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K-stability of Casagrande–Druel varieties

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Abstract. We introduce a new subclass of Fano varieties (Casagrande–Druel varieties) that are n-dimensional varieties constructed from Fano double covers of dimension $n-1$. We conjecture that a Casagrande–Druel variety is K-polystable if the double cover and its base space are K-polystable. We prove this for smoothable Casagrande–Druel threefolds, and for Casagrande–Druel varieties constructed from double covers of $\mathbb{P}^{n-1}$ ramified over smooth hypersurfaces of degree $2d$ with $n > d > \frac{n}{2} > 1$. As an application, we describe the connected components of the K-moduli space parametrizing smoothable K-polystable Fano threefolds in the families № 3.9 and № 4.2 in the Mori–Mukai classification.

Contents

Throughout this paper, all varieties are defined over $\mathbb{C}$.

1. Introduction

Let V be a Fano variety with Kawamata log terminal singularities, and let L be a line bundle on V such that the divisor $-(K_V+L)$ is ample, and $|2L|$ contains a non-zero effective

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divisor. Let R be a divisor in $|2L|$, and let $\eta\colon B \to V$ be the double cover ramified over R. Then B can be explicitly constructed as follows. Let $Y = \mathbb{P}(\mathcal{O}_V \oplus \mathcal{O}_V(L))$, let $\pi\colon Y \to V$ be the natural projection, and let ξ be the tautological line bundle on Y. Set $H = \pi^*(L)$. Then we have the isomorphisms

$$H^0(Y, \mathcal{O}_Y(\xi)) \cong H^0(V, \mathcal{O}_V) \oplus H^0(V, \mathcal{O}_V(L)),$$
$$H^0(Y, \mathcal{O}_Y(\xi - H)) \cong H^0(V, \mathcal{O}_V) \oplus H^0(V, \mathcal{O}_V(-L)).$$

Using these isomorphisms, fix sections $u^+ \in H^0(Y, \mathcal{O}_Y(\xi))$ and $u^- \in H^0(Y, \mathcal{O}_Y(\xi - H))$ that correspond to $1 \in H^0(V, \mathcal{O}_V)$ under the isomorphisms above. Set $S^\pm = \{u^\pm = 0\}$. Then we have $S^- \cap S^+ = \varnothing$ and $S^+ \sim S^- + H$. Take $f \in H^0(V, \mathcal{O}_V(2L))$ that defines R. Then we can identify B with the divisor $\{\pi^*(f)(u^-)^2 = (u^+)^2\} \in |2S^+|$, where the double cover η is induced by π.

Remark 1.1. We allow R to be singular, so B can be very singular (and even reducible). However, if the log pair $(V, \frac{1}{2}R)$ has Kawamata log terminal singularities, then the double cover B is a Fano variety with Kawamata log terminal singularities [29]. So, for simplicity, we will always say that B is a Fano double cover (even if B is non-normal or reducible).

Let $F = \pi^*(R)$, and let $\phi\colon X \to Y$ be the blow up of the intersection $S^+ \cap F$. Then

$$X \text{ is smooth} \iff Y \text{ and } B \text{ are smooth} \iff V \text{ and } R \text{ are smooth.}$$

Moreover, the variety X is also a Fano variety (see Section 2).

Definition 1.2. If the Fano variety X has at most Kawamata log terminal singularities, then X is called *the Casagrande–Druel variety* constructed from $\eta\colon B \to V$ (or from the ramification divisor $R \subset V$). Note that $L \in \operatorname{Pic} V$ is uniquely determined by R.

The group $\operatorname{Aut}(Y)$ contains a subgroup $\Gamma \cong \mathbb{G}_m$ that fixes both S^- and S^+ pointwise, and the action of Γ lifts to $\operatorname{Aut}(X)$, so we can identify Γ with a subgroup in $\operatorname{Aut}(X)$. In Section 2, we will show that $\operatorname{Aut}(X)$ also contains an involution ι such that

$$\langle \Gamma, \iota \rangle \cong \mathbb{G}_m \rtimes \boldsymbol{\mu}_2,$$

and ι swaps the proper transforms of the sections S^- and S^+. Set $G = \langle \Gamma, \iota \rangle$ and $\theta = \pi \circ \phi$. Then we have the commutative diagram

$$\begin{tikzcd}
 & X \arrow[r, "\iota"] \arrow[dl, "\phi"'] \arrow[d, "\theta"] & X \arrow[dr, "\phi"] \arrow[d, "\theta"'] & \\
Y \arrow[dr, "\pi"'] & & & Y \arrow[dl, "\pi"] \\
 & V \arrow[r, equal] & V &
\end{tikzcd}$$

and the composition θ is a G-equivariant conic bundle such that G acts trivially on V.

Remark 1.3. Our construction of Casagrande–Druel varieties is inspired by the paper [12]. See [12, Lemma 3.1 (iii)]. But it goes back to the construction of de Jonquieres involutions using hyperelliptic curves instead of Fano double covers. See also [11, 20, 33, 43].

The del Pezzo surface of degree 6 (blow up of $\mathbb{P}^2$ at three general points) is the unique smooth Casagrande–Druel surface. Smooth Casagrande–Druel threefolds form 3 families. To present them, we use labeling of smooth Fano threefolds from [7].

Example 1.4. Let $V = \mathbb{P}^2$, let $L = \mathcal{O}_{\mathbb{P}^2}(1)$, let R be an arbitrary smooth conic in $|2L|$. Then $B \cong \mathbb{P}^1 \times \mathbb{P}^1$, and X is the unique smooth Fano threefold in the family № 3.19.

Example 1.5. Let $V = \mathbb{P}^2$, let $L = \mathcal{O}_{\mathbb{P}^2}(2)$, let R be any smooth quartic curve in $|2L|$. Then B is a del Pezzo surface of degree 2, and X is a Fano threefold in the family № 3.9.

Example 1.6. Let $V = \mathbb{P}^1 \times \mathbb{P}^1$, let $L = \mathcal{O}_V(1,1)$, let R be any smooth curve in $|2L|$. Then B is a del Pezzo surface of degree 4, and X is a Fano threefold in the family № 4.2.

All smooth Casagrande–Druel threefolds are K-polystable; see [27, Theorem 6.1] and [7]. In fact, K-polystable Casagrande–Druel varieties exist in every dimension.

Example 1.7 ([16, 17]). Suppose that $V = \mathbb{P}^{n-1}$, $L = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$, R is smooth, $n \geqslant 2$. Then X can be obtained by blowing up the n-dimensional smooth quadric at two points. The variety X is spherical, and it is known that X is K-polystable [17, §4.4.2].

In this paper, we prove the following theorem.

Theorem 1.8. *Suppose that $V = \mathbb{P}^{n-1}$, $L = \mathcal{O}_{\mathbb{P}^{n-1}}(r)$, R is smooth, $n > r > \frac{n}{2} > 1$. Then X is K-polystable.*

We obtain this result as an application of the following K-polystability criteria.

Theorem 1.9. *Suppose that both V and R are smooth (or equivalently X is smooth), and $-K_V \sim_{\mathbb{Q}} aL$, where $a \in \mathbb{Q}_{>0}$ such that $a > 1$. Let μ be the smallest rational number such that μL is very ample. Set $n = \dim(X)$ (so $\dim(V) = n-1$), set $d = L^{n-1}$, set*

$$k_n(a,d,\mu) = \frac{a^{n+1} - (a-1)^{n+1}}{(n+1)(a^n - (a-1)^n)} d\mu^{n-2} + \frac{a^{n+1} - (a+n)(a-1)^n}{2(n+1)(a^n - (a-1)^n)}$$

and set

$$\gamma = \min\left\{\frac{1}{k_n(a,d,\mu)}, \frac{(n+1)(a^n - (a-1)^n)}{(n+1-a)a^n + (a-1)^{n+1}}, \frac{a\delta(V)(n+1)(a^n - (a-1)^n)}{n(a^{n+1} - (a-1)^{n+1})}\right\},$$

where $\delta(V)$ is the δ-invariant of the Fano variety V. If $n \geqslant 3$, $d\mu^{n-2} \geqslant 2$ and $\gamma > 1$, then the Casagrande–Druel variety X is K-polystable.

Remark 1.10. In the notation of Theorem 1.9, if $n \geqslant 2$ and $d\mu^{n-2} < 2$, then we have $d\mu^{n-2} = 1$, which gives $V = \mathbb{P}^{n-1}$ and $L = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$, so X is K-polystable; see Example 1.7.

In this paper, we also prove the following two theorems about K-polystability of several singular Casagrande–Druel 3-folds.

Theorem 1.11. *Suppose $V = \mathbb{P}^1 \times \mathbb{P}^1$, $L = \mathcal{O}_V(1,1)$, and R is one of the following curves:*

(1) *$C_1 + C_2$, where C_1 and C_2 are smooth curves in $|L|$ such that $|C_1 \cap C_2| = 2$;*

(2) *$\ell_1 + \ell_2 + \ell_3 + \ell_4$, where ℓ_1 and ℓ_2 are two distinct smooth curves of degree $(1,0)$, and ℓ_3 and ℓ_4 are two distinct smooth curves of degree $(0,1)$;*

(3) *$2C$, where C is a smooth curve in $|L|$.*

Then X is K-polystable.

Theorem 1.12. *Suppose $V = \mathbb{P}^2$, $L = \mathcal{O}_{\mathbb{P}^2}(2)$, and R is one of the following curves:*

(1) *a singular reduced curve in $|2L|$ with at most $\mathbb{A}_1$ or $\mathbb{A}_2$ singularities;*

(2) *$C_1 + C_2$, where C_1 and C_2 are smooth conics that are tangent at two points;*

(3) *$C + \ell_1 + \ell_2$, where C is a smooth conic, ℓ_1 and ℓ_2 are distinct lines tangent to C;*

(4) *$2C$, where C is a smooth conic.*

Then X is K-polystable.

To present their applications, let $\mathcal{M}^{\mathrm{Kss}}_{n,v}$ be the K-moduli functor of Fano varieties that have dimension n and anticanonical volume $v \in \mathbb{Q}_{>0}$ in the sense of [47, Theorem 2.17]. Then $\mathcal{M}^{\mathrm{Kss}}_{n,v}$ is an Artin stack of finite type [9, 28, 45]. Moreover, as in [30, Theorem 1.3], it admits a separated good moduli space (see [5, 10]) $\mathcal{M}^{\mathrm{Kss}}_{n,v} \to M^{\mathrm{Kps}}_{n,v}$ in the sense of [4], where $M^{\mathrm{Kps}}_{n,v}$ is a proper [8, 30] and projective [14, 47] scheme whose points parametrize K-polystable Fano varieties of dimension n and anticanonical volume v. Let $M^{\mathrm{Kps}}_{(3.9)}$ and $M^{\mathrm{Kps}}_{(4.2)}$ be the closed subvarieties of $M^{\mathrm{Kps}}_{3,26}$ and $M^{\mathrm{Kps}}_{3,28}$ whose general points parametrize smooth Fano threefolds in the families № 3.9 and № 4.2, respectively. Then Theorems 1.11 and 1.12 imply the following two results (see Section 6 and cf. [24]).

Corollary 1.13. *Let $V = \mathbb{P}^1 \times \mathbb{P}^1$, $L = \mathcal{O}_V(1,1)$, $\Gamma = (\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})) \rtimes \boldsymbol{\mu}_2$ and $T = \mathbb{P}(H^0(V, \mathcal{O}_V(2,2))^\vee)$. Let $T^{\mathrm{ss}} \subset T$ be the GIT semistable open subset with respect to the natural Γ-action, and let M be the GIT quotient $T^{\mathrm{ss}} /\!\!/ \Gamma$. Then there is a morphism*

$$\begin{array}{ccc} \Phi\colon M & \to & M^{\mathrm{Kps}}_{3,28} \\ \Psi & & \Psi \\ [f] & \mapsto & [X_f], \end{array}$$

where X_f is the Casagrande–Druel threefold that is constructed from $R = \{f = 0\} \in |2L|$. Furthermore, the morphism Φ is an isomorphism onto $M^{\mathrm{Kps}}_{(4.2)}$, and $M^{\mathrm{Kps}}_{(4.2)}$ is a connected component of the scheme $M^{\mathrm{Kps}}_{3,28}$.

Corollary 1.14. *Let $V = \mathbb{P}^2$, $L = \mathcal{O}_{\mathbb{P}^2}(2)$, $\Gamma = \mathrm{SL}_3(\mathbb{C})$, $T = \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4))^\vee)$. Let $T^{\mathrm{ss}} \subset T$ be the GIT semistable open subset with respect to the natural Γ-action, and let M be the GIT quotient $T^{\mathrm{ss}} /\!\!/ \Gamma$. Then there exists a morphism*

$$\begin{array}{ccc} \Phi\colon M & \to & M^{\mathrm{Kps}}_{3,26} \\ \Psi & & \Psi \\ [f] & \mapsto & [X_f], \end{array}$$

where X_f is the Casagrande–Druel threefold that is constructed from $R = \{f = 0\} \in |2L|$. Furthermore, the morphism Φ is an isomorphism onto $M^{\mathrm{Kps}}_{(3.9)}$, and $M^{\mathrm{Kps}}_{(3.9)}$ is a connected component of the scheme $M^{\mathrm{Kps}}_{3,26}$.

If B is the smooth del Pezzo surface from Examples 1.4, 1.5, 1.6, then B is K-polystable. If B is the Fano manifold from Theorem 1.8, then B is K-polystable [19, Theorem 1.1]. If B is the singular del Pezzo surface from Theorems 1.11 and 1.12 such that R is reduced, then B is also K-polystable [35]. Inspired by this, we pose the following conjecture.

Conjecture 1.15. *If V and B are K-polystable Fano varieties, then X is K-polystable.*

If B is a K-polystable Fano variety, the log Fano pair $(V, \frac{1}{2}R)$ is also K-polystable [31]. Thus, our conjecture is closely related to the following recent result.

Theorem 1.16 ([32]). *Suppose that $-K_V \sim_{\mathbb{Q}} aL$, where $a \in \mathbb{Q}_{>0}$ such that $a > 1$. Set*

$$\lambda_n(a) = \frac{a^{n+1} - (a+n)(a-1)^n}{2(n+1)(a^n - (a-1)^n)},$$

where $n = \dim X$. Then X is K-semistable if and only if $(V, \lambda_n(a)R)$ is K-semistable.

The K-polystability of V in Conjecture 1.15 is necessary.

Example 1.17 (Yuchen Liu). Let $V = \mathbb{P}(1,1,4)$, let $L = \mathcal{O}_V(4)$, let R be a general curve in $|2L|$, and let $\lambda \in (0, \frac{3}{4}) \cap \mathbb{Q}$. Then $(V, \lambda R)$ is a log Fano pair. One can show that

$$\delta(V, \lambda R) \geqslant 1 \ (\delta(V, \lambda R) > 1, \text{ respectively}) \iff \lambda \geqslant \frac{3}{8} \left(\lambda > \frac{3}{8}, \text{ respectively}\right),$$

so that the singular del Pezzo surface B is K-polystable, but $(V, \frac{9}{52}R)$ is not K-semistable. Hence, the threefold X is not K-semistable by Theorem 1.16.

Let us say few words about the proofs of Theorems 1.9 and 1.12. In Section 2, we will show that $X/\iota \cong Y$, and we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "\rho"] \arrow[dr, "\theta"'] & & Y \arrow[dl, "\pi"] \\ & V, & \end{tikzcd}$$

where ρ is the quotient map, which is a double cover ramified over our divisor $B \in |2S^+|$. Thus, using [31], we see that

$$X \text{ is K-polystable} \iff \text{the log Fano pair } \left(Y, \frac{1}{2}B\right) \text{ is K-polystable.}$$

In Section 3, we will prove the following result, which implies Theorem 1.9.

Theorem 1.18. *Suppose that V and R are smooth (so B is smooth), and $-K_V \sim_{\mathbb{Q}} aL$, where $a \in \mathbb{Q}_{>0}$ such that $a > 1$. Let μ be a rational number such that μL is very ample.*

Set $n = \dim Y$ *(so* $\dim V = n-1$*) and* $d = L^{n-1}$. *Suppose* $n \geqslant 3$ *and* $d\mu^{n-2} \geqslant 2$. *Then*

$$\delta\Big(Y, \frac{1}{2}B\Big) \geqslant \min\bigg\{\frac{1}{k_n(a,d,\mu)}, \frac{(n+1)(a^n-(a-1)^n)}{(n+1-a)a^n+(a-1)^{n+1}}, \frac{a\delta(V)(n+1)(a^n-(a-1)^n)}{n(a^{n+1}-(a-1)^{n+1})}\bigg\},$$

where $k_n(a,d,\mu)$ *is defined in Theorem* 1.9.

Proof of Theorem 1.9. Indeed, notice that the right-hand side of the inequality in Theorem 1.18 is precisely γ as defined in Theorem 1.9. By assumption, $\gamma > 1$, and so, by [31], it follows that X is K-polystable. □

We refer the reader to the excellent survey [46] for an overview on K-stability and to [7, 22] for extensive applications of the celebrated Abban–Zhuang theory introduced in [2]. In these applications (especially in Sections 4 and 5, we will make extensive use of Zhuang's result [49] that equivariant K-polystability for reductive groups implies K-polystability. We will also make frequent use of the result in [31] to determine K-stability of branched covers.

Let us describe the structure of this paper. First, in Section 2, we will prove a few basic properties of Casagrande–Druel varieties. Then, in Section 3, we will prove Theorem 1.18. In Sections 4 and 5, we will give proofs of Theorem 1.11 and Theorem 1.12, respectively. Finally, in Section 6, we will prove Corollary 1.13, and we will show that $M^{\mathrm{Kps}}_{(4.2)} \cong \mathbb{P}(1,2,3)$. We will omit the proof of Corollary 1.14, since it is similar to the proof of Corollary 1.13.

2. Preliminaries

Let V be a (possibly non-projective) variety, let L_1 and L_2 be line bundles on V such that $L_1 + L_2 \nsim 0$ and $|L_1 + L_2| \neq \varnothing$, and let $f \in H^0(V, \mathcal{O}_V(L_1+L_2))$ that defines a non-zero effective divisor R on V. Set

$$Y_1 = \mathbb{P}(\mathcal{O}_V \oplus \mathcal{O}(L_1)), \quad Y_2 = \mathbb{P}(\mathcal{O}_V \oplus \mathcal{O}(L_2)).$$

Now, let $\pi_1: Y_1 \to V$ and $\pi_2: Y_2 \to V$ be the natural projections, and let ξ_1 and ξ_2 be the tautological line bundles on Y_1 and Y_2, respectively. We have the isomorphisms

$$\begin{aligned} H^0(Y_1, \mathcal{O}_{Y_1}(\xi_1)) &\cong H^0(V, \mathcal{O}_V) \oplus H^0(V, \mathcal{O}_V(L_1)), \\ H^0\big(Y_1, \mathcal{O}_{Y_1}(\xi_1 - \pi_1^*(L_1))\big) &\cong H^0(V, \mathcal{O}_V) \oplus H^0(V, \mathcal{O}_V(-L_1)), \\ H^0(Y_2, \mathcal{O}_{Y_2}(\xi_2)) &\cong H^0(V, \mathcal{O}_V) \oplus H^0(V, \mathcal{O}_V(L_2)), \\ H^0\big(Y_2, \mathcal{O}_{Y_2}(\xi_2 - \pi_2^*(L_2))\big) &\cong H^0(V, \mathcal{O}_V) \oplus H^0(V, \mathcal{O}_V(-L_2)). \end{aligned}$$

Using these isomorphisms, fix sections

$$\begin{aligned} &u_1^+ \in H^0(Y_1, \mathcal{O}_{Y_1}(\xi_1)), \quad u_1^- \in H^0\big(Y_1, \mathcal{O}_{Y_1}(\xi_1 - \pi_1^*(L_1))\big), \\ &u_2^+ \in H^0(Y_2, \mathcal{O}_{Y_2}(\xi_2)), \quad u_2^- \in H^0\big(Y_2, \mathcal{O}_{Y_2}(\xi_2 - \pi_2^*(L_2))\big) \end{aligned}$$

that correspond to the section $1 \in H^0(V, \mathcal{O}_V)$. Let

$$\begin{aligned} S_1^- &= \{u_1^- = 0\} \subset Y_1, \quad S_1^+ = \{u_1^+ = 0\} \subset Y_1, \\ S_2^- &= \{u_2^- = 0\} \subset Y_2, \quad S_2^+ = \{u_2^+ = 0\} \subset Y_2. \end{aligned}$$

For $i \in \{1,2\}$, the divisors S_i^- and S_i^+ are disjoint sections of the natural projection π_i such that $S_i^-|_{S_i^-} \sim -L_i \sim -S_i^+|_{S_i^+}$, where we use isomorphisms $S_i^- \cong V \cong S_i^+$ induced by π_i.

