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On Priority-Proportional Payments in Financial Networks[★]

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ABSTRACT

We study financial systems from a game-theoretic standpoint. A financial system is represented by a network, where nodes correspond to banks, and directed labeled edges correspond to debt contracts between them. The existence of cycles in the network indicates that a payment of a bank to one of its lenders might affect the bank's incoming payments. So, if a bank cannot fully repay its debt, then the exact payments it makes to each of its lenders can affect the cash inflow back to itself. We naturally assume that the banks are interested in their financial well-being (utility) which is aligned with the amount of incoming payments they receive from the network. This defines a game among the banks, that can be seen as utility-maximizing agents who strategize over their payments.

We introduce a class of financial network games that arise under some natural payment strategies called priority-proportional payments. We compute valid payment profiles for fixed payment strategies and we investigate existence and (in)efficiency of equilibrium strategies, under different variations of the game that capture several financial aspects that commonly arise in practice. We conclude with examining the computational complexity of a variety of related problems.

1. Introduction

A financial system comprises a set of institutions, such as banks, that engage in financial transactions. The interconnections showing the *liabilities* (financial obligations or debts) among the banks are represented by a network and can be highly complex. Hence, the wealth and financial well-being of a bank depends on the entire network and not just on the well-being of its immediate debtors (or *borrowers*). For example, a possible bankruptcy of a bank and the corresponding damage to its immediate creditors (or *lenders*), resulting from the bank's failure to repay (known as credit risk), can be propagated through the financial network by causing the lenders' (and other banks', in sequence) inability to repay their debts, thus having a global effect.

In this study, we examine the global effect arising from the intricate payment decisions made by individual banks within the financial networks. We assume that each bank has a fixed amount of *external assets* (not affected by the network) which are measured in the same currency as the liabilities. A bank's *total assets* comprise its external assets and its incoming payments, and can be used for (outgoing) payments to its lenders. A bank's payment decision, for example, can specify the priority to be given to each of its debts or lenders. An authority (e.g., the government or the financial regulator) is assumed to monitor the decisions of the banks in order to guarantee that they comply with general regulatory principles, such as the absolute priority and the limited liability ones (see, e.g., [10]). According to these principles, a bank can leave a liability (partially) unpaid only if its assets are not enough to fully repay its liabilities. In particular, the *absolute priority principle* requires that a bank that does not pay its liabilities in full, has to spend all its remaining assets to pay its lenders, and the *limited liability principle* implies that a bank that does not have enough assets to pay its liabilities in full, will spend no more than all its remaining assets to pay its lenders. Surprisingly, not all decisions of the banks lead to valid solutions (or payment profiles)[1], or there can be ambiguity in the overall solution despite the individual fixed payment decisions of the banks [10]. This gives rise to the *clearing problem* aiming to determine the final payment profile of the network for fixed individual payment decisions. Such payments are called

[★]This paper is an extended version of the authors' conference paper [19]. The research was carried out when Hao Zhou was a PhD student at University of Essex. Authors are ranked alphabetically.

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[1]An example of such a network appears in [29] and assumes the presence of default costs and Credit Default Swaps (definitions appear in Section 2).

clearing payments and are compatible with the given individual payment decisions. Computing clearing payments is a necessary step before performing a game-theoretic analysis.

We investigate the consequences of individual bank decisions in clearing from a game-theoretic standpoint. Following the recent work of Bertschinger et al. [4], we diverge from the common assumption of proportionality in payments, that dictates that a bank pays its lenders proportionally to their respective liabilities. We assume that banks strategize about their payments by appropriately deciding the priority of their payment actions, consistently with a predefined payment scheme (also known as a *payment rule*) and the current regulation; the chosen priorities may have an effect on the incoming payments of a given bank. We consider a natural payment scheme that allows allocations of lenders to priority-classes independently of the available assets. We refer to this payment scheme as priority-proportional scheme, and the corresponding strategies as *priority-proportional strategies*.

Our paper initiates the game-theoretic analysis of priority-proportional payments in financial networks. Payments with priorities are simple to express as well as quite common and very well-motivated in the financial world. Indeed, bankruptcy codes allow for assigning priorities to the payout to different lenders, in case an entity is not able to repay all its obligations. Such a distribution of payments can be part of a reorganization plan ordered by the court [3]. Priority classes in bankruptcy law have also been considered in [12, 18], among others. Moving beyond regulated financial contexts, similar behavior is common in everyday transactions between individuals with pairwise debt relations. This is the first time that the impact of priority-proportional payments is assessed in a strategic setting, as are other elements of our analysis, even though such payments have been considered in the past (most recently by Papp and Wattenhofer [25]). Our work demonstrates that, despite their simplicity, priority-proportional strategies exhibit certain desired properties with respect e.g., to equilibria existence and quality. They are also not very restrictive, which can lead to unattractive instances. Our work can be seen as an attempt to examine whether such priority-based payment plans are indeed stable equilibria; this can be useful from a mechanism design perspective both in theoretical terms as well as in practical applications.

Our game-theoretic analysis considers two different definitions of utility motivated by the financial literature, namely *total assets*, computed as the sum of external assets and incoming payments, or *equity*, respectively. Traditionally, the financial health of a bank has been measured by its equity, which is equal to the amount of remaining assets after payments (total assets minus liabilities) if this is positive, and 0 otherwise. This means that all banks that have more debt than assets have equity 0, so equities fail to capture the potentially different available assets these banks might have. Total assets can be seen as a refinement of the equities: indeed, for any financially healthy bank (that can repay all its debt) its total assets equal its equity plus its liabilities (a fixed term), while for other banks, total assets allow them to distinguish among different states that are indistinguishable in terms of equity. Note that, since by the absolute priority principle a bank is obliged by law to fully repay all its lenders if it has enough assets, it is only the payment decisions of banks whose debt is more than their assets that can have an effect on the network. For this reason, computing the banks' utility as their total assets is a suitable approach in a game-theoretic context. Overall, both definitions are aligned with the individual financial wealth and welfare of a bank, so maximizing the utility is a bank's natural individual objective.

Our model is based on the seminal and widely adopted work of Eisenberg and Noe [10] who provide a basic financial network model, i.e., proportional payments, non-negative external assets, and debt-only liabilities. In addition to considering a different payment scheme, we enhance the basic model by also considering financial features commonly arising in practice, such as default costs [26], Credit Default Swap (CDS) contracts [29], and negative external assets [9] (definitions appear in Section 2). We analyze the efficiency of the states arising from various clearing payments of financial networks, with a focus on ones consistent with equilibrium strategies. Note that even though, as stated above, both utility definitions are aligned with the individual financial wealth and welfare of a bank, it turns out that these notions of individual utility are not always aligned with the welfare of the whole financial system.

Our contribution. Overall, we aim to quantify the extent to which strategic behavior of the banks affects the welfare of the society, by analyzing *financial network games* under priority-proportional strategies, defined for different utility functions, such as total assets or equities, and which potentially allow CDS contracts, default costs, or negative external assets. We derive structural results that have to do with the existence, the computation, and the properties of clearing payments for fixed payment decisions in a non-strategic setting, and/or the existence, quality and computation of equilibrium strategies. Specifically, in Section 3 we prove the existence of maximal clearing payments under priority-proportional strategies, even in the presence of default costs, and provide an algorithm that computes them efficiently. We are then able to prove existence of equilibria when the utility is defined as the equity, but show that equilibria are not guaranteed to exist when the utility is captured by the total assets. We then turn our attention to the efficiency of equilibria and provide an almost complete picture of the price of anarchy [22] and the price of stability [30, 2]. Our results for total assets appear in Section 4, while the case of equities is treated in Section 5. Finally, in Section 6, we study computational aspects of computing payment profiles that achieve desired financial objectives, and we investigate the computational challenges associated with equilibrium-related issues within the context of financial network games under priority-proportional payment rules.

Related work. Financial networks and their related properties have been analyzed in various works that follow the standard (non-strategic) model developed by Eisenberg and Noe [10]. They introduce a financial network model allowing debt-only contracts among banks with non-negative external assets that make proportional payments. Among other results, they prove that there always exist maximum and minimum clearing payments, and show that the uniqueness of clearing payments can be guaranteed in *regular* networks where the risk orbit of each node, defined as the set of all accessible nodes from that node, must contain at least one node with positive external assets; they also design an efficient iterative algorithm that computes them.

Stachurski [31] shows that the condition of regularity is not necessary to guarantee the existence of unique clearing payments under the model by Eisenberg and Noe [10]. In particular, [31] shows that unique clearing payments exist if each node in the network is *cash accessible*[2], which is a weaker condition than regularity; a node is cash accessible if it is accessible from a node with positive external assets. A similar approach was taken in [5] and [6] at the same time and independently. In the conference version of our manuscript, which appeared no later than the aforementioned papers, we introduce the notion of *proper payments*, which are defined as clearing payments such that all the money circulating in the financial network have originated from some bank with positive external assets. The cash accessibility condition in [31], can be seen as the equivalent of proper payments for network structures. Our work, besides introducing proper payments, provides an algorithmic approach on how to compute such unique proper clearing payments.

Following the model of Eisenberg and Noe [10], a series of papers extend and enrich the model by adding default costs [26], cross-ownership [11, 32], liabilities with various maturities [1, 23], negative external assets [9] and CDS contracts [25, 29]. In [26], Rogers and Veraart prove the existence of maximal clearing payments in the presence of default costs and provide an algorithm that computes them. Schuldenzucker et al. [28] also consider CDS and present that finding clearing payments when CDS contracts are allowed is PPAD-complete. Furthermore, in [29], they show that, in general, there can be zero or many clearing payments in the networks with CDS and provide sufficient conditions for the existence of unique clearing payments. Ioannidis et al. [16] show that computing strongly approximate clearing payments in financial networks with CDS contracts is FIXP-complete, while in [17] they further study the computational complexity of the clearing problem in financial networks with derivatives, whenever payment priorities among lenders are applied. Most recently, Calafiore et al. [5] study the clearing problem in dynamic financial networks, and show that optimal dynamic clearing payments in the pro-rata case are unique, and can be determined via a time-decoupled sequential optimization approach.

The strategic aspects of financial networks have been considered very recently. Most relevant to our setting is the paper by Bertschinger et al. [4] who also study the inefficiency of equilibria in financial networks. They follow the standard model of [10] and focus mainly on two payment schemes, namely coin-ranking and edge-ranking strategies, while also considering the total assets (as opposed to equity) as a measure of the individual utility of each bank. Apart from defining the graph-theoretic version of the clearing problem, they present a large range of results on the existence and quality of equilibria. Our work extends this line of research by considering a different payment scheme (which extends edge-ranking strategies by allowing ties in the ranking) and the possibility of additional common financial features (default costs, CDS contracts, and negative external assets). In a similar spirit, Papp and Wattenhofer [25] consider the impact of individual bank actions, such as removing an incoming debt, donating extra funds to another bank, or investing more external assets, when CDS contracts are also allowed. They mostly focus on the case where banks have predefined priorities over their lenders and they remark that even by redefining such priorities, a bank cannot affect its equity. Hoefer and Wilhelmi [15] study financial network games under minimal clearing and priorities over debt contracts (see also [7] for minimal clearing payments). For completeness we note that the proper clearing payments notion introduced in this paper is different to both maximal clearing payments and minimal clearing payments mentioned above.[3] Kanellopoulos et al. [20] study edge-removal games where each bank wants to maximize its own total asset by strategically removing a part of incoming edges, and provide results about the properties of equilibria,

[2] An example of a cash accessible yet irregular network is presented in [31]
[3] Examples demonstrating this are presented in Section 2.

i.e., the existence and inefficiency, as well as the computational aspects. More recently, the same group of authors investigate debt transfer games where banks can be strategic about whether or not to transfer their debt claims, and complement theoretic study with empirical game analysis [21].

Schuldenzucker and Seuken [27] consider the problem of portfolio compression, where a set of liabilities forming a directed cycle in the financial network may be simultaneously removed, if all participating banks approve it. They consider questions related to the banks' incentives to participate in such a compression. Mayo and Wellman [24] also investigate portfolio compression but from an empirical game-theoretic perspective. Additional strategic considerations, albeit less related to our setting, are the focus of Allouch and Jalloul [1] who consider liabilities with two different maturity dates and study how banks may strategically deposit some amount from their first-period endowment in order to increase their assets in the second period, while Csóka and Herings [8] consider liability games and study how to distribute the assets of a bank in default, among its lenders and the bank itself.

2. Preliminaries

In the following, we denote by $[n]$ the set of integers $\{1, \dots, n\}$. The key notions we use in this paper and their graphical representation are presented in an example (see Figure 1), at the end of this section.

Financial networks. Consider a set $N = \{v_1, \dots, v_n\}$ of n banks, where each bank v_i initially has some *external assets* e_i corresponding to income received from entities outside the financial system; note that e_i may also be negative and corresponds to financial obligations towards entities outside the system in consideration.

Banks have payment obligations, i.e., *liabilities*, among themselves. These are in the form of a simple debt contract or of a Credit Default Swap. A *debt contract* creates a liability l^0_{ij} of bank v_i (the borrower) to bank v_j (the lender); we assume that $l^0_{ij} \geq 0$ and $l^0_{ii} = 0$. Note that both $l^0_{ij} > 0$ and $l^0_{ji} > 0$ may hold. Banks with sufficient funds to pay their obligations in full are called *solvent* banks, while ones that cannot are *in default*. The *recovery rate*, r_k, of a bank v_k that is in default, is defined as the fraction of its total liabilities that it can fulfill. A *Credit Default Swap* (CDS) is a conditional liability of $(1 - r_k)l^k_{ij}$ of the borrower v_i to the lender v_j, subject to the default of v_k, called the *reference entity*. Overall, the total liability of bank v_i to bank v_j is $l_{ij} = l^0_{ij} + \sum_{k\in[n]} (1 - r_k)l^k_{ij}$. Let $L_i = \sum_j l_{ij}$ be the total liabilities of bank v_i and set $\mathbf{l} = (L_1, \dots, L_n)$. We use $\mathcal{N}$ to refer to the resulting financial network.

Let p_{ij} denote the actual payment from v_i to v_j; we assume that $p_{ii} = 0$. These payments define a payment matrix $\mathbf{P} = (p_{ij})$ with $i, j \in [n]$. Then, $p_i = \sum_{j\in[n]} p_{ij}$ represents the total outgoing payments of bank v_i, while $\mathbf{p} = (p_1, \dots, p_n)$ is the payment vector; this should not be confused with the breakdown of individual payments of bank v_i that is denoted by $\mathbf{p}_i = (p_{i1}, \dots, p_{in})$. A bank in default may need to liquidate its external assets or make payments to entities outside the financial system (e.g., to pay wages). This is modeled using *default costs* defined by values $\alpha, \beta \in [0, 1]$. A bank in default can only use an α fraction of its external assets (when this is positive) and a β fraction of its incoming payments.

The absolute priority and limited liability regulatory principles, discussed in the introduction, imply that a solvent bank must repay all its obligations to all its lenders, while a bank in default must repay as much of its debt as possible, taking default costs also into account.

A *payment rule* $\pi_{ij} : \mathbb{R}_+ \to [0, l_{ij}]$ denotes how much bank v_i pays to v_j based on v_i's existing funds for payments. Note that the choice of a payment rule only matters for banks in default. Furthermore, by the previous discussion, for a bank v_i that is in default and can only use a budget $b_i < L_i$ for payments, it must hold $\sum_{j\in[n]} \pi_{ij}(b_i) = b_i$. Summarizing, it must hold that $p_{ij} \in [0, l_{ij}]$ and, furthermore, $\mathbf{P} = \Phi(\mathbf{P})$, where

$$\Phi(\mathbf{X})_{ij} = \begin{cases} l_{ij}, & L_i \leq e_i + \sum_{k=1}^n x_{ki} \\ \pi_{ij}\left(\alpha e_i + \beta \sum_{k=1}^n x_{ki}\right), & 0 \leq e_i + \sum_{k=1}^n x_{ki} < L_i \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Payments $\mathbf{P}$ (captured by x_{ij} variables in Equation (1)) that satisfy these constraints are called *clearing payments*[4]. We define the notion of *proper* clearing payments, which are clearing payments that satisfy that all the money

[4]Clearing payments are not necessarily unique.

circulating in the financial network have originated from some bank with positive external assets. In the following, we only consider proper clearing payments.[5]

Financial network games. These games arise naturally when we view the banks as strategic agents. We denote the strategy of bank v_i by $s_i(\cdot)$ that dictates how v_i allocates its existing funds for any possible value these might have. The strategy profile of all players is denoted by $\mathbf{s} = (s_1(\cdot), \dots, s_n(\cdot))$, while we denote by $\mathbf{s}_{-i}$ the strategy profile of all players except v_i.

