where $\dfrac{\delta}{\delta u}$ is defined as in equation (2.33). The adjoint equation of equation

(5.3), where in this case, is simplified as

$$F = u_t + \left(r - \frac{1}{2}\beta^2 y\right) u_x + \frac{1}{2}\beta^2 y u_{xx} + \frac{1}{2}\sigma^2\beta^2 y u_{yy} + \rho\sigma\beta^2 y u_{xy} + \kappa(\theta - \beta^2 y)u_y - ru,$$

is given by

$$\begin{aligned}
F^* =& \frac{\delta}{\delta u}(vF) \\
=& \left(\frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} - D_y\frac{\partial}{\partial u_y} + D_x^2\frac{\partial}{\partial u_{xx}} + D_y^2\frac{\partial}{\partial u_{yy}} + D_{xy}^2\frac{\partial}{\partial u_{xy}}\right)(vF) \\
=& - D_t(v) - D_x\left[\left(r - \frac{1}{2}\beta^2 y\right) v\right] - D_y[\kappa(\theta - \beta^2 y)v] + D_x^2\left(\frac{1}{2}\beta^2 y v\right) \\
&+ D_y^2\left(\frac{1}{2}\sigma^2\beta^2 y v\right) + D_{xy}^2[\rho\sigma\beta^2 y v] - rv \\
=& - v_t - \left(r - \frac{1}{2}\beta^2 y\right) v_x - \kappa[(\theta - \beta^2 y)v_y - \beta^2 v] + \frac{1}{2}\beta^2 y v_{xx} \\
&+ \frac{1}{2}\sigma^2\beta^2(2v_y + y v_{yy}) + \rho\sigma\beta^2(v_x + y v_{xy}) - rv \\
=& - v_t - \left(r - \frac{1}{2}\beta^2 y - \rho\sigma\beta^2\right) v_x - \kappa(\theta - \sigma^2\beta^2 - \beta^2 y)v_y + \frac{1}{2}\beta^2 y v_{xx} \\
&+ \frac{1}{2}\sigma^2\beta^2 y v_{yy} + \rho\sigma\beta^2 y v_{xy} + (\kappa\beta^2 - r)v.
\end{aligned} \tag{5.33}$$

The conserved vectors as given by [42] are

$$
\begin{aligned}
C^t =& \tau\mathcal{L} + W\left(\frac{\partial \mathcal{L}}{\partial u_t}\right),\\
C^x =& \xi\mathcal{L} + W\left[\frac{\partial \mathcal{L}}{\partial u_x} - D_x\left(\frac{\partial \mathcal{L}}{\partial u_{xx}}\right) - D_y\left(\frac{\partial \mathcal{L}}{\partial u_{xy}}\right)\right] + D_x(W)\left[\frac{\partial \mathcal{L}}{\partial u_{xx}}\right]\\
&+ D_y(W)\left[\frac{\partial \mathcal{L}}{\partial u_{xy}}\right],\\
C^y =& \varphi\mathcal{L} + W\left[\frac{\partial \mathcal{L}}{\partial u_y} - D_y\left(\frac{\partial \mathcal{L}}{\partial u_{yy}}\right) - D_x\left(\frac{\partial \mathcal{L}}{\partial u_{xy}}\right)\right] + D_y(W)\left[\frac{\partial \mathcal{L}}{\partial u_{yy}}\right]\\
&+ D_x(W)\left[\frac{\partial \mathcal{L}}{\partial u_{xy}}\right],
\end{aligned} \tag{5.34}
$$

where W and $\mathcal{L}$, the Lagrangian form, are defined as

$$W = \eta - \xi u_x - \tau u_t - \varphi u_y, \quad \mathcal{L} = vF. \tag{5.35}$$