Now, we set $Q = Y_1 \times_V Y_2$. Then we have the canonical isomorphisms

$$\mathbb{P}\big(\mathcal{O}_{Y_1} \oplus \mathcal{O}_{Y_1}(\pi_1^*(L_2))\big) \cong Q \cong \mathbb{P}\big(\mathcal{O}_{Y_2} \oplus \mathcal{O}_{Y_2}(\pi_2^*(L_1))\big),$$

so that we have the commutative Cartesian diagram

$$\begin{tikzcd} & Q \arrow[dl, "\rho_1"'] \arrow[dr, "\rho_2"] & \\ Y_1 \arrow[dr, "\pi_1"'] & & Y_2 \arrow[dl, "\pi_2"] \\ & V, & \end{tikzcd}$$

where ρ_1 and ρ_2 are natural projections. Set $\vartheta = \pi_1 \circ \rho_1 = \pi_2 \circ \rho_2$.

Set $F_1 = \pi_1^*(R) \subset Y_1$. Let $\phi_1\colon X \to Y_1$ be the blow up along the intersection $F_1 \cap S_1^+$, and let E_1 be the ϕ_1-exceptional divisor. Note that $F_1 + S_1^-$ corresponds to

$$\pi_1^*(f)u_1^- \in H^0\big(Y_1, \mathcal{O}_{Y_1}(\xi_1 + \pi_1^*(L_2))\big)$$

and S_1^+ corresponds to $u_1^+ \in H^0(Y_1, \mathcal{O}_{Y_1}(\xi_1))$. Thus, the ideal sheaf $\mathcal{I} \subset \mathcal{O}_{Y_1}$ of $F_1 \cap S_1^+$ admits the surjection

$$\mathcal{O}_{Y_1}(\xi_1 + \pi_1^*(L_2)) \oplus \mathcal{O}_{Y_1}(\xi_1) \to \mathcal{I} \to 0.$$

Therefore, there is a natural closed embedding $X \hookrightarrow Q$ over V such that its image is the effective divisor defined by the zeroes of the section

$$\vartheta^*(f)u_1^-u_2^- - u_1^+u_2^+ \in H^0\big(Q, \mathcal{O}_Q(\rho_1^*(\xi_1) + \rho_2^*(\xi_2))\big),$$

where we identified $H^0(Q, \mathcal{O}_Q(\rho_i^*(D))) = H^0(Y_i, \mathcal{O}_{Y_i}(D))$ for every $D \in \mathrm{Pic}(Y_i)$.

Let us identify X with its image in Q. Set $\theta = \pi_1 \circ \phi_1$. Then θ is induced by ϑ, it is a conic bundle, and R is its discriminant divisor. Set

$$S_1 = \phi_1^*(S_1^-), \quad S_2 = \phi_1^*(S_1^+) - E_1, \quad E_2 = \phi_1^*(F_1) - E_1.$$

Then S_1, S_2, E_2 are effective Cartier divisors on the variety X; these are the proper transforms of the divisors S_1^-, S_1^+, F_1, respectively. Moreover, the divisors S_1 and S_2 are mutually disjoint sections of the conic bundle θ. Furthermore, we have

$$S_1|_{S_1} \sim -L_1 \quad \text{and} \quad S_2|_{S_2} \sim -L_2,$$

where we use isomorphisms $S_1 \cong V$ and $S_2 \cong V$ induced by θ. Similarly, we see that the divisor $E_1 + E_2$ is given by zeroes of the section

$$\theta^*(f) \in H^0\big(X, \mathcal{O}_X(\theta^*(L_1 + L_2))\big) \cong H^0(V, \mathcal{O}_V(L_1 + L_2)).$$

Set $F_2 = \pi_2^*(R) \subset Y_2$, and let $\phi_2\colon X \to Y_2$ be the morphism induced by $\rho_2\colon Q \to Y_2$. Since the defining equation of $X \subset Q$ is symmetric, we conclude that ϕ_2 is the blow up along the scheme-theoretic intersection $F_2 \cap S_2^+$, the ϕ_2-exceptional divisor is E_2, and there exists the following commutative diagram:

$$\begin{tikzcd}
 & X \arrow[dl, "\phi_1"'] \arrow[dd, "\theta"] \arrow[dr, "\phi_2"] & \\
Y_1 \arrow[dr, "\pi_1"'] & & Y_2 \arrow[dl, "\pi_2"] \\
 & V. &
\end{tikzcd}$$

This is an elementary transformation of the $\mathbb{P}^1$-bundle π_1 in the sense of Maruyama [33]. Now, using [33, Theorem 1.4] and [33, Proposition 1.6], we see that

$$S_1 = \phi_2^*(S_2^+) - E_2, \quad S_2 = \phi_2^*(S_2^-), \quad E_1 = \phi_2^*(F_1) - E_2.$$

Remark 2.1. Let $U = \mathbb{P}(\mathcal{O}_V \oplus \mathcal{O}_V(-L_1) \oplus \mathcal{O}_V(-L_2))$, let ξ_U be the tautological line bundle on the variety U, let $\pi_U\colon U \to V$ be the natural projection. We have the isomorphisms

$$\begin{aligned}
H^0(U, \mathcal{O}_U(\xi_U)) &\cong H^0(V, \mathcal{O}_V) \oplus H^0(V, \mathcal{O}_V(-L_1)) \oplus H^0(V, \mathcal{O}_V(-L_2)),\\
H^0\big(U, \mathcal{O}_U(\xi_U + \pi_U^*(L_1))\big) &\cong H^0(V, \mathcal{O}_V) \oplus H^0(V, \mathcal{O}_V(L_1)) \oplus H^0(V, \mathcal{O}_V(L_1 - L_2)),\\
H^0\big(U, \mathcal{O}_U(\xi_U + \pi_U^*(L_2))\big) &\cong H^0(V, \mathcal{O}_V) \oplus H^0(V, \mathcal{O}_V(L_2)) \oplus H^0(V, \mathcal{O}_V(L_2 - L_1)).
\end{aligned}$$

Using these isomorphisms, fix sections

$$\begin{aligned}
&v_0 \in H^0(U, \mathcal{O}_U(\xi_U)),\\
&v_1 \in H^0\big(U, \mathcal{O}_U(\xi_U + \pi_U^*(L_1))\big),\\
&v_2 \in H^0\big(U, \mathcal{O}_U(\xi_U + \pi_U^*(L_2))\big),
\end{aligned}$$

which correspond to the section $1 \in H^0(V, \mathcal{O}_V)$. Recall that $Q/V \cong \mathbb{P}^1 \times \mathbb{P}^1$. Projecting from the section $u_1^- = u_2^- = 0$, we get a birational map $Q \dashrightarrow U$. Since X/V is a $(1,1)$ divisor on Q/V which does not pass through the point (section) we project from, the map restricts to an isomorphism of X on its image. The image of X on U is a conic given by the equation

$$\pi_U^*(f)v_0^2 - v_1v_2 = 0,$$

so that we can identify X with a Cartier divisor on U such that $X \sim 2\xi_U + \pi_U^*(L_1 + L_2)$.

Proposition 2.2. *Suppose that V is normal and projective, and K_V is $\mathbb{Q}$-Cartier. Then X is normal, and K_X is $\mathbb{Q}$-Cartier. Moreover, the following assertion holds:*

$$-K_X \textit{ is ample} \iff -K_V, -K_V - L_1, -K_V - L_2 \textit{ are ample}.$$

Proof. The normality of the variety X follows from Remark 2.1 and [25, Proposition 5.24]. Similarly, using notation introduced in Remark 2.1, we see that

$$K_U \sim_{\mathbb{Q}} -3\xi_U + \pi_U^*(K_V - L_1 - L_2),$$

so K_X is $\mathbb{Q}$-Cartier by the adjunction formula, because X is a Cartier divisor on U.

To prove the remaining assertion, suppose that $-K_V, -K_V - L_1, -K_V - L_2$ are ample. Then $\xi_U + \pi_U^*(-K_V)$ in Remark 2.1 is ample. Then so is $-K_X \sim_{\mathbb{Q}} (\xi_U + \pi_U^*(-K_V))|_X$. Alternatively, we can prove the ampleness of $-K_X$ directly. Namely, observe that

$$-K_X \sim_{\mathbb{Q}} S_1 + S_2 + \theta^*(-K_V). \tag{2.1}$$

Moreover, applying the adjunction formula to the sections S_1 and S_2, we get

$$-K_X|_{S_1} \sim_{\mathbb{Q}} -K_V - L_1, \quad -K_X|_{S_2} \sim_{\mathbb{Q}} -K_V - L_2,$$

where we used $S_1 \cong V$ and $S_2 \cong V$. Hence, if $-K_V, -K_V - L_1, -K_V - L_2$ are ample, then the divisor $-K_X$ is also ample by Kleiman's ampleness criterion.

This also shows that both divisors $-K_V - L_1, -K_V - L_2$ are ample if $-K_X$ is ample. Observe that $E_1 \cap E_2 \cong R$. Using this isomorphism and (2.1), we get $-K_V|_R \sim -K_X|_R$. On the other hand, we have

$$-2K_V \sim_{\mathbb{Q}} (-K_V - L_1) + (-K_V - L_2) + R.$$

Hence, using Kleiman's criterion again, we see that $-K_V$ is ample if $-K_X$ is ample. □

From now on, we assume, in addition, that V is normal and projective.

Example 2.3. Suppose $V = \mathbb{P}^1 \times \mathbb{P}^1$, and L_1 and L_2 are divisors of degrees $(1, 0)$ and $(0, 1)$, and R is a smooth divisor in $|L_1 + L_2|$. Then X is a smooth Fano 3-fold by Proposition 2.2. One can show that X is the unique smooth Fano 3-fold in the deformation family №4.7. Note that X is K-polystable [7, §3.3].

Remark 2.4 ([21, Lemma 9.8]). Suppose that V is a smooth Fano variety, and assume $-K_V \sim_{\mathbb{Q}} aL$, where L is an ample divisor in $\mathrm{Pic}(V)$, and $a \in \mathbb{Q}_{>0}$. Suppose R and X are smooth, and

$$L_1 \sim_{\mathbb{Q}} a_1 L, \quad L_2 \sim_{\mathbb{Q}} a_2 L,$$

where a_1 and a_2 are rational numbers such that $a_1 \geqslant a_2$. It follows from Proposition 2.2 that X is a Fano variety if and only if $a > a_1$. Further, if X is a Fano variety, then it follows from the proof of [21, Lemma 9.8] that

$$\beta(S_2) < 0 \iff a_1 > a_2.$$

Therefore, if $a > a_1 > a_2$, then X is a K-unstable Fano variety.

From now on, we also assume that $L_1 = L_2$. Set $L = L_1$. Then $R \in |2L|$. Set

$$Y = \mathbb{P}(\mathcal{O}_V \oplus \mathcal{O}(L)).$$

let $\pi: Y \to V$ be the natural projection, and let ξ be the tautological line bundle on Y. Note that $Y \cong Y_1 \cong Y_2$. Using the isomorphisms

$$H^0(Y, \mathcal{O}_Y(\xi)) \cong H^0(V, \mathcal{O}_V) \oplus H^0(V, \mathcal{O}_V(L)),$$
$$H^0\big(Y, \mathcal{O}_Y(\xi - \pi^*(L))\big) \cong H^0(V, \mathcal{O}_V) \oplus H^0(V, \mathcal{O}_V(-L)),$$

fix $u^+ \in H^0(Y, \mathcal{O}_Y(\xi))$ and $u^- \in H^0(Y, \mathcal{O}_Y(\xi - \pi^*(L)))$ that correspond to $1 \in H^0(V, \mathcal{O}_V)$. Let $S^- = \{u^- = 0\}$ and $S^+ = \{u^+ = 0\}$. Then $S^+ \sim S^- + \pi^*(L)$.

Proposition 2.5. *There is a double cover $X \to Y$ ramified in a divisor $B \in |2S^+|$ such that the projection π induces a double cover $B \to V$ that is ramified in R.*

Proof. Let $T = \mathbb{P}(\mathcal{O}_V \oplus \mathcal{O}_V(-L)) \oplus \mathcal{O}_V(-2L))$, let $\varpi\colon T \to V$ be the natural projection, and let ξ_T be the tautological line bundle on T. Observe that

$$\begin{aligned} H^0(T, \mathcal{O}_T(\xi_T)) &\cong H^0(V, \mathcal{O}_V) \oplus H^0(V, \mathcal{O}_V(-L)) \oplus H^0(V, \mathcal{O}_V(-2L)), \\ H^0\big(T, \mathcal{O}_T(\xi_T + \varpi^*(L))\big) &\cong H^0(V, \mathcal{O}_V) \oplus H^0(V, \mathcal{O}_V(L)) \oplus H^0(V, \mathcal{O}_V(-L)), \\ H^0\big(T, \mathcal{O}_T(\xi_T + \varpi^*(2L))\big) &\cong H^0(V, \mathcal{O}_V) \oplus H^0(V, \mathcal{O}_V(2L)) \oplus H^0(V, \mathcal{O}_V(L)). \end{aligned}$$

Using these isomorphisms, fix sections

$$\begin{aligned} &t_0 \in H^0(T, \mathcal{O}_T(\xi_T)), \\ &t_1 \in H^0\big(T, \mathcal{O}_T(\xi_T + \varpi^*(L))\big), \\ &t_2 \in H^0\big(T, \mathcal{O}_T(\xi_T + \varpi^*(2L))\big), \end{aligned}$$

which corresponds to $1 \in H^0(V, \mathcal{O}_V)$. Then

$$\begin{aligned} \{t_0 = 0\} &\cong \mathbb{P}(\mathcal{O}_V(-L) \oplus \mathcal{O}_V(-2L)), \\ \{t_1 = 0\} &\cong \mathbb{P}(\mathcal{O}_V \oplus \mathcal{O}_V(-2L)), \\ \{t_2 = 0\} &\cong \mathbb{P}(\mathcal{O}_V \oplus \mathcal{O}_V(-L)). \end{aligned}$$

Now, we consider the homomorphism

$$\mathcal{O}_Q \oplus \mathcal{O}_Q(\vartheta^*(L)) \oplus \mathcal{O}_Q(\vartheta^*(2L)) \to \mathcal{O}_Q(\rho_1^*(\xi_1) + \rho_2^*(\xi_2)) \tag{2.2}$$

defined by the composition of

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \colon \mathcal{O}_Q \oplus \mathcal{O}_Q(\vartheta^*(L)) \oplus \mathcal{O}_Q(\vartheta^*(2L)) \\ &\qquad \to \mathcal{O}_Q \oplus \mathcal{O}_Q(\vartheta^*(L)) \oplus \mathcal{O}_Q(\vartheta^*(L)) \oplus \mathcal{O}_Q(\vartheta^*(2L)) \end{aligned}$$

and the surjection

$$\mathcal{O}_Q \oplus \mathcal{O}_Q(\vartheta^*(L)) \oplus \mathcal{O}_Q(\vartheta^*(L)) \oplus \mathcal{O}_Q(\vartheta^*(2L)) \twoheadrightarrow \mathcal{O}_Q(\rho_1^*(\xi_1) + \rho_2^*(\xi_2))$$

obtained by the tensor product of the pullbacks of the following natural surjections:

$$\begin{aligned} \mathcal{O}_{Y_1} \oplus \mathcal{O}_{Y_1}(\pi_1^*(L_1)) &\twoheadrightarrow \mathcal{O}_{Y_1}(\xi_1), \\ \mathcal{O}_{Y_2} \oplus \mathcal{O}_{Y_2}(\pi_2^*(L_2)) &\twoheadrightarrow \mathcal{O}_{Y_2}(\xi_2). \end{aligned}$$

Then (2.2) is surjective. This gives the morphism $\rho\colon Q \to T$ over V with

$$\begin{aligned} \rho^*(t_0) &= u_1^- u_2^-, \\ \rho^*(t_1) &= \frac{1}{2}(u_1^+ u_2^- + u_1^- u_2^+), \\ \rho^*(t_2) &= u_1^+ u_2^+, \end{aligned}$$

where we identified $H^0(Q, \mathcal{O}_Q(\rho_i^*(D))) = H^0(Y_i, \mathcal{O}_{Y_i}(D))$ for $D \in \mathrm{Pic}(Y_i)$.

Using the local criterion for flatness, we see that ρ is flat. Further, ρ is finite of degree 2. Now, using [23, I (6.11)] and [23, I (6.12)], we see that the morphism ρ is branched over the divisor $B_T \in |2(\xi_T + \varpi^*(L))|$ that is given by $t_1^2 - t_0 t_2 = 0$.

Let Y_0 be the divisor in $|\xi_T + \varpi^*(2L)|$ that is given by $\varpi^*(f)t_0 - t_2 = 0$, and let $\pi_0\colon Y_0 \to V$ be the morphism induced by ϖ. Then $X = \rho^*(Y_0)$ as Cartier divisors, so that the restriction $X \to Y_0$ is a double cover branched over $B_T|_{Y_0}$. Moreover, using the exact sequence

$$0 \to \mathcal{O}_V(-2L) \xrightarrow{\begin{pmatrix} f \\ 0 \\ -1 \end{pmatrix}} \mathcal{O}_V \oplus \mathcal{O}_V(-L) \oplus \mathcal{O}_V(-2L) \xrightarrow{\begin{pmatrix} 1 & 0 & f \\ 0 & 1 & 0 \end{pmatrix}} \mathcal{O}_V \oplus \mathcal{O}_V(-L) \to 0,$$

we get an isomorphism $Y_0 \cong Y$ over V. Hence, we identify $Y = Y_0$.

Set $B = B_T|_Y$. Then B is defined by

$$(u^+)^2 - \pi^*(f)(u^-)^2 = 0,$$

which implies the remaining assertions of the proposition. □

Let $\iota \in \mathrm{Aut}(X)$ be the Galois involution of the double cover $X \to Y$ in Proposition 2.5. Then $\iota(S_1) = S_2$ and $\iota(E_1) = E_2$, and it follows from the proof of Proposition 2.5 that the conic bundle $\theta\colon X \to V$ is $\langle\iota\rangle$-equivariant with ι acting trivially on V.

Proposition 2.6. *Suppose that V is smooth, L is nef, X has Kawamata log terminal singularities, and $-K_X$ is ample. Then the deformations of X are unobstructed.*

Proof. By Remark 2.1, X can be embedded into $U = \mathbb{P}_V(\mathcal{O}_V \oplus \mathcal{O}_V(-L) \oplus \mathcal{O}_V(-L))$ such that $X \in |2\xi_U + 2\pi_U^*(L)|$, where ξ_U is the tautological line bundle and π_U is the natural projection. Therefore, since U is smooth, the variety X has at worst canonical singularities, and X has at worst local complete intersection singularities. Hence, it follows from [42, Theorem 2.3.2], [42, Theorem 2.4.1], [42, Corollary 2.4.2] that the deformation functor

$$\mathrm{Def}_X\colon \mathcal{A} \to (\mathrm{Sets})$$

has a semi-universal formal element in the sense of [42, Definition 2.2.6], where $\mathcal{A}$ is the category of local $\mathbb{C}$-algebras with the residue field $\mathbb{C}$. Thus, by [41, Proposition 2.4] and [41, Proposition 2.6], the deformations of X are unobstructed if $\mathrm{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) = 0$.

Let us show that $\mathrm{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) = 0$. Set $n = \dim(X)$. As in [41, §1.2], we have

$$\mathrm{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \simeq \mathrm{Ext}^2_{\mathcal{O}_X}(\Omega^1_X \otimes \omega_X, \omega_X) \simeq H^{n-2}(X, \Omega^1_X \otimes \omega_X)^\vee.$$

Since $-K_V$ and $-K_V - L$ are ample by Proposition 2.2 and L is nef, we see that

$$\xi_U + \pi_U^*(-K_V)$$

is ample, and $\xi_U + \pi_U^*(L)$ is nef. In particular, both divisors

$$\begin{aligned} -K_U &\sim 3\xi_U + \pi_U^*(-K_V + 2L), \\ -K_U - X &\sim \xi_U + \pi_U^*(-K_V) \end{aligned}$$

are ample. On the other hand, using the exact sequence of sheaves

$$0 \to \mathcal{O}_U(-X)|_X \to \Omega^1_U|_X \to \Omega^1_X \to 0,$$

we get the following exact sequence:

$$H^{n-2}(X, \Omega^1_U|_X \otimes \omega_X) \to H^{n-2}(X, \Omega^1_X \otimes \omega_X) \to H^{n-1}(X, \mathcal{O}_U(-X)|_X \otimes \omega_X).$$

Moreover, using the Kawamata–Viehweg vanishing theorem, we get

$$H^{n-1}(X, \mathcal{O}_U(-X)|_X \otimes \omega_X) \simeq H^1(X, K_X + (-K_U)|_X)^\vee = 0.$$

Furthermore, using the exact sequence of sheaves

$$0 \to \Omega^1_U \otimes \omega_U \to \Omega^1_U \otimes \omega_U(X) \to \Omega^1_U|_X \otimes \omega_X \to 0,$$

we get the exact sequence

$$H^{n-2}(U, \Omega^1_U \otimes \omega_U(X)) \to H^{n-2}(X, \Omega^1_U|_X \otimes \omega_X) \to H^{n-1}(U, \Omega^1_U \otimes \omega_U).$$

Since both ω_U and $\omega_U(X)$ are anti-ample, the Akizuki–Nakano vanishing theorem gives

$$H^{n-2}(U, \Omega^1_U \otimes \omega_U(X)) = H^{n-1}(U, \Omega^1_U \otimes \omega_U) = 0.$$

This gives $\mathrm{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) = 0$, which completes the proof. □

3. K-polystability criteria

The goal of this section is to prove Theorem 1.18. To do so, we will apply the theory of Abban–Zhuang [2], as applied in [7, §1.7] and [22], consisting on bounding delta-invariants below by picking a specific flag.