Given clearing payments $\mathbf{P}$, we define bank v_i's *utility*, denoted by $u_i(\mathbf{P})$, using either of the following two notions. The *total assets* $a_i(\mathbf{P})$ (see also [4, 7]) are calculated as the sum of external assets (fixed) and the total incoming payments. Formally,

$$a_i(\mathbf{P}) = e_i + \sum_{j\in[n]} p_{ji}. \tag{2}$$

The *equity* $E_i(\mathbf{P})$ (see also [25]) measures the net remaining assets after clearing and is calculated as follows.

$$E_i(\mathbf{P}) = \max\{0, a_i(\mathbf{P}) - L_i\}. \tag{3}$$

Proportional payments have been frequently studied in the financial literature (e.g., in [9, 10, 26]). Given clearing payments $\mathbf{P}$ (see Equation (1)), the payment rule is such that $p_{ij} = l_{ij}$ when $e_i + \sum_{k\in[n]} p_{ki} \geq L_i$ and $p_{ij} = \left(\alpha e_i + \beta \sum_{k\in[n]} p_{ki}\right) \frac{l_{ij}}{L_i}$, otherwise. In words, solvent banks pay their liabilities in full, while banks in default split their total assets among their lenders, proportionally, relative to their respective liabilities. Note that, when constrained to use proportional payments, there is no strategic decision making involved.

We focus on *priority-proportional payments*, where a bank's strategy is independent of its total assets and consists of a complete ordering of its lenders allowing for ties. Lenders of higher priority must be fully repayed before any payments are made towards lenders of lower priority, while lenders of equal priority are treated as in proportional payments. For example, a bank v_i having banks a, b, c, and d as lenders may select strategy $s_i = (a, b|c|d)$ that has banks a, b in the top priority class, followed by c and, finally, with d in the lowest priority class.

Let $L_i^{(m)}$ denote the total liability of bank v_i to banks in its m-th priority class. We use parameters k_{ij} to imply that bank v_j is in the k_{ij}-th priority class of bank v_i and denote by π'_{ij} the relative liability of v_i towards v_j in the corresponding priority class, i.e., $\pi'_{ij} = \frac{l_{ij}}{L_i^{(k_{ij})}}$ if $L_i^{(k_{ij})} > 0$, and $\pi'_{ij} = 0$ if $L_i^{(k_{ij})} = 0$. For given priority-proportional strategies for all banks, the clearing payments $\mathbf{P}$ (see Equation (1)) must also satisfy

$$p_{ij} = \min\left\{\max\left\{\left(p_i - \sum_{m=1}^{k_{ij}-1} L_i^{(m)}\right) \cdot \pi'_{ij}, 0\right\}, l_{ij}\right\}. \tag{4}$$

That is, the payment of v_i to a lender in priority class $L_i^{(k_{ij})}$ occurs only after all payments to lenders of higher priority have been guaranteed. Then, payments to lenders in $L_i^{(k_{ij})}$ are made proportionally to their claims in that priority class. Finally, we also have $0 \leq p_{ij} \leq l_{ij}$.

We will now define the notion of *Nash equilibrium* in a financial network game. First, let us stress that a strategy profile has consistent clearing payments which are not necessarily unique. It is standard practice (see, e.g. [10, 26]) to focus attention to maximal clearing payments (such payments point-wise maximize all corresponding payments) to avoid this ambiguity. So, given a strategy profile $\mathbf{s_{eq}}$ and the corresponding clearing payments $\mathbf{P_{eq}}$, we say $\mathbf{s_{eq}}$ is a pure Nash equilibrium if for each bank v_i with $i \in [n]$ and any alternative strategy $s_i(\cdot)$ of v_i, it holds that $u_i(\mathbf{P_{eq}}) \geq u_i(\mathbf{P})$, where $\mathbf{P}$ denotes the clearing payments under the profile $\mathbf{s} = (s_i(\cdot), s_{-i}^{eq}(\cdot))$, where player v_i chooses strategy $s_i(\cdot)$ while all other players keep using their equilibrium strategy. We only consider pure Nash equilibria and clarify that the utility of a bank for a given strategy profile is computed based on the assumption that the maximal clearing payments will be realized every time. In Section 3, we show how we can compute such payments efficiently. Extending strategy deviations to coalitions and joint deviations, we are interested in *strong equilibria* where no coalition can cooperatively deviate so that all coalition members obtain strictly greater utility.

[5]We note that we use the term "originate" to imply that some bank is reached through the network by some external asset through payments in $\mathbf{P}$, even though we do not assume clearing payments are made sequentially.

Social welfare: Given clearing payments $\mathbf{P}$, the social welfare $SW(\mathbf{P})$ is the sum of the banks' utilities; the particular utility notion (total assets or equities) will be clear from the context. The optimal social welfare is denoted by OPT. To measure the quality of equilibria, the Price of Anarchy (PoA) of all possible instances is defined as the worst-case ratio of the optimal social welfare over the social welfare achieved at any equilibrium over all possible networks. In contrast, the Price of Stability (PoS) measures how far the highest social welfare that can be achieved at equilibrium is from the optimal social welfare over all possible instances. These are defined as follows mathematically, where $\mathbf{P}_{eq}$ is the set of all equilibrium strategy (payment) profiles in the corresponding network.

$$\text{PoA} = \max_{\mathcal{N}} \max_{\mathbf{P}\in\mathbf{P_{eq}}} \frac{OPT}{SW(\mathbf{P})} \qquad \text{PoS} = \max_{\mathcal{N}} \min_{\mathbf{P}\in\mathbf{P_{eq}}} \frac{OPT}{SW(\mathbf{P})}.$$

Example 1. We represent a financial network by a graph as follows. Nodes correspond to banks and black edges correspond to debt-liabilities; a directed edge from node v_i to node v_j with label l_{ij}^0 implies that bank v_i owes bank v_j an amount of money equal to l_{ij}^0. Nodes are also labeled, their label appears in a rectangle and denotes their external assets; we omit these labels for banks with external assets equal to 0. A pair of red edges (one solid and one dotted) represents a CDS contract: a solid directed edge from node v_i to node v_j with label l_{ij}^k and a dotted undirected edge connecting this edge with a third node v_k implies that bank v_i owes v_j an amount of money equal to $(1 - r_k)l_{ij}^k$.

Figure 1 depicts a financial network with five banks having external assets $e_1 = e_4 = 1$ and $e_2 = e_3 = e_5 = 0$. There exist four debt contracts, i.e., bank v_1 owes v_2 and v_3 two coins and one coin, respectively; v_2 owes v_1 one coin, and v_3 owes v_5 one coin. There is also a CDS contract between v_4, v_5, and v_3, with nominal liability $l_{45}^3 = 1$, with v_4 being the borrower, v_5 the lender, and v_3 the reference entity. v_4 will only need to pay v_5 if v_3 is in default; the amount owed would be equal to $1(1 - r_3)$. Regarding default costs, we assume $\alpha = \beta = 1$.

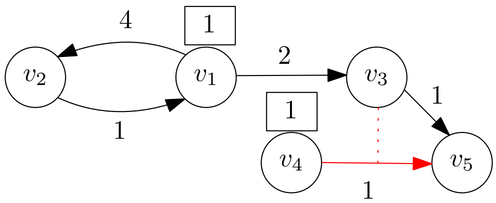


Figure 1: An example of a financial network.

In this network, only v_1 can strategize about its payments, as it is the only bank with more than one lenders, hence we focus just on v_1's strategy:

- Let v_1 select the priority-proportional strategy $s_1 = (v_2|v_3)$. Then, the payment vector would be $\mathbf{p} = (2, 1, 0, 1, 0)$ with $\mathbf{p}_1 = (0, 2, 0, 0, 0)$. Note that this is valid since $r_3 = \frac{p_3}{L_3} = \frac{p_{35}}{l_{35}} = \frac{0}{1} = 0$ and $p_{45} = (1 - r_3)l_{45}^3 = 1$. The total assets of the banks are $a_1(\mathbf{P}) = 2$, $a_2(\mathbf{P}) = 2$, $a_3(\mathbf{P}) = 0$, $a_4(\mathbf{P}) = 1$ and $a_5(\mathbf{P}) = 1$, and the social welfare is $SW(\mathbf{P}) = 6$.

- If v_1's strategy is $s_1' = (v_3|v_2)$, the only consistent payment vector would be $\mathbf{p}' = (1, 0, 1, 0, 0)$ with $\mathbf{p}_1' = (0, 0, 1, 0, 0)$. Then, $a_1(\mathbf{P}') = 1$, $a_2(\mathbf{P}') = 0$, $a_3(\mathbf{P}') = 1$, $a_4(\mathbf{P}') = 1$ and $a_5(\mathbf{P}') = 1$, resulting in $SW(\mathbf{p}') = 4$.

- If v_1 decides to pay proportionally, that is $s_1'' = (v_2, v_3)$, the payment vector would be $\mathbf{p}'' = (2, 1, 2/3, 1/3, 0)$ with $\mathbf{p}_1'' = (0, 4/3, 2/3, 0, 0)$; note that $p_{45}'' = l_{45}^3(1 - r_3) = 1(1 - \frac{p_{35}}{l_{35}}) = 1/3$. Then, $a_1(\mathbf{P}'') = 2$, $a_2(\mathbf{P}'') = 4/3$, $a_3(\mathbf{P}'') = 2/3$, $a_4(\mathbf{P}'') = 1$ and $a_5(\mathbf{P}'') = 1$, resulting in $SW(\mathbf{P}'') = 6$.

Comparing the total assets under the different strategies discussed above, v_1 would select either strategy $(v_2|v_3)$ or (v_2, v_3), returning the maximum possible total assets (i.e., 2). Therefore, any strategy profile $\mathbf{s}$ where either $s_1 = (v_2|v_3)$ or $s_1 = (v_2, v_3)$ is a Nash equilibrium.

By slightly modifying Figure 1 we can also demonstrate that proper clearing payments are different than minimal clearing payments. Indeed, assume that $e_1 = 2$ and v_1's strategy is $s_1^* = (v_3|v_2)$. In this case, minimal clearing payments would have no money circulating between v_1 and v_2, i.e., $\mathbf{p}^* = (2, 0, 1, 0, 0)$ with $\mathbf{p}_1^* = (0, 0, 2, 0, 0)$, while maximal proper ones would allow money circulating that cycle, i.e., $\mathbf{p}^* = (3, 1, 1, 0, 0)$ with $\mathbf{p}_1^* = (0, 1, 2, 0, 0)$. Note that, in this particular example, maximal clearing payments are also proper, but this is not always the case. To see that, consider a network of two banks with zero external assets and unit liabilities from each bank to the other. Maximal clearing payments would have payments of 1 traversing the cycle, but, as there are no external assets, the only proper clearing payments are 0 payments.

3. Existence and computation of clearing payments

This section contains our results relating to the existence and the properties of (proper) clearing payments in financial network games under priority-proportional payments. We begin by arguing that, given a strategy profile, clearing payments always exist even in the presence of default costs. Furthermore, in case there are multiple clearing payments, there exist maximal payments, i.e., ones that point-wise maximize all corresponding payments, and we provide a polynomial-time algorithm that computes them. Note that this result is in a non-strategic context, but it is necessary in order to perform our game-theoretic analysis as it allows us to argue about well-defined deviations by considering clearing payments consistently among different strategy profiles.

Lemma 1. *In financial network games with default costs, there always exist maximal clearing payments under a given strategy profile.*

Proof. We first focus on the case of not necessarily proper payments. The proof follows by Tarski's fixed-point theorem, along similar lines to [10, 26]. Tarski's fixed-point theorem states that a monotone function on a complete lattice has a set of fixed points that form a complete lattice, which in turn implies that it has a (least and a) greatest fixed point. To prove our claim, it suffices to note that the set of payments form a complete lattice and that Φ is monotone. Indeed, any payment p_{ij} is lower-bounded by 0 and upper-bounded by l_{ij} and for any two clearing payments $\mathbf{P}$ and $\mathbf{P}'$ such that $\mathbf{P} \geq \mathbf{P}'$ ($\mathbf{P}$ is pointwise at least as big as $\mathbf{P}'$) it holds that $\Phi(\mathbf{P}) \geq \Phi(\mathbf{P}')$, where Φ is defined in (1). Therefore, $\Phi(\cdot)$ has a greatest fixed-point and a least fixed-point. We now claim that the existence of maximal clearing payments implies the existence of maximal proper clearing ones. Indeed, consider some maximal clearing payments $\mathbf{P}$ and the resulting proper payments $\mathbf{P}'$ obtained by running Algorithm 1 below with input $\mathbf{P}$. Informally, Algorithm 1, starting with $\mathbf{P}$, identifies, by working in rounds, all banks that are reached by external assets through payments and deletes all outgoing payments of any other bank. We note that the resulting $\mathbf{P}'$ payments are indeed clearing ones, since the payments that are deleted, in the first place only reached banks whose outgoing payments are decreased to zero as well. In other words, a bank that makes payments utilizing money that has not originated from some positive external asset, cannot have received money that originated from some external asset; otherwise none of the payments would have been improper. Clearly, if $\mathbf{P}'$ are not maximal proper clearing payments, then there exist proper clearing payments $\tilde{\mathbf{P}}$ with $p'_{ij} < \tilde{p}_{ij}$ for some banks v_i, v_j. If $p'_{ij} = p_{ij}$ we obtain a contradiction to the maximality of $\mathbf{P}$, otherwise if $p'_{ij} < p_{ij}$, this means that $p'_{ij} = 0$ and there cannot be $\tilde{p}_{ij} > 0$ that belongs to a proper payment vector $\tilde{\mathbf{P}}$. □

Algorithm 1: PROPER($\mathbf{x}$)

/* The algorithm takes in input payments **x** and returns proper payments. */

Set MARKED $= \{v_i : e_i > 0\}$ and CHECKED $= \emptyset$;

while MARKED $\neq \emptyset$ **do**

  Pick $v_i \in$ MARKED;

  for $v_j \notin$ MARKED $\cup$ CHECKED *with* $x_{ij} > 0$ **do**

    MARKED = MARKED $\cup \{v_j\}$;

  MARKED = MARKED $\setminus \{v_i\}$ and CHECKED = CHECKED $\cup \{v_i\}$;

for $v_i \notin$ CHECKED **do**

  Set all outgoing payments from v_i in $\mathbf{x}$ to 0;

Return $\mathbf{x}$

We now show how such maximal clearing payments can be computed. Given a strategy profile in a financial network game under priority-proportional payments, Algorithm 2 below, that extends related algorithms in [10, 26], computes the maximal (proper) clearing payments in polynomial time. In particular, and since the strategy profile is fixed, we will argue about the payment vector consisting of the total outgoing payments for each bank; the detailed payments then follow by the strategy profile.

The algorithm proceeds in rounds. In each round μ, tentative vectors of payments, $\mathbf{p}^{(\mu)} = (p_1^{(\mu)}, \dots, p_n^{(\mu)})$, and effective equities, $E^{(\mu)}$, are computed. At the beginning of round 0, all banks are marked as tentatively solvent, which we denote by $\mathcal{D}_{-1} = \emptyset$; $\mathcal{D}_\mu$ is used to denote the banks in default after the μ-th round of the algorithm. The algorithm works so that once a bank is in default in some round, then it remains in default until the termination of the algorithm. Indeed, by Lemma 2 below, the vectors of payments are non-increasing between rounds and the strategies are fixed. Algorithm PROPER is called when $\mathcal{D}_\mu = \mathcal{D}_{\mu-1}$ and outputs proper clearing payments.

Algorithm 2: MCP

```
/* The algorithm assumes given strategies (priority classes). By abusing notation
   and for ease of exposition, we denote by p_i^(κ) the total outgoing payments of bank
   v_i at round κ and by p^(κ) the vector of total outgoing payments at round κ.    */
```

1 Set $\mu = 0, \mathbf{p}^{(-1)} = \mathbf{l}$ and $\mathcal{D}_{-1} = \emptyset$;

2 Compute $E_i^{(\mu)} := e_i + \sum_{j\in[n]} p_{ji}^{(\mu-1)} - L_i$, for $i = 1, \dots, n$;

3 $\mathcal{D}_\mu = \left\{ v_i : E_i^{(\mu)} < 0 \right\}$;

4 **if** $\mathcal{D}_\mu \neq \mathcal{D}_{\mu-1}$ **then**

5 Compute $\mathbf{p}^{(\mu)}$ that is consistent with Equation (4) and satisfies

$$p_i^{(\mu)} = \begin{cases} \alpha e_i + \beta \left(\sum_{j\in\mathcal{D}_\mu} p_{ji}^{(\mu)} + \sum_{j\in\mathcal{N}\setminus\mathcal{D}_\mu} l_{ji} \right), & \forall v_i \in \mathcal{D}_\mu \\ L_i & \forall v_i \in \mathcal{N} \setminus \mathcal{D}_\mu. \end{cases};$$

6 Set $\mu = \mu + 1$;

7 **go to** Line 2;

8 **else**

9 Run PROPER($\mathbf{p}^{(\mu-1)}$)

Lemma 2. *The payment vectors computed in each round of Algorithm 2 are pointwise non-increasing, i.e.,* $\mathbf{p}^{(\mu)} \leq \mathbf{p}^{(\mu-1)}$ *for any round* $\mu \geq 0$.

Proof. We prove the lemma by induction. The base of our induction is $\mathbf{p}^{(0)} \leq \mathbf{p}^{(-1)} = \mathbf{l}$. It holds that $p_i^{(0)} = L_i$ if $v_i \in \mathcal{N} \setminus \mathcal{D}_0$, so it suffices to compute $p_i^{(0)}$ and show that $p_i^{(0)} \leq L_i$ for $v_i \in \mathcal{D}_0$.