5.5.1 Case 1: $X_1 = \dfrac{\partial}{\partial x}$

For $X_1 = \dfrac{\partial}{\partial x}$, where $\eta = \tau = \varphi = 0, \xi = 1, W = -u_x$. Hence, the conserved vectors

$$C_1^t = \tau\mathcal{L} + W\left(\frac{\partial \mathcal{L}}{\partial u_t}\right) = -u_x\left(\frac{\partial [vF]}{\partial u_t}\right) = -u_x v, \tag{5.36}$$

$$
\begin{aligned}
C_1^x =& vF - u_x\left[\frac{\partial vF}{\partial u_x} - D_x\left(\frac{\partial vF}{\partial u_{xx}}\right) - D_y\left(\frac{\partial vF}{\partial u_{xy}}\right)\right] - D_x(u_x)\left[\frac{\partial vF}{\partial u_{xx}}\right] \\
&- D_y(u_x)\left[\frac{\partial vF}{\partial u_{xy}}\right] \\
=& vF - u_x\left[\left(r - \frac{1}{2}\beta^2 y\right)v - D_x\left(\frac{1}{2}\beta^2 yv\right) - D_y(\rho\sigma\beta^2 yv)\right] - D_x(u_x)\left(\frac{1}{2}\beta^2 yv\right) \\
&- D_y(u_x)(\rho\sigma\beta^2 yv) \\
=& u_t v + \frac{1}{2}\sigma^2\beta^2 y u_{yy} v - ruv + \kappa(\theta - \beta^2 y)u_y v + \frac{1}{2}\beta^2 y u_x v_x + \rho\sigma\beta^2 u_x v \\
&+ \rho\sigma\beta^2 y u_x v_y.
\end{aligned} \tag{5.37}
$$

Similarly,

$$
\begin{aligned}
C_1^y =& \left(\frac{1}{2}\sigma^2\beta^2 - \kappa(\theta - \beta^2 y)\right)u_x v + \frac{1}{2}\sigma^2\beta^2 y u_x v_y + \rho\sigma\beta^2 y u_x v_x - \frac{1}{2}\sigma^2\beta^2 y u_{xy} v \\
&- \rho\sigma\beta^2 y u_{xx} v.
\end{aligned} \tag{5.38}
$$

5.5.2 Case 2: $X_2 = \dfrac{\partial}{\partial t}$

For the case when $X_2 = \dfrac{\partial}{\partial t}$, where $\eta = \xi = \varphi = 0, \tau = 1$, $W = -u_t$. Using (5.34), we have

$$\begin{aligned} C_2^t =& vF - u_t \frac{\partial(vF)}{\partial u_t} = vF - u_t v \\ =& \Bigg[\left(r - \frac{1}{2}\beta^2 y \right) u_x + \frac{1}{2}\beta^2 y u_{xx} + \frac{1}{2}\sigma^2\beta^2 y u_{yy} + \rho\sigma\beta^2 y u_{xy} \\ & + \kappa(\theta - \beta^2 y)u_y - ru \Bigg] v. \end{aligned} \tag{5.39}$$

Similarly,

$$\begin{aligned} C_2^x =& \left(\frac{1}{2}\beta^2 y - r + \rho\sigma\beta^2 \right) u_t v + \frac{1}{2}\beta^2 y u_t v_x + \rho\sigma\beta^2 y u_t v_y \\ & - \beta^2 \left(\frac{1}{2} + \rho\sigma \right) y u_{xt} v, \end{aligned} \tag{5.40}$$

and

$$\begin{aligned} C_2^y =& \left[\kappa(\beta^2 y - \theta) + \frac{1}{2}\sigma^2\beta^2 \right] u_t v + \frac{1}{2}\sigma^2\beta^2 y u_t v_y + \rho\sigma\beta^2 y u_t v_x \\ & - \sigma\beta^2 \left(\frac{1}{2}\sigma + \rho \right) y u_{xt} v. \end{aligned} \tag{5.41}$$