Fix a positive integer $n \geqslant 3$. Let V be a smooth projective variety of dimension $n-1$, and let L be an ample Cartier divisor on V. Set $d = L^{n-1}$. Fix $\mu \in \mathbb{Q}_{>0}$ such that μL is very ample. Let $Y = \mathbb{P}(\mathcal{O}_V \oplus \mathcal{O}_V(L))$, and let $\pi\colon Y \to V$ be the natural projection. Set $H = \pi^*(L)$. Let S^- and S^+ be disjoint sections of the projection π such that $S^+ \sim S^- + H$.

Remark 3.1. Unlike Section 1, we do not assume that V is a Fano variety.

Fix a positive rational number $a \geqslant 1$. Let $D(a) = S^- + aH$. Then $D(a)$ is nef and big. Moreover, if $a > 1$, then $D(a)$ is ample.

Lemma 3.2 (cf. [48]). *Let P be a point in S^-. Then*

$$\delta_P(Y; D(a)) \geqslant \min\left\{\frac{(n+1)(a^n - (a-1)^n)}{(n+1-a)a^n + (a-1)^{n+1}}, \frac{\delta(V; L)(n+1)(a^n - (a-1)^n)}{n(a^{n+1} - (a-1)^{n+1})}\right\},$$

where $\delta_P(Y; D(a))$ is the (local) δ-invariant of the variety Y polarized by the divisor $D(a)$, and $\delta(V; L)$ is the δ-invariant of V polarized by L. Further, if $\delta(V; L) \leqslant a$, then

$$\delta_P(Y; D(a)) \geqslant \frac{\delta(V; L)(n+1)(a^n - (a-1)^n)}{n(a^{n+1} - (a-1)^{n+1})}.$$

Proof. It follows from [2, 7] that

$$\delta_P(Y; D(a)) \geqslant \min\left\{\frac{1}{S_{D(a)}(S^-)}, \inf_{\substack{F/S^- \\ P \in C_{S^-}(F)}} \frac{A_{S^-}(F)}{S(W^{S^-}_{\bullet,\bullet}; F)}\right\},$$

where $S(W^{S^-}_{\bullet,\bullet}; F)$ is defined in [7, Section 1.7], and the infimum is taken over all prime divisors over S^- whose centers on S^- contain P. This easily implies the required assertion.

Indeed, take $u \in \mathbb{R}_{\geqslant 0}$. Then $D(a) - uS^- \sim_{\mathbb{R}} (1-u)S^- + aH$, so that

$$D(a) - uS^- \text{ is nef } \iff D(a) - uS^- \text{ is pseudo-effective } \iff u \leqslant 1.$$

Thus, since $\mathrm{vol}(D(a)) = D(a)^n = d(a^n - (a-1)^n)$, we have

$$\begin{aligned} S_{D(a)}(S^-) &= \frac{1}{D(a)^n}\int_0^\infty \mathrm{vol}(D(a) - uS^-)\,du \\ &= \frac{1}{d(a^n-(a-1)^n)}\int_0^1 ((1-u-a)^n(-1)^{n+1}d + a^n d)\,du \\ &= \frac{(n+1-a)a^n + (a-1)^{n+1}}{(n+1)(a^n-(a-1)^n)}. \end{aligned}$$

Using $S^- \cong V$, we get $(D(a) - uS^-)|_{S^-} \sim_{\mathbb{R}} (a+u-1)H|_{S^-} \sim_{\mathbb{R}} (a+u-1)L$.

Let F be any prime divisor over S^-. Then it follows from [7, Section 1.7] that

$$\begin{aligned} S(W^{S^-}_{\bullet,\bullet}; F) &= \frac{n}{D(a)^n}\int_0^1\int_0^\infty \mathrm{vol}((D(a)-uS^-)|_{S^-} - vF)\,dv\,du \\ &= \frac{n}{D(a)^n}\int_0^1\int_0^\infty \mathrm{vol}((a+u-1)L - vF)\,dv\,du \\ &= \frac{n}{D(a)^n}\int_0^1 (a+u-1)^n \int_0^\infty \mathrm{vol}(L - vF)\,dv\,du \\ &= \frac{n}{d(a^n-(a-1)^n)}\cdot\frac{a^{n+1}-(a-1)^{n+1}}{n+1}\int_0^\infty \mathrm{vol}(L-vF)\,dv \\ &= \frac{n}{n+1}\,\frac{a^{n+1}-(a-1)^{n+1}}{d(a^n-(a-1)^n)}\cdot L^{n-1}S_L(F) \\ &= \frac{n}{n+1}\,\frac{a^{n+1}-(a-1)^{n+1}}{a^n-(a-1)^n}S_L(F). \end{aligned}$$

This gives

$$\begin{aligned} \frac{A_{S^-}(F)}{S(W^{S^-}_{\bullet,\bullet}; F)} &= \frac{A_{S^-}(F)}{S_L(F)}\cdot\frac{n+1}{n}\cdot\frac{a^n-(a-1)^n}{a^{n+1}-(a-1)^{n+1}} \\ &\leqslant \delta_P(V;L)\cdot\frac{n+1}{n}\cdot\frac{a^n-(a-1)^n}{a^{n+1}-(a-1)^{n+1}}, \end{aligned}$$

which implies the first part of the assertion.

We now assume $\delta(V;L) \leqslant a$ and we want to show

$$\frac{(n+1)(a^n-(a-1)^n)}{(n+1-a)a^n + (a-1)^{n+1}} \geqslant \frac{\delta(V;L)(n+1)(a^n-(a-1)^n)}{n(a^{n+1}-(a-1)^{n+1})}.$$

This inequality is equivalent to

$$\delta(V;L) \leqslant \frac{n(a^{n+1}-(a-1)^{n+1})}{(n+1-a)a^n + (a-1)^{n+1}}.$$

We must show that the right-hand side of the inequality above is at least a. But

$$\frac{n(a^{n+1}-(a-1)^{n+1})}{(n+1-a)a^n+(a-1)^{n+1}} > a \iff a^{n+1}(a-1)-(a-1)^{n+1}(a+n) > 0,$$

which is clearly true. □

Now, fix a smooth divisor $B \in |2S^+|$. Let $\eta\colon B \to V$ be the morphism induced by π. Suppose that η is the double cover ramified over a smooth divisor $R \in |2L|$. Set $\Delta = \frac{1}{2}B$. Note that $B \cap S^- = \varnothing$. Let $k_n(a,d,\mu)$ be the number defined in Theorem 1.9.

Proposition 3.3. *Let P be a point in $Y \setminus S^-$. Suppose that $d\mu^{n-2} \geqslant 2$. Then*

$$\delta_P(Y,\Delta;D(a)) \geqslant \frac{1}{k_n(a,d,\mu)},$$

where $\delta_P(Y,\Delta;D(a))$ is the (local) δ-invariants of the pair (Y,Δ) polarized by $D(a)$.

This result together with Lemma 3.2 implies Theorem 1.18.

Proof of Theorem 1.18. Note that V is a Fano variety and $-K_V \sim_{\mathbb{Q}} aL$. Then

$$-K_Y \sim 2S^+ - \pi^*(K_V+L) \sim_{\mathbb{Q}} 2S^+ + (a-1)H,$$

which gives

$$-(K_Y+\Delta) \sim_{\mathbb{Q}} S^+ + (a-1)H \sim_{\mathbb{Q}} S^- + aH = D(a),$$

so that (Y,Δ) is the log Fano pair and

$$\delta(Y,\Delta) = \delta(Y,\Delta;D(a)),$$

where $\delta(Y,\Delta)$ is the δ-invariant of the log Fano pair (Y,Δ). Now, we can apply Lemma 3.2 and Proposition 3.3 to get the required assertion. □

In the remaining part of the section, we will prove Proposition 3.3 by induction on n.

3.1. Base of induction. Let V be a smooth projective surface, let L be an ample Cartier divisor on V, let μ be the smallest rational number such that μL is very ample, let

$$Y = \mathbb{P}(\mathcal{O}_V \oplus \mathcal{O}_V(L)),$$

and let $\pi\colon Y \to V$ be the natural projection. Set $H = \pi^*(L)$. Let S^- and S^+ be disjoint sections of the projection π such that $S^+ \sim S^- + H$, and let B be an irreducible normal surface in $|2S^+|$ such that π induces a double cover $B \to V$ which is ramified in a reduced curve $R \in |2L|$. Fix $a \in \mathbb{Q}$ such that $a \geqslant 1$. Let $D(a) = S^- + aH$. Then $D(a)$ is nef and big, and $D(a)$ is ample for $a > 1$. Set $\Delta = \frac{1}{2}B$ and $d = L^2$.

Remark 3.4. Since μL is very ample and L is Cartier, we have $d\mu = (\mu L)\cdot L \in \mathbb{Z}_{>0}$ and

$$d\mu^2 = (\mu L)^2 \in \mathbb{Z}_{>0}.$$

Moreover, if $d\mu = 1$, then $\mu = 1$, $d = L^2 = 1$, $V = \mathbb{P}^2$ and $L = \mathcal{O}_{\mathbb{P}^2}(1)$.

Suppose, in addition, that $d\mu \geqslant 2$. Set

$$k_3(a,d,\mu) = \frac{8d\mu a^3 + 6(1-2d\mu)a^2 + 8(d\mu-1)a - 2d\mu + 3}{8(3a^2-3a+1)}.$$

Let P be a point in Y such that $P \notin S^-$ and $P \notin \mathrm{Sing}(B)$.

Proposition 3.5. *One has $\delta_P(Y,\Delta;D(a)) \geqslant \frac{1}{k_3(a,d,\mu)}$.*

In the remaining part of this subsection, we will prove this result. We will only consider the case $P \in B$, because the case $P \notin B$ is much simpler.

Let V_1 be a general curve in $|\mu L|$ that contains the point $\pi(P)$, and let $Y_1 = \pi^*(V_1)$. Then V_1 is a smooth curve, and Y_1 is a smooth surface. For simplicity, we set $D = D(a)$. Take $u \in \mathbb{R}_{\geqslant 0}$. Then $D - uY_1 \sim_{\mathbb{R}} S^- + (a-\mu u)H$, so that $D - uY_1$ is pseudo-effective if and only if $u \leqslant \frac{a}{\mu}$. We have

$$(D-uY_1)|_{S^-} \sim_{\mathbb{R}} (S^- + (a-\mu u)H)|_{S^-} \sim_{\mathbb{R}} (a-1-\mu u)L,$$

where we use isomorphism $S^- \cong V$ induced by π. Hence, the divisor $D - uY_1$ is nef if and only if $u \leqslant \frac{a-1}{\mu}$. Moreover, the Zariski decomposition of $D - uY_1$ is

$$P(u) \sim_{\mathbb{R}} \begin{cases} S^- + (a-\mu u)H & \text{if } u \in [0, \frac{a-1}{\mu}], \\ (a-\mu u)(S^- + H) = (a-\mu u)S^+ & \text{if } u \in [\frac{a-1}{\mu}, \frac{a}{\mu}], \end{cases}$$

and

$$N(u) = \begin{cases} 0 & \text{if } u \in [0, \frac{a-1}{\mu}], \\ (\mu u + 1 - a)S^- & \text{if } u \in [\frac{a-1}{\mu}, \frac{a}{\mu}], \end{cases}$$

where $P(u)$ is the positive part, and $N(u)$ is the negative part.

Note that $H^3 = 0$, $H^2 \cdot S^- = d$, $H \cdot (S^-)^2 = -d$, $(S^-)^3 = d$. Then

$$\begin{aligned} S_D(Y_1) &= \frac{1}{D^3}\int_0^{\frac{a}{\mu}} \mathrm{vol}(D-uY_1)\,du \\ &= \frac{1}{(S^-+aH)^3}\Bigg(\int_0^{\frac{a-1}{\mu}} (S^- + (a-\mu u)H)^3\,du \\ &\qquad + \int_{\frac{a-1}{\mu}}^{\frac{a}{\mu}} ((a-\mu u)(S^-+H))^3\,du\Bigg) \\ &= \frac{(2a-1)(2a^2-2a+1)}{4\mu(3a^2-3a+1)}. \end{aligned}$$

Let f be the fiber of the $\mathbb{P}^1$-bundle π that contains P. Then there are two cases to consider: either B intersects f transversely at P or tangentially. For each case, we consider an appropriate plt blow up $h\colon \widetilde{Y}_1 \to Y_1$ at the point P with smooth exceptional curve E. We let $\Delta_1 = \Delta|_{Y_1}$, and we denote by $\widetilde{\Delta}_1$ the proper transform on $\widetilde{Y}_1$ of the divisor Δ_1. Then it follows from [2, 7, 22] that

$$\delta_P(Y,\Delta) \geqslant \min\left\{\frac{1}{S_D(Y_1)}, \frac{A_{Y_1,\Delta_1}(E)}{S(V^{Y_1}_{\bullet,\bullet};E)}, \inf_{Q\in E}\frac{A_{E,\Delta_E}(Q)}{S(V^{\widetilde{Y}_1,E}_{\bullet,\bullet,\bullet};Q)}\right\}.$$

where $S(V^{Y_1}_{\bullet,\bullet}; E)$ and $S(V^{\widetilde{Y}_1,E}_{\bullet,\bullet,\bullet}; Q)$ are defined in [7, Section 1.7], and Δ_E is the different computed via the adjunction formula

$$K_E + \Delta_E = (K_{\widetilde{Y}_1} + \widetilde{\Delta}_1 + E)|_E.$$

For instance, if h is the ordinary blow up at the point P, then $\Delta_E = \widetilde{\Delta}_1|_E$. For simplicity, we rewrite the last inequality as

$$\frac{1}{\delta_P(Y,\Delta)} \leqslant \max\left\{S_D(Y_1), \frac{S(V^{Y_1}_{\bullet,\bullet}; E)}{A_{Y_1,\Delta_1}(E)}, \sup_{Q\in E} \frac{S(V^{\widetilde{Y}_1,E}_{\bullet,\bullet,\bullet}; Q)}{A_{E,\Delta_E}(Q)}\right\}. \tag{3.1}$$

Thus, to prove Proposition 3.5, it is enough to bound each term in (3.1) by $k_3(a,d,\mu)$.

We set $S_1^- = S^-|_{Y_1}$, $H_1 := H|_{Y_1}$, $B_1 := B|_{Y_1}$, $D_1 = P(u)|_{Y_1}$. Note that $H_1 \equiv d\mu f$ and

$$D_1 \equiv \begin{cases} S_1^- + (a-\mu u)d\mu f & \text{if } u \in [0, \frac{a-1}{\mu}], \\ (a-\mu u)(S_1^- + d\mu f) & \text{if } u \in [\frac{a-1}{\mu}, \frac{a}{\mu}]. \end{cases}$$

We denote by $\widetilde{S}_1^-$, $\widetilde{B}_1$, $\widetilde{f}$ the proper transforms on $\widetilde{Y}_1$ of the curves S_1^-, B_1, f, respectively. Recall that Y_1 is a $\mathbb{P}^1$-bundle over the smooth curve V_1. In Lemmas 3.6 and 3.7, we estimate $\delta_P(Y,\Delta; D(a))$ when B and f intersect transversely or tangentially, respectively. Notice that $\widetilde{Y}_1$ has Picard rank 3 and its Mori cone is generated by the divisors $\widetilde{S}_1^-$, $\widetilde{f}$ and E.

Lemma 3.6. *Suppose B intersects f transversally. Then $\delta_P(Y,\Delta; D(a)) \geqslant \frac{1}{k_3(a,d,\mu)}$.*

Proof. Let $h\colon \widetilde{Y}_1 \to Y_1$ be the ordinary blow up at P, where E is the h-exceptional curve. We have $\widetilde{S}_1^- \sim h^*(S_1^-)$ and $\widetilde{f} \sim h^*(f) - E$. Take $v \in \mathbb{R}_{\geqslant 0}$. Then

$$h^*(D_1) - vE \equiv \begin{cases} \widetilde{S}_1^- + (a-\mu u)d\mu \widetilde{f} + ((a-\mu u)d\mu - v)E & \text{if } u \in [0, \frac{a-1}{\mu}], \\ (a-\mu u)(\widetilde{S}_1^- + d\mu \widetilde{f}) + ((a-\mu u)d\mu - v)E & \text{if } u \in [\frac{a-1}{\mu}, \frac{a}{\mu}]. \end{cases}$$

We have the following intersection numbers:

$\bullet$	$\widetilde{S}_1^-$	$\widetilde{f}$	E
$\widetilde{S}_1^-$	$-d\mu$	1	0
$\widetilde{f}$	1	-1	1
E	0	1	-1

This shows that $h^*(D_1) - vE$ is pseudo-effective if and only if $v \leqslant (a-\mu u)d\mu$.

If $u \in [0, \frac{a-1}{\mu}]$, the positive part of the Zariski decomposition of $h^*(D_1) - vE$ is

$$\widetilde{P}(u,v) \equiv \begin{cases} \widetilde{S}_1^- + (a-\mu u)d\mu \widetilde{f} + ((a-\mu u)d\mu - v)E \\ \qquad\qquad \text{if } v \in [0,1], \\ \widetilde{S}_1^- + ((a-\mu u)d\mu + 1 - v)\widetilde{f} + ((a-\mu u)d\mu - v)E \\ \qquad\qquad \text{if } v \in [1, 1 - d\mu^2 u + ad\mu - d\mu], \\ \dfrac{-d\mu^2 u + ad\mu - v}{d\mu - 1}(\widetilde{S}_1^- + d\mu \widetilde{f} + (d\mu - 1)E) \\ \qquad\qquad \text{if } v \in [1 - d\mu^2 u + ad\mu - d\mu, (a-\mu u)d\mu], \end{cases}$$

and the negative part is

$$\widetilde{N}(u,v)=\begin{cases}0 & \text{if } v\in[0,1],\\ (v-1)\widetilde{f} & \text{if } v\in[1,1-d\mu^2u+ad\mu-d\mu],\\ \dfrac{d\mu(\mu u-a+v)}{d\mu-1}\widetilde{f}+\dfrac{d\mu^2u-ad\mu+d\mu+v-1}{d\mu-1}\widetilde{S}_1^- & \\ & \text{if } v\in[1-d\mu^2u+ad\mu-d\mu,(a-\mu u)d\mu].\end{cases}$$

Similarly, if $u\in[\frac{a-1}{\mu},\frac{a}{\mu}]$, the positive part of the Zariski decomposition of $h^*(D_1)-vE$ is

$$\widetilde{P}(u,v)\equiv\begin{cases}(a-\mu u)(\widetilde{S}_1^-+d\mu\widetilde{f})+((a-\mu u)d\mu-v)E & \\ & \text{if } v\in[0,a-\mu u],\\ \dfrac{1}{d\mu-1}(-d\mu^2u+ad\mu-v)(\widetilde{S}_1^-+d\mu\widetilde{f}+(d\mu-1)E) & \\ & \text{if } v\in[a-\mu u,(a-\mu u)d\mu].\end{cases}$$

and the negative part is

$$\widetilde{N}(u,v)=\begin{cases}0 & \text{if } v\in[0,a-\mu u],\\ \dfrac{1}{d\mu-1}(d\mu(\mu u-a+v)\widetilde{f}+(\mu u-a+v)\widetilde{S}_1^-) & \\ & \text{if } v\in[a-\mu u,(a-\mu u)d\mu].\end{cases}$$

Now, using results from [7, Section 1.7], we compute

$$\begin{aligned}S(W^{\widetilde{Y}_1}_{\bullet,\bullet};E)&=\frac{3}{D^3}\int_0^{\frac{a}{\mu}}\int_0^{(a-\mu u)d\mu}\mathrm{vol}(D_1-vF)\,dv\,du\\&=\frac{3}{(S^-+aH)^3}\int_0^{\frac{a}{\mu}}\int_0^{(a-\mu u)d\mu}\widetilde{P}(u,v)^2\,dv\,du\\&=\frac{4a^3d\mu+6(1-d\mu)a^2+4(d\mu-2)a-d\mu+3}{4(3a^2-3a+1)}.\end{aligned}$$

Moreover, we have $A_{Y_1,\Delta_1}(E)=2-\frac{1}{2}=\frac{3}{2}$, so that

$$\frac{S(W^{Y_1}_{\bullet,\bullet};E)}{A_{Y_1,\Delta_1}(E)}=\frac{4a^3d\mu+6(1-d\mu)a^2+4(d\mu-2)a-d\mu+3}{6(3a^2-3a+1)}.$$

Let Q be a point in E. Then, using results from [7, Section 1.7], we compute

$$\begin{aligned}S(W^{\widetilde{Y}_1,E}_{\bullet,\bullet,\bullet};Q)&=\frac{3}{(S^-+aH)^3}\int_0^{\frac{a}{\mu}}\int_0^{(a-\mu u)d\mu}(\widetilde{P}(u,v)\cdot E)^2\,dv\,du+F_q(W^{\widetilde{Y}_1,E}_{\bullet,\bullet,\bullet})\\&=\frac{6a^2-8a+3}{4(3a^2-3a+1)}+F_Q(W^{\widetilde{Y}_1,E}_{\bullet,\bullet,\bullet}),\end{aligned}$$

where

$$F_Q(W^{\widetilde{Y}_1,E}_{\bullet,\bullet,\bullet})=\frac{6}{(S^-+aH)^3}\int_0^{\frac{a}{\mu}}\int_0^{(a-\mu u)d\mu}(\widetilde{P}(u,v)\cdot E)\cdot\mathrm{ord}_Q(\widetilde{N}(u,v)|_E)\,dv\,du,$$

because $P\notin\mathrm{Supp}(N(u))$ for $u\in[0,\frac{a}{\mu}]$. Notice that

$$F_Q(W^{\widetilde{Y}_1,E}_{\bullet,\bullet,\bullet})\neq 0$$

only when $Q\in\widetilde{f}$.