We wish to find the solution $\mathbf{x}$ to the following system of equations

$$\begin{aligned} x_i &= \alpha e_i + \beta \left(\textstyle\sum_{j\in\mathcal{D}_0} x_{ji} + \sum_{j\in\mathcal{N}\setminus\mathcal{D}_0} l_{ji} \right), && \forall v_i \in \mathcal{D}_0, \\ x_i &= L_i, && \forall v_i \in \mathcal{N} \setminus \mathcal{D}_0. \end{aligned} \tag{5}$$

We compute $\mathbf{x}$ using a recursive method and starting from $\mathbf{x}^{(0)} = \mathbf{p}^{(-1)} = \mathbf{l}$. We define $x^{(k)}$, $k \geq 1$, recursively by

$$x_i^{(k+1)} = \alpha e_i + \beta \left(\sum_{j\in\mathcal{D}_0} x_{ji}^{(k)} + \sum_{j\in\mathcal{N}\setminus\mathcal{D}_0} l_{ji} \right).$$

Now for $v_i \in \mathcal{D}_0$, we have

$$x_i^{(1)} = \alpha e_i + \beta \left(\sum_{j\in\mathcal{D}_0} x_{ji}^{(0)} + \sum_{j\in\mathcal{N}\setminus\mathcal{D}_0} l_{ji} \right) \leq e_i + \sum_{j\in\mathcal{D}_0} x_{ji}^{(0)} + \sum_{j\in\mathcal{N}\setminus\mathcal{D}_0} l_{ji} < L_i = x_i^{(0)},$$

where the first inequality holds because $\alpha, \beta \leq 1$, and the second inequality holds by our assumption that $v_i \in \mathcal{D}_0$. Hence, sequence $\mathbf{x}^{(k)}$ is decreasing. Since the solution to Equation (5) is non-negative, $\mathbf{x}$ can be computed as $\mathbf{x} = \lim_{k\to\infty} \mathbf{x}^{(k)}$, which completes the base of our induction.

Now assume that $\mathbf{p}^{(\mu)} \leq \mathbf{p}^{(\mu-1)}$ for some $\mu \geq 0$. We will prove that $\mathbf{p}^{(\mu+1)} \leq \mathbf{p}^{(\mu)}$. Similarly to before, $p_i^{(\mu+1)} = p_i^{(\mu)} = L_i$ if $v_i \in \mathcal{N} \setminus \mathcal{D}_{\mu+1}$, so it suffices to compute $p_i^{(\mu+1)}$ and show that $p_i^{(\mu+1)} \leq p_i^{(\mu)}$ for $v_i \in \mathcal{D}_{\mu+1}$.

The desired $p_i^{(\mu+1)}$ is the solution $\mathbf{x}$ to the following system of equations

$$x_i = \begin{cases} \alpha e_i + \beta\left(\sum_{j\in\mathcal{D}_{\mu+1}} x_{ji} + \sum_{j\in\mathcal{N}\setminus\mathcal{D}_{\mu+1}} l_{ji}\right), & \forall i \in \mathcal{D}_{\mu+1}, \\ L_i, & \forall v_i \in \mathcal{N} \setminus \mathcal{D}_{\mu+1}. \end{cases} \tag{6}$$

We compute $\mathbf{x}$ recursively starting with $\mathbf{x}^{(0)} = \mathbf{p}^{(\mu)}$. We define $x^{(k)}$, $k \geq 1$, recursively by

$$x_i^{(k+1)} = \alpha e_i + \beta\left(\sum_{j\in\mathcal{D}_{\mu+1}} x_{ji}^{(k)} + \sum_{j\in\mathcal{N}\setminus\mathcal{D}_{\mu+1}} l_{ji}\right). \tag{7}$$

For $v_i \in \mathcal{D}_{\mu+1}$, we have

$$\begin{aligned} x_i^{(1)} &= \alpha e_i + \beta\left(\sum_{j\in\mathcal{D}_{\mu+1}} x_{ji}^{(0)} + \sum_{j\in\mathcal{N}\setminus\mathcal{D}_{\mu+1}} l_{ji}\right) \\ &= \alpha e_i + \beta\left(\sum_{j\in\mathcal{D}_{\mu+1}} p_{ji}^{(\mu)} + \sum_{j\in\mathcal{N}\setminus\mathcal{D}_{\mu+1}} l_{ji}\right) \\ &= \alpha e_i + \beta\left(\left(\sum_{j\in\mathcal{D}_{\mu}} p_{ji}^{(\mu)} + \sum_{j\in\mathcal{D}_{\mu+1}\setminus\mathcal{D}_{\mu}} p_{ji}^{(\mu)}\right) + \sum_{j\in\mathcal{N}\setminus\mathcal{D}_{\mu+1}} l_{ji}\right) \\ &= \alpha e_i + \beta\left(\left(\sum_{j\in\mathcal{D}_{\mu}} p_{ji}^{(\mu)} + \sum_{j\in\mathcal{D}_{\mu+1}\setminus\mathcal{D}_{\mu}} l_{ji}\right) + \sum_{j\in\mathcal{N}\setminus\mathcal{D}_{\mu+1}} l_{ji}\right) \\ &= \alpha e_i + \beta\left(\sum_{j\in\mathcal{D}_{\mu}} p_{ji}^{(\mu)} + \sum_{j\in\mathcal{N}\setminus\mathcal{D}_{\mu}} l_{ji}\right), \end{aligned}$$

where we note that our assumption $\mathbf{p}^{(\mu)} \leq \mathbf{p}^{(\mu-1)}$ implies that $\mathcal{D}_{\mu+1} \supseteq \mathcal{D}_{\mu}$. Now we can split the set $\mathcal{D}_{\mu+1}$ into $\mathcal{D}_{\mu}$ and $\mathcal{D}_{\mu+1} \setminus \mathcal{D}_{\mu}$. For $v_i \in \mathcal{D}_{\mu}$ we have $x_i^{(1)} = \mathbf{p}^{(\mu)} = x_i^{(0)}$. For $v_i \in \mathcal{D}_{\mu+1} \setminus \mathcal{D}_{\mu}$ we have

$$\begin{aligned} x_i^{(1)} &= \alpha e_i + \beta\left(\sum_{j\in\mathcal{D}_{\mu}} p_{ji}^{(\mu)} + \sum_{j\in\mathcal{N}\setminus\mathcal{D}_{\mu}} l_{ji}\right) \\ &\leq e_i + \sum_{j\in\mathcal{D}_{\mu}} p_{ji}^{(\mu)} + \sum_{j\in\mathcal{N}\setminus\mathcal{D}_{\mu}} l_{ji} \\ &= e_i + \sum_{j\in[n]} p_{ji}^{(\mu)} \\ &< L_i = p_i^{(\mu)} = x_i^{(0)}, \end{aligned}$$

which implies that the sequence $x^{(k)}$ is decreasing. Since the solution to Equation (6) is non-negative, $\mathbf{x} = \mathbf{p}^{\mu+1}$ can be computed as $\mathbf{x} = \lim_{k\to\infty} \mathbf{x}^{(k)}$, which completes our claim that $\mathbf{p}^{(\mu)} \leq \mathbf{p}^{(\mu-1)}$ for any round $\mu \geq 0$. □

We now present the main result of this section; recall the discussion immediately above Algorithm 2.

Theorem 3. *Algorithm 2 computes the maximal clearing payments under priority-proportional strategies in polynomial time.*

Proof. Regarding the running time of Algorithm 2, observe that algorithm PROPER runs in polynomial time as it requires at most n rounds and, clearly, each round requires polynomial time. Indeed, note that each bank can enter set MARKED at most once and will leave MARKED to join set CHECKED after each other bank is examined at most once. To establish that Algorithm 2 runs also in polynomial time, observe that, by Lemma 2, at each round the set of banks in default can only increase and the algorithm terminates after at most n rounds.

Regarding the correctness of the Algorithm 2, we start by proving by induction that the payment vector provided as input to Algorithm 1 is at least equal (pointwise) to the maximal clearing vector $\mathbf{p}^*$. As a base of our induction, it is easy to see that $\mathbf{p}^{(-1)} = \mathbf{l} \geq \mathbf{p}^*$. Now assume that $\mathbf{p}^{(\mu-1)} \geq \mathbf{p}^*$ for some $\mu \geq 0$; we will prove that $\mathbf{p}^{(\mu)} \geq \mathbf{p}^*$. We denote by $\mathcal{D}_*$ the banks in default under the maximal clearing vector $\mathbf{p}^*$, i.e., $\mathcal{D}_* = \left\{ v_i : e_i + \sum_{j\in[n]} p^*_{ji} < L_i \right\}$. Our inductive hypothesis $\mathbf{p}^{(\mu-1)} \geq \mathbf{p}^*$ implies $\mathcal{D}_\mu \subseteq \mathcal{D}_*$. Hence, for banks $v_i \in \mathcal{N} \setminus \mathcal{D}_\mu$, we have $\mathbf{p}^{(\mu)}_i = L_i \geq p^*_i$. For $v_i \in \mathcal{D}_\mu$ we refer to the proof of Lemma 2 above and consider Equation (7) again while starting the recursive solution with $x^{(0)}_i = p^{(\mu)}_i$. For $v_i \in \mathcal{D}_\mu$, we observe

$$\begin{aligned}
x^{(1)}_i &= \alpha e_i + \beta \left(\sum_{j\in\mathcal{D}_\mu} p^{(\mu)}_{ji} + \sum_{j\in\mathcal{N}\setminus\mathcal{D}_\mu} l_{ji} \right) \\
&\geq \alpha \cdot e_i + \beta \cdot \left(\sum_{j\in\mathcal{D}_\mu} p^*_{ji} + \sum_{j\in\mathcal{N}\setminus\mathcal{D}_\mu} l_{ji} \right) \\
&\geq \alpha e_i + \beta \left(\sum_{j\in[n]} p^*_{ji} \right) \\
&= L^{(0)}_i.
\end{aligned}$$

Recursion (7) then implies that $x^{(k)} \geq \mathbf{p}^*$ for all k and hence $\mathbf{p}^{(\mu+1)} = \mathbf{x} = \lim_{k\to\infty} \mathbf{x}^k \geq \mathbf{p}^*$.

We have now proved that the input to Algorithm 1 is at least equal (pointwise) to the desired maximal clearing vector $\mathbf{p}^*$. However, by Lemma 2 we also know that $\mathbf{p}^{(\mu)} \leq \mathbf{p}^{(\mu-1)}$ for all $\mu \geq 0$. It holds by design that the input of Algorithm 1 is a clearing vector, so $\mathbf{p}^*$ is the only possible such input. By the arguments in the proof of Lemma 1, Algorithm 1 with input the maximal clearing payments computes the maximal proper clearing payments, and the claim follows. □

4. Maximizing the total assets

We now turn our attention to financial network games under priority-proportional strategies when the utility is defined as the total assets. We note that in this case, the maximal clearing payments, computed in Section 3 are weakly preferred by all banks among all clearing payments of the given strategy profile; indeed, the utility is computed as the sum between a fixed term (external assets) and the incoming payments which are by definition maximized. So, in case of various clearing payments, it is reasonable to limit our attention to the (unique) maximal clearing payments computed in Section 3.

We begin with some results for the important special case where $\alpha = \beta = 1$. We refer to this as the case *without default costs* and, in the following, we avoid referring to default costs in the statements that correspond to this case. We begin with a negative result regarding the existence of Nash equilibria.

Theorem 4. *Nash equilibria are not guaranteed to exist when banks aim to maximize their total assets.*

Sketch of Proof. Consider the financial network depicted in Figure 2 where M is an arbitrarily large integer and σ is a positive constant strictly less than $\frac{1}{4} - \frac{15\epsilon}{4}$. Only banks v_1, v_2 and v_3 have more than one available strategies and, hence, it suffices to argue about them. The instance is inspired by an equivalent result in [4] regarding edge-ranking strategies, however the instance used in the proof of that result admits an equilibrium under priority-proportional strategies.

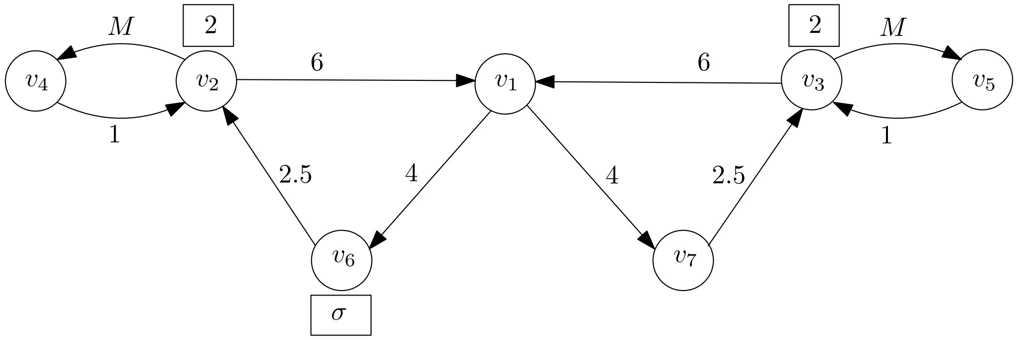


Figure 2: A game that does not admit Nash equilibria.

Observe that since M is large, whenever v_2 has v_4 in its top priority class, either alone or together with v_1, then the payment towards v_1 is at most $(5.5 + \sigma)\epsilon$, where $\epsilon = \frac{6}{M+6}$, i.e., a very small payment. Similarly, whenever v_3 has v_5 in its top priority class, the payment towards v_1 is at most 5.5ϵ. Furthermore, v_2 and v_3 can never fully repay their liabilities to any of their lenders; this implies that any (non-proportional) strategy that has a single lender in the topmost priority class will never allow for payments to the second lender.

If none of v_2 and v_3 has v_1 as their single topmost priority lender, then, by the remark above about payments towards v_1, at least one of them has an incentive to deviate and set v_1 as their top priority lender. Indeed, each of v_2, v_3 has utility at most $\frac{3+\sigma+3\epsilon}{1-\epsilon}$, and at least one of them would receive utility at least 4 by deviating. Furthermore, it cannot be that both v_2 and v_3 have v_1 as their single top priority lender, as at least one of them is in the top priority class of v_1. This bank, then, wishes to deviate and follow a proportional strategy so as to receive also the payment from its other borrower.

It remains to consider the case where one of v_2, v_3, let it be v_2, has v_1 as the single top priority lender and the remaining bank, let it be v_3, either has a proportional strategy or has v_5 as its single top priority lender. In this setting, if v_1 follows a proportional strategy, v_1 wishes to deviate and select v_2 as its single top priority lender. Otherwise, if v_1 has v_2 as its top priority lender, then v_3 deviates and sets v_1 as its top lender. Finally, if v_1 has v_3 as its single top priority lender, then v_3 deviates to a proportional strategy.

The full table of utilities appears in Appendix A (Table 1). □

Remark 1. Note that the network shown in Figure 2 does not admit a Nash equilibrium if $\sigma < \frac{1}{4} - \frac{15\epsilon}{4}$. However, once this condition does not hold, i.e., $\sigma \geq \frac{1}{4} - \frac{15\epsilon}{4}$, then the strategy profile where v_1 and v_3 pay proportionally while v_2 plays $(v_1|v_4)$ forms a Nash equilibrium since nobody has an incentive to deviate unilaterally.

We now aim to quantify the social welfare loss in Nash equilibria when each bank aims to maximize its total assets. While the focus is on financial network games under priority-proportional payments, we warm-up by considering the well-studied case of proportional payments, where we show that these may lead to outcomes where the social welfare can be far from optimal.

Theorem 5. *Proportional payments can lead to arbitrarily bad social welfare loss with respect to total assets. In acyclic financial networks, the social welfare loss is at most a factor of $n/2$ and this is almost tight.*

Proof. Consider the financial network between banks v_1, v_2, and v_3 that is shown in Figure 3, where M is arbitrarily large. Observe that paying proportionally leads to clearing payments $\mathbf{p}_1 = (0, 1/2, 1), \mathbf{p}_2 = (1/2, 0, 0), \mathbf{p}_3 = (0, 0, 0)$. Hence, the total assets are $a_1(\mathbf{P}) = 3/2$, $a_2(\mathbf{P}) = 1/2$, and $a_3(\mathbf{P}) = 1$, and, therefore, $SW(\mathbf{P}) = 3$. However, if bank v_1 chooses to pay bank v_2, the resulting clearing payments would be $\mathbf{p}'_1 = (0, M, 1), \mathbf{p}'_2 = (M, 0, 0), \mathbf{p}'_3 = (0, 0, 0)$ with total assets $a_1(\mathbf{P}') = 1 + M$, $a_2(\mathbf{P}') = M$, and $a_3(\mathbf{P}') = 1$, that sum up to $SW(\mathbf{P}') = 2M + 2$. Since $OPT \geq SW(\mathbf{P}')$ and M can be very large, we conclude that the social welfare achieved by proportional payments can be arbitrarily smaller than the optimal social welfare.