5.5.3 Case 3: $X_3 = u\dfrac{\partial}{\partial u}$

For the case $X_3 = u\dfrac{\partial}{\partial u}$, $W = u$. Hence,

$$C_3^t = u\frac{\partial vF}{\partial u_t} = uv. \tag{5.42}$$

Similarly, we have

$$C_3^x = \left(r - \frac{1}{2}\beta^2 y - \rho\sigma\beta^2\right) uv - \frac{1}{2}\beta^2 yuv_x - \rho\sigma\beta^2 yuv_y + \frac{1}{2}\beta^2 yu_x v \\ + \rho\sigma\beta^2 yu_y v, \tag{5.43}$$

and

$$C_3^y = \left(\kappa(\theta - \beta^2 y) - \frac{1}{2}\sigma^2\beta^2\right) uv - \frac{1}{2}\sigma^2\beta^2 yuv_y + \rho\sigma\beta^2 yuv_x + \frac{1}{2}\sigma^2\beta^2 yu_y v \\ + \rho\sigma\beta^2 yu_x v. \tag{5.44}$$

5.6 Conclusion

The power option model (5.3) emits the infinitesimal generators (5.9) which potentially generate infinite numbers of solutions. The optimal system obtained in (5.10) helps to narrow the number down to three. The invariant solutions of equation (5.3) are then obtained as equations (5.20,5.26,5.32).

To illustrate the invariant solutions obtained, we present the graphical representation of the solutions with the values of the parameters chosen as follows

- mean reversion speed, $\kappa = 1.5$,
- volatility of volatility, $\sigma = 0.8$,
- positive constant, $\beta = 2$

- average level of volatility, $\theta = 0.15$,
- risk-free rate, $r = 0.2$,
- constant, $a = 1$,
- constant, $b = 2$,
- Brownian motions correlation coefficient, $\rho = 2$.

With the parameters chosen above, the values of the constants involved in the invariant solutions above are given as

$$\mathcal{A} = 0.8242, \quad \mathcal{B} = 0.6906, \quad \mathcal{C} = 4.6875, \quad \mathcal{D} = -1.6758, \quad \mathcal{E} = -0.3125,$$
$$\mathcal{F} = 1.2689, \quad \mathcal{G} = -4.6388, \quad \mathcal{H} = 1.0890, \quad \mathcal{I} = 3.9652.$$

With the values of the constants above, the graphical representation of $u_1(t, y)$ in equation (5.20) is illustrated in figure 5.1. The value of u_1 rises with the increase of y but the changes of t have a minimum impact on the value of u_1.

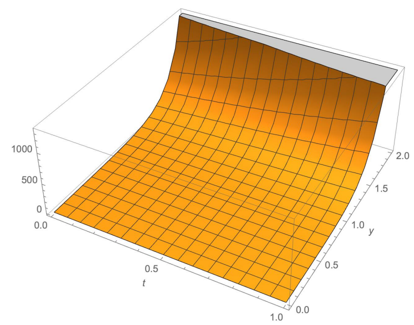
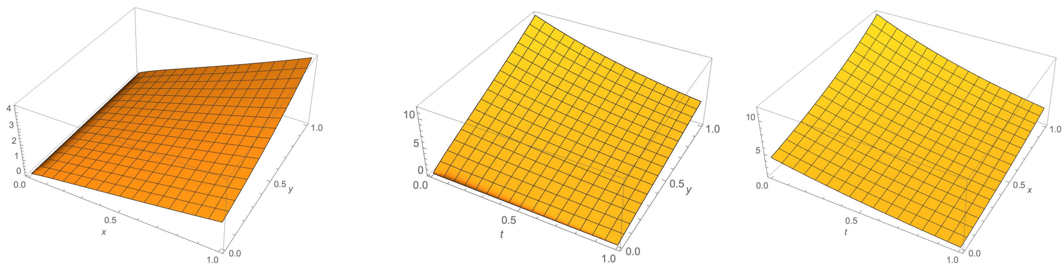