Thus, there are three cases to consider.

- $Q = E \cap \widetilde{f}$. Then

$$F_Q(W^{\widetilde{Y}_1,E}_{\bullet,\bullet,\bullet}) = \frac{3 - 8a + 6a^2 + d\mu - 4ad\mu + 6a^2d\mu - 4a^3d\mu}{4(3a^2 - 3a + 1)}$$

and $A_{E,\Delta_E}(Q) = 1$ since $Q \notin \widetilde{B}_1$. Hence, we have

$$\frac{S(W^{\widetilde{Y}_1,E}_{\bullet,\bullet,\bullet}; Q)}{A_{E,\Delta_E}(Q)} = \frac{d\mu(2a-1)(2a^2-2a+1)}{4(3a^2-3a+1)}.$$

- $Q \in E \cap \widetilde{B}_1$. Then $A_{E,\Delta_E}(Q) = \frac{1}{2}$, so that

$$\frac{S(W^{\widetilde{Y}_1,E}_{\bullet,\bullet,\bullet}; Q)}{A_{E,\Delta_E}(Q)} = \frac{6a^2 - 8a + 3}{2(3a^2-3a+1)}.$$

- $Q \in E$ away from $\widetilde{f}$ and $\widetilde{B}_1$. Then $A_{E,\Delta_E}(Q) = 1$, so that

$$\frac{S(W^{\widetilde{Y}_1,E}_{\bullet,\bullet,\bullet}; Q)}{A_{E,\Delta_E}(Q)} = \frac{6a^2 - 8a + 3}{4(3a^2-3a+1)}.$$

The third case is smaller than the previous one (exactly half), so we do not consider it. So, using (3.1), we obtain the inequality

$$\frac{1}{\delta_P(Y,\Delta)} \leqslant \max\left\{\frac{(2a-1)(2a^2-2a+1)}{4\mu(3a^2-3a+1)}, \frac{4a^3d\mu + 6(1-d\mu)a^2 + 4(d\mu-2)a - d\mu + 3}{6(3a^2-3a+1)}, \frac{d\mu(2a-1)(2a^2-2a+1)}{4(3a^2-3a+1)}, \frac{6a^2-8a+3}{2(3a^2-3a+1)}\right\}. \tag{3.2}$$

Recall from Remark 3.4 that $d\mu^2 \geqslant 1$. This allows us to conclude

$$\frac{d\mu(2a-1)(2a^2-2a+1)}{4(3a^2-3a+1)} \geqslant \frac{(2a-1)(2a^2-2a+1)}{4\mu(3a^2-3a+1)},$$

so we can discard the first term in (3.2). Moreover, since $d\mu \geqslant 2$, we have

$$\frac{4a^3d\mu + 6(1-d\mu)a^2 + 4(d\mu-2)a - d\mu + 3}{6(3a^2-3a+1)} \leqslant k_3(a,d,\mu),$$
$$\frac{d\mu(2a-1)(2a^2-2a+1)}{4(3a^2-3a+1)} \leqslant k_3(a,d,\mu),$$
$$\frac{6a^2-8a+3}{2(3a^2-3a+1)} \leqslant k_3(a,d,\mu),$$

which gives $\delta_P(Y,\Delta; D(a)) \geqslant \frac{1}{k_3(a,d,\mu)}$. □

Now, we deal with the case when f is tangent to B at the point P.

Lemma 3.7. *Suppose B and f are tangent at P. Then $\delta_P(Y, \Delta; D(a)) \geqslant \frac{1}{k_3(a,d,\mu)}$.*

Proof. Now, we let $h\colon \widetilde{Y}_1 \to Y_1$ be the $(1,2)$-weighted blow up of the point P such that the curves $\widetilde{B}_1$ and $\widetilde{f}$ are disjoint. Then $\widetilde{f} = h^*(f) - 2E$. Take $v \in \mathbb{R}_{\geqslant 0}$. Then

$$h^*(D_1) - vE \equiv \begin{cases} \widetilde{S}_1^- + (a-\mu u)d\mu\widetilde{f} + (2(a-\mu u)d\mu - v)E & \text{if } u \in [0, \frac{a-1}{\mu}], \\ (a-\mu u)(\widetilde{S}_1^- + d\mu\widetilde{f}) + (2(a-\mu u)d\mu - v)E & \text{if } u \in [\frac{a-1}{\mu}, \frac{a}{\mu}]. \end{cases}$$

Moreover, we have the following intersection numbers:

$\bullet$	$\widetilde{S}_1^-$	$\widetilde{f}$	E
$\widetilde{S}_1^-$	$-d\mu$	1	0
$\widetilde{f}$	1	-2	1
E	0	1	$-\frac{1}{2}$

Thus, the divisor $h^*(D_1) - vE$ is pseudo-effective if and only if $v \leqslant 2(a-\mu u)d\mu$.

If $u \in [0, \frac{a-1}{\mu}]$, the positive part of the Zariski decomposition of $h^*(D_1) - vE$ is

$$\widetilde{P}(u,v) \equiv \begin{cases} \widetilde{S}_1^- + (a-\mu u)d\mu\widetilde{f} + (2(a-\mu u)d\mu - v)E \\ \qquad \text{if } v \in [0,1], \\ \widetilde{S}_1^- + \Big((a-\mu u)d\mu + \dfrac{1-v}{2}\Big)\widetilde{f} + (2(a-\mu u)d\mu - v)E \\ \qquad \text{if } v \in [1, -2d\mu^2 u + 2ad\mu - 2d\mu + 1], \\ \dfrac{-2d\mu^2 u + 2ad\mu - v}{2d\mu - 1}(\widetilde{S}_1^- + d\mu\widetilde{f} + (2d\mu - 1)E) \\ \qquad \text{if } v \in [-2d\mu^2 u + 2ad\mu - 2d\mu + 1, 2(a-\mu u)d\mu], \end{cases}$$

and the negative part is

$$\widetilde{N}(u,v) = \begin{cases} 0 & \text{if } v \in [0,1], \\ \dfrac{v-1}{2}\widetilde{f} & \text{if } v \in [1, -2d\mu^2 u + 2ad\mu - 2d\mu + 1], \\ \dfrac{d\mu(\mu u - a + v)}{2d\mu - 1}\widetilde{f} + \dfrac{2d\mu^2 u - 2ad\mu + 2d\mu + v - 1}{2d\mu - 1}\widetilde{S}_1^- \\ \qquad \text{if } v \in [-2d\mu^2 u + 2ad\mu - 2d\mu + 1, 2(a-\mu u)d\mu]. \end{cases}$$

Similarly, if $u \in [\frac{a-1}{\mu}, \frac{a}{\mu}]$, the positive part of the Zariski decomposition of $h^*(D_1) - vE$ is

$$\widetilde{P}(u,v) \equiv \begin{cases} (a-\mu u)(\widetilde{S}_1^- + d\mu\widetilde{f}) + (2(a-\mu u)d\mu - v)E \\ \qquad \text{if } v \in [0, a-\mu u], \\ \dfrac{-2d\mu^2 u + 2ad\mu - v}{2d\mu - 1}(\widetilde{S}_1^- + d\mu\widetilde{f} + (2d\mu - 1)E) \\ \qquad \text{if } v \in [a-\mu u, 2(a-\mu u)d\mu], \end{cases}$$

and the negative part is

$$\widetilde{N}(u,v) = \begin{cases} 0 & \text{if } v \in [0, a-\mu u], \\ \dfrac{d\mu(\mu u - a + v)}{2d\mu - 1}\widetilde{f} + \dfrac{\mu u - a + v}{2d\mu - 1}\widetilde{S}_1^- & \text{if } v \in [a-\mu u, 2(a-\mu u)d\mu]. \end{cases}$$

Now, using results from [7, Section 1.7], we compute

$$
\begin{aligned}
S(W^{Y_1}_{\bullet,\bullet};E) &= \frac{3}{D^3}\int_0^{\frac{a}{\mu}}\int_0^{2(a-\mu u)d\mu}\mathrm{vol}(D_1-vF)\,dv\,du\\
&= \frac{3}{(S^-+aH)^3}\int_0^{\frac{a}{\mu}}\int_0^{2(a-\mu u)d\mu}\widetilde{P}(u,v)\,dv\,du\\
&= \frac{1}{4}\cdot\frac{8a^3d\mu+6(1-2d\mu)a^2+8(d\mu-1)a-2d\mu+3}{3a^2-3a+1}
\end{aligned}
$$

Moreover, since $A_{Y_1,\Delta_1}(E)=2$, we have

$$
\frac{S(W^{Y_1}_{\bullet,\bullet};E)}{A_{Y_1,\Delta_1}(E)}=\frac{1}{8}\cdot\frac{8a^3d\mu+6(1-2d\mu)a^2+8(d\mu-1)a-2d\mu+3}{3a^2-3a+1}.
$$

Let Q be a point in E. Using results from [7, Section 1.7], we get

$$
\begin{aligned}
S(W^{\widetilde{Y}_1,E}_{\bullet,\bullet,\bullet};Q) &= \frac{3}{(S^-+aH)^3}\int_0^{\frac{a}{\mu}}\int_0^{2(a-\mu u)d\mu}(\widetilde{P}(u,v)\cdot E)^2\,dv\,du+F_Q(W^{\widetilde{Y}_1,E}_{\bullet,\bullet,\bullet})\\
&= \frac{1}{8}\cdot\frac{6a^2-8a+3}{3a^2-3a+1}+F_Q(W^{\widetilde{Y}_1,E}_{\bullet,\bullet,\bullet}),
\end{aligned}
$$

where

$$
F_Q(W^{\widetilde{Y}_1,E}_{\bullet,\bullet,\bullet})=\frac{6}{(S^-+aH)^3}\int_0^{\frac{a}{\mu}}\int_0^{2(a-\mu u)d\mu}(\widetilde{P}(u,v)\cdot E)\cdot\mathrm{ord}_Q(\widetilde{N}(u,v)|_E)\,dv\,du.
$$

There are three cases to consider.

- $Q=E\cap\widetilde{f}$. Then

$$
F_Q(W^{\widetilde{Y}_1,E}_{\bullet,\bullet,\bullet})=\frac{1}{8}\frac{8a^3d\mu-6(2d\mu-1)a^2+8(d\mu+1)a-2d\mu-3}{3a^2-3a+1}
$$

  and $A_{E,\Delta_E}(Q)=1$ since $Q\notin\widetilde{B}_1$. Hence, we have

$$
\frac{S(W^{\widetilde{Y}_1,E}_{\bullet,\bullet,\bullet};Q)}{A_{E,\Delta_E}(Q)}=\frac{d\mu}{4}\cdot\frac{(2a-1)(2a^2-2a+1)}{3a^2-3a+1}.
$$

- $Q\in E\cap\widetilde{B}$. Then $A_{E,\Delta_E}(Q)=\frac{1}{2}$, so that

$$
\frac{S(W^{\widetilde{Y}_1,E}_{\bullet,\bullet,\bullet};Q)}{A_{E,\Delta_E}(Q)}=\frac{1}{4}\cdot\frac{6a^2-8a+3}{3a^2-3a+1}.
$$

- $Q\in E$ is the $\mathbb{A}_1$ singularity. Then $A_{E,\Delta_E}(Q)=\frac{1}{2}$, and so

$$
\frac{S(W^{\widetilde{Y}_1,E}_{\bullet,\bullet,\bullet};Q)}{A_{E,\Delta_E}(Q)}=\frac{1}{4}\cdot\frac{6a^2-8a+3}{3a^2-3a+1}.
$$

We have the inequality

$$\frac{1}{\delta_P(Y,\Delta)} \leqslant \max\left\{\frac{(2a-1)(2a^2-2a+1)}{4\mu(3a^2-3a+1)}, \frac{1}{8}\cdot\frac{8a^3d\mu+6(1-2d\mu)a^2+8(d\mu-1)a-2d\mu+3}{3a^2-3a+1}, \frac{d\mu}{4}\cdot\frac{(2a-1)(2a^2-2a+1)}{3a^2-3a+1}, \frac{1}{4}\cdot\frac{6a^2-8a+3}{3a^2-3a+1}\right\}.$$

Now, arguing as in the end of the proof of Lemma 3.6, we find

$$\frac{1}{\delta_P(Y,\Delta)} \leqslant \frac{1}{8}\cdot\frac{8a^3d\mu+6(1-2d\mu)a^2+8(d\mu-1)a-2d\mu+3}{3a^2-3a+1},$$

and the result follows. □

Proof of Proposition 3.5. This is a combination of Lemmas 3.6 and 3.7. □

3.2. The induction. Let us use all assumptions and notation introduced in Section 3. Recall that μ is the smallest rational number for which μL is a very ample Cartier divisor on the variety V and $d = L^{n-1}$. Then $\mu^{n-1}d = (\mu L)^{n-1} \geqslant 1$. Let us prove Proposition 3.3 by induction on $\dim(Y) = n \geqslant 3$; the base of induction (the case when $n = 3$) is done in Section 3.1.

Therefore, we suppose that Proposition 3.3 holds for varieties of dimension $n-1 \geqslant 3$. Let P be a point in Y such that $P \notin S^-$. We must prove that

$$\delta_P(Y,\Delta; D(a)) \geqslant \frac{1}{k_n(a,d,\mu)},$$

where $k_n(a,d,\mu)$ is presented in Theorem 1.9. We will only consider the case when $P \in B$, since the case $P \notin B$ is simpler and similar. Thus, we suppose that $P \in B$.

Let V_{n-1} be a general divisor in $|\mu L|$ that contains the point $\pi(P)$. Set

$$Y_{n-1} = \pi^*(V_{n-1}).$$

For simplicity, set $D = D(a)$. First, let us compute $S_D(Y_{n-1})$. Take $u \in \mathbb{R}_{\geqslant 0}$. Then

$$D(a) - uY_{n-1} \sim_{\mathbb{R}} S^- + (a-\mu u)H,$$

so $D(a) - uY_{n-1}$ is pseudo-effective if and only if $u \leqslant \frac{a}{\mu}$. For $u \in [0, \frac{a}{\mu}]$, let $P(u)$ be the positive part of the Zariski decomposition of $D(a) - uY_{n-1}$, and let $N(u)$ be its negative part. Then

$$P(u) \equiv \begin{cases} S^- + (a-\mu u)H = D(a-\mu u) & \text{if } u \in [0, \frac{a-1}{\mu}],\\ (a-\mu u)(S^- + H) = (a-\mu u)D(1) & \text{if } u \in [\frac{a-1}{\mu}, \frac{a}{\mu}],\end{cases}$$

and

$$N(u) = \begin{cases} 0 & \text{if } u \in [0, \frac{a-1}{\mu}],\\ (\mu u + 1 - a)S^- & \text{if } u \in [\frac{a-1}{\mu}, \frac{a}{\mu}].\end{cases}$$

Recall that $S^- \cap S^+ = \varnothing$. Note that $(S^-)^n = (-1)^{n+1}d$ and $(S^+)^n = d$. Hence, we have

$$D(a)^n = (S^- + aH)^n = ((1-a)S^- + aS^+)^n = d(a^n - (a-1)^n).$$

Now, we compute

$$\begin{aligned}
S_D(Y_{n-1}) &= \frac{1}{D(a)^n}\int_0^\infty \operatorname{vol}(D(a) - uY_{n-1})\,du \\
&= \frac{1}{D(a)^n}\int_0^{\frac{a-1}{\mu}} (S^- + (a-\mu u)H)^n\,du \\
&\quad + \frac{1}{D(a)^n}\int_{\frac{a-1}{\mu}}^{\frac{a}{\mu}} ((a-\mu u)(S^- + H))^n\,du \\
&= \frac{1}{D(a)^n}\int_0^{\frac{a-1}{\mu}} d((-1)^{n+1}(1-a+\mu u)^n + (a-\mu u)^n)\,du \\
&\quad + \frac{1}{D(a)^n}\int_{\frac{a-1}{\mu}}^{\frac{a}{\mu}} d(a-\mu u)^n\,du \\
&= \frac{a^{n+1} - (a-1)^{n+1}}{\mu(n+1)(a^n - (a-1)^n)}.
\end{aligned}$$

Set

$$\operatorname{Res}_n(a) = \frac{a^{n+1} - (a+n)(a-1)^n}{2(n+1)(a^n - (a-1)^n)}.$$

Lemma 3.8. *One has* $k_n(a,d,\mu) = S_{D(a)}(Y_{n-1})d\mu^{n-1} + \operatorname{Res}_n(a)$ *and* $\operatorname{Res}_n(a) > 0$.

Proof. The equality follows from the formulas for $k_n(a,d,\mu)$ and $S_{D(a)}(Y_{n-1})$. Let us show that $\operatorname{Res}_n(a) > 0$. We may assume that $a > 1$. The denominator is clearly positive. Hence, we only need to verify that $a^{n+1} - (a+n)(a-1)^n > 0$. But

$$\left(\frac{a}{a-1}\right)^n = \left(1 + \frac{1}{a-1}\right)^n = \sum_{i=0}^n \binom{n}{i}\left(\frac{1}{a-1}\right)^i > 1 + \frac{n}{a-1} > 1 + \frac{n}{a} = \frac{a+n}{a},$$

which gives $a^{n+1} - (a+n)(a-1)^n > 0$. This shows that $\operatorname{Res}_n(a) > 0$. □

Set $\Delta_{n-1} = \Delta|_{Y_{n-1}}$. Then $S_D(Y_{n-1}) \leqslant k_n(a,d,\mu)$ by Lemma 3.8, since $d\mu^{n-1} \geqslant 1$. Therefore, using [2], we see that $\delta_P(Y,\Delta;D) \geqslant \frac{1}{k_n(a,d,\mu)}$ provided that

$$S(V^{Y_{n-1}}_{\bullet,\bullet}; E) \leqslant k_n(a,d,\mu)A_{Y_{n-1},\Delta_{n-1}}(E) \tag{3.3}$$

for every prime divisor E over the variety Y_{n-1} such that its center on Y_{n-1} contains P, where $A_{Y_{n-1},\Delta_{n-1}}(E)$ is the log discrepancy, and $S(V^{Y_{n-1}}_{\bullet,\bullet}; E)$ is defined in [7, Section 1.7].