For the case of acyclic financial networks, let ξ_1 and ξ_2 denote the total external assets of non-leaf and leaf nodes, respectively; note that a leaf node has no lenders in the network. Clearly, since there are no default costs, a non-leaf bank

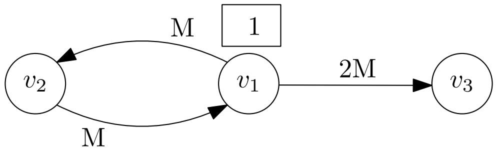


Figure 3: A financial network where proportional payments lead to low social welfare with respect to total assets. M is arbitrarily large.

v_i with external assets e_i and total liabilities L_i will generate additional revenue of at least $\min\{e_i, L_i\}$ through payments to its neighboring banks. Hence, for any clearing payments $\mathbf{P}$ we have that $SW(\mathbf{P}) \geq \xi_1 + \sum_i \min\{e_i, L_i\} + \xi_2$. In the optimal clearing payments, each non-leaf bank v_i with external assets e_i may generate additional revenue of at most $(n-1)\cdot\min\{e_i, L_i\}$, since the network is acyclic, and, therefore, we have that $OPT \leq \xi_1 + (n-1)\sum_i \min\{e_i, L_i\} + \xi_2$, i.e., the social welfare loss is at most a factor of $n/2$ as the ratio is maximized when, for each v_i we have $e_i = L_i$.

To see that this social welfare loss factor is almost tight, consider an acyclic financial network where bank v_1 with external asset $e_1 = 1$ has two lenders, v_2 and v_3, with $l_{12} = l^0_{12} = M$ and $l_{13} = l^0_{13} = 1$, where M is an arbitrarily large integer. Bank v_3 is, then, the first bank along a path from v_3 to v_n, where for $i \in \{4, \dots, n\}$ v_i is a lender of v_{i-1} and all liabilities equal 1. Under proportional payments, we obtain that $\mathbf{p}_1 = (0, M/(M+1), 1/(M+1), \dots, 0)$, $\mathbf{p}_2 = \mathbf{p}_n = (0, \dots, 0)$ and for any $i \in \{3, \dots, n-1\}$, $\mathbf{p}_i = (0, \dots, 1/(M+1), \dots, 0)$ where $1/(M+1)$ is the $(i+1)^{th}$ entry, thus $SW(\mathbf{P}) = 2 + (n-3)/(M+1)$. In the optimal clearing payments, bank v_1 fully repays its liability towards v_3 and this payment propagates along the path from v_3 to v_n resulting to $OPT = n-1$, leading to a social welfare loss factor of $(n-1)/2 - \epsilon$, where ϵ goes to zero as M tends to infinity. □

We remark that, given clearing payments with proportional payments, a bank may wish to deviate.

Remark 2. Proportional payments may not form a Nash equilibrium when banks aim to maximize their total assets.

Proof. Consider the instance shown in Figure 3 and the proof of Theorem 5; bank v_1 has more total assets when playing strategy $(v_2|v_3)$ than when using the proportional strategy (v_2, v_3). □

We now turn our attention to financial network games under priority-proportional strategies. To avoid text repetitions, we omit referring to these games in our statements. We start with a positive result on the quality of equilibria when allowing for default costs in the (extreme) case $\alpha = \beta = 0$. In the more general case, however, the price of stability may be unbounded. Finally, we show that the price of stability may also be unbounded in the absence of default costs, if negative external assets are allowed.

Theorem 6. *The price of stability when banks aim to maximize their total assets, is:*

(i) 1 *if default costs* $\alpha = \beta = 0$ *apply,*

(ii) unbounded if default costs $\alpha > 0$ *or* $\beta > 0$ *apply, and*

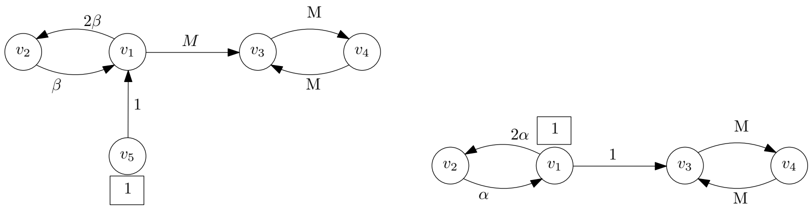
(iii) unbounded if negative external assets are allowed even if default costs do not apply ($\alpha = \beta = 1$).

Proof. Regarding the first statement, consider the equilibrium resulting to the optimal social welfare. Clearly, each solvent bank pays all its liabilities to its lenders and, hence, its strategy is irrelevant. Furthermore, each bank that is in default cannot make any payments towards its lenders, due to the default costs, therefore its strategy is irrelevant as well. Since no bank can increase its total assets, any strategy profile that leads to optimal social welfare is a Nash equilibrium and the claim follows.

With respect to case (ii), we begin with the case where $\beta > 0$ and consider the financial network shown in Figure 4(a) where M is arbitrarily large. Bank v_1 is the only bank that can strategize about its payments. Its strategy set comprises $(v_2|v_3)$, $(v_3|v_2)$, and (v_2, v_3), which result in utility $1+\beta$, 1, and $1 + \frac{2\beta^3}{M+2\beta-2\beta^3}$, respectively; note that, unless v_1 selects strategy $(v_2|v_3)$, v_2 is also in default as M is arbitrarily large. For sufficiently large M, we observe that any Nash equilibrium must have v_1 choosing strategy $(v_2|v_3)$, leading to clearing payments $\mathbf{p}_1 = (0, \beta^2+\beta, 0, 0, 0)$, $\mathbf{p}_2 = (\beta, 0, 0, 0, 0)$, $\mathbf{p}_3 = \mathbf{p}_4 = (0, 0, 0, 0, 0)$, and $\mathbf{p}_5 = (1, 0, 0, 0, 0)$ with $SW(\mathbf{P}) = 2+\beta^2+2\beta$. Now, when v_1 chooses strategy

$(v_3|v_2)$, we obtain clearing payments $\mathbf{p}'_1 = (0,0,\beta,0,0), \mathbf{p}'_2 = (0,0,0,0,0), \mathbf{p}'_3 = (0,0,0,M,0), \mathbf{p}'_4 = (0,0,M,0,0)$, and $\mathbf{p}'_5 = (1,0,0,0,0)$ with $SW(\mathbf{P}') = 2M + 2 + \beta$. The claim follows since $OPT \geq SW(\mathbf{P}')$.

Now, let us assume that $\alpha > 0$ and $\beta = 0$ and consider the financial network shown in Figure 4(b).



(a) The case where $\beta > 0$ (b) The case where $\alpha > 0$ and $\beta = 0$

Figure 4: The instances used in the proof of the unbounded price of stability when default costs satisfy $\alpha > 0$ or $\beta > 0$

.

Again, bank v_1 is the only bank that can strategize about its payments. v_1's total assets when choosing strategies $(v_2|v_3)$, $(v_3|v_2)$, and (v_2, v_3) are $1+\alpha$, 1, and 1, respectively; note that when v_1 chooses strategy (v_2, v_3), bank v_2 is also in default as it receives a payment of $\frac{2\alpha^2}{1+2\alpha}$ which is strictly less than α for any $\alpha > 0$. Hence, in any Nash equilibrium v_1 chooses strategy $(v_2|v_3)$, resulting in clearing payments $\mathbf{p}_1 = (0,\alpha,0,0), \mathbf{p}_2 = (\alpha,0,0,0), \mathbf{p}_3 = \mathbf{p}_4 = (0,0,0,0)$ with social welfare $SW(\mathbf{P}) = 1 + 2\alpha$. The optimal social welfare, however, is achieved when v_1 chooses strategy $(v_3|v_2)$, resulting in clearing payments $\mathbf{p}'_1 = (0,0,\alpha,0), \mathbf{p}'_2 = (0,0,0,0), \mathbf{p}'_3 = (0,0,0,M)$, and $\mathbf{p}'_4 = (0,0,M,0)$ with $OPT = 2M + 1 + \alpha$.

Finally, regarding case (iii), consider the financial network shown in Figure 5. Only bank v_1 can strategize about

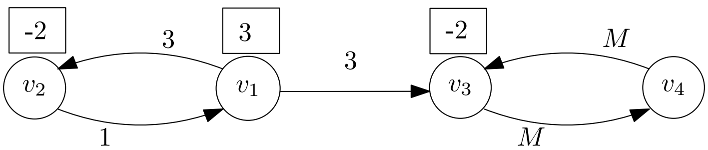


Figure 5: A financial network with negative external assets yields an unbounded price of stability.

its payments. Clearly, v_1's total assets equal 3, unless v_2 pays its debt (even partially). This is only possible when v_1 chooses strategy $(v_2|v_3)$ and prioritizes the payment of v_2 (note v_2's negative external assets will "absorb" any payment that is at most 2). Therefore, the resulting Nash equilibrium leads to the clearing payments $\mathbf{p}_1 = (0,3,1,0), \mathbf{p}_2 = (1,0,0,0), \mathbf{p}_3 = (0,0,0,0)$, and $\mathbf{p}_4 = (0,0,0,0)$ with social welfare $SW(\mathbf{P}) = 4$. However, when v_1 chooses strategy $(v_3|v_2)$ we obtain the clearing payments $\mathbf{p}'_1 = (0,0,3,0), \mathbf{p}'_2 = (0,0,0,0), \mathbf{p}'_3 = (0,0,0,M)$, and $\mathbf{p}'_4 = (0,0,M,0)$ with social welfare $SW(\mathbf{P}') = 2M + 2$. Hence, since $OPT \geq SW(\mathbf{P}')$ we obtain the theorem. □

The proof of Theorem 6(iii) in fact holds for any type of strategies as v_1 always prefers to pay in full its liability to v_2. This includes the case of a very general payment strategy scheme, namely coin-ranking strategies [4] that are known to have price of stability 1 with non-negative external assets.

Next, we show that the price of anarchy can be unbounded even in the absence of default costs, CDS contracts, and negative externals. Bertschinger et al. [4] have shown a similar result for coin-ranking strategies, albeit for a network that has no external assets; our result extends to the case of coin-ranking strategies and strengthens the result of [4] to capture the case of proper clearing payments.

Theorem 7. *The price of anarchy is unbounded when banks aim to maximize their total assets.*

Proof. Consider the financial network between banks v_i, $i \in [4]$, that is shown in Figure 6, where M is arbitrarily large. Clearly, bank v_1 is the only bank that can strategize about its payments, and observe that $a_1 = 1$ regardless of v_1's strategy. Hence, any strategy profile in this game is a Nash equilibrium. Consider the clearing payments $\mathbf{p}_1 = (0,1,0,0), \mathbf{p}_2 = \mathbf{p}_3 = \mathbf{p}_4 = (0,0,0,0)$ that are obtained when v_1's strategy is $s_1 = (v_2|v_3)$ and note that $SW(\mathbf{P}) = 2$. If, however, v_1 selects strategy $s'_1 = (v_3|v_2)$, we end up with the clearing payment $\mathbf{p}'_1 = (0,0,1,0), \mathbf{p}'_2 =$

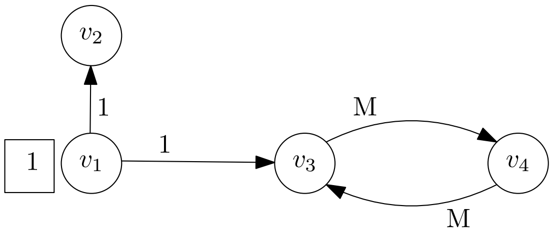


Figure 6: A financial network that yields unbounded price of anarchy with respect to total assets.

$(0,0,0,0)$, $\mathbf{p}'_3 = (0,0,0,M)$, and $\mathbf{p}'_4 = (0,0,M,0)$ and we obtain $SW(\mathbf{P}') = 2M + 2$. Hence, PoA $\geq \frac{2M+2}{2} = M + 1$ which can become arbitrarily large. □

5. Maximizing the equity

In this section, we consider the setting where a bank's individual utility as well as the social welfare are expressed in terms of equities. We present interesting properties of clearing payments and observe that Nash equilibria always exist in such games, contrary to the case of total assets.

5.1. Existence and properties of equilibria

We warm up with a known statement in absence of default costs; the short proof is included here for completeness. In particular, each bank obtains the same equity under all clearing payments, so it does not have a preference; this provides additional justification to our assumption to limit our attention to maximal clearing payments computed in Section 3, in case of various clearing payments.

Lemma 8 ([14]). *Each bank obtains the same equity under different clearing payments, given a strategy profile. That is, given the banks' strategies, for any two different (not necessarily maximal) clearing payments* $\mathbf{P}$ *and* $\mathbf{P}'$*, it holds* $E_i(\mathbf{P}) = E_i(\mathbf{P}')$ *for each bank* v_i*.*

Proof. Let $\mathbf{P}^*$ be the maximal clearing payments and let $\mathbf{P}$ be any other clearing payments. The corresponding equities for each bank v_i are

$$E_i(\mathbf{P}^*) = \max(0, e_i + \sum_{j\in[n]} p^*_{ji} - L_i) = e_i + \sum_{j\in[n]} p^*_{ji} - \sum_{j\in[n]} p^*_{ij} \tag{8}$$

and

$$E_i(\mathbf{P}) = \max(0, e_i + \sum_{j\in[n]} p_{ji} - L_i) = e_i + \sum_{j\in[n]} p_{ji} - \sum_{j\in[n]} p_{ij}, \tag{9}$$

respectively. The rightmost equalities above hold, since each bank either pays all its liabilities, if it is solvent, or uses all its external and internal assets to pay part of its liabilities, if it is in default.

Using (8) and (9) and by summing over all banks, we obtain

$$\sum_{i\in[n]} E_i(\mathbf{P}^*) - \sum_{i\in[n]} E_i(\mathbf{P}) = \sum_{i\in[n]}\sum_{j\in[n]} \left(p^*_{ji} - p^*_{ij}\right) - \sum_{i\in[n]}\sum_{j\in[n]} \left(p_{ji} - p_{ij}\right) = 0,$$

as, for any clearing payments, the total incoming payments equal the total outgoing payments.

We remark that, since $\mathbf{P}^*$ are maximal clearing payments we get that $\sum_{j\in[n]} p^*_{ij} \geq \sum_{j\in[n]} p_{ij}$ for each bank v_i and, hence, by (8) and (9) it can only be that $E_i(\mathbf{P}^*) \geq E_i(\mathbf{P})$. Therefore, since we have shown that $\sum_{i\in[n]} E_i(\mathbf{P}^*) = \sum_{i\in[n]} E_i(\mathbf{P})$, we conclude that $E_i(\mathbf{P}^*) = E_i(\mathbf{P})$ for each bank v_i. □

Lemma 8 also indicates that, for any given strategy profile, any bank has the same equity in all resulting clearing payments. We exploit this property to obtain the following result; this extends a result by Papp and Wattenhofer (Theorem 7 in [25]) which holds for the maximal clearing payments.

Theorem 9. *Even with CDS contracts, any strategy profile is a Nash equilibrium, when banks aim to maximize their equity. This holds even if the clearing payments that are realized are not maximal.*

Proof. Assume otherwise that there exists a strategy profile which is not a Nash equilibrium. Let v_i be a bank that wishes to deviate and s_i be its strategy. Clearly, if v_i is solvent it can repay all its liabilities in full and, hence, the payment priorities are irrelevant. So, let us assume that v_i is in default. If, by deviating, v_i remains in default, then its equity remains 0. Therefore, assume that v_i deviates to another strategy s_i' where it is solvent. In that case, however, v_i could fully repay its liabilities regardless of the payment priorities and, therefore, it can still repay its liabilities when playing s_i; a contradiction. □

Note that Lemma 8 no longer holds once default costs are introduced; see e.g., Example 3.3 in [26] where both banks, each having a singleton strategy set, may be in default or solvent depending on the clearing payments. The next result extends Theorem 7 in [25] to the setting with default costs, and guarantees the existence of Nash equilibria (and actually strong ones) when banks wish to maximize their equity.

Theorem 10. *Even with default costs and negative external assets, any strategy profile is a strong equilibrium when banks aim to maximize their equity.*

Proof. We begin by transforming an instance $\mathcal{I}$ with negative external assets into another instance $\mathcal{I}'$ without negative external assets, albeit with a slightly restricted strategy space for each bank. In particular, we add an auxiliary bank t and define liabilities and assets as follows. For any pair of banks v_i, v_j which does not include t, we set $l'^0_{ij} = l^0_{ij}$. For each bank v_i with $e_i < 0$, we set $e_i' = 0$ and set liability $l'^0_{it} = -e_i$, while for any other bank i' we set $e'_{i'} = e_{i'}$ and $l'^0_{i't} = 0$. Furthermore, we restrict the strategy space of each bank in $\mathcal{I}'$ so that their topmost priority class includes only bank t. An example of this process is shown in Figure 7.

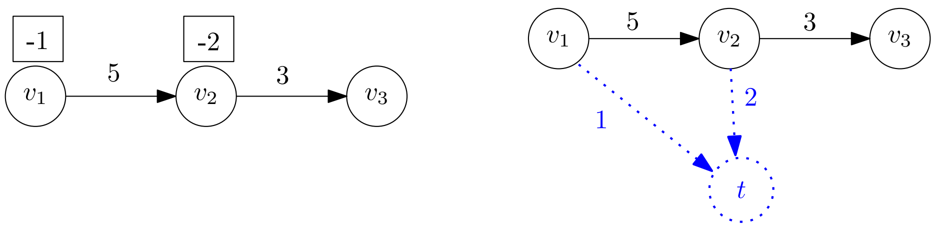


Figure 7: Transforming an instance with negative external assets into an instance without negative external assets.