Figure 5.1: $u_1(t, y)$

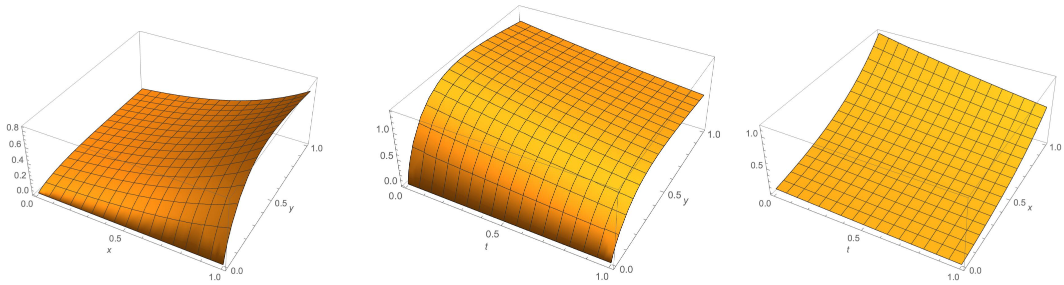


(a) $u_2(1, x, y)$ (b) $u_2(t, 1, y)$ (c) $u_2(t, x, 1)$

Figure 5.2: $u_2(t, x, y)$ with different fixed variables

The invariant solution of the generator $aX_1 + X_2$, on the other hand, is a function with three variables t, x, and y. This makes equation (5.26), $u_2(t, x, y)$, impossible to be displayed as a surface unless we fix one of the variables. In figure 5.2, we illustrate the surface of $u_2(t, x, y)$ with different variables set fixed. Figure 5.2a shows that when t is fixed, the value of u_2 rises with the increase of x and y. Furthermore, it increases at a greater rate when both x and y grow. Referring to figure 5.2b, the value of the option decreases with time but increases with y when x is fixed. We see a similar trend in figure 5.2c when we fix y.

In figure 5.3, the surfaces of $u_3(t, x, y)$ are displayed with different variables set fixed. When time is fixed, the value of u_3 increases at different rates when x and y grow, as shown in figure 5.3a. The value of the option turn an interesting turn when we fix x. Figure 5.3b shows the value of the option increases rapidly at first but slows down to a lower rate when y grows. When y is fixed, the value of the option increases at a moderate rate along the x-axis, see figure 5.3c. Both figures 5.3b and 5.3c show that the value of the option fades with



(a) $u_3(1, x, y)$ (b) $u_3(t, 1, y)$ (c) $u_3(t, x, 1)$

Figure 5.3: $u_3(t, x, y)$ with different fixed variables

time.

Generally, the value of the power option increases with the growth of x and y. On the other hand, we see the depreciation of the power option value over time. The value of an option is influenced by the price of the underlying asset, the strike price, the volatility of the underlying asset's price, as well as the time until expiration. The longer the time until expiration, the higher the value of an option, as it allows more time for the underlying asset's price to move in the direction favorable to the option holder. There is a greater probability for the option to end up in the money.

On the contrary, as the expiration date approaches, the time value of an option decreases, and it becomes increasingly influenced by intrinsic value. This is because the likelihood of the option ending up "out of the money" increases as the expiration date nears, which decreases the option's overall value.

Chapter 6

Conclusion

Since the introduction of the Black-Scholes-Merton model [12, 61], differential equations were deeply embedded in financial models. The mysterious behaviors of the financial market were finally revealed. Various financial models derived from the BSM models [19, 35, 94] were introduced. With the increasing applications in the financial market, the ability of fractional calculus to capture complex phenomena, which the traditional models are incapable of, is getting more prominent [93]. The involvement of calculus in these rising financial models had called for the need for a tool to decrypt complex partial or fractional differential equations.

The application of Lie symmetries in computational finance aims to understand the behavior of financial instruments under various market conditions, which is crucial for risk management and decision-making. In this thesis, financial models involving time-fractional differential equations were studied, as well as a power options model under Heston dynamic.

Chapter 1 briefly introduced the background of computational finance, including the adoption of mathematics in studying the financial and economic behavior of the stocks and options market. The chapter ended with a discussion of the research purpose of this study as well as the contribution of the author to the field of computational finance.