Suppose that $n \geqslant 4$. Let us prove (3.3) using Proposition 3.3 applied to (Y_{n-1}, Δ_{n-1}). Let E be a prime divisor over Y_{n-1} whose center in Y_{n-1} contains P. Since $P \notin S^-$, it follows from [7, Corollary 1.108] that

$$S(V^{Y_{n-1}}_{\bullet,\bullet}; E) = \frac{n}{D^n}\int_0^{\frac{a}{\mu}}\left(\int_0^\infty \operatorname{vol}(P(u)|_{Y_{n-1}} - vE)\,dv\right)du$$

$$
\begin{aligned}
&= \frac{n}{D^n}\int_0^{\frac{a-1}{\mu}}\int_0^{\infty}\mathrm{vol}(S^- + (a-\mu u)H - vE)\,dv\,du \\
&\qquad + \frac{n}{D^n}\int_{\frac{a-1}{\mu}}^{\frac{a}{\mu}}\int_0^{\infty}\mathrm{vol}((a-\mu u)(S^- + H) - vE)\,dv\,du \\
&= \frac{n}{D^n}\int_0^{\frac{a-1}{\mu}}\int_0^{\infty}\mathrm{vol}(S^- + (a-\mu u)H - vE)\,dv\,du \\
&\qquad + \frac{n}{D^n}\int_{\frac{a-1}{\mu}}^{\frac{a}{\mu}}(a-\mu u)^n\int_0^{\infty}\mathrm{vol}(S^- + H - vE)\,dv\,du.
\end{aligned}
$$

Now, applying Proposition 3.3 (induction step), we get

$$
\begin{aligned}
&\int_0^{\infty}\mathrm{vol}(S^- + (a-\mu u)H - vE)\,dv \\
&\qquad \leqslant k_{n-1}(a-\mu u, d\mu, \mu)(S^- + (a-\mu u)H)^{n-1}A_{Y_{n-1},\Delta_{n-1}}(E)
\end{aligned}
$$

and

$$
\int_0^{\infty}\mathrm{vol}(S^- + H - vE)\,dv \leqslant k_{n-1}(1, d\mu, \mu)(S^- + H)^{n-1}A_{Y_{n-1},\Delta_{n-1}}(E).
$$

Hence, combining, we obtain

$$
\begin{aligned}
S(V^{Y_{n-1}}_{\bullet,\bullet}; E) &\leqslant \frac{n}{D^n}\int_0^{\frac{a-1}{\mu}} k_{n-1}(a-\mu u, d\mu, \mu)(S^- + (a-\mu u)H)^{n-1}A_{Y_{n-1},\Delta_{n-1}}(E)\,du \\
&\qquad + \frac{n}{D^n}\int_{\frac{a-1}{\mu}}^{\frac{a}{\mu}}(a-\mu u)^n k_{n-1}(1, d\mu, \mu)(S^- + H)^{n-1}A_{Y_{n-1},\Delta_{n-1}}(E)\,du \\
&= A_{Y_{n-1},\Delta_{n-1}}(E)\frac{n}{D^n}\int_0^{\frac{a-1}{\mu}} k_{n-1}(a-\mu u, d\mu, \mu)(S^- + (a-\mu u)H)^{n-1}\,du \\
&\qquad + A_{Y_{n-1},\Delta_{n-1}}(E)\frac{n}{D^n}\int_{\frac{a-1}{\mu}}^{\frac{a}{\mu}}(a-\mu u)^n k_{n-1}(1, d\mu, \mu)(S^- + H)^{n-1}\,du.
\end{aligned}
$$

Let us compute these two integrals separately. We have

$$
\begin{aligned}
A_1 &:= \int_0^{\frac{a-1}{\mu}} k_{n-1}(a-\mu u, d\mu, \mu)(S^- + (a-\mu u)H)^{n-1}\,du \\
&= d\mu^{n-1}\int_0^{\frac{a-1}{\mu}}\frac{d\mu((-1)^{n-1}(1-a+\mu u)^n + (a-\mu u)^n)}{\mu n}\,du \\
&\qquad + \int_0^{\frac{a-1}{\mu}}\frac{d\mu((a-\mu u)^n - (a-\mu u+n-1)(a-\mu u-1)^{n-1})}{2n}\,du \\
&= \frac{d^2\mu^{n-1}}{\mu n(n+1)}(a^{n+1} - (a-1)^{n+1} - 1) + \frac{d}{2n(n+1)}(a^{n+1} - (a+n)(a-1)^n - 1)
\end{aligned}
$$

and

$$
\begin{aligned}
A_2 := \int_{\frac{a-1}{\mu}}^{\frac{a}{\mu}}(a_n-\mu u)^n k_{n-1}(1, d\mu, \mu)(S^- + H)^{n-1}\,du &= \frac{d(2d\mu^{n-2} + 1)}{2n(n+1)} \\
&= \frac{d^2\mu^{n-1}}{\mu n(n+1)} + \frac{d}{2n(n+1)}.
\end{aligned}
$$

Adding these two integrals, we get

$$\frac{n}{D(a)^n}(A_1 + A_2) = \frac{d\mu^{n-1}}{\mu(n+1)}\frac{a^{n+1} - (a-1)^{n+1}}{a^n - (a-1)^n} + \frac{1}{2(n+1)}\frac{a^{n+1} - (a+n)(a-1)^n}{a^n - (a-1)^n}$$
$$= S_{D(a)}(Y_{n-1})d\mu^{n-1} + \mathrm{Res}_n(a).$$

This gives $S(V^{Y_{n-1}}_{\bullet,\bullet}; E) \leqslant k_n(a,d,\mu)A_{Y_{n-1},\Delta_{n-1}}(E)$ by Lemma 3.8, which proves (3.3) and completes the proof of Proposition 3.3.

3.3. Applications. The only application of Theorem 1.9 we could find is Theorem 1.8. Let us use assumptions and notation of Theorem 1.9. Let $V = \mathbb{P}^{n-1}$ and $L = \mathcal{O}_{\mathbb{P}^{n-1}}(r)$. Suppose that $1 < \frac{n}{2} < r < n$. Then $\mu = \frac{1}{r}$, $d = r^{n-1}$ and $a = \frac{n}{r}$.

Lemma 3.9. *One has $k_n(a,d,\mu) < 1$.*

Proof. One has

$$k_n(a,d,\mu) = \frac{(2d\mu^{n-2}+1)a^{n+1} - (a+n)(a-1)^n - 2d\mu^{n-2}(a-1)^{n+1}}{2(n+1)(a^n - (a-1)^n)}.$$

Thus, it is enough to show that

$$2(n+1)(a^n - (a-1)^n)$$
$$- \big((2d\mu^{n-2}+1)a^{n+1} - (a+n)(a-1)^n - 2d\mu^{n-2}(a-1)^{n+1}\big) > 0.$$

Substituting $\mu = \frac{1}{r}$, $d = r^{n-1}$, $a = \frac{n}{r}$, and multiplying by r^{n+1}, we get the inequality

$$(n^n - (n-r)^n(r+1))(2r-n) > 0,$$

which holds since $2r - n > 0$ and $n > r > \frac{n}{2}$ by assumption. □

Lemma 3.10. *One has*

$$\frac{(n+1)(a^n - (a-1)^n)}{(n+1-a)a^n + (a-1)^{n+1}} > 1.$$

Proof. The inequality is equivalent to

$$(n+1)(a^n - (a-1)^n) > (n+1-a)a^n + (a-1)^{n+1}.$$

Substituting $a = \frac{n}{r}$, multiplying by r^n, and dividing by n, we get $n^n - (r+1)(n-r)^n > 0$, which holds since $1 < \frac{n}{2} < r < n$. □

Lemma 3.11. *One has*

$$\frac{a\delta(V)(n+1)(a^n - (a-1)^n)}{n(a^{n+1} - (a-1)^{n+1})} > 1.$$

Proof. We have $\delta(V) = \delta(\mathbb{P}^{n-1}) = 1$. Thus, the required inequality is equivalent to

$$n(a^{n+1} - (a-1)^{n+1}) - a(n+1)(a^n - (a-1)^n) < 0.$$

Substituting $a = \frac{n}{r}$, multiplying by r^{n+1}, and dividing by n, we get $n^n - (r+1)(n-r)^n > 0$, which holds since $1 < \frac{n}{2} < r < n$. □

Theorem 1.8 follows from Lemmas 3.9, 3.10, 3.11 and Theorem 1.9.

4. Proof of Theorem 1.11

The goal of this section is to prove Theorem 1.11 and describe singular K-polystable limits of smooth Fano 3-folds in the deformation family №4.2. We start with the following (probably well-known) result, which we fail to find in the literature.

Proposition 4.1. *Let C be a $(2,2)$-curve in $\mathbb{P}^1 \times \mathbb{P}^1$. Then C is*

- *GIT stable for $\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})$-action if and only if it is smooth,*
- *GIT strictly polystable if and only if it is one of the curves in Theorem 1.11.*

Proof. Choose homogeneous coordinates $x,\ y$ of degree $(1,0)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, and choose homogeneous coordinates $u,\ v$ of degree $(0,1)$. Then C is given by

$$\sum_{i=0}^{2}\sum_{j=0}^{2} a_{ij} x^{2-i} y^i u^{2-j} v^j = 0.$$

Observe that any one-parameter subgroup $\lambda\colon \mathbb{C}^* \to \mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$ is conjugate to a diagonal one of the form

$$t \mapsto \left(\begin{pmatrix} t^{r_0} & 0 \\ 0 & t^{-r_0}\end{pmatrix}, \begin{pmatrix} t^{r_1} & 0 \\ 0 & t^{-r_1}\end{pmatrix}\right)$$

for some integers $r_1 \geqslant r_0 \geqslant 0$ and $r_1 > 0$, which we will write as $\lambda = (r_0, -r_0, r_1, -r_1)$. Then the Hilbert–Mumford function is

$$\mu(f,\lambda) = \max\{r_0(2-2i) + r_1(2-2j),\ a_{ij} \neq 0\}.$$

Clearly, if $\mu(f,\lambda) \leqslant 0$, then $a_{00} = a_{10} = a_{01} = 0$. Moreover, if this inequality is strict, then we additionally have $a_{11} = 0$. Furthermore, we have $\mu(x^2v^2,\lambda) = -\mu(y^2u^2,\lambda)$. So at least one of a_{20} and a_{02} is zero. Without loss of generality, we assume that $a_{20} = 0$. Therefore, if $\mu(f,\lambda) < 0$, then $a_{00} = a_{10} = a_{01} = a_{11} = a_{20} = 0$.

Suppose that C is singular at the point $([1:0],[1:0])$, so that $a_{00} = a_{10} = a_{01} = 0$, and consider the one-parameter subgroup $\lambda = (1,-1,1,-1)$. Then $\mu(f,\lambda) = 4 - 2(i+j)$, which is non-positive if and only if $i + j \geqslant 2$. But, since $a_{ij} = 0$ whenever $i + j < 2$, we conclude that $\mu(f,\lambda) \leqslant 0$ and C is not stable.

Conversely, suppose there exists a one-parameter subgroup λ for which $\mu(f,\lambda) \leqslant 0$. Note that $\mu(x^{2-i}y^iu^{2-j}v^j,\lambda) > 0$ for any one-parameter subgroup λ provided that $i + j < 2$. This gives $a_{00} = a_{10} = a_{01} = 0$, so that the curve C is singular at $([1:0],[1:0])$.

Now, let us describe the unstable locus. Suppose $a_{00} = a_{10} = a_{01} = a_{11} = a_{20} = 0$. Consider the one-parameter subgroup $\lambda = (1,-1,2,-2)$. Then

$$\mu(f,\lambda) = 6 - 2(i+2j),$$

which is negative if and only if $i + 2j > 3$. But since $a_{ij} = 0$ whenever $i + 2j \leqslant 3$, it follows that $\mu(f,\lambda) < 0$. Similarly, one can show that C is GIT-unstable if it can be given by

$$a_{02}x^2v^2 + a_{12}xyv^2 + a_{21}y^2uv + a_{22}y^2v^2 = 0.$$

This describes all possibilities for the curve C to be GIT-semistable, which easily implies the description of GIT-polystable $(2,2)$-curves. $\square$

Now, we set $V=\mathbb{P}^1\times\mathbb{P}^1$. Let $L=\mathscr{O}_V(1,1)$, let R be a curve in $|2L|$, set

$$Y=\mathbb{P}(\mathscr{O}_V\oplus\mathscr{O}_V(L)),$$

let $\pi\colon Y\to V$ be the natural projection, let S^- and S^+ be disjoint sections of π such that

$$S^+\sim S^-+\pi^*(L).$$

Finally, we set $F=\pi^*(R)$, and let $\phi\colon X\to Y$ be the blow up at the intersection $S^+\cap F$. If R is smooth, then X is K-polystable [7]. Theorem 1.11 says that X is also K-polystable in the case when R is one of the following singular curves:

(1) C_1+C_2, where C_1 and C_2 are smooth curves in $|L|$ such that $|C_1\cap C_2|=2$;

(2) $\ell_1+\ell_2+\ell_3+\ell_4$, where ℓ_1 and ℓ_2 are two distinct smooth curves of degree $(1,0)$, and ℓ_3 and ℓ_4 are two distinct smooth curves of degree $(0,1)$;

(3) $2C$, where C is a smooth curve in $|L|$.

Now, let us prove Theorem 1.11. We start with the following remark.

Remark 4.2. Suppose that $R=\ell_1+\ell_2+\ell_3+\ell_4$, where ℓ_1 and ℓ_2 are two distinct smooth curves in V of degree $(1,0)$, and ℓ_3 and ℓ_4 are two distinct smooth curves of degree $(0,1)$. Then X is toric, and it corresponds to the moment polytope in $M_{\mathbb{R}}$ whose vertices are

$$\begin{array}{llll}(0,0,1), & (1,0,1), & (1,1,1), & (0,1,1),\\ (1,1,0), & (-1,1,0), & (-1,-1,0), & (1,-1,0),\\ (0,0,-1), & (-1,0,-1), & (-1,-1,-1), & (0,-1,-1).\end{array}$$

The barycenter of the moment polytope is the origin, so X is K-polystable. See also [24].

Our next step is the following simple lemma.

Lemma 4.3. *Suppose $R=2C$ for a smooth curve $C\in|L|$. Then X is K-polystable.*

Proof. Here, the morphism ϕ is a weighted blow up at the intersection $\pi^*(C)\cap S^+$, and X has non-isolated singularities along a smooth curve, which we will denote by $\overline{C}$. The threefold X can be obtained in a slightly different way. Let us describe it.

Set $W=V\times\mathbb{P}^1$, let $\varpi\colon W\to V$ be the natural projection, let $\widetilde{S}^-$ and $\widetilde{S}^+$ be its disjoint sections, and let $\widetilde{E}=\varpi^*(C)$. Then there exists commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "\alpha"'] \arrow[dr, "\psi"] & \\ W \arrow[dr, "\varpi"'] & & X \arrow[dl, "\pi\circ\phi"] \\ & V & \end{tikzcd}$$

such that

- α is the blow up along the intersection curves $\widetilde{E}\cap\widetilde{S}^-$ and $\widetilde{E}\cap\widetilde{S}^+$,
- ψ contracts the proper transform of the surface $\widetilde{E}$ to the curve $\overline{C}$,
- $\phi\circ\psi$ maps the proper transforms of the surfaces $\widetilde{S}^-$ and $\widetilde{S}^+$ to the surfaces S^- and S^+, respectively.

Let $\widehat{E}$ be the proper transform on the threefold U of the surface $\widetilde{E}$. We may assume that the curve C is the diagonal curve in $V=\mathbb{P}^1\times\mathbb{P}^1$. Using this, we see that

$$\mathrm{Aut}(X)\cong\mathrm{Aut}(U)\cong\mathrm{Aut}(W,\widetilde{E}+\widetilde{S}^-+\widetilde{S}^+)\cong\mathrm{PGL}_2(\mathbb{C})\times(\mathbb{G}_m\rtimes\boldsymbol{\mu}_2)\times\boldsymbol{\mu}_2.$$

Indeed, we have that $\mathrm{Aut}(X)$ lifts to U since ψ is a blow up along the singular locus. In particular, ψ is $\mathrm{Aut}(U)$-equivariant. On the other hand, α is $\mathrm{Aut}(U)$-equivariant as well. By construction, $\mathrm{Aut}(X)\to\mathrm{Aut}(W,\widetilde{E}+\widetilde{S}^-+\widetilde{S}^+)$ is an isomorphism. Finally, W is a product and the last isomorphism follows.

Observe that $\widehat{E}$ is the only $\mathrm{Aut}(X)$-invariant prime divisor over X. Thus, using [49], we conclude that the threefold X is K-polystable if $\beta(\widehat{E})>0$. Let us compute $\beta(\widehat{E})$.

We let F^- and F^+ be α-exceptional surfaces such that $\alpha(F^-)\subset\widetilde{S}^-$ and $\alpha(F^+)\subset\widetilde{S}^+$, let $\widehat{S}^-$ and $\widehat{S}^+$ be the proper transforms on U of the surfaces S^- and S^+, respectively. Further, set $H_1=(\mathrm{pr}_1\circ\alpha)^*(\mathcal{O}_{\mathbb{P}^1}(1))$, $H_2=(\mathrm{pr}_2\circ\alpha)^*(\mathcal{O}_{\mathbb{P}^1}(1))$, $H_3=(\mathrm{pr}_3\circ\alpha)^*(\mathcal{O}_{\mathbb{P}^1}(1))$, where $\mathrm{pr}_1,\mathrm{pr}_2,\mathrm{pr}_3$ are projections $W\to\mathbb{P}^1$ such that pr_1 and pr_2 factors through ϖ. Then

$$\psi^*(-K_X)\sim -K_U\sim 2(H_1+H_2+H_3)-F^--F^+\sim 2\widehat{E}+\widehat{S}^-+\widehat{S}^++2(F^-+F^+).$$

Now, we take $u\in\mathbb{R}_{\geqslant 0}$. Then the divisor $\psi^*(-K_X)-u\widehat{E}$ is $\mathbb{R}$-rationally equivalent to

$$(2-u)(H_1+H_2)+2H_3+(u-1)(F^-+F^+)\sim_{\mathbb{R}}(2-u)\widehat{E}+\widehat{S}^-+\widehat{S}^++2(F^-+F^+),$$

and $\widehat{S}^-+\widehat{S}^++2(F^-+F^+)$ is not big, so $\psi^*(-K_X)-u\widehat{E}$ is pseudo-effective if and only if $u\leqslant 2$. Moreover, if $u\in[0,1]$, then the divisor $\psi^*(-K_X)-u\widehat{E}$ is nef. Furthermore, if $u\in[1,2]$, then the Zariski decomposition of the divisor $\psi^*(-K_X)-u\widehat{E}$ is given by

$$\psi^*(-K_X)-u\widehat{E}\sim_{\mathbb{R}}\underbrace{(2-u)(H_1+H_2)+2H_3}_{\text{positive part}}+\underbrace{(u-1)(F^-+F^+)}_{\text{negative part}}.$$

Hence, we have

$$\begin{aligned}\beta(\widehat{E})&=1-\frac{1}{(-K_X)^3}\int_0^2\mathrm{vol}(\psi^*(-K_X)-u\widehat{E})\,du\\&=1-\frac{1}{28}\int_0^1((2-u)(H_1+H_2)+2H_3+(u-1)(F^-+F^+))^3\,du\\&\qquad-\frac{1}{28}\int_1^2((2-u)(H_1+H_2)+2H_3)^3\,du\\&=1-\int_0^1 8u^3-24u^2+28\,du-\int_1^2 12(2-u)^2\,du=\frac{1}{14}>0,\end{aligned}$$

which implies that X is K-polystable. □

To complete the proof of Theorem 1.11, let us present X as a codimension two complete intersection in a toric variety. Let $T=(\mathbb{C}^7\setminus Z(I))/\mathbb{G}_m^2$, where the $\mathbb{G}_m^2$-action is given by

$$\begin{pmatrix}x&y&z&w&u&v&s\\1&1&1&1&1&0&2\\0&0&0&0&1&1&0\end{pmatrix},$$

and I is the irrelevant ideal $\langle x,y,z,w,s\rangle \cap \langle u,v\rangle$. Let $\widetilde{\mathbb{P}} = \mathrm{Proj}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1))$. Then we can identify $\widetilde{\mathbb{P}}$ with the hypersurface in T given by $s = f(x,y,z,w)$, where $f(x,y,z,w)$ is any non-zero homogeneous polynomial of degree 2. Since Y can be obtained by blowing up the quadric cone over the surface $\{xy = zw\} \subset \mathbb{P}^3$ at the vertex, we can identify Y with the complete intersection in T given by

$$\begin{cases} xy = zw, \\ \ \ s = f(x,y,z,w). \end{cases}$$

Then the projection $\pi\colon T \to V$ is given by $(x,y,z,w,u,v,s) \mapsto (x,y,z,w)$, where we identify V with $\{xy = zw\} \subset \mathbb{P}^3$. Then the surface S^- is cut out on Y by $v = 0$. Moreover, we can assume that S^+ is cut out on Y by $u = 0$, and we can identify R with the curve in S^+ that is cut out by $s = 0$.

Let $\varphi\colon \overline{T} \to T$ be the blow up of T along $u = s = 0$. Then $\overline{T} = (\mathbb{C}^8 \setminus Z(\overline{I}))/\mathbb{G}_m^3$, where the torus action is given by the matrix

$$\begin{pmatrix} x & y & z & w & u & v & s & t \\ 1 & 1 & 1 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \end{pmatrix}$$

and the irrelevant ideal

$$\overline{I} = \langle x,y,z,w,s\rangle \cap \langle x,y,z,w,t\rangle \cap \langle u,v\rangle \cap \langle u,s\rangle \cap \langle v,t\rangle.$$

Then φ induces the blow up of Y along R. Thus, we can identify X with the complete intersection in the toric variety $\overline{T}$ given by

$$\begin{cases} xy = zw, \\ \ st = f(x,y,z,w). \end{cases}$$

Now, the subgroup $\Gamma \cong \mathbb{G}_m$ of the group $\mathrm{Aut}(X)$ mentioned in Section 1 can be explicitly seen – it consists of all automorphisms $(x,y,z,w,u,v,s,t) \mapsto (x,y,z,w,\lambda u,v,s,t)$, where $\lambda \in \mathbb{C}^*$. Similarly, we can choose the involution $\iota \in \mathrm{Aut}(X)$ to be the involution

$$(x,y,z,w,u,v,s,t) \mapsto (x,y,z,w,v,u,t,s).$$

Note that ι is not canonically defined, since we can conjugate it with an element in Γ.