Given a strategy profile for instance $\mathcal{I}$, we create the corresponding strategy profile for instance $\mathcal{I}'$ by having bank t as the single topmost priority lender and, then, append the initial strategy profile. It holds that, for any given strategy profile, the maximal clearing payments for instance $\mathcal{I}$ corresponds to maximal clearing payments in $\mathcal{I}'$ for the new strategy profile; this can be easily proved by contradiction. We can now proceed with the proof by assuming non-negative external assets, without loss of generality.

Consider maximal clearing payments $\mathbf{P}^*$ and the associated strategy profile $\mathbf{s}$. Let us assume that there is a coalition of banks $C = \{v_{C_1}, \dots, v_{C_k}\}$ where each member of the coalition can strictly increase its equity after a joint deviation. In particular, let v_{C_i}, for $i = 1, \dots, k$ change its strategy from s_{C_i} to s'_{C_i}. Clearly, each v_{C_i} must have a strictly positive equity in the resulting new maximal clearing payments $\mathbf{P}'$, since its equity was 0 before. But then, each v_{C_i} should remain solvent under strategy s_{C_i} as well, and therefore v_{C_i}'s actual payment priorities are irrelevant, for $i = 1, \dots, k$. Note that $\mathbf{P}'$ should also be maximal clearing payments under the initial strategy profile; a contradiction to the maximality of $\mathbf{P}^*$. □

5.2. (In)efficiency of equilibria

We start by noting that Lemma 8 together with Theorem 9 imply the following, which we note holds for any payment scheme.

Corollary 11. *The price of anarchy in financial network games with CDS contracts is* 1 *when banks aim to maximize their equity.*

The above positive result, however, no longer holds when default costs or negative external assets exist. For these cases we derive the following results.

Theorem 12. *The price of anarchy with default costs (α, β) when banks aim to maximize their equity, is*

(i) 1 *when* $\alpha = \beta = 0$,

(ii) *at most* $1/\alpha$*, if* $\beta = 1$ *and* $\alpha \in (0, 1)$ *, and this is tight,*

(iii) *unbounded when: a)* $\beta \in (0, 1)$*, b)* $\beta = 0$ *and* $\alpha \in (0, 1]$*, or c)* $\beta = 1$ *and* $\alpha = 0$,

(iv) *unbounded when negative external assets are allowed, even if default costs do not apply* ($\alpha = \beta = 1$).

Proof. We begin with the case $\alpha = \beta = 0$. We claim that all strategy profiles correspond to the same clearing payments hence admit the same social welfare. It suffices to observe that neither the strategy of a solvent bank, nor the strategy of a bank in default, affect the set of banks in default and consequently the clearing payments. Consider two strategy profiles $\mathbf{s}$ and $\mathbf{t}$. A bank i that is solvent under $\mathbf{s}$, will be solvent and continue to make payments $p_i = L_i$ under any strategy (given the strategies of everyone else), i.e., will be solvent at the clearing payments consistent with strategy vector $(t_i, \mathbf{s}_{-i})$, derived by $\mathbf{s}$ if bank i alone changes her strategy from s_i to t_i. On the other hand, a bank i that is in default under $\mathbf{s}$, will similarly remain in default under $(t_i, \mathbf{s}_{-i})$ and continue to make 0 payments since $\alpha = \beta = 0$, thus not affecting the set of banks in default. The claim follows by considering the individual deviations from s_i to t_i of all banks i sequentially and observing that the set of banks in default and, hence, the clearing payments are unaffected at each step.

Now, consider case (ii). For the upper bound, consider any clearing payments $\mathbf{P}$ and let $S(\mathbf{P})$ and $D(\mathbf{P})$ be the set of solvent and in default banks under $\mathbf{P}$. Recall that for any bank v_i, we have $E_i(\mathbf{P}) = \max\{0, a_i(\mathbf{P}) - L_i\}$. For any bank $v_i \in S(\mathbf{P})$, it holds that $E_i(\mathbf{P}) = e_i + \sum_{j\in[n]} p_{ji} - \sum_{j\in[n]} p_{ij}$, as $\sum_{j\in[n]} p_{ij} = \min\{e_i + \sum_{j\in[n]} p_{ji}, L_i\}$. Similarly, for any $v_i \in D(\mathbf{P})$, we have $E_i(\mathbf{P}) = \alpha e_i + \sum_{j\in[n]} p_{ji} - \sum_{j\in[n]} p_{ij}$, since $\beta = 1$. By summing over all banks, we get

$$
\begin{aligned}
E(\mathbf{P}) &= \sum_{i:v_i\in S(\mathbf{P})} E_i(\mathbf{P}) + \sum_{i:v_i\in D(\mathbf{P})} E_i(\mathbf{P}) \\
&= \sum_{i\in[n]} \left(e_i + \sum_{j\in[n]} p_{ji} - \sum_{j\in[n]} p_{ij} \right) - (1-\alpha) \sum_{i:v_i\in D(\mathbf{P})} e_i \\
&= \sum_{i\in[n]} e_i - (1-\alpha) \sum_{i:v_i\in D(\mathbf{P})} e_i.
\end{aligned}
$$

Clearly, the social welfare is maximized when $D(\mathbf{P}) = \emptyset$, i.e., $OPT \le \sum_{i\in[n]} e_i$, while it is minimized when $D(\mathbf{P})$ includes all banks, i.e., $E(\mathbf{P}) \ge \alpha \sum_{i\in[n]} e_i$; this completes the proof of the upper bound.

With respect to the lower bound for the case $\beta = 1$ and $\alpha \in [0, 1)$ (note this captures cases (ii) as well as (iii)c), consider the financial network in Figure 8. As before, v_2 is always in default. If v_2 selects strategy $(v_3|v_6)$, then v_5 ends up having equity $M + 2\alpha$. However, $(v_6|v_3)$ is also an equilibrium strategy for v_2, but it results in equity 2α for v_6, equity αM for v_5 and equity 0 for the remaining banks. The claim follows by straightforward calculations.

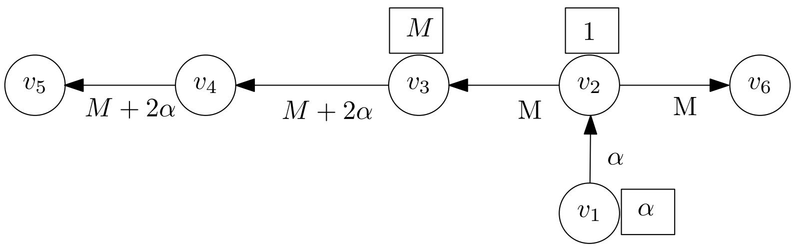


Figure 8: The instance in the proof of Theorem 12, when $\beta \in \{0, 1\}$ and $\alpha \neq \beta$.

Regarding the case (iii)a, i.e., $\beta \in (0, 1)$, consider the financial network in Figure 9.

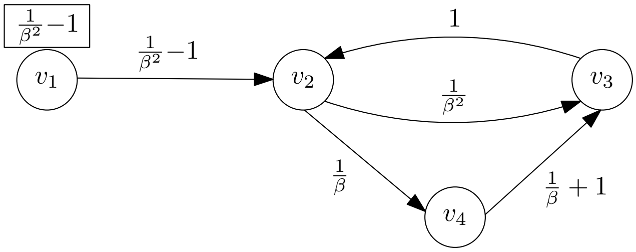


Figure 9: The instance in the proof of Theorem 12, where we assume $\beta \in (0, 1)$.

Clearly, only bank v_2 can strategize and observe that it is always in default irrespective of its strategy. When v_2 selects strategy $s_2 = (v_4|v_3)$, we obtain the clearing payments $\mathbf{p}_1 = (0, \frac{1}{\beta^2} - 1, 0, 0)$, $\mathbf{p}_2 = (0, 0, 0, \frac{1}{\beta})$, $\mathbf{p}_3 = (0, 1, 0, 0)$, and $\mathbf{p}_4 = (0, 0, 1, 0)$. Notice that v_2 and v_4 are in default, while v_3 is solvent with zero equity, thus we have $SW(\mathbf{P}) = 0$.

When, however, v_2 selects strategy $s'_2 = (v_3|v_4)$, we obtain the clearing payments $\mathbf{p}'_1 = (0, \frac{1}{\beta^2} - 1, 0, 0)$, $\mathbf{p}'_2 = (0, 0, \frac{1}{\beta}, 0)$, $\mathbf{p}'_3 = (0, 1, 0, 0)$, and $\mathbf{p}'_4 = (0, 0, 0, 0)$. Now, v_2 and v_4 are in default and we have $SW(\mathbf{P}') = 1/\beta - 1$. Since $OPT \geq SW(\mathbf{P}')$, the claim follows.

Regarding the case (iii)b, i.e., $\beta = 0$ and $\alpha \in (0, 1]$, consider the financial network in Figure 8 and note that v_2 is always in default irrespective of its strategy. If v_2 selects strategy $s_2 = (v_6|v_3)$, then v_6 ends up having equity α. However, strategy $(v_3|v_6)$ is also an equilibrium strategy for v_2, but it results in each bank having equity 0.

Finally, regarding case (iv), consider the financial network in Figure 10. Clearly, only v_1 can strategize but, irrespective of its strategy, v_1 is always in default and $E_1 = 0$; hence, any strategy profile is a Nash equilibrium. When v_1 selects strategy $s_1 = (v_2|v_3)$ we obtain the clearing payments $\mathbf{p}_1 = (0, 1, 0)$, $\mathbf{p}_2 = (0, 0, 0)$, $\mathbf{p}_3 = (0, 0, 0)$. However, when v_1 selects strategy $s'_1 = (v_3|v_2)$ we obtain the clearing payments $\mathbf{p}'_1 = (0, 0, 1)$, $\mathbf{p}'_2 = (0, 0, 0)$, $\mathbf{p}'_3 = (0, 0, 0)$ with $SW(\mathbf{P}') = 1$. Since $OPT \geq SW(\mathbf{P}')$, the claim follows and the proof of the theorem is complete. □

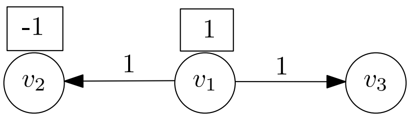


Figure 10: A financial network with negative external assets and unbounded price of anarchy.

Still, in the presence of default costs or negative external assets, Theorem 10 leads to the next positive result.

Corollary 13. *The strong price of stability is* 1 *even with default costs and negative external assets, when banks aim to maximize their equity.*

If we relax the stability notion and consider the price of stability in super-strong equilibria, where a coalition of banks deviates if at least one strictly improves its utility and no bank suffers a decrease in utility, then we obtain a negative result.

Theorem 14. *The price of stability in super-strong equilibria is unbounded when negative external assets are allowed and banks aim to maximize their equity.*

Proof. Consider the financial network as shown in Figure 11, where $\varepsilon > 0$ is a small constant. Observe that v_1 is always in default since even its maximal possible total asset $a_1 = 2 - \varepsilon$ is still less than its total liabilities $L_1 = 3$, which means $E_1 = 0$ in any scenario. If v_1's strategy is other than $s_1 = (v_2|v_3)$, then v_2 is always in default no matter what strategies it chooses, that is $E_2 = 0$. Thus, any strategy profile where v_1 does not prioritize the payment of v_2 is a Nash equilibrium with $E_1 = E_2 = 0$. However, if v_1 and v_2 form a coalition, then the only superstrong equilibrium occurs when v_1 and v_2 prioritize the payment of each other, resulting in the clearing payments $\mathbf{p}_1 = (0, 1, 1 - \epsilon, 0)$, $\mathbf{p}_2 = (1 - \epsilon, 0, 0, 1)$, $\mathbf{p}_3 = \mathbf{p}_4 = (0, 0, 0, 0)$ with $SW(\mathbf{P}) = \varepsilon$. Furthermore, when v_1 follows the strategy $s'_1 = (v_3|v_2)$ and v_2 follows strategy $s'_2 = (v_1|v_4)$, we obtain the clearing payments

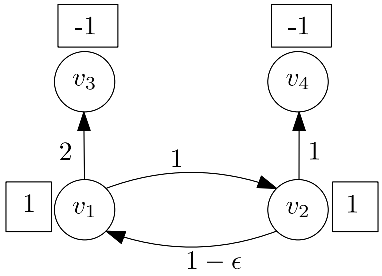


Figure 11: A financial network with negative external assets and unbounded price of stability in super-strong equilibria.

$\mathbf{p}'_1 = (0, 0, 2-\epsilon, 0), \mathbf{p}'_2 = (1-\epsilon, 0, 0, \epsilon), \mathbf{p}'_3 = \mathbf{p}'_4 = (0, 0, 0, 0)$ with $SW(\mathbf{P}') = 1-\varepsilon$. Since $OPT \geq SW(\mathbf{P}')$, we obtain that the superstrong price of stability is at least $\frac{1-\varepsilon}{\varepsilon}$ and the theorem follows since ε can be arbitrarily small. □

6. Computational complexity

We conclude this study by delving into computational aspects, focusing either on addressing equilibrium-related issues that emerge in financial network games where banks strategize about priority-proportional payments to maximize their total assets, or centrally computing payment profiles that maximize social welfare, i.e., the sum of the total assets or the equities.

In particular, we prove that the following four problems are NP-hard; see below for the formal proofs.

- Theorem 15: Verify if a given strategy profile is a pure Nash equilibrium, when banks aim to maximize their total assets and equilibria are guaranteed to exist.
- Theorem 16: Decide if a Nash equilibrium exists when banks aim to maximize their total assets.
- Theorem 17: Compute a strategy profile that maximizes the sum of total assets.
- Theorem 19: Compute a strategy profile that maximizes the sum of equities for a network with default costs $\alpha \in (0, 1)$ and $\beta \in [0, 1)$.

Theorem 15. *It is NP-hard to verify if a given strategy profile is a pure Nash equilibrium, when banks aim to maximize their total assets and equilibria are guaranteed to exist.*

Proof. The proof relies on a reduction from the NP-complete problem RXC3 [13], a variant of EXACT COVER BY 3-SETS (X3C). In RXC3, we are given an element set X, with $|X| = 3k$ for an integer k, and a collection C of subsets of X where each such subset contains exactly three elements. Furthermore, each element in X appears in exactly three subsets in C, that is $|C| = |X| = 3k$. The question is if there exists a subset $C' \subseteq C$ of size k that contains each element of X exactly once.

Given an instance $\mathcal{I}$ of RXC3, we construct an instance $\mathcal{I}'$ as follows. There is a central bank v, having external assets of $3k$, and we add a bank t_i for each element i of X, as well as two banks u_i, g_i for each element i in C; all banks except v have zero external assets. Bank v has a liability of M to each u_i, where M is an arbitrarily large integer. Each u_i, corresponding to set $(x, y, z) \in C$, has liability M to the three banks t_x, t_y, t_z respectively corresponding to the three elements $x, y, z \in X$. Also, u_i has liability 6 towards g_i, while g_i has liability 3 towards u_i. Finally, each bank t_i has liability 1 towards bank v. Note that this construction requires polynomial time; see also Figure 12.

In our proof, we will first show the existence of Nash equilibria and, then, we will identify a condition on bank v's total assets in an equilibrium that will allow us to decide instance $\mathcal{I}$ of RXC3. For any fixed strategy profile, we denote by $\mathcal{T}_v$ the set of u_i banks belonging to v's top priority class. We exploit the fact that, in any equilibrium, any u_i in $\mathcal{T}_v$ has just g_i in its top priority class. We prove this claim below.

To prove that an equilibrium exists, observe that any bank t_i or g_i, for $i \in \{1, \dots, 3k\}$, has only one outgoing edge in the financial network, so it cannot strategize about its payments. Since M is arbitrarily large and $a_v \leq 6k$, any u_i not in $\mathcal{T}_v$ has zero incoming payments and, hence, its strategy is irrelevant. The equilibrium existence follows by arguing that the strategy profile maximizing a_v is a Nash equilibrium. To see that, observe that if this strategy profile is not an

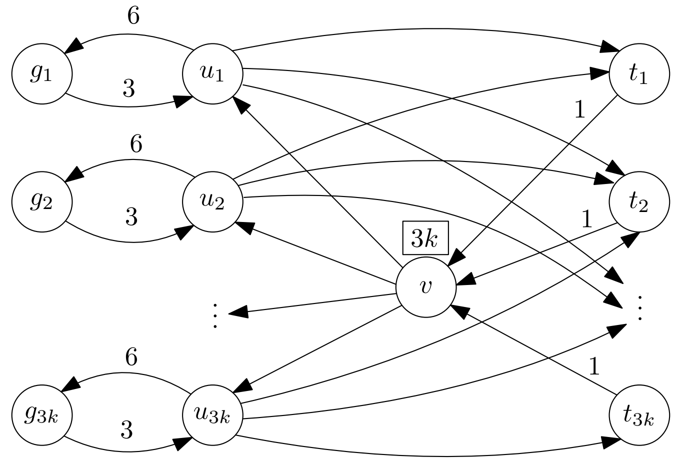


Figure 12: The reduction used to prove hardness of computing a Nash equilibrium when one is guaranteed to exist. All edges with missing labels correspond to liability M where M is an arbitrary large constant.

equilibrium, then some u_i from $\mathcal{T}_v$ can deviate profitably. By our claim above, any such u_i has only g_i in its top priority class and, since $a_v \geq 3k$ it holds that $p_{g_i,u_i} = 3$. Therefore, any increase in u_i's total assets must necessarily occur due to an increase in a_v; a contradiction to our choice of strategy profile.