The literature review in Chapter 2 discussed the Lie symmetry analysis in length, including the definitions and applications of Lie points symmetries, group invariant solutions, optimal systems, the law of conservation, etc. In view of the involvement of fractional differential equations in this study, the application of Gamma, Beta, and Mittag-Leffler functions in solving fractional differential equations was demonstrated using some examples.

Motivated by Fallahgoul *et al.* [25], a time-fractional Black-Scholes-Merton model was studied in Chapter 3. Using Lie's method, three infinitesimal generators and their corresponding Lie group of transformations were obtained. The commutators and adjoint representations were then used to present an optimal system of the Lie group. Finally, two invariant solutions of the time-fractional Black-Scholes-Merton equation were suggested, with their visual representations shown and discussed in the final conclusion of Chapter 3.

Chapter 4 started with a discussion of an arbitrage-free stock price model.

The discussion was then extended to the Lie symmetry analysis of a time-fractional arbitrage-free stock price model. At the end of Chapter 4, an invariant solution of the time-fractional model was suggested. This solution was graphically compared with a solution of the non-fractional model.

The final chapter of the thesis demonstrated the search for invariant solutions as well as the law of conservation of power options under the Heston dynamic. The invariant solutions found were illustrated and discussed with some assumptions.

The thesis shows the application of Lie symmetry analysis in solving complex partial and fractional differential equations. In most cases, multiple solutions of the differential equations will be acquired. However, the search of the invariant solutions of fractional differential equations is often interrupted at the final stage. The lack of "basic ingredients" of ordinary fractional differential equations has forced stop the search. For instance, the search of the invariant solutions of a time-fractional Fisher equation, which is

$$u_t^\alpha - \frac{\sigma^2}{2}u_{xx} + a(u - u^2) = 0,$$

was forced stop for the reduced fractional equation $D_t^\alpha F(t) + a(F - F^2) = 0$ has no known solution.

Our attempt in solving time-fractional Vasicek and CIR models, which sound

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} + \kappa(\theta - x)\frac{\partial u}{\partial x} - xu = 0 \tag{6.1}$$

and

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{1}{2}\sigma^2 x \frac{\partial^2 u}{\partial x^2} + \kappa(\theta - x)\frac{\partial u}{\partial x} - xu = 0 \tag{6.2}$$

went smoothly at the beginning. The infinitesimal generators of equations (6.1) and (6.2) were obtained as

$$\begin{aligned}
X_1 &= t\frac{\partial}{\partial t} + \frac{1}{2}\alpha x\frac{\partial}{\partial x} + \frac{1}{2}K\alpha x(x-\theta)u\frac{\partial}{\partial u},\\
X_2 &= \frac{\partial}{\partial x} + Kxu\frac{\partial}{\partial u},\\
X_3 &= u\frac{\partial}{\partial u},\\
X_\infty &= B(x,t)\frac{\partial}{\partial u},
\end{aligned} \tag{6.3}$$

and

$$\begin{aligned}
X_1 &= t\frac{\partial}{\partial t} + \alpha x\frac{\partial}{\partial x} + K\alpha xu\frac{\partial}{\partial u},\\
X_2 &= \sqrt{x}\frac{\partial}{\partial x} + \left(\frac{K}{\sqrt{x}}(x-\theta) + \frac{1}{4\sqrt{x}}\right)u\frac{\partial}{\partial u},\\
X_3 &= u\frac{\partial}{\partial u},\\
X_\infty &= B(x,t)\frac{\partial}{\partial u},
\end{aligned} \tag{6.4}$$

respectively, where $K = \kappa/\sigma^2$. Unfortunately, the search for invariant solutions ended fruitlessly for the reduced fractional differential equations were unsolvable.

Future work plans to extend this research to apply Lie symmetry analysis to time-fractional power options under the Heston dynamic. Furthermore, more invariant solutions could be generated based on the infinitesimal generators obtained in previous chapters. The inclusion of the initial conditions into the suggested invariant solutions for the financial models in the previous chapters is part of the future works. Finally, the relationship between financial models and their fractional versions is worth investigating.

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