Suppose that $R = C_1 + C_2$, where C_1 and C_2 are smooth curves in $|L|$ that meet transversally at two points. Then, up to a change of coordinates, we may assume that

$$f(x,y,z,t) = xy - \lambda(z^2 + w^2),$$

where $\lambda \in \mathbb{C}$ such that $\lambda \notin \{0, 2, -2\}$. Then X is the complete intersection in $\overline{T}$ given by

$$\begin{cases} xy = zw, \\ \ st = xy - \lambda(z^2 + w^2). \end{cases}$$

We can see from the equation $f = 0$ of $R = C_1 + C_2$ that the group $\mathrm{Aut}(V, C_1 + C_2)$ contains $\mathbb{G}_m \rtimes \boldsymbol{\mu}_2^2$, where the two involutions swap coordinates x, y or z, w, and the $\mathbb{G}_m$ is

defined by
$$(x, y, z, w) \mapsto \left(\mu x, \frac{y}{\mu}, z, w\right).$$
It follows that $\mathrm{Aut}(X)$ contains automorphisms
$$(x, y, z, w, u, v, s, t) \mapsto \left(\mu x, \frac{y}{\mu}, z, w, u, v, s, t\right),$$
where $\mu \in \mathbb{C}^*$. Similarly, the group $\mathrm{Aut}(X)$ contains two involutions
$$(x, y, z, w, u, v, s, t) \mapsto (y, x, z, w, u, v, s, t),$$
$$(x, y, z, w, u, v, s, t) \mapsto (x, y, w, z, u, v, s, t).$$
Let G be the subgroup in $\mathrm{Aut}(X)$ that is generated by all automorphisms described above. Then $G \cong \mathbb{G}_m^2 \rtimes \boldsymbol{\mu}_2^3$, and we have the following result.

Lemma 4.4. *The following assertions hold:*

(a) *X does not contain G-fixed points,*

(b) *X does not contain G-invariant irreducible curves,*

(c) *X contains two G-invariant irreducible surfaces – they are cut out by $z \pm w = 0$.*

Proof. Left to the reader. □

Now, we can complete the proof of Theorem 1.11. Suppose that X is not K-polystable. Using [49], we see that there is a G-invariant prime divisor $\mathbf{F}$ over X such that $\beta(\mathbf{F}) \leqslant 0$. Let Z be the center of this divisor on X. By Lemma 4.4, Z is a surface and $Z \sim (\pi \circ \phi)^*(L)$. Then, as in [21], we compute $\beta(\mathbf{F}) = \beta(Z) > 0$. This shows that X is K-polystable.

5. Proof of Theorem 1.12

In this section, we prove Theorem 1.12. This result describes all singular K-polystable limits of smooth Fano 3-folds in the family № 3.9. To show this, we need the following theorem.

Theorem 5.1 ([26, Theorem 2], [34, Example 7.13], [3]). *Let C be a quartic curve in $\mathbb{P}^2$. Then the curve C is*

- *GIT stable for* $\mathrm{PGL}_3(\mathbb{C})$*-action if and only if it is smooth or has $\mathbb{A}_1$ or $\mathbb{A}_2$-singularities,*
- *GIT strictly polystable if and only if it is one of the remaining curves in Theorem* 1.12.

Let us prove Theorem 1.12. Set $V = \mathbb{P}^2$, $L = \mathcal{O}_{\mathbb{P}^2}(2)$ and $Y = \mathbb{P}(\mathcal{O}_V \oplus \mathcal{O}_V(L))$. Let $\pi: Y \to V$ be the natural projection, set $H = \pi^*(L)$, let S^- and S^+ be disjoint sections of π such that $S^+ \sim S^- + H$, and let R be one of the following curves:

(1) a reduced quartic curve with at most $\mathbb{A}_1$ or $\mathbb{A}_2$ singularities;

(2) $C_1 + C_2$, where C_1 and C_2 are smooth conics that are tangent at two points;

(3) $C + \ell_1 + \ell_2$, where C is a smooth conic, ℓ_1 and ℓ_2 are distinct lines tangent to C;

(4) $2C$, where C is a smooth conic in $|L|$.

Set $F = \pi^*(R)$, and let $\phi\colon X \to Y$ be the blow up at the complete intersection $S^+ \cap F$. Then X is a singular Fano threefold, and our Theorem 1.12 claims that X is K-polystable. To prove this, we start with the most singular (and the most symmetric case).

Lemma 5.2. *Suppose that $R = 2C$ for a smooth conic $C \subset \mathbb{P}^2$. Then X is K-polystable.*

Proof. In this case, the threefold X has non-isolated singularities along a smooth curve, and the proof is very similar to the proof of Lemma 4.3. Namely, we have

$$\mathrm{Aut}(X) \cong \mathrm{PGL}_2(\mathbb{C}) \times (\mathbb{G}_m \rtimes \boldsymbol{\mu}_2), \tag{5.1}$$

and there exists exactly one $\mathrm{Aut}(X)$-invariant prime divisor over X, the exceptional divisor of the blow up of X along the curve $\mathrm{Sing}(X)$. So, to check that X is K-polystable, it is enough to compute the β-invariant of this prime divisor. Let us give details.

As in the proof of Lemma 4.3, we set $W = V \times \mathbb{P}^1$. Let $\varpi\colon W \to V$ be the natural projection, let $\widetilde{S}^-$ and $\widetilde{S}^+$ be its disjoint sections, and let $\widetilde{E} = \varpi^*(C)$. Then there exists the commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "\alpha"'] \arrow[dr, "\psi"] & \\ W \arrow[dr, "\varpi"'] & & X \arrow[dl, "\pi\circ\phi"] \\ & V & \end{tikzcd} \tag{5.2}$$

such that

- α is a blow up along the curves $\widetilde{E} \cap \widetilde{S}^-$ and $\widetilde{E} \cap \widetilde{S}^+$,
- ψ is a contraction of the proper transform of $\widetilde{E}$ to the curve $\mathrm{Sing}(X)$,
- $\phi \circ \psi$ maps the proper transforms of $\widetilde{S}^-$ and $\widetilde{S}^+$ to S^- and S^+, respectively.

This easily implies (5.1). Similarly, we see that (5.2) is $\mathrm{Aut}(X)$-equivariant.

Let $\widehat{E}$ be the ψ-exceptional divisor. Then $\widehat{E}$ is the only $\mathrm{Aut}(X)$-invariant prime divisor over the threefold X. Thus, if $\beta(\widehat{E}) > 0$, them X is K-polystable [49].

We let F^- and F^+ be α-exceptional surfaces such that $\alpha(F^-) \subset \widetilde{S}^-$ and $\alpha(F^+) \subset \widetilde{S}^+$, let $\widehat{S}^-$ and $\widehat{S}^+$ be the proper transforms on U of the surfaces S^- and S^+, respectively. Set $H_1 = (\mathrm{pr}_1 \circ \alpha)^*(\mathcal{O}_{\mathbb{P}^1}(1))$ for the projection $\mathrm{pr}_1\colon W \to \mathbb{P}^1$, set $H_2 = (\varpi \circ \alpha)^*(\mathcal{O}_V(1))$. Then $\widehat{E} \sim 2H_2 - F^- - F^+$, which gives

$$\psi^*(-K_X) \sim -K_U \sim 2H_1 + 3H_2 - F^- - F^+ \sim_{\mathbb{Q}} 2H_1 + \frac{3}{2}\widehat{E} + \frac{1}{2}(F^- + F^+).$$

Take $u \in \mathbb{R}_{\geqslant 0}$. Then

$$\begin{aligned} \psi^*(-K_X) - u\widehat{E} &\sim_{\mathbb{R}} 2H_1 + (3-2u)H_2 + (u-1)(F^- + F^+) \\ &\sim_{\mathbb{R}} 2H_1 + \frac{3-2u}{2}\widehat{E} + \frac{1}{2}(F^- + F^+). \end{aligned}$$

This shows that $\psi^*(-K_X) - u\widehat{E}$ is pseudo-effective if and only if $u \leqslant \frac{3}{2}$. Moreover, if we have $u \in [0, 1]$, then the divisor $\psi^*(-K_X) - u\widehat{E}$ is nef. If $1 < u \leqslant \frac{3}{2}$, its Zariski decomposition is

$$\psi^*(-K_X) - u\widehat{E} \sim_{\mathbb{R}} \underbrace{2H_1 + (3-2u)H_2}_{\text{positive part}} + \underbrace{(u-1)(F^- + F^+)}_{\text{negative part}}.$$

Hence, we have

$$\begin{aligned}
\beta(\widehat{E}) &= 1 - \frac{1}{(-K_X)^3}\int_0^{\frac{3}{2}} \operatorname{vol}(\psi^*(-K_X) - u\widehat{E})\,du \\
&= 1 - \frac{1}{26}\int_0^1 (2H_1 + (3-2u)H_2 + (u-1)(F^- + F^+))^3\,du \\
&\quad - \frac{1}{26}\int_1^{\frac{3}{2}} (2H_1 + (3-2u)H_2)^3\,du \\
&= 1 - \frac{1}{26}\int_0^1 16u^3 - 36u^2 + 26\,du - \frac{1}{26}\int_1^{\frac{3}{2}} 24u^2 - 72u + 54\,du = \frac{7}{26} > 0,
\end{aligned}$$

which implies that X is K-polystable. □

Similarly, we can show that X is K-polystable if $R = C_1 + C_2$, where C_1 and C_2 are smooth conics that are tangent at two points. Indeed, in this case, the full automorphism group $\operatorname{Aut}(X)$ contains a subgroup G such that $G \cong (\mathbb{G}_m)^2 \rtimes \boldsymbol{\mu}_2^2$, the threefold X does not contains G-fixed points, and the only G-invariant irreducible curve in X is a smooth fiber of the conic bundle $\pi \circ \phi$. Therefore, arguing exactly as in the proofs of [7, Lemma 4.64] and [7, Lemma 4.66], we see that X is K-polystable.

However, this approach fails in the case when R has a singular point of type $\mathbb{A}_1$ or $\mathbb{A}_2$, since, in general, X would not have as many symmetries. To overcome this difficulty, we will use another approach described in the end of Section 1. Namely, we proved in Section 2 that $\operatorname{Aut}(X)$ contains an involution ι such that ι swaps the proper transforms of S^- and S^+, $X/\iota \cong Y$, and the following diagram commutes:

$$\begin{tikzcd}
 & X \arrow[dl, "\phi"'] \arrow[dr, "\rho"] & \\
Y \arrow[dr, "\pi"'] & & Y \arrow[dl, "\pi"] \\
 & V, &
\end{tikzcd}$$

where ρ is the quotient map. Moreover, we also proved that the double cover ρ is ramified over a divisor $B \in |2S^+|$ such that the morphism $B \to V$ induced by π is a double cover ramified in the curve R. Set $\Delta = \frac{1}{2}B$. Then $-K_X \sim_{\mathbb{Q}} \rho^*(K_Y + \Delta)$, and (Y, Δ) has Kawamata log terminal singularities. Therefore, (Y, Δ) is a log Fano pair. Moreover, it follows from [31] that

$$X \text{ is K-polystable } \iff \Big(Y, \frac{1}{2}B\Big) \text{ is K-polystable.}$$

However, everything in life comes with a price: the action of the group $\Gamma \cong \mathbb{G}_m$ described earlier in Section 1 does not descent to Y via ρ, because Γ does not commute with ι. Thus, the group $\operatorname{Aut}(Y, \Delta)$ is much smaller than the group $\operatorname{Aut}(X)$.

To explicitly describe $B \subset Y$, consider Y as the toric variety $(\mathbb{C}^5 \setminus Z(I))/\mathbb{G}_m^2$ such that the torus action is given by the matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

with irrelevant ideal $I = \langle x_1, x_2, x_3\rangle \cap \langle x_4, x_5\rangle$. Let us also consider x_1, x_2, x_3 as coordinates on $V = \mathbb{P}^2$, so that the projection π is given by $(x_1, x_2, x_3, x_4, x_5) \mapsto (x_1, x_2, x_3)$. Then $S^- = \{x_5 = 0\}$. Moreover, we may assume that $S^+ = \{x_4 = 0\}$, and B is given by

$$x_4^2 - f_4(x_1, x_2, x_3)x_5^2 = 0,$$

where $f_4(x_1, x_2, x_3)$ is a quartic polynomial such that $R = \{f_4(x_1, x_2, x_3) = 0\}$.

In the remaining part of the section, we will prove that the pair (Y, Δ) is K-polystable. Recall that $H = \pi^*(L)$. Note also that

$$-(K_Y + \Delta) \sim_{\mathbb{Q}} S^- + \frac{3}{2}H.$$

We will split the proof in several lemmas and propositions. We start with the following lemma.

Lemma 5.3. *Let P be a point in S^-. Then $\delta_P(Y, \Delta) > 1$.*

Proof. Let us apply Lemma 3.2. We have

$$\delta_P(Y, \Delta) = \delta_P(Y; D(a)) \geqslant \min\left\{\frac{4(a^3 - (a-1)^3)}{(4-a)a^3 + (a-1)^4}, \frac{4(a^3 - (a-1)^3)}{3(a^4 - (a-1)^4)}\delta(V; L)\right\},$$

where $D(a) = -(K_Y + \Delta)$ and $a = \frac{3}{2}$. Thus, we have

$$\delta_P(Y, \Delta) \geqslant \min\left\{\frac{26}{17}, \frac{13}{15}\delta(V; L)\right\}.$$

But

$$\delta(V; L) = \delta\Big(V; \frac{2}{3}(-K_V)\Big) = \frac{3}{2}\delta(V; -K_V) = \frac{3}{2}\delta(V) = \frac{3}{2}\delta(\mathbb{P}^2) = \frac{3}{2},$$

so that $\delta_P(Y, \Delta) \geqslant \frac{13}{10}$. □

Similarly, applying Proposition 3.5, we obtain the following lemma.

Lemma 5.4. *Let P be a point Y such that $P \notin \mathrm{Sing}(B)$. Then $\delta_P(Y, \Delta) > 1$.*

Proof. By Lemma 5.3, we may assume that $P \notin S^-$. Then Proposition 3.5 gives

$$\delta_P(Y, \Delta) = \delta_P(Y; D(a)) \geqslant \frac{8(3a^2 - 3a + 1)}{8d\mu a^3 + 6(1 - 2d\mu)a^2 + 8(d\mu - 1)a - 2d\mu + 3},$$

where $D(a) = -(K_Y + \Delta)$, $a = \frac{3}{2}$, $d = L^2 = 4$, $\mu = \frac{1}{2}$. This gives $\delta_P(Y, \Delta) \geqslant \frac{52}{49}$. □

The two most difficult parts of the proof that (Y, Δ) is K-polystable are the following two propositions, which will be proved in Sections 5.1 and 5.2 later.

Proposition 5.5. *Let P be a point in B such that B has singular point of type $\mathbb{A}_1$ at P, and let $\mathbf{F}$ be a prime divisor over Y such that $P = C_Y(\mathbf{F})$. Then $\beta_{Y,\Delta}(\mathbf{F}) > 0$.*

Proposition 5.6. *Let P be a point in B such that B has singular point of type $\mathbb{A}_2$ at P, and let $\mathbf{F}$ be a prime divisor over Y such that $P = C_Y(\mathbf{F})$. Then $\beta_{Y,\Delta}(\mathbf{F}) > 0$.*

By Lemmas 5.3 and 5.4 and Propositions 5.5 and 5.6, the log pair (Y, Δ) is K-stable in the case when R is a reduced plane quartic curve that has at most $\mathbb{A}_1$ or $\mathbb{A}_2$ singularities.

Therefore, to complete the proof, we may assume that R is one of the following curves:

(2) $C_1 + C_2$, where C_1 and C_2 are smooth conics that are tangent at two points;

(3) $C + \ell_1 + \ell_2$, where C is a smooth conic, ℓ_1 and ℓ_2 are distinct lines tangent to C;

(4) $2C$, where C is a smooth conic in $|L|$.

Hence, appropriately changing coordinates x_1, x_2, x_3, we may assume that

$$f_4(x_1, x_2, x_3) = (x_1x_2 - x_3^2)(x_1x_2 - \lambda x_3^2),$$

where one of the following three cases holds:

(2) $\lambda \notin \{0, 1\}$, $R = C_1 + C_2$, where $C_1 = \{x_1x_2 = x_3^2\}$ and $C_2 = \{x_1x_2 = \lambda x_3^2\}$;

(3) $\lambda = 0$, $R = C + \ell_1 + \ell_2$, where $C = \{x_1x_2 = x_3^2\}$, $\ell_1 = \{x_1 = 0\}$ and $\ell_2 = \{x_2 = 0\}$;

(4) $\lambda = 1$, $R = 2C$, where $C = \{x_1x_2 = x_3^2\}$.

In each case, the group $\mathrm{Aut}(Y, \Delta)$ contains an involution τ such that

$$\tau(x_1, x_2, x_3, x_4, x_5) = (x_2, x_1, x_3, x_4, x_5).$$

Lemma 5.7. *Suppose that $\lambda \notin \{0, 1\}$. Then (Y, Δ) is K-polystable.*

Proof. Suppose (Y, Δ) is not K-polystable. It follows from [49] that there is a $\langle\tau\rangle$-invariant prime divisor $\mathbf{F}$ over Y such that $\beta_{Y,\Delta}(\mathbf{F}) \leqslant 0$. Let P be a general point in $C_Y(\mathbf{F})$. Then $\delta_P(Y, \Delta) \leqslant 1$. But $P \notin \mathrm{Sing}(B)$, since $\mathrm{Sing}(B)$ consists of two singular points that are swapped by τ. Then $\delta_P(Y, \Delta) > 1$ by Lemmas 5.3 and 5.4, which is a contradiction. □

Lemma 5.8. *Suppose $\lambda = 0$. Then (Y, Δ) is K-polystable.*

Proof. The surface B has a singular point of type $\mathbb{A}_1$, and two singular points of type $\mathbb{A}_3$, that are swapped by τ. Arguing as in the proof of Lemma 5.7 and using Propositions 5.5, we see that X is K-polystable. □

Lemma 5.9 (cf. Lemma 5.2). *Suppose $\lambda = 1$. Then (Y, Δ) is K-polystable.*

Proof. In this case, we have $R = 2C$, where C is an irreducible conic. Then we have $B = B_1 + B_2$, where B_1 and B_2 are smooth surfaces in $|S^+|$ that intersect transversally along a smooth curve such that $\pi(B_1 \cap B_2) = C$.

We already know from Lemma 5.2 that the threefold X is K-polystable in this case, so that (Y, Δ) is also K-polystable [31]. Let us prove this directly for consistency.

Let $W = V \times \mathbb{P}^1$, let $\varpi\colon W \to V$ be the natural projection, let $\widetilde{S}^-$, $\widetilde{B}_1$, $\widetilde{B}_2$ be its disjoint sections, and let $\widetilde{E} = \varpi^*(C)$. Then there exists the commutative diagram

$$\begin{tikzcd}
& U \arrow[ld, "\alpha"'] \arrow[rd, "\psi"] & \\
W \arrow[rd, "\varpi"'] & & Y \arrow[ld, "\pi"] \\
& V &
\end{tikzcd}$$

such that α is a blow up along the curve $\widetilde{E}\cap\widetilde{S}^-$, the morphism ψ is a contraction of the proper transform of the surface $\widetilde{E}$ to the intersection curve $B_1\cap B_2$ such that ψ maps the proper transforms of the surfaces $\widetilde{S}^-$, $\widetilde{B}_1$, $\widetilde{B}_2$ to the surfaces S^-, B_1, B_2, respectively. Then

$$\mathrm{Aut}(Y,\Delta)\cong\mathrm{Aut}(U)\cong\mathrm{Aut}(W,\widetilde{B}_1+\widetilde{B}_2+\widetilde{E}+\widetilde{S}^-)\cong\mathrm{PGL}_2(\mathbb{C})\times\boldsymbol{\mu}_2.$$

Note that the commutative diagram above is $\mathrm{Aut}(Y,\Delta)$-equivariant.

Let F be α-exceptional surface, let $\widehat{E}$ be the ψ-exceptional surface, let $\widehat{B}_1$ and $\widehat{B}_2$ be the proper transforms on U of the surfaces B_1 and B_2, respectively. Set $\widehat{\Delta}=\frac{1}{2}(\widehat{B}_1+\widehat{B}_2)$. Then $K_U+\widehat{\Delta}\sim_{\mathbb{Q}}\psi^*(K_Y+\Delta)$, so that ψ is log crepant for $(U,\widehat{\Delta})$. Then $A_{Y,\Delta}(\widehat{E})=1$.