Now, consider any Nash equilibrium; we claim that there exists a solution to instance $\mathcal{I}'$ of RXC3 if and only if $a_v = 6k$ in the equilibrium. Let us assume that there exists a solution to $\mathcal{I}'$, which means that there exists $C' \subseteq C$ of size k that contains each element of X exactly once. In this case, the strategy profile where the top priority class $\mathcal{T}_v$ of bank v includes the u_i's corresponding to $i \in C'$, while any u_i chooses strategy $(g_i | t_x, t_y, t_z)$, forms a Nash equilibrium where $a_v = 6k$. Indeed, observe that, under this strategy profile, each t_i receives a payment of 1 from each neighboring u_i, while, similarly, v receives a payment of 1 from each t_i; hence, $a_v = 6k$. Clearly, v cannot improve its total assets, while g_i's and t_i's cannot strategize. It remains to argue about the u_i's. Note that for those u_i's corresponding to $i \in C/C'$, their strategy is irrelevant as they receive no payment from v, and hence from g_i as well. For those u_i's corresponding to $i \in C'$, it suffices to note that, since v already receives the maximum possible incoming payments, any deviating strategy for u_i cannot increase u_i's incoming payments.

On the other direction, we now show that any Nash equilibrium with $a_v = 6k$ leads to a solution for instance $\mathcal{I}'$. We begin by showing that, in order for $a_v = 6k$ to hold, it must be that $|\mathcal{T}_v| = k$. Indeed, if $|\mathcal{T}_v| < k$, then at most $3 \cdot |\mathcal{T}_v| < 3k$ t_i's receive any payment from the u_i's and, hence, it cannot be the case that v receives a total payment of $3k$ from the t_i's; a contradiction to $a_v = 6k$. Otherwise, when $|\mathcal{T}_v| > k$, since each u_i in $\mathcal{T}_v$ strictly prioritizes g_i, it holds that the payment from each such u_i to its neighboring t_i's is $\max\left\{0, a_{u_i} - 6\right\} = \max\{0, \frac{a_v}{|\mathcal{T}_v|} + 3 - 6\}$, as each such u_i receives the same payment from v. So, the total payments from the u_i's in $\mathcal{T}_v$ to the t_i's equals $\sum_{i \in \mathcal{T}_v} \max\left\{0, \frac{a_v}{|\mathcal{T}_v|} - 3\right\} = \max\{0, a_v - 3 \cdot |\mathcal{T}_v|\} < a_v - 3k = 3k$, as $|\mathcal{T}_v| > k$. That is, the total payments received by the t_i's are strictly less than $3k$ and, therefore, so are the payments received by v; a contradiction to $a_v = 6k$.

It suffices to prove our claim that, in any equilibrium, any u_i in $|\mathcal{T}_v|$ has just g_i in its topmost priority class, i.e., this choice maximizes a_{u_i}. Assume that there exists a strategy profile $\mathbf{s}$ where this does not hold for at least one u_i in $\mathcal{T}_v$. We will prove that $\mathbf{s}$ is not an equilibrium; we will first treat the case where, under $\mathbf{s}$, g_i is not in $\mathcal{T}_{u_i}$ and, then, the case where g_i is not the only bank in $\mathcal{T}_{u_i}$.

Recall that only bank v and the u_i's can strategize about their payments. For any u_i, we denote by $\mathcal{O}_{u_i}$ the three neighbors of u_i among the t_i's.

Case 1. When g_i is not in $\mathcal{T}_{u_i}$, then $p_{u_i,g_i} = p_{g_i,u_i} = 0$, since M is an arbitrarily large integer. So, $a_{u_i}(\mathbf{s}) = \frac{a_v(\mathbf{s})}{|\mathcal{T}_v|}$. We now consider the deviation of u_i to a strategy having g_i as the only bank in $\mathcal{T}_{u_i}$, regardless of how the remaining priority class are formed; let $\mathbf{s}'$ be the resulting profile. In that case, since u_i is in $\mathcal{T}_v$, we have $p_{g_i,u_i} = 3$, as any incoming payment from v to u_i in $\mathcal{T}_v$ will necessarily travel along the cycle with g_i until the liability of g_i to u_i is fully paid. So, the total assets of u_i under its new strategy is $a_{u_i}(\mathbf{s}') = \frac{a_v(\mathbf{s}')}{|\mathcal{T}_v|} + 3$ and it remains to argue about

$a_v(\mathbf{s}')$. Observe that the maximum incoming payment to v from banks in $\mathcal{O}_{u_i}$ is at most 3 and this bounds the impact on v's total assets caused by u_i's deviation, i.e., $a_v(\mathbf{s}) - a_v(\mathbf{s}') \leq 3$.

Therefore, we can get

$$a_{u_i}(\mathbf{s}) = \frac{a_v(\mathbf{s})}{|\mathcal{T}_v|} \leq \frac{a_v(\mathbf{s}') + 3}{|\mathcal{T}_v|} = \frac{a_v(\mathbf{s}')}{|\mathcal{T}_v|} + \frac{3}{|\mathcal{T}_v|} < \frac{a_v(\mathbf{s}')}{|\mathcal{T}_v|} + 3 = a_{u_i}(\mathbf{s}').$$

Note that last strict inequality clearly holds when $|\mathcal{T}_v| > 1$. When $|\mathcal{T}_v| = 1$, there is only bank in $\mathcal{T}_v$ and, then, $a_{u_i}(\mathbf{s}) \leq 3k + 3$ if g_i is not in $\mathcal{T}_{u_i}$. However, if g_i is in $\mathcal{T}_{u_i}$, then $a_{u_i}(\mathbf{s}') \geq 3k + 4 > 3k + 3 \geq a_{u_i}(\mathbf{s})$.

Case 2. When g_i is not the only bank in $\mathcal{T}_{u_i}$, due to the arbitrarily large liability M between u_i and the bank(s) from $\mathcal{O}_{u_i}$ belonging to $\mathcal{T}_{u_i}$, we have that $p_{u_i,g_i} = p_{g_i,u_i} < 3$.

Let $\lambda \in \{1, 2, 3\}$ be the number of banks from $\mathcal{O}_{u_i}$ that are in $\mathcal{T}_{u_i}$, together with g_i. Since $p_{u_i,g_i} = p_{g_i,u_i} < 3$ we obtain

$$p_{g_i,u_i} + \frac{a_v(\mathbf{s})}{|\mathcal{T}_v|} = p_{u_i,g_i} + \frac{\lambda M}{6} \cdot p_{u_i,g_i}. \tag{10}$$

As $p_{u_i,g_i} = p_{g_i,u_i}$, we get $p_{g_i,u_i} = \frac{6a_v(\mathbf{s})}{\lambda|\mathcal{T}_v|\cdot M}$, therefore the total assets of u_i are $a_{u_i}(\mathbf{s}) = \frac{a_v(\mathbf{s})}{|\mathcal{T}_v|} + \frac{6a_v(\mathbf{s})}{\lambda|\mathcal{T}_v|\cdot M}$. When u_i deviates to a strategy having g_i as the only bank in $\mathcal{T}_{u_i}$, we obtain the strategy profile $\mathbf{s}'$ and u_i's total assets become

$$a_{u_i}(\mathbf{s}') = \frac{a_v(\mathbf{s}')}{|\mathcal{T}_v|} + 3.$$

As before, we have $a_v(\mathbf{s}) - a_v(\mathbf{s}') \leq 3$, and we obtain

$$\begin{aligned} a_{u_i}(\mathbf{s}) &= \frac{a_v(\mathbf{s})}{|\mathcal{T}_v|} + \frac{6a_v(\mathbf{s})}{\lambda|\mathcal{T}_v| \cdot M} \\ &\leq \frac{a_v(\mathbf{s}') + 3}{|\mathcal{T}_v|} + \frac{6a_v(\mathbf{s})}{\lambda|\mathcal{T}_v| \cdot M} \\ &= \frac{a_v(\mathbf{s}')}{|\mathcal{T}_v|} + \frac{3}{|\mathcal{T}_v|} + \frac{6a_v(\mathbf{s})}{\lambda|\mathcal{T}_v| \cdot M} \\ &< \frac{a_v(\mathbf{s}')}{|\mathcal{T}_v|} + 3 \\ &= a_{u_i}(\mathbf{s}'), \end{aligned} \tag{11}$$

where the last (strict) inequality holds since $|\mathcal{T}_v| > 1$ and since M is arbitrarily large.

Observe that when $|\mathcal{T}_v| = 1$, we have $a_{u_i}(\mathbf{s}) \leq 3k + 3 + \frac{18k+18}{\lambda M}$ when g_i shares the first priority with λ banks from $\mathcal{O}_{u_i}$. However, if u_i selects just g_i to be in $\mathcal{T}_{u_i}$, then $a_{u_i}(\mathbf{s}') \geq 3k + 4 > a_{u_i}(\mathbf{s})$.

This concludes the proof for our claim that, at any equilibrium, each u_i in $\mathcal{T}_v$ has g_i as the only bank in $\mathcal{T}_{u_i}$, and, hence, the theorem follows. □

Note that the above theorem also implies NP-hardness for the problems of computing a Nash equilibrium when one is guaranteed to exist as well as verifying whether a given strategy is a best-response strategy. The first statement is an immediate conclusion, while for the second, about computing best responses, the proof follows along (almost) identical lines to the previous one. Indeed, it suffices to fix the strategy of each u_i so that g_i is the only entry in $\mathcal{T}_{u_i}$ while all banks in $\mathcal{O}_{u_i}$ are in the second priority class. Then, using identical arguments as before, we can show that v has a strategy that leads to $a_v = 6k$ if and only if the instance in RXC3 admits a solution.

Theorem 16. *Deciding if a Nash equilibrium exists is NP-hard when banks aim to maximize their total assets.*

Proof. Our proof follows from a reduction from the RXC3 problem, similarly to the proof of Theorem 15 for the hardness of computing a Nash equilibrium, when one is guaranteed to exist. In fact, we modify the construction used in the proof of Theorem 15 and combine it with the network used in the proof of Theorem 4 that does not admit Nash Equilibria.

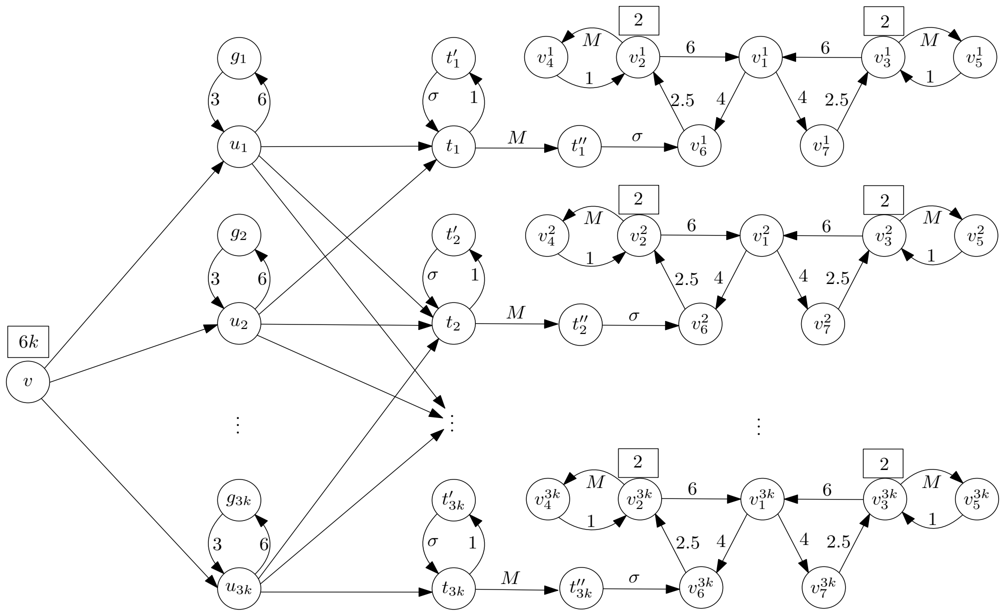


Figure 13: The reduction used to show hardness of determining if pure Nash equilibrium exists where M is an arbitrary large constant, and $\varepsilon = \frac{6}{M+6}$ as well as $\sigma = \frac{1}{4} - \frac{15}{4}\varepsilon$. All edges with missing labels correspond to liability M.

Recall that in RXC3, we are given an element set X, with $|X| = 3k$ for an integer k, and a collection C of subsets of X where each such subset contains exactly three elements. Furthermore, each element in X appears in exactly three subsets in C, that is $|C| = |X| = 3k$. The question is if there exists a subset $C' \subseteq C$ of size k that contains each element of X exactly once.

Given an instance $\mathcal{I}$ of RXC3, we construct an instance $\mathcal{I}'$ as follows. There is a central bank v, having external assets of $6k$, and we add banks t_i, t'_i and t''_i for each element i of X, as well as two banks u_i, g_i for each element i in C; all banks except v have zero external assets. Bank v has a liability of M to each u_i, where M is an arbitrarily large integer. Each u_i, corresponding to set $(x, y, z) \in C$, has liability M to the three banks t_x, t_y, t_z respectively corresponding to the three elements $x, y, z \in X$. Also, u_i has liability 6 towards g_i, while g_i has liability 3 towards u_i. Furthermore, each t_i has liabilities of 1 and M towards t'_i and t''_i respectively, while bank t'_i has liability σ to t_i. Lastly, we add $3k$ copies of the instance depicted in Figure 2 and introduce a liability of σ from each t''_i to bank v^i_6 that is the corresponding v_6 bank from the i-th copy; see also Figure 13.

As before, let $\mathcal{T}_j$ denote the top priority of bank j, while $\mathcal{O}_{u_i}$ stands for the set of those t_i's with an incoming edge from u_i. Note that, as in the proof of Theorem 15, the (weakly) dominant strategy of each u_i is to prioritize g_i alone, while the (weakly) dominant strategy of each t_i is to prioritize t'_i alone. Observe that any u_i in $\mathcal{T}_v$ that collects a payment of x from v will pay $\max\{0, x-3\}$ to the t_i's, while, similarly, any t_i that collects a payment of x from the u_i's will pay $\max\{0, x-(1-\sigma)\}$ to the t''_i.

According to Remark 1, an additional incoming payment of (at least) σ to v_6 in Figure 2 is a sufficient and necessary condition for a Nash equilibrium to exist. This implies that a necessary and sufficient condition for a Nash equilibrium to exist in Figure 13 is that each t''_i makes a payment of at least σ to the corresponding v^i_6 bank. This, in turn, is equivalent to each t_i receiving payments of at least 1 from the u_i's.

Now, we will claim that there exists a Nash equilibrium if and only if there is a solution to $\mathcal{I}'$ of RXC3. If there is a solution to $\mathcal{I}'$ of RXC3, then we let the top priority class $\mathcal{T}_v$ of bank v include just the u_i's corresponding to $i \in C'$, while any u_i chooses the strategy where g_i is alone in $\mathcal{T}_{u_i}$ while all banks in $\mathcal{O}_{u_i}$ are in the second priority class. This eventually ensures that each t^i receives an incoming payment of 1 from the u_i's, as required for a Nash equilibrium to exist.

On the other hand, if there does not exist a solution to $\mathcal{I}'$ of RXC3, we show that at least one t_i receives a payment of strictly less than 1 and, hence, no equilibrium exists. To see that, we argue about $|\mathcal{T}_v|$. If $|\mathcal{T}_v| < k$, then at most $3|\mathcal{T}_v| < 3k$ t_i's can receive incoming payments from the u_i's, so at least one t_i bank receives less than 1. If $|\mathcal{T}_v| > k$, then the aggregate payment received by the t_i's from the u_i's is at most $6k - 3|\mathcal{T}_v| < 3k$, i.e., again at least one t_i receives less than 1. It remains to argue about the case where $|\mathcal{T}_v| = k$. Then, the only case where k u_i's can cover all $3k$ t_i's is when the RXC3 instance admits a solution; a contradiction to our assumption. □

Theorem 17. *Computing a strategy profile that maximizes the sum of total assets is NP-hard.*

Proof. Our proof follows by a reduction from the PARTITION problem. Recall that in PARTITION, an instance $\mathcal{I}$ consists of a set X of positive integers $\{x_1, x_2, ..., x_k\}$ and the question is whether there exists a subset X^* of X such that $\sum_{i\in X^*} x_i = \frac{\sum_{i\in X} x_i}{2} = \frac{S}{2}$; we restrict attention to non-trivial instances where S is even. Starting from $\mathcal{I}$, we build an instance $\mathcal{I}'$ as follows. Let $e_i = x_i$ and $M = 3k \max_i {e_i}^2$. We add three banks v_i, v_i^1 and v_i^2 for each element $x_i \in X$. For each i, we add an edge with liability $2M$ from v_i to v_i^1, we add an edge with liability M from v_i^1 to v_i^2, and we add an edge with liability e_i from v_i^2 to v_i. We add an extra bank s that has liability e_i towards each bank v_i, for $i = 1, \dots, k$. Bank s has external assets equal to $\frac{S}{2}$. Moreover, we add a sequence of nodes $t_1, t_2, \dots, t_n$, where $n \geq 10$. We add edges of liability $2e_i$ from each v_i to t_1, as well as edges of liability $\frac{S}{2}$ from each t_i to t_{i+1}, for $i = 1, \dots, n-1$. Finally, there is an edge of liability $\frac{S}{2}$ from t_n to s; see also Figure 14. Clearly, the reduction requires polynomial time.

We will prove that there exists a solution to instance $\mathcal{I}$ of PARTITION if and only if a social welfare of $\frac{n+9}{2}S = (n+9)\frac{\sum_{i\in X} x_i}{2}$ can be achieved. Let us assume that instance $\mathcal{I}$ admits a solution, i.e. there exists a subset X^* in $\mathcal{I}$ with $\sum_{i\in X^*} x_i = \frac{S}{2}$. Let each v_i that corresponds to $x_i \in X^*$ prioritize payments towards t_1 (pay according to $(t_1|v_i^1)$), while the remaining v_i's corresponding to $x_i \in X \setminus X'$ prioritize v_i^1 (pay according to $(v_i^1|t_1)$). This implies that the total assets of t_1 are $a_{t_1} \geq \frac{\sum_{i\in X} x_i}{2}$, since all edges in the path from t_1 to s as well as all outgoing edges of s are saturated. Overall, the social welfare of this assignment, measured as the sum of total assets of all banks, satisfies

$$\begin{aligned}
\mathcal{SW}(\mathbf{s}_Y^{opt}) &\geq e_s + \sum_{i\in X} p(s, v_i) + \sum_{i\in X^*} p(v_i, t) + \sum_{i\in X\setminus X^*} \{p(v_i, v_i^1) + p(v_i^1, v_i^2) + p(v_i^2, v_i)\} + \sum_{i=1}^{n-1} p(t_i, t_{i+1}) + p(t_n, s) \\
&= \frac{S}{2} + S + \sum_{i\in X^*} x_i + \sum_{i\in X\setminus X^*} \{2x_i + 2x_i + x_i\} + (n-1)\frac{S}{2} + \frac{S}{2} \\
&= (n+3)\frac{S}{2} + \sum_{i\in X^*} x_i + \sum_{i\in X^*} \{2x_i + 2x_i + x_i\} \\
&= (n+9)\frac{S}{2},
\end{aligned}$$

where the last two inequalities hold since X^* is a solution to instance $\mathcal{I}$. In fact, the inequality above is actually an equality, as the flow on the edges that we seemingly ignore is always zero, but this is not important at this point.

We will now prove that if there does not exist a solution to instance $\mathcal{I}$ of PARTITION then the optimal social welfare is less than $\frac{n+9}{2}S = (n+9)\frac{\sum_{i\in X} x_i}{2}$. We note that each bank v_i has three possible strategies, namely to prioritize t_1, to prioritize v_i^1 and to pay proportionally. The strategy each of these banks, v_i, chooses, determines the payments between itself, v_i^1, v_i^2 and t_1, for a fixed incoming payment. Figure 15 shows the actual payments on the respective subgraph based on different strategies of the v_is, if we assume an incoming payment of e_i to v_i; which will be useful later in the proof.

Our proof uses the following claim.

Claim 18. *If $\mathcal{SW}(\mathbf{s}) \geq (n+9)\frac{S}{2}$ then*

(i) t_1 is solvent,

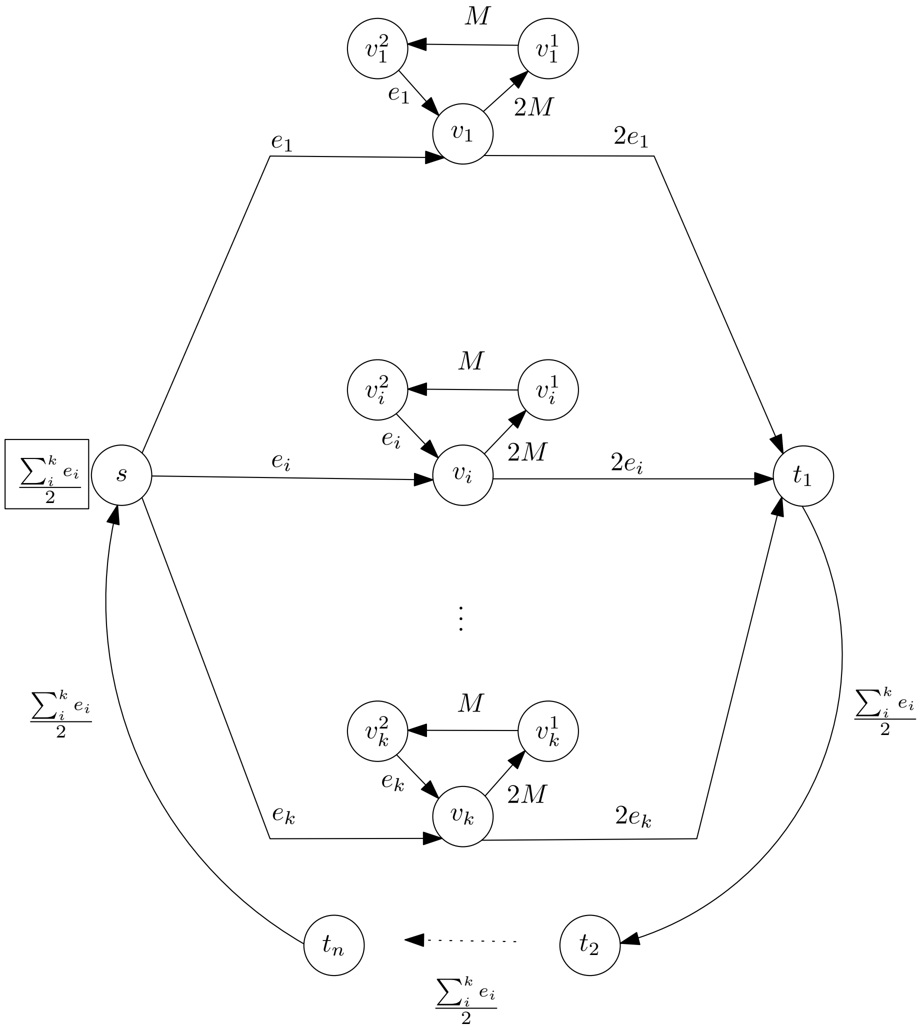


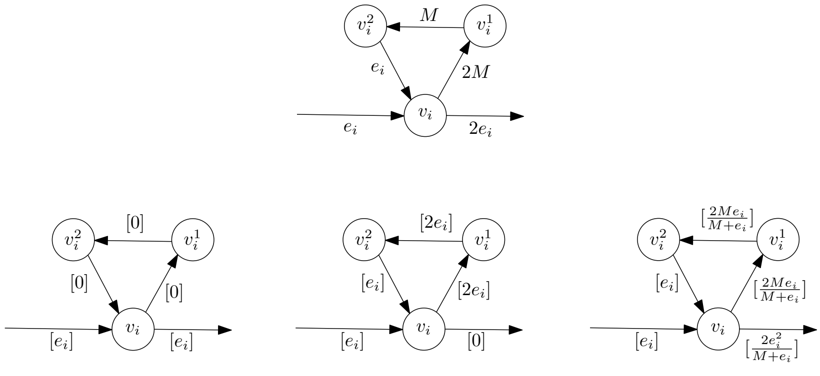
Figure 14: The reduction used to show the hardness of computing a strategy profile that maximizes the sum of total assets.

(ii) $\sum_{v_i \in T} e_i \geq \frac{S}{2}$*, where* T *is the set of* v_i*s prioritising payments to* t_1*, and*

(iii) none of the v_i*s pay proportionally at the strategy profile that maximizes the sum of total assets.*

Proof. We begin with some definitions that will be useful in the proof. Let ξ_1 be the set of v_is that prioritize payments towards t_1 (pay according to $(t_1|v_i^1)$), let ξ_2 be the set of v_i's that prioritize payments towards v_i^1 (pay according to $(v_i^1|t_1)$), and let ξ_3 be the set of v_i's paying proportionally (pay according to (t_1, v_i^1)); clearly ξ_1, ξ_2 and ξ_3 form a partition of the v_is. We prove part (i) by showing that if t_1 is insolvent then the optimal social welfare is less than $\frac{n+9}{2}S = (n+9)\frac{\sum_{i \in X} x_i}{2}$. If we denote the actual payment through the edges on the path from t_1 to s by F, then the social welfare under any strategy profile $\mathbf{s}$, satisfies the following (the calculations are shown in Figure 15).

$$\begin{aligned}\mathcal{SW}(\mathbf{s}) &\leq e_s + n \cdot F + 2\sum_{v_i \in \xi_1} e_i + 6\sum_{v_i \in \xi_2} e_i + \sum_{v_i \in \xi_3}\left(6e_i - 2\frac{e_i^2}{M + e_i}\right) \\ &\leq \frac{S}{2} + n \cdot F + 2\sum_{v_i \in \xi_1} e_i + 6\sum_{v_i \in \xi_2 \cup \xi_3} e_i - 2\sum_{v_i \in \xi_3}\frac{e_i^2}{M + e_i}. \end{aligned} \tag{12}$$



$v_i : (t_1|v_i^1)$ $v_i : (v_i^1|t_1)$ $v_i : (v_i^1, t_1)$

Figure 15: The network on the top is the original sub-network of Figure 14 which we focus on. The following three subgraphs show the different payments respectively depending on different strategies v_i uses. The numbers on the square brackets are the corresponding payments.

If we assume that t_1 is insolvent, then we can show that the above expression is strictly less than $(n+9)\frac{S}{2}$; we distinguish between two cases depending on the quantity $\sum_{\xi_2\cup\xi_3} e_i$. If $\sum_{\xi_2\cup\xi_3} e_i \le \frac{S}{2}$, then $\mathcal{SW}(\mathbf{s})$ is maximized for $\sum_{\xi_1} e_i = \sum_{\xi_2\cup\xi_3} e_i = \frac{S}{2}$. To see this note that ξ_1, ξ_2 and ξ_3 form a partition of the v_is and $M = 5k\cdot(max\{e_1, \cdots e_k\})^2$ by definition. Since, by assumption, t_1 is insolvent, i.e. $F < \frac{S}{2}$, we get that $\mathcal{SW}(\mathbf{s}) < (n+9)\frac{S}{2}$. Now, if $\sum_{\xi_2\cup\xi_3} e_i > \frac{S}{2}$, then we can conclude that $\sum_{\xi_2\cup\xi_3} e_i = \frac{S}{2} + \lambda$ for some $\lambda \ge 1$, since S is even and the e_is are integers. Then $\sum_{\xi_1} e_i = \frac{S}{2} - \lambda$ and the total incoming payments of t_1 are $\frac{S}{2} - \lambda + 2\sum_{\xi_3}\frac{e_i^2}{M+e_i}$, implying F is upper-bounded by this quantity. Therefore, from (12), we get that

$$
\begin{aligned}
\mathcal{SW}(\mathbf{s}) &\le \frac{S}{2} + n\cdot\left(\frac{S}{2} - \lambda + 2\sum_{\xi_3}\frac{e_i^2}{M+e_i}\right) + 2\cdot(\frac{S}{2} - \lambda) + 6\cdot(\frac{S}{2} + \lambda) - 2\sum_{\xi_3}\frac{e_i^2}{M+e_i} \\
&\le (n+9)\frac{S}{2} + 2\cdot(n-1)\sum_{\xi_3}\frac{e_i^2}{M+e_i} - (n-4)\cdot\lambda \\
&< (n+9)\frac{S}{2} + 2\cdot(n-1)k\frac{e_i^2}{3ke_i^2} - (n-4) \\
&\le (n+9)\frac{S}{2} + \frac{2}{3}\cdot(n-1) - (n-4) \\
&\le (n+9)\frac{S}{2} - \frac{n-10}{3} \\
&\le (n+9)\frac{S}{2},
\end{aligned}
$$

where the strict inequality holds since $|\xi_3| \le k$, $e_i > 0$ and $M = 3k\max_i {e_i}^2$, while the last inequality holds since $n \ge 10$ by assumption. We can conclude that if $\mathcal{SW}(\mathbf{s}) \ge (n+9)\frac{S}{2}$, then t_1 is solvent.

Next, we show part (ii), i.e. that if $\mathcal{SW}(\mathbf{s}) \ge (n+9)\frac{S}{2}$ then $\sum_{\xi_1} e_i \ge \frac{S}{2}$. From Figure (15), we can see that the total assets (incoming payments) of t_1 satisfy the following

$$
\begin{aligned}
a_{t_1} &= \sum_{\xi_1} e_i + 0 + \sum_{\xi_3}\frac{2e_i^2}{M+e_i} \\
&\le \sum_{\xi_1} e_i + 0 + k\frac{2e_i^2}{3ke_i^2 + e_i}
\end{aligned}
$$

$$< \sum_{\xi_1} e_i + 0 + \frac{2}{3},$$

where the strict inequality holds since $e_i > 0$ and $M = 3k \max_i {e_i}^2$. By the integrality of the e_is we can conclude that $\sum_{\xi_1} e_i < \frac{S}{2}$ is equivalent to $\sum_{\xi_1} e_i \leq \frac{S}{2} - 1$, which would imply that ${}_{t_1} < \frac{S}{2}$. However, we have proved that our assumption that $\mathcal{SW}(\mathbf{s}) \geq (n+9)\frac{S}{2}$ implies that t_1 is solvent, hence reaching a contradiction. We can conclude that $\sum_{\xi_1} e_i \geq \frac{S}{2}$.

Finally, we prove part (iii), by showing that at the strategy profile that maximizes the social welfare, none of the v_is pay proportionally, under the assumptions that t_1 is solvent and $\sum_{\xi_1} e_i \geq \frac{S}{2}$, also using parts (i) and (ii). Since the payments from banks in ξ_1 to t_1 are enough to saturate the edges of the path starting from t_1 to s, then these, as well as the outgoing edges of s are all saturated regardless of the set of banks in ξ_2 and ξ_3. By looking at Figure 15, we can see that the total flow on the edges $(v_i, v_i^1), (v_i^1, v_i^2), (v_i^2, v_i)$ and (v_i, t_1) is higher for $v_i \in \xi_2$ as opposed to $v_i \in \xi_3$. We can conclude that at the strategy profile that maximizes social welfare, if t_1 is solvent and $\sum_{\xi_1} e_i \geq \frac{S}{2}$, then no bank v_i pays proportionally, i.e. $\xi_3 = \emptyset$. □

We continue with the proof of our main claim and prove that if there does not exist a solution to instance $\mathcal{I}$ of PARTITION then the optimal social welfare is less than $\frac{n+9}{2}S = (n+9)\frac{\sum_{i \in X} x_i}{2}$. We assume otherwise, that there does not exist a solution to instance $\mathcal{I}$ of PARTITION but the optimal social welfare is greater than $\frac{n+9}{2}S = (n+9)\frac{\sum_{i \in X} x_i}{2}$. From Claim 18(iii), we know that none of the v_i's pays proportionally at the strategy profile that maximizes social welfare, so we denote by X' be the set of v_i that prioritize payments towards t_1 (pay according to $(t_1 | v_i^1)$), while the remaining v_i's corresponding to $x_i \in X \setminus X'$ prioritize payments towards v_i^1 (pay according to $(v_i^1 | t_1)$). From Claim 18(i) we know that the edges on the path between t_1 and s, as well as the outgoing edges of s are saturated. Overall, we get that the optimal social welfare $\mathcal{SW}(\mathbf{s}^{opt})$, under our assumptions, satisfies

$$\begin{aligned}
\mathcal{SW}(\mathbf{s}^{opt}) &= e_s + n \cdot \frac{S}{2} + S + \sum_{i \in X'} e_i + \sum_{i \in X \setminus X'} 5e_i \\
&= (n+3)\frac{S}{2} + \sum_{i \in X} e_i + 4 \sum_{i \in X \setminus X'} e_i \\
&< (n+3)\frac{S}{2} + 3 \sum_{i \in X} e_i \\
&= (n+3)\frac{S}{2} + 3S \\
&= (n+9)\frac{S}{2},
\end{aligned} \tag{13}$$

where the first strict inequality is derived by Lemma 18(ii) and our assumption that there does not exist a solution to instance $\mathcal{I}$. This implies that$\sum_{X'} e_i > \frac{S}{2}$, hence $\sum_{i \in X \setminus X'} e_i < \frac{S}{2}$. We have reached a contradiction to our assumption that the optimal social welfare is at least equal to $\frac{n+9}{2}S = (n+9)\frac{\sum_{i \in X} x_i}{2}$, as desired. The proof is complete. □

As we claimed before, in the network without default cost, the sum of equities (also called *market value* in finance terminology) remains unchanged and is equal to $\sum_i e_i$. However, once we take default costs into account, then things are much different.