First, we compute $\beta_{Y,\Delta}(\widehat{E})$. Set $H_1=(\mathrm{pr}_1\circ\alpha)^*(\mathcal{O}_{\mathbb{P}^1}(1))$ and $H_2=(\varpi\circ\alpha)^*(\mathcal{O}_V(1))$, where pr_1 is the natural projection $W\to\mathbb{P}^1$. Then $\widehat{\Delta}\sim_{\mathbb{Q}}H_1$ and $\widehat{E}\sim 2H_2-F$, so that

$$\psi^*(K_Y+\Delta)\sim_{\mathbb{Q}}K_U+\widehat{\Delta}\sim_{\mathbb{Q}}H_1+3H_2-F\sim_{\mathbb{Q}}H_1+\frac{3}{2}\widehat{E}+\frac{1}{2}F.$$

Let u be a non-negative real number. Then

$$\psi^*(K_Y+\Delta)-u\widehat{E}\sim_{\mathbb{R}}H_1+(3-2u)H_2+(u-1)F\sim_{\mathbb{R}}H_1+\frac{3-2u}{2}\widehat{E}+\frac{1}{2}F,$$

and this divisor is pseudo-effective if and only if $u\leqslant\frac{3}{2}$. For $u\in[0,\frac{3}{2}]$, let $P(u)$ be the positive part of the Zariski decomposition of $\psi^*(K_Y+\Delta)-u\widehat{E}$, and let $N(u)$ be the negative part. Then

$$P(u)\sim_{\mathbb{R}}\begin{cases}H_1+(3-2u)H_2+(u-1)F & \text{if } 0\leqslant u\leqslant 1,\\ H_1+(3-2u)H_2 & \text{if } 1\leqslant u\leqslant\frac{3}{2},\end{cases}$$

and

$$N(u)=\begin{cases}0 & \text{if } 0\leqslant u\leqslant 1,\\ (u-1)F & \text{if } 1\leqslant u\leqslant\frac{3}{2}.\end{cases}$$

This gives

$$\begin{aligned}\beta_{Y,\Delta}(\widehat{E})&=A_{Y,\Delta}(\widehat{E})-\frac{1}{(-K_Y-\Delta)^3}\int_0^{\frac{3}{2}}(P(u))^3\,du\\&=1-\frac{1}{13}\int_0^1(2H_1+(3-2u)H_2+(u-1)F)^3\,du\\&\qquad-\frac{1}{13}\int_1^{\frac{3}{2}}(2H_1+(3-2u)H_2)^3\,du\\&=1-\int_0^1 8u^3-18u^2+13\,du-\int_1^{\frac{3}{2}}12u^2-36u+27\,du=\frac{7}{26}>0.\end{aligned}$$

Suppose that (Y,Δ) is not K-polystable. By [49], there exists an $\mathrm{Aut}(Y,\Delta)$-invariant prime divisor $\mathbf{F}$ over Y such that $\beta_{Y,\Delta}(\mathbf{F})\leqslant 0$. Let Z be its center on Y. Then $\delta_P(Y,\Delta)\leqslant 1$ for every point $P\in Z$. Hence, it follows from Lemmas 5.3 and 5.4 that $Z\subset B_1\cap B_2$. Hence, since Z is a $\mathrm{Aut}(Y,\Delta)$-invariant irreducible subvariety, we see that $Z=B_1\cap B_2$.

Let $\widehat{Z}$ be the center of the divisor $\mathbf{F}$ on the threefold U. Then $\widehat{Z}\neq\widehat{E}$, since $\beta(\widehat{E})>0$. Moreover, since $\widehat{Z}\subset\widehat{E}$ and $\widehat{Z}$ is $\mathrm{Aut}(U)$-invariant, we see that $\widehat{Z}$ is a $\mathrm{Aut}(U)$-invariant section of the natural projection $\widehat{E}\to Z$. Set $A=K_U+\widehat{\Delta}$. Then

$$0\geqslant\beta_{Y,\Delta}(\mathbf{F})=A_{Y,\Delta}(\mathbf{F})-S_A(\mathbf{F})=A_{U,\widehat{\Delta}}(\mathbf{F})-S_A(\mathbf{F}),$$

because $K_U + \widehat{\Delta} \sim_{\mathbb{Q}} \psi^*(K_Y + \Delta)$. Moreover, it follows from [2, 7, 22] that

$$1 \geqslant \frac{A_{U,\widehat{\Delta}}(\mathbf{F})}{S_A(\mathbf{F})} \geqslant \min\left\{\frac{1}{S_A(\widehat{E})}, \frac{1}{S_A(W^{\widehat{E}}_{\bullet,\bullet}; \widehat{Z})}\right\},$$

where $S_A(W^{\widehat{E}}_{\bullet,\bullet}; \widehat{Z})$ is defined in [7, Section 1.7]. But $S_A(\widehat{E}) = \frac{19}{26}$, so $S_A(W^{\widehat{E}}_{\bullet,\bullet}; \widehat{Z}) \geqslant 1$.

Let us compute $S_A(W^{\widehat{E}}_{\bullet,\bullet}; \widehat{Z})$. Using [7, Corollary 1.109], we see that

$$\begin{aligned} S_A(W^{\widehat{E}}_{\bullet,\bullet}; \widehat{Z}) = \frac{3}{A^3}\int_0^{\frac{3}{2}} (P(u)|_{\widehat{E}})^2 \operatorname{ord}_{\widehat{Z}}(N(u)|_{\widehat{E}}) \\ + \frac{3}{A^3}\int_0^{\frac{3}{2}}\int_0^{\infty} \operatorname{vol}(P(u)|_{\widehat{E}} - v\widehat{Z})\, dv\, du, \end{aligned}$$

which is easy to compute, because $\widehat{E} \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let us do this.

Let $\mathbf{s} = F \cap \widehat{E}$. Then $\mathbf{s}$ is a section of the projection $\widehat{E} \to Z$. Let $\mathbf{f}$ be a fiber of this projection. Then

$$P(u)|_{\widehat{E}} = \begin{cases} (6-4u)\mathbf{f} + u\mathbf{s} & \text{if } 0 \leqslant u \leqslant 1, \\ (6-4u)\mathbf{f} + \mathbf{s} & \text{if } 1 \leqslant u \leqslant \frac{3}{2}, \end{cases}$$

and

$$N(u)|_{\widehat{E}} = \begin{cases} 0 & \text{if } 0 \leqslant u \leqslant 1, \\ (u-1)\mathbf{s} & \text{if } 1 \leqslant u \leqslant \frac{3}{2}. \end{cases}$$

Thus, we see that $S_A(W^{\widehat{E}}_{\bullet,\bullet}; \widehat{Z}) \leqslant S_A(W^{\widehat{E}}_{\bullet,\bullet}; \mathbf{s})$ and

$$\begin{aligned} S_A(W^{\widehat{E}}_{\bullet,\bullet}; \mathbf{s}) &= \frac{3}{13}\int_1^{\frac{3}{2}} ((6-4u)\mathbf{f} + \mathbf{s})^2 (u-1)\, du \\ &\quad + \frac{3}{13}\int_0^1\int_0^u ((6-4u)\mathbf{f} + (u-v)\mathbf{s})^2\, dv\, du \\ &\quad + \frac{3}{13}\int_1^{\frac{3}{2}}\int_0^1 ((6-4u)\mathbf{f} + (1-v)\mathbf{s})^2\, dv\, du \\ &= \frac{3}{13}\int_1^{\frac{3}{2}} 2(6-4u)(u-1)\, du \\ &\quad + \frac{3}{13}\int_0^1\int_0^u 2(6-4u)(u-v)\, dv\, du \\ &\quad + \frac{3}{13}\int_1^{\frac{3}{2}}\int_0^1 2(6-4u)(1-v)\, dv\, du = \frac{5}{13} < 1, \end{aligned}$$

which is a contradiction. □

In the remaining part of this sections, we will prove Proposition 5.5 and 5.6.

5.1. Proof of Proposition 5.5. Let us use notation introduced earlier in this section before Proposition 5.5, and let P be an isolated ordinary double point of the surface B. Then, up to a change of coordinates, we may assume that $P = (0, 0, 1, 0, 1)$ and

$$f_4(x_1, x_2, 1) = x_1^2 + x_2^2 + \text{higher order terms.}$$

Let $\rho\colon Y_0 \to Y$ be the blow up at P; note that ρ is a log resolution of (Y, B). Then Y_0 is the toric variety $(\mathbb{C}^6 \setminus Z(I_0))/\mathbb{G}_m^3$ for the torus action given by

$$M = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

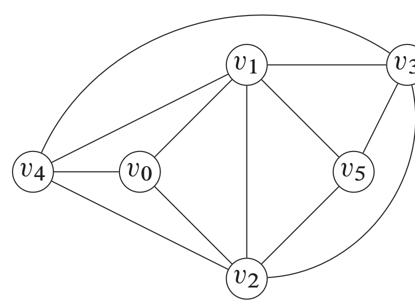
with irrelevant ideal

$$I_0 = \langle x_1, x_2, x_3 \rangle \cap \langle x_1, x_2, x_4 \rangle \cap \langle x_4, x_5 \rangle \cap \langle x_0, x_3 \rangle \cap \langle x_0, x_5 \rangle.$$

To describe its fan, denote the vector generating the ray corresponding to x_i by v_i. Then

$$\begin{aligned} &v_0 = (1,1,1), && v_1 = (1,0,0), && v_2 = (0,1,0), \\ &v_3 = (-1,-1,-2), && v_4 = (0,0,1), && v_5 = (0,0,-1). \end{aligned}$$

The cone structure can be derived from the irrelevant ideal I_0, and it can be visualized via the following diagram:



Let $F_i = \{x_i = 0\} \subset Y_0$, and let $C_{ij} = F_i \cap F_j$ for $i \neq j$ such that $\dim(F_i \cap F_j) = 1$. Geometrically, the divisors F_i are as follows.

- F_0 is the exceptional divisor of the blow up $\rho\colon Y_0 \to Y$.
- Let $D \sim \pi^* C$ be a pullback of a line and suppose D contains P; then strict transform $\rho_*^{-1} D$ of D on Y_0 is linearly equivalent to F_1 and F_2.
- And for pullback, we have $\rho^* D \sim F_3$.
- Divisors F_4 and F_5 are the proper transforms of the positive and negative sections of π on Y_0, respectively.

Consider the $\mathbb{Z}^3$-grading of $\mathrm{Pic}(Y_0)$ given by M. If D_1 and D_2 are two divisors in $\mathrm{Pic}(Y_0)$, then it follows from [15, Chapter 5] that

$$D_1 \sim D_2 \iff \deg_M(D_1) = \deg_M(D_2).$$

Moreover, we have

$$\overline{\mathrm{Eff}(Y_0)} = \langle F_0, F_1, F_5 \rangle \quad \text{and} \quad \overline{\mathrm{NE}(Y_0)} = \langle C_{12}, C_{15}, C_{01} \rangle.$$

In particular, a divisor D with $\deg_M(D) = (a, b, c)$ is effective if and only if all $a, b, c \geqslant 0$. Note that curve C_{01} is a line in the exceptional divisor F_0, C_{12} is the proper transform of a fiber of π passing through P, and C_{15} is a pullback of the negative section of $\rho(F_1) \cong \mathbb{F}_2$.

Lemma 5.10. *Intersections of divisors F_0, F_1, F_5 are given by the following table:*

F_0^3	$F_0^2F_1$	$F_0^2F_5$	$F_0F_1^2$	$F_0F_1F_5$	$F_0F_5^2$	F_1^3	$F_1^2F_5$	$F_1F_5^2$	F_5^3
1	−1	0	1	0	0	−1	1	−2	4

Proof. Recall that, for distinct torus-invariant divisors F_i, F_j, F_k, we may compute their intersection using the fan and the cone structure (or the irrelevant ideal)

$$F_iF_jF_k = \begin{cases} 0, & x_ix_jx_k \in I_0, \\ \dfrac{1}{|\det\{v_i, v_j, v_k\}|} & \text{otherwise.} \end{cases}$$

This fact together with the linear equivalences implies the required assertion. □

Using Lemma 5.10, we obtain the following intersection table:

$\bullet$	F_0	F_1	F_5
C_{12}	1	−1	1
C_{15}	0	1	−2
C_{01}	−1	1	0

Now, we set $A = -(K_Y + \Delta)$. Take $u \in \mathbb{R}_{\geqslant 0}$. Set

$$L(u) = \rho^*(A) - uF_0.$$

Then $L(u) \sim_{\mathbb{R}} (3-u)F_0 + 3F_1 + F_5$. So the divisor $L(u)$ is pseudo-effective if and only if $u \leqslant 3$. Let us find a Zariski decomposition of the divisor $L(u)$ for $u \in [0, 3]$.

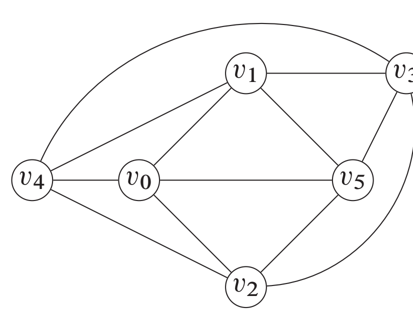
The divisor $L(u)$ is nef for $u \in [0, 1]$. We have $L(1) \cdot C_{12} = 0$. Since C_{12} is a flopping curve, we have to consider a small $\mathbb{Q}$-factorial modification $Y_0 \dashrightarrow Y_1$ such that

$$Y_1 = (\mathbb{C}^6 \setminus Z(I_1))/\mathbb{G}_m^3,$$

where the torus action is the same (given by the matrix M) and the irrelevant ideal

$$I_1 = \langle x_1, x_2\rangle \cap \langle x_4, x_5\rangle \cap \langle x_0, x_3\rangle,$$

which is obtained from I_0 by replacing $\langle x_0, x_5\rangle$ with $\langle x_1, x_2\rangle$. The fan of Y_1 is generated by the same vectors, but the cone structure is different:



Abusing our previous notation, we denote the divisor $\{x_i = 0\} \subset Y_1$ also by F_i, and we let $C_{ij} = F_i \cap F_j$ for $i \neq j$ such that $F_i \cap F_j$ is a curve. As above, we see that

$$\overline{\mathrm{NE}(Y_1)} = \langle C_{01}, C_{15}, C_{05} \rangle.$$

Moreover, intersections of divisors on Y_1 are described in the following table:

F_0^3	$F_0^2 F_1$	$F_0^2 F_5$	$F_0 F_1^2$	$F_0 F_1 F_5$	$F_0 F_5^2$	F_1^3	$F_1^2 F_5$	$F_1 F_5^2$	F_5^3
0	0	−1	0	1	−1	0	0	−1	3

Using these intersections, we obtain the following intersection table:

$\bullet$	F_0	F_1	F_5
C_{05}	−1	1	−1
C_{15}	1	0	−1
C_{01}	0	0	1

The proper transform on Y_1 of the divisor $L(u)$ is nef for $u \in [1, 2]$, and it intersects the curve C_{15} trivially for $u = 2$. Note that $C_{15} \sim C_{25}$ on the surface F_5, which implies that the divisor F_5 is contained in the negative part of the Zariski decomposition of the proper transform of the divisor $L(u)$. In fact, we have $N(u) = (u - 2)F_5$ and

$$P(u) = (3 - u)(F_0 + F_5) + 3F_1,$$

where $N(u)$ is the negative part of the decomposition, and $P(u)$ is the positive part.

Lemma 5.11. *One has $A_{Y,\Delta}(F_0) = 2$ and $S_A(F_0) = \frac{49}{26}$, so that*

$$\frac{A_{Y,\Delta}(F_0)}{S_A(F_0)} = \frac{52}{49}.$$

Proof. The equality $A_{Y,\Delta}(F_0) = 2$ is obvious. Moreover, we have

$$\mathrm{vol}(L(u)) = \begin{cases} -u^3 + 13, & u \in [0, 1], \\ -3u^2 + 3u + 12, & u \in [1, 2], \\ 3u^3 - 18u^2 + 27u, & u \in [2, 3]. \end{cases}$$

Thus, we compute

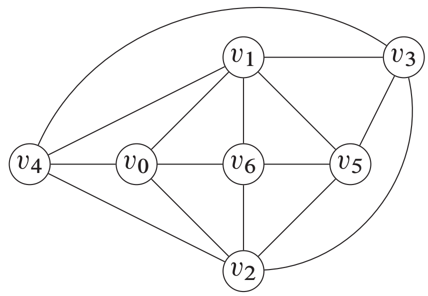
$$S_A(F_0) = \frac{1}{A^3} \int_0^3 \mathrm{vol}(L(u))\, du = \frac{49}{26},$$

as claimed. □

Now, we construct a common toric resolution $\widetilde{Y}$ for Y_0 and Y_1. Such variety is easy to see from the fans of Y_0 and Y_1; we want to add the following ray:

$$v_6 = (1, 1, 0) \in \langle v_1, v_2 \rangle \cap \langle v_0, v_5 \rangle.$$

Set $\widetilde{Y}$ to be the toric variety corresponding to $v_0, \dots, v_6$ with the following cone structure:



Let $\varphi_0 \colon \widetilde{Y} \to Y_0$ and $\varphi_1 \colon \widetilde{Y} \to Y_1$ be the corresponding toric birational maps. Then

- φ_0 is the blow up of Y_0 along the curve C_{12},
- φ_1 is the blow up of Y_1 along the curve C_{05}.

Set $\widetilde{F}_i = \{x_i = 0\} \subset \widetilde{Y}$. Then $\widetilde{F}_6$ is the exceptional divisor of φ_0 and φ_1.

The Zariski decomposition of the divisor $\varphi_0^*(L(u))$ can be described as follows:

$$\widetilde{P}(u) \sim_{\mathbb{R}} \begin{cases} (3-u)\widetilde{F}_0 + 3\widetilde{F}_1 + \widetilde{F}_5 + 3\widetilde{F}_6, & u \in [0,1], \\ (3-u)\widetilde{F}_0 + 3\widetilde{F}_1 + \widetilde{F}_5 + (4-u)\widetilde{F}_6, & u \in [1,2], \\ (3-u)(\widetilde{F}_0 + \widetilde{F}_5) + 3\widetilde{F}_1 + (6-2u)\widetilde{F}_6, & u \in [2,3], \end{cases}$$

and

$$\widetilde{N}(u) = \begin{cases} 0, & u \in [0,1], \\ (u-1)\widetilde{F}_6, & u \in [1,2], \\ (u-2)\widetilde{F}_5 + (2u-3)\widetilde{F}_6, & u \in [2,3], \end{cases}$$

where $\widetilde{P}(u)$ is the positive part, and $\widetilde{N}(u)$ is the negative part.

Let $\sigma \colon \widetilde{F}_0 \to F_0$ be the morphism induced by φ_0. Recall that F_0 is the exceptional divisor of the blow up ρ at a smooth point P. Then, since σ is a blow up at one point, we have $\widetilde{F}_0 \cong \mathbb{F}_1$. Let $\mathbf{e}$ be the σ-exceptional curve, and let $\mathbf{f}$ be a fiber of the natural projection $\widetilde{F}_0 \to \mathbb{P}^1$. Then $\widetilde{F}_0|_{\widetilde{F}_0} \sim -\mathbf{e} - \mathbf{f}$, $\widetilde{F}_1|_{\widetilde{F}_0} \sim \mathbf{f}$, $\widetilde{F}_5|_{\widetilde{F}_0} \sim 0$, $\widetilde{F}_6|_{\widetilde{F}_0} = \mathbf{e}$, which gives

$$\widetilde{P}(u)|_{\widetilde{F}_0} = \begin{cases} u(\mathbf{f} + \mathbf{e}), & u \in [0,1], \\ u\mathbf{f} + \mathbf{e}, & u \in [1,2], \\ u\mathbf{f} + (3-u)\mathbf{e}, & u \in [2,3], \end{cases}$$

and

$$\widetilde{N}(u)|_{\widetilde{F}_0} = \begin{cases} 0, & u \in [0,1], \\ (u-1)\mathbf{e}, & u \in [1,2], \\ (2u-3)\mathbf{e}, & u \in [2,3]. \end{cases}$$

We are ready to apply [2,7,22]. Set $B_{F_0} = \rho_*^{-1}(B)|_{F_0}$; since B has a node at P, we see that B_{F_0} is a conic. We set $\Delta_{F_0} = \frac{1}{2} B_{F_0}$ and we set

$$\delta(F_0, \Delta_{F_0}; V_{\bullet,\bullet}^{\widetilde{F}_0}) = \inf_{E/\widetilde{F}_0} \frac{A_{F_0, \Delta_{F_0}}(E)}{S(W_{\bullet,\bullet}^{\widetilde{F}_0}; E)},$$

where the infimum is taken over all prime divisors E over $\widetilde{F}_0$, and

$$S(W^{\widetilde{F}_0}_{\bullet,\bullet}; E) = \frac{3}{A^3}\int_0^3 (\widetilde{P}(u)|_{\widetilde{F}_0})^2 \operatorname{ord}_E(\widetilde{N}(u)|_{\widetilde{F}_0})\,du + \frac{3}{A^3}\int_0^3\int_0^\infty \operatorname{vol}(\widetilde{P}(u)|_{\widetilde{F}_0} - vE)\,dv\,du.$$

Let $\mathbf{F}$ be a prime divisor over Y such that $P = C_Y(\mathbf{F})$. Recall that

$$\beta_{Y,\Delta}(\mathbf{F}) = A_{Y,\Delta}(\mathbf{F}) - S_A(\mathbf{F}) = A_{Y,\Delta}(\mathbf{F}) - \frac{1}{A^3}\int_0^\infty \operatorname{vol}(A - u\mathbf{F})\,du.$$

It follows from [22, Theorem 4.8] and [22, Corollary 4.9] that

$$\frac{A_{Y,\Delta}(\mathbf{F})}{S_A(\mathbf{F})} \geqslant \delta_P(Y,\Delta) \geqslant \min\left\{\frac{A_{Y,\Delta}(F_0)}{S_A(F_0)}, \delta(F_0, \Delta_{F_0}; V^{\widetilde{F}_0}_{\bullet,\bullet})\right\}. \tag{5.3}$$

Suppose $\beta_{Y,\Delta}(\mathbf{F}) \leqslant 0$. Then it follows from (5.3) and Lemma 5.11 that there is a prime divisor E over $\widetilde{F}_0$ such that

$$S(W^{\widetilde{F}_0}_{\bullet,\bullet}; E) \geqslant A_{F_0,\Delta_{F_0}}(E). \tag{5.4}$$

Let Z be the center of the divisor E on the surface $\widetilde{F}_0$. Note that $\sigma(\mathbf{e}) \notin B_{F_0}$.

Lemma 5.12. *One has $Z \cap \mathbf{e} = \varnothing$.*

Proof. Note that $A_{F_0,\Delta_{F_0}}(\mathbf{e}) = 2$. Let us compute $S(W^{\widetilde{F}_0}_{\bullet,\bullet}; \mathbf{e})$. For $u \in [0,3]$, let

$$t(u) = \sup\{v \in \mathbb{R}_{\geqslant 0} \mid \widetilde{P}(u)|_{\widetilde{F}_0} - v\mathbf{e} \text{ is pseudo-effective}\}.$$

For every $v \in [0, t(u)]$, let us denote by $P(u,v)$ and $N(u,v)$ the positive and the negative parts of the Zariski decompositions of the divisor $\widetilde{P}(u)|_{\widetilde{F}_0} - v\mathbf{e}$, respectively. Then

$$S(W^{\widetilde{F}_0}_{\bullet,\bullet}; \mathbf{e}) = \frac{3}{A^3}\int_0^3 (P(u,0))^2 \operatorname{ord}_{\mathbf{e}}(\widetilde{N}(u)|_{\widetilde{F}_0})\,du + \frac{3}{A^3}\int_0^3\int_0^{t(u)} (P(u,v))^2\,dv\,du.$$

Observe that

$$\operatorname{ord}_{\mathbf{e}}(\widetilde{N}(u)|_{\widetilde{F}_0}) = \begin{cases} 0, & u \in [0,1],\\ u-1, & u \in [1,2],\\ 2u-3, & u \in [2,3].\end{cases}$$

Moreover, we have

$$t(u) = \begin{cases} u, & u \in [0,1],\\ 1, & u \in [1,2],\\ 3-u, & u \in [2,3].\end{cases}$$

Furthermore, we have $N(u,v) = 0$ for every $u \in [0,3]$ and $v \in [0, t(u)]$. Finally, we have

$$P(u,v) = \begin{cases} uf + (u-v)e, & u \in [0,1], v \in [0,u],\\ uf + (1-v)e, & u \in [1,2], v \in [0,1],\\ uf + (3-u-v)e, & u \in [2,3], v \in [0, 3-u],\end{cases}$$

which gives

$$(P(u,v))^2 = \begin{cases} u^2 - v^2, & u \in [0,1], v \in [0,u], \\ u^2 - (1-v-u)^2, & u \in [1,2], v \in [0,1], \\ u^2 - (3-2u-v)^2, & u \in [2,3], v \in [0,3-u]. \end{cases}$$

Integrating, we get $S(W^{\widetilde{F}_0}_{\bullet,\bullet}; \mathbf{e}) = \frac{20}{13} < 2 = A_{F_0,\Delta_{F_0}}(\mathbf{e})$, so that $Z \neq \mathbf{e}$ by (5.4).

Suppose that $Z \cap \mathbf{e} \neq \varnothing$. Let O be a point of the intersection $Z \cap \mathbf{e}$. Then it follows from [22, Theorem 4.17] and [22, Corollary 4.18] that

$$\frac{A_{F_0,\Delta_{F_0}}(E)}{S(W^{\widetilde{F}_0}_{\bullet,\bullet}; E)} \geqslant \min\left\{\frac{2}{S(W^{\widetilde{F}_0}_{\bullet,\bullet}; \mathbf{e})}, \frac{1}{S(W^{\widetilde{F}_0,\mathbf{e}}_{\bullet,\bullet,\bullet}; O)}\right\} = \min\left\{\frac{13}{10}, \frac{1}{S(W^{\widetilde{F}_0,\mathbf{e}}_{\bullet,\bullet,\bullet}; O)}\right\},$$

where

$$S(W^{\widetilde{F}_0,\mathbf{e}}_{\bullet,\bullet,\bullet}; O) = \frac{3}{A^3}\int_0^3 \int_0^{t(u)} (P(u,v)\cdot \mathbf{e})^2 \, dv\, du.$$

Integrating, we get $S(W^{\widetilde{F}_0,\mathbf{e}}_{\bullet,\bullet,\bullet}; O) = \frac{20}{13}$, which contradicts (5.4). □

Thus, we see that Z is disjoint from $\mathbf{e}$. In particular, we see that

$$Z \cap \operatorname{Supp}(\widetilde{N}(u)|_{\widetilde{F}_0}) = \varnothing$$

for every $u \in [0,3]$. This will simplify some formulas in the following.

Let $B_{\widetilde{F}_0}$ be the strict transform on $\widetilde{F}_0$ of the curve B_{F_0}. Then $B_{\widetilde{F}_0}$ is a smooth irreducible curve in $|2(\mathbf{e}+\mathbf{f})|$. Set $\Delta_{\widetilde{F}_0} = \frac{1}{2}B_{\widetilde{F}_0}$. Let O be a point in Z. We may assume that $O \in \mathbf{f}$. Then there are three cases to consider:

(1) $O \notin B_{\widetilde{F}_0}$,

(2) $O \in B_{\widetilde{F}_0} \cap \mathbf{f}$, and $\mathbf{f}$ intersects $B_{\widetilde{F}_0}$ transversely at the point O,

(3) $O = B_{\widetilde{F}_0} \cap \mathbf{f}$, and $\mathbf{f}$ is tangent to $B_{\widetilde{F}_0}$ at the point O.

Let $\theta\colon \widehat{F}_0 \to \widetilde{F}_0$ be a plt blow up of the point O defined as follows:

- the map θ is an ordinary blow up in the case when $O \notin B_{\widetilde{F}_0}$, or when $O \in B_{\widetilde{F}_0} \cap \mathbf{f}$, and the fiber $\mathbf{f}$ intersects the curve $B_{\widetilde{F}_0}$ transversely at the point O,
- the map θ is a weighted blow up at the point $O = B_{\widetilde{F}_0} \cap \mathbf{f}$ with weights $(1,2)$ such that the proper transforms on $\widehat{F}_0$ of the curves $B_{\widetilde{F}_0}$ and $\mathbf{f}$ are disjoint in the case when the fiber $\mathbf{f}$ is tangent to the curve $B_{\widetilde{F}_0}$ at the point O.

Let C be the θ-exceptional curve. We have $C \cong \mathbb{P}^1$. Let $B_{\widehat{F}_0}$ be the proper transform on the surface $\widehat{F}_0$ of the curve $B_{\widehat{F}_0}$. Set $\Delta_{\widehat{F}_0} = \frac{1}{2}B_{\widehat{F}_0}$. Let Δ_C be the effective $\mathbb{Q}$-divisor on the curve C known as the different, which can be defined via the adjunction formula

$$K_C + \Delta_C = (K_{\widehat{F}_0} + \Delta_{\widehat{F}_0})|_C.$$

If θ is a usual blow up, then $\Delta_C = \Delta_{\widehat{F}_0}|_C$. Similarly, if θ is a weighted blow up, then

$$\Delta_C = \Delta_{\widehat{F}_0}|_C + \frac{1}{2}\mathbf{o},$$

where $\mathbf{o}$ is the singular point of the surface $\widehat{F}_0$ contained in C ($\mathbf{o}$ is an ordinary double point, which is not contained in the proper transforms of the curves $B_{\widetilde{F}_0}$ and $\mathbf{f}$).

Now, for $u \in [0, 3]$, we let

$$\hat{t}(u) = \sup\{v \in \mathbb{R}_{\geqslant 0} \mid \theta^*(\widetilde{P}(u)|_{\widetilde{F}_0}) - vC \text{ is pseudo-effective}\}.$$

For every $v \in [0, \hat{t}(u)]$, let us denote by $\widehat{P}(u, v)$ and $\widehat{N}(u, v)$ the positive and the negative parts of the Zariski decompositions of the divisor $\theta^*(\widetilde{P}(u)|_{\widetilde{F}_0}) - vC$, respectively. Then

$$1 \geqslant \frac{A_{F_0,\Delta_{F_0}}(E)}{S(W^{\widetilde{F}_0}_{\bullet,\bullet}; E)} \geqslant \min\left\{\frac{A_{F_0,\Delta_{F_0}}(C)}{S(W^{\widetilde{F}_0}_{\bullet,\bullet}; C)}, \inf_{Q\in C} \frac{A_{C,\Delta_C}(Q)}{S(W^{\widehat{F}_0,C}_{\bullet,\bullet,\bullet}; Q)}\right\} \tag{5.5}$$

by (5.4) and [22, Corollary 4.18], where the infimum is taken by all points $Q \in C$, and

$$S(W^{\widehat{F}_0,C}_{\bullet,\bullet,\bullet}; Q) = \frac{3}{A^3} \int_0^3 \int_0^{\hat{t}(u)} (\widehat{P}(u, v) \cdot C)^2 \, dv \, du + F_Q(W^{\widehat{F}_0,C}_{\bullet,\bullet,\bullet})$$

for

$$F_Q(W^{\widehat{F}_0,C}_{\bullet,\bullet,\bullet}) = \frac{6}{A^3} \int_0^3 \int_0^{\hat{t}(u)} (\widehat{P}(u, v) \cdot C) \operatorname{ord}_Q(\widehat{N}(u, v)|_C) \, dv \, du.$$

Denote by $\widehat{\mathbf{e}}$ and $\widehat{\mathbf{f}}$ the proper transforms of the curves $\mathbf{e}$ and $\mathbf{f}$, respectively.

Lemma 5.13. *Suppose that θ is an ordinary blow up. Let Q be a point in C. Then*

$$\frac{A_{F_0,\Delta_{F_0}}(C)}{S(W^{\widetilde{F}_0}_{\bullet,\bullet}; C)} \geqslant \frac{39}{29} \quad \textit{and} \quad \frac{A_{C,\Delta_C}(Q)}{S(W^{\widehat{F}_0,C}_{\bullet,\bullet,\bullet}; Q)} \geqslant \frac{13}{10}.$$

Proof. One has

$$\theta^*(\widetilde{P}(u)|_{\widetilde{F}_0}) \sim_{\mathbb{R}} \begin{cases} u(\widehat{\mathbf{f}} + \widehat{\mathbf{e}} + C), & u \in [0, 1], \\ u(\widehat{\mathbf{f}} + C) + \widehat{\mathbf{e}}, & u \in [1, 2], \\ u(\widehat{\mathbf{f}} + C) + (3 - u)\widehat{\mathbf{e}}, & u \in [2, 3]. \end{cases}$$

This easily implies that $\hat{t}(u) = u$ and

$$\widehat{N}(u, v) = \begin{cases} 0, & u \in [0, 1], v \in [0, u], \\ 0, & u \in [1, 2], v \in [0, 1], \\ (v - 1)\widehat{\mathbf{f}}, & u \in [1, 2], v \in [1, u], \\ 0, & u \in [2, 3], v \in [0, 3 - u], \\ (v + u - 3)\widehat{\mathbf{f}}, & u \in [2, 3], v \in [3 - u, u], \end{cases}$$

so that

$$\widehat{P}(u, v) = \begin{cases} u(\widehat{\mathbf{f}} + \widehat{\mathbf{e}}) + (u - v)C, & u \in [0, 1], v \in [0, u], \\ u\widehat{\mathbf{f}} + (u - v)C + \widehat{\mathbf{e}}, & u \in [1, 2], v \in [0, 1], \\ (u - v + 1)\widehat{\mathbf{f}} + (u - v)C + \widehat{\mathbf{e}}, & u \in [1, 2], v \in [1, u], \\ u\widehat{\mathbf{f}} + (u - v)C + \widehat{\mathbf{e}}, & u \in [2, 3], v \in [0, 3 - u], \\ (3 - v)\widehat{\mathbf{f}} + (u - v)C + (3 - u)\widehat{\mathbf{e}}, & u \in [2, 3], v \in [3 - u, u], \end{cases}$$

which gives

$$(\widehat{P}(u,v))^2 = \begin{cases} u^2 - v^2, & u \in [0,1], v \in [0,u], \\ -v^2 + 2u - 1, & u \in [1,2], v \in [0,1], \\ 2u - 2v, & u \in [1,2], v \in [1,u], \\ -3u^2 - v^2 + 12u - 9, & u \in [2,3], v \in [0,3-u], \\ -2u^2 + 2uv + 6u - 6v, & u \in [2,3], v \in [3-u,u]. \end{cases}$$

Thus, integrating, we get $S(W^{\widetilde{F}_0}_{\bullet,\bullet}; C) = \frac{29}{26}$. Note that

$$A_{F_0,\Delta_{F_0}}(C) = \begin{cases} \frac{3}{2}, & O \in B_{\widetilde{F}_0}, \\ 2, & O \notin B_{\widetilde{F}_0}. \end{cases}$$

This gives the first required inequality. Similarly, we compute

$$S(W^{\widehat{F}_0,C}_{\bullet,\bullet,\bullet}; Q) = \frac{9}{26} + F_Q(W^{\widehat{F}_0,C}_{\bullet,\bullet,\bullet}), \quad \text{where } F_Q(W^{\widehat{F}_0,C}_{\bullet,\bullet,\bullet}) = \begin{cases} \frac{11}{26}, & Q = \widehat{\mathbf{f}} \cap C, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that

$$A_{C,\Delta_C}(Q) = \begin{cases} \frac{1}{2}, & Q \in B_{\widehat{F}_0}, \\ 1, & Q \notin B_{\widehat{F}_0}. \end{cases}$$

Moreover, if $O \in B_{\widetilde{F}_0} \cap \mathbf{f}$, the intersection $C \cap \widehat{\mathbf{f}}$ consists of a single point, which is not contained in $B_{\widehat{F}_0}$. Thus, we have

$$\frac{A_{C,\Delta_C}(Q)}{S(W^{\widehat{F}_0,C}_{\bullet,\bullet,\bullet}; Q)} = \begin{cases} \frac{13}{10}, & Q = C \cap \widehat{\mathbf{f}}, \\ \frac{13}{9}, & Q = C \cap B_{\widehat{F}_0}, \\ \frac{26}{9} & \text{otherwise,} \end{cases}$$

which implies the second required inequality. □

Thus, it follows from (5.5) and Lemma 5.13 that $O = B_{\widetilde{F}_0} \cap \mathbf{f}$, so $\mathbf{f}$ and $B_{\widetilde{F}_0}$ are tangent at the point O. Then θ is a weighted blow up with weights $(1,2)$. We have

$$\theta^*(\widetilde{P}(u)|_{\widetilde{F}_0}) \sim_{\mathbb{R}} \begin{cases} u(\widehat{\mathbf{f}} + \widehat{\mathbf{e}} + 2C), & u \in [0,1], \\ u(\widehat{\mathbf{f}} + 2C) + \widehat{\mathbf{e}}, & u \in [1,2], \\ u(\widehat{\mathbf{f}} + 2C) + (3-u)\widehat{\mathbf{e}}, & u \in [2,3]. \end{cases}$$

This gives $\widehat{t}(u) = 2u$. Moreover, we have

$$\widehat{N}(u,v) = \begin{cases} 0, & u \in [0,1], v \in [0,u], \\ (v-u)(\widehat{\mathbf{f}} + \widehat{\mathbf{e}}), & u \in [0,1], v \in [u,2u], \\ 0, & u \in [1,2], v \in [0,1], \\ \frac{v-1}{2}\widehat{\mathbf{f}}, & u \in [1,2], v \in [1,2u-1], \\ (v-u)\widehat{\mathbf{f}} + (v-2u+1)\widehat{\mathbf{e}}, & u \in [1,2], v \in [1,2u-1], \\ 0, & u \in [2,3], v \in [0,3-u], \\ \frac{v+u-3}{2}\widehat{\mathbf{f}}, & u \in [2,3], v \in [0,3u-3], \\ (v-u)\widehat{\mathbf{f}} + (v+3-3u)\widehat{\mathbf{e}}, & u \in [2,3], v \in [3u-3,2u], \end{cases}$$

and

$$\widehat{P}(u,v)=\begin{cases}(2u-v)C+u\widehat{\mathbf{f}}+u\widehat{\mathbf{e}}, & u\in[0,1], v\in[0,u],\\ (2u-v)(C+\widehat{\mathbf{f}}+\widehat{\mathbf{e}}), & u\in[0,1], v\in[u,2u],\\ (2u-v)C+u\widehat{\mathbf{f}}+\widehat{\mathbf{e}}, & u\in[1,2], v\in[0,1],\\ (2u-v)C+\frac{2u-v+1}{2}\widehat{\mathbf{f}}+\widehat{\mathbf{e}}, & u\in[1,2], v\in[1,2u-1],\\ (2u-v)(C+\widehat{\mathbf{f}}+\widehat{\mathbf{e}}), & u\in[1,2], v\in[1,2u-1],\\ (2u-v)C+u\widehat{\mathbf{f}}+(3-u)\widehat{\mathbf{e}}, & u\in[2,3], v\in[0,3-u],\\ (2u-v)C+\frac{u-v+3}{2}\widehat{\mathbf{f}}+(3-u)\widehat{\mathbf{e}}, & u\in[2,3], v\in[0,3u-3],\\ (2u-v)(C+\widehat{\mathbf{f}}+\widehat{\mathbf{e}}), & u\in[2,3], v\in[3u-3,2u].\end{cases}$$

Then

$$(\widehat{P}(u,v))^2=\begin{cases}u^2-\frac{v^2}{2}, & u\in[0,1], v\in[0,u],\\ \frac{(2u-v)^2}{2}, & u\in[0,1], v\in[u,2u],\\ 2u-1-\frac{v^2}{2}, & u\in[1,2], v\in[0,1],\\ 2u-v-\frac{1}{2}, & u\in[1,2], v\in[1,2u-1],\\ \frac{(2u-v)^2}{2}, & u\in[1,2], v\in[1,2u-1],\\ 12u-9-3u^2-\frac{v^2}{2}, & u\in[2,3], v\in[0,3-u],\\ \frac{(5u-2v-3)(u-3)}{2}, & u\in[2,3], v\in[0,3u-3],\\ \frac{(2u-v)^2}{2}, & u\in[2,3], v\in[3u-3,2u].\end{cases}$$

Now, integrating, we get $S(W^{\widetilde{F}_0}_{\bullet,\bullet};C)=\frac{49}{26}$. Thus, since $A_{F_0,\Delta_{F_0}}(C)=2$, we get

$$\frac{A_{F_0,\Delta_{F_0}}(C)}{S(W^{\widetilde{F}_0}_{\bullet,\bullet};C)}=\frac{52}{49},$$

so it follows from (5.5) that there is a point $Q\in C$ such that $S(W^{\widehat{F}_0,C}_{\bullet,\bullet,\bullet};Q)\geqslant A_{C,\Delta_C}(Q)$. On the other hand, we compute

$$S(W^{\widehat{F}_0,C}_{\bullet,\bullet,\bullet};Q)=\frac{9}{52}+F_Q(W^{\widehat{F}_0,C}_{\bullet,\bullet,\bullet}),$$

where

$$F_Q(W^{\widehat{F}_0,C}_{\bullet,\bullet,\bullet})=\begin{cases}\frac{3}{4}, & Q=C\cap\widehat{\mathbf{f}},\\ 0 & \text{otherwise.}\end{cases}$$

Recall that $B_{\widehat{F}_0}$ and $\widehat{\mathbf{f}}$ are disjoint and do not contain the singular point of the surface $\widehat{F}_0$. Moreover, we have

$$A_{C,\Delta_C}(Q)=\begin{cases}\frac{1}{2}, & Q=C\cap B_{\widehat{F}_0},\\ \frac{1}{2}, & Q=\mathrm{Sing}(\widehat{F}_0),\\ 1 & \text{otherwise.}\end{cases}$$

Thus, summarizing, we get

$$\frac{A_{C,\Delta_C}(Q)}{S(W^{\widehat{F}_0,C}_{\bullet,\bullet,\bullet};Q)}=\begin{cases}\frac{13}{12} & Q=C\cap\widehat{\mathbf{f}},\\ \frac{26}{9}, & Q=C\cap B_{\widehat{F}_0},\\ \frac{26}{9}, & Q=\mathrm{Sing}(\widehat{F}_0),\\ \frac{