Theorem 19. *In a network with default costs $\alpha \in (0,1)$ and $\beta \in [0,1)$, computing a strategy profile that maximizes the sum of equities is NP-hard.*

Proof. Our proof follows by a reduction from the PARTITION problem. Recall that in PARTITION, an instance $\mathcal{I}$ consists of a set X of positive integers $\{x_1, x_2, ..., x_k\}$ and the question is whether there exists a subset X^* of X such that $\sum_{i \in X^*} x_i = \frac{\sum_{i \in X} x_i}{2}$; we restrict attention to non-trivial instances where the sum of x_i is even. Starting from $\mathcal{I}$, we build an instance $\mathcal{I}'$ as follows. For each element $x_i \in X$, we add three banks v_i, v_i' and v_i'', and allocate external assets of $\frac{e_i}{\alpha} = \frac{x_i}{\alpha}$ to each v_i for any given α. For each i, we add edges with liability $\frac{e_i}{\alpha}$ from v_i to v_i' and to v_i'' respectively. Furthermore, we add four extra banks G, G', T and T' where both G and T have external assets of $\frac{\sum_i e_i}{2}$. Moreover, for each i, v_i' and v_i'' have a liability of $e_i = x_i$ to G and T, respectively; while G and T have a liability of $\sum_i e_i$ to G' and T' respectively. See also Figure 16. Clearly, the reduction requires polynomial time.

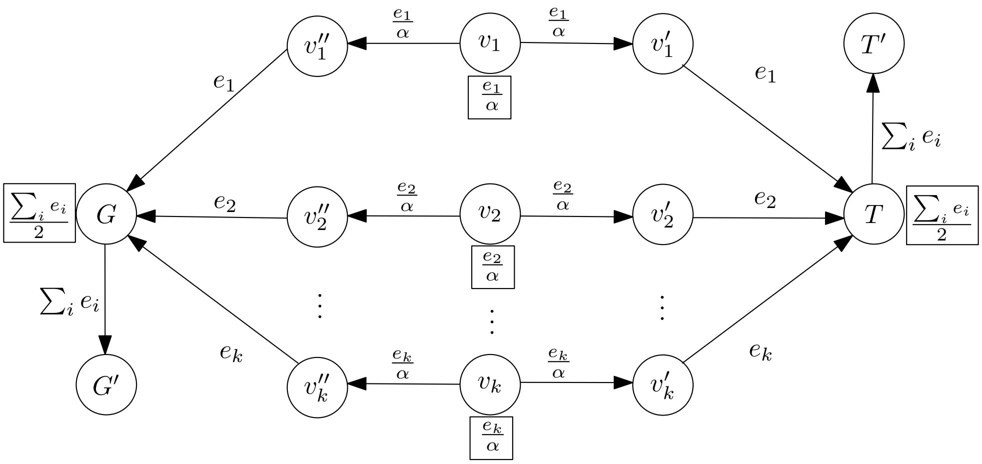


Figure 16: The reduction used to show the hardness of computing a strategy profile maximizing the sum of equities.

Note that each v_i would be always in default since its total liabilities of $\frac{2e_i}{\alpha}$ is strictly larger than its total assets of $\frac{e_i}{\alpha}$, and the total outgoing payments from each single v_i is exactly equal to $\frac{e_i}{\alpha} \cdot \alpha = e_i$. Additionally, since $\frac{e_i}{\alpha} > e_i$, then the other v_i's lender would receive nothing if v_i prioritizes the payment towards one of v_i' and v_i'' uniquely.

We will prove that there exists a solution to instance $\mathcal{I}$ of PARTITION if and only if the sum of equities of $2\sum_i e_i$ can be achieved. Let us assume that instance $\mathcal{I}$ admits a solution, i.e. there exists a subset X^* in $\mathcal{I}$ with $\sum_{i \in X^*} x_i$. Let each v_i that corresponds to $x_i \in X^*$ prioritize payments towards v_i' uniquely, while the remaining v_i's corresponding to $x_i \in X \setminus X^*$ prioritize v_i'' alone. In this case, all the v_i's corresponding to $x_i \in X^*$ are exactly solvent with sending out a payment of e_i to T, thereby making T exactly solvent and resulting with $E_{T'} = \sum_i e_i$. The similar thing occurs to the remaining v_i's corresponding to $X \setminus X^*$, which returns $E_{G'} = \sum_i e_i$. Now all the banks except G' and T' have zero equities, therefore, this strategy profile results in the sum of equities of $2\sum_i e_i$.

We will now prove that if there does not exist a solution to instance $\mathcal{I}$ of PARTITION then the maximal sum of equities is less than $2\sum_i e_i$. Before continuing the detailed proof, we first present a claim which is necessary afterwards.

Claim 20. *If at least one v_i pays proportionally, then the sum of equities must be strictly less than $2\sum_i e_i$. Namely, none of v_is would pay proportionally under the strategy profile that maximizes the sum of equities.*

Proof. Let V' and V'' be the set of v_is who uniquely prioritize v_i' and v_i'' respectively, while V^* stands for the remaining v_is who pay proportionally. As claimed before, regardless of what strategy that v_i plays, the total payments that v_i can make is exactly e_i, i.e., $p_{v_i,v_i'} + p_{v_i,v_i''} \equiv e_i$. According to the rule of proportionality, if v_i pays proportionally, then we have $p_{v_i,v_i'} = p_{v_i,v_i''} = \frac{e_i}{2} < e_i$, which would make corresponding v_i' and v_i'' insolvent and trigger the loss of equities. In particular, the total incoming payments from all v_i's to T is $\beta \cdot \sum_{v_i \in V*} p_{v_i,v_i'} + \sum_{v_i \in V'} p_{v_i,v_i'} = \beta \cdot \sum_{v_i \in V*} \frac{e_i}{2} + \sum_{v_i \in V'} e_i$, resulting in $a_T = \beta \cdot \sum_{v_i \in V*} \frac{e_i}{2} + \sum_{v_i \in V'} e_i + \frac{\sum_i e_i}{2}$. Similarly, we can achieve $a_G = \beta \cdot \sum_{v_i \in V*} \frac{e_i}{2} + \sum_{v_i \in V''} e_i + \frac{\sum_i e_i}{2}$. Since the sum of liabilities for G and T is $L_G + L_T = 2\sum_i e_i$, but their total asset is

$$\begin{aligned} a_T + a_G &= (\beta \cdot \sum_{v_i \in V*} \frac{e_i}{2} + \sum_{v_i \in V'} e_i + \frac{\sum_i e_i}{2}) + (\beta \cdot \sum_{v_i \in V*} \frac{e_i}{2} + \sum_{v_i \in V''} e_i + \frac{\sum_i e_i}{2}) \\ &= \sum_i e_i + \beta \cdot \sum_{v_i \in V*} e_i + \sum_{v_i \in V'} e_i + \sum_{v_i \in V''} e_i \end{aligned}$$

$$< 2\sum_i e_i = L_G + L_T.$$

This implies that at least one bank between G and T would be in default. Without loss of generality, let's assume that G is in default. Then the sum of equities satisfies

$$\begin{aligned}\sum_i E_i &= E_{G'} + E_{T'} \\ &\leq \max\{\alpha, \beta\} \cdot a_G + a_T \\ &< a_T + a_G \\ &< 2\sum_i e_i.\end{aligned}$$

The strict inequalities above are due to both α and β are strictly less than 1. □

We continue with the proof of our main claim and prove that if there does not exist a solution to instance $\mathcal{I}$ of PARTITION then the optimal market value is less than $2\sum_i e_i$. We assume otherwise, that there does not exist a solution to instance $\mathcal{I}$ of PARTITION but the maximal sum of equities is greater than $2\sum_i e_i$. From Lemma 20, we know that v_i prioritizes either v_i' or v_i'' alone at the strategy profile that maximizes the sum of equities, so we denote by X' the set of v_i that prioritizes payments towards v_i', while the remaining v_i's corresponding to $x_i \in X \setminus X'$ prioritize payments towards v_i'' alone. Since $\sum_i p_{v_i,v_i'} + \sum_i p_{v_i,v_i''} = \sum_i e_i$ and there does not exist a solution to instance $\mathcal{I}$ of PARTITION, therefore, without loss of generality, let us assume that $\sum_i p_{v_i,v_i'} > \frac{\sum_i e_i}{2} > \sum_i p_{v_i,v_i''}$, which implies that bank G would be in default but T solvent. Hence, we get that the maximal sum of equities, under our assumptions, satisfies

$$\begin{aligned}\sum_i E_i &= E_{G'} + E_{T'} \\ &\leq \max\{\alpha, \beta\} \cdot a_G + a_T \\ &= \max\{\alpha, \beta\} \cdot (\sum_i p_{v_i,v_i''} + \frac{\sum_i e_i}{2}) + (\sum_i p_{v_i,v_i'} + \frac{\sum_i e_i}{2}) \\ &< \sum_i e_i + \left[\max\{\alpha, \beta\} \cdot \sum_i p_{v_i,v_i'} + \sum_i p_{v_i,v_i''}\right] \\ &< \sum_i e_i + \left[\sum_i p_{v_i,v_i'} + \sum_i p_{v_i,v_i''}\right] \\ &= \sum_i e_i + \sum_i e_i = 2\sum_i e_i.\end{aligned}$$

The proof is complete.

□

7. Conclusions and discussion

We have studied strategic payment games in financial networks with priority-proportional payments and we have presented an almost full picture both with respect to structural properties of clearing payments as well as the quality of Nash equilibria. We have also studied the computational complexity of various equilibrium-related problems, like determining whether pure Nash equilibria exist, computing the best response, and verifying if a given strategy profile is a pure Nash equilibrium when equilibria always exist. We note that most of our results can be straightforwardly generalized to the setting of maximal (but non-necessarily proper) clearing payments, while some theorems, such as Theorem 6(ii) and Theorem 7, strongly rely on the notion of proper clearing payments.

Our work reveals several interesting open questions. In particular, an interesting restriction on the strategy space is to always prioritize strategies that could lead to incoming payments, i.e., when each firm i always ranks higher a payment that might create a cash flow through a directed cycle (in the liabilities graph) back to i, than a payment that does not. While some of our negative results would still hold, e.g., Theorem 4, others, such as Theorem 6, crucially rely on the reverse ranking. One can also consider from a mechanism design angle, i.e., to design strategy-proof policies where banks weakly prefer proportional payments.

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$s_1 = (v_6 \| v_7)$			
	$s_3 = (v_1 \| v_5)$	$s_3 = (v_1, v_5)$	$s_3 = (v_5 \| v_1)$
$s_2 = (v_1 \| v_4)$	$9, 4.5, 4.5$	$\frac{4.5-\varepsilon}{1-\varepsilon}, 4.5, \frac{3.5}{1-\varepsilon}$	$4.5, 4.5, 3.5$
$s_2 = (v_1, v_4)$	$2 + \frac{(5+\sigma)\varepsilon}{1-\varepsilon}, \frac{5+\sigma}{1-\varepsilon}, 2$	$\frac{(6+\sigma)\varepsilon}{1-\varepsilon}, \frac{3+3\varepsilon+\sigma}{1-\varepsilon}, 3$	$\frac{(3+\sigma)\varepsilon}{1-\varepsilon}, \frac{3+\sigma}{1-\varepsilon}, 3$
$s_2 = (v_4 \| v_1)$	$2, 5 + \sigma, 2$	$3\varepsilon, 3 + 3\varepsilon + \sigma, 3$	$0, 3, 3$

$s_1 = (v_7 \| v_6)$			
	$s_3 = (v_1 \| v_5)$	$s_3 = (v_1, v_5)$	$s_3 = (v_5 \| v_1)$
$s_2 = (v_1 \| v_4)$	$9, 4.5, 4.5$	$2 + \frac{(5\varepsilon+\sigma)}{1-\varepsilon}, 2 + \sigma, \frac{(5+\sigma)}{1-\varepsilon}$	$2 + \sigma, 2 + \sigma, 5 + \sigma$
$s_2 = (v_1, v_4)$	$\frac{4.5-1.5\varepsilon+\sigma\varepsilon}{1-\varepsilon}, \frac{3.5+\sigma}{1-\varepsilon}, 4.5$	$\frac{(6+\sigma)\varepsilon}{1-\varepsilon}, 3 + \sigma, \frac{3+3\varepsilon+\sigma\varepsilon}{1-\varepsilon}$	$3\varepsilon + 3\sigma, 3 + \sigma, 3 + 3\varepsilon + \sigma\varepsilon$
$s_2 = (v_4 \| v_1)$	$4.5, 3.5 + \sigma, 4.5$	$\frac{3\varepsilon}{1-\varepsilon}, 3 + \sigma, \frac{3}{1-\varepsilon}$	$0, 3 + \sigma, 3$

$s_1 = (v_6, v_7)$			
	$s_3 = (v_1 \| v_5)$	$s_3 = (v_1, v_5)$	$s_3 = (v_5 \| v_1)$
$s_2 = (v_1 \| v_4)$	$9, 4.5, 4.5$	$\frac{4+6\varepsilon+2\sigma}{1-\varepsilon}, \frac{4+\varepsilon+2\sigma-\sigma\varepsilon}{1-\varepsilon}, \frac{5+\sigma}{1-\varepsilon}$	$4 + 2\sigma, 4 + 2\sigma, 5 + \sigma$
$s_2 = (v_1, v_4)$	$\frac{4+6\varepsilon+2\sigma}{1-\varepsilon}, \frac{5+\sigma}{1-\varepsilon}, \frac{4+\varepsilon+\sigma\varepsilon}{1-\varepsilon}$	$\frac{6\varepsilon+\sigma\varepsilon}{1-\varepsilon}, \frac{6+\sigma\varepsilon}{2(1-\varepsilon)} + \sigma, \frac{6+\sigma\varepsilon}{2(1-\varepsilon)}$	$\frac{6\varepsilon+2\sigma\varepsilon}{2-\varepsilon}, \frac{6+2\sigma}{2-\varepsilon}, \frac{6+\sigma\varepsilon}{2-\varepsilon}$
$s_2 = (v_4 \| v_1)$	$4, 5 + \sigma, 5$	$\frac{6\varepsilon}{2-\varepsilon}, \frac{6\varepsilon}{2-\varepsilon} + \sigma, \frac{6\varepsilon}{2-\varepsilon}$	$0, 3 + \sigma, 3$

Table 1
The table of utilities for the network without Nash equilibria; note that $\varepsilon = \frac{6}{M+6}$ and $\sigma < \frac{1}{4} - \frac{15\varepsilon}{4}$. Each cell entry contains the utilities of v_1, v_2, v_3 in that order. Each 3×3 subtable corresponds to a fixed strategy for v_1.

A. Proof of Theorem 4 (cont'd)

Proof. The full table of utilities appears in Table 1.

- Case 1: Given v_1 plays $(v_6|v_7)$, corresponding to the sub table on the top, the strategy $(v_4|v_1)$ is dominated by (v_1, v_4) for bank v_2. Now if we start from the strategy profile where v_2 plays (v_1, v_4) and v_3 plays $(v_5|v_1)$, then v_2 has incentive to deviate to $(v_1|v_4)$. Afterwards, v_3 would benefit by moving to (v_1, v_5). Now starting from the strategy profile where v_2 plays $(v_1|v_4)$ and v_3 plays (v_1, v_5), a cycle of best response is formed between the strategy profiles where v_2 plays $(v_1|v_4)$ and v_3 plays (v_1, v_5), where v_2 plays $(v_1|v_4)$ and v_3 plays $(v_1|v_5)$, where v_2 plays (v_1, v_4) and v_3 plays $(v_1|v_5)$, and where v_2 plays (v_1, v_4) and v_3 plays (v_1, v_5) sequentially, eventually back to strategy profile where v_2 plays $(v_1|v_4)$ and v_3 plays (v_1, v_5). Therefore, there does not exist a Nash equilibrium whenever v_1 plays $(v_6|v_7)$.
- Case 2: Although the network is not fully symmetric, the additional v_6's external assets σ are too small to affect the symmetry. Therefore, the case where v_1 plays $(v_7|v_6)$ does not admit any Nash equilibrium as well.
- Case 3: Given v_1 plays (v_6, v_7), corresponding to the sub table at the bottom, strategy (v_1, v_4) is always preferable to bank v_2 than strategy $(v_4|v_1)$, and the strategy profiles where v_2 plays (v_1, v_4) and v_3 plays $(v_5|v_1)$ and where v_2 plays $(v_1|v_4)$ and v_3 plays $(v_5|v_1)$ are not stable for the same reasons as Case 1. In addition, given strategy profile where v_2 plays (v_1, v_4) and v_3 plays $(v_1|v_5)$, v_1 has incentive to move from (v_6, v_7) to $(v_7|v_6)$. Similarly, given strategy profile $< (v_1|v_4), (v_1, v_5) >$, v_1 has incentive to move from (v_6, v_7) to $(v_6|v_7)$. Note that the last two subitems hold since $\sigma < \frac{1}{4} - \frac{15\epsilon}{4}$. Otherwise, it implies that $\frac{4+6\epsilon+2\sigma}{1-\epsilon} \leq \frac{4.5-1.5\epsilon}{1-\epsilon}$, then at least the strategy profile where v_1, v_2 and v_3 play $(v_6, v_7), (v_1|v_4)$ and (v_1, v_5) respectively forms a Nash equilibrium.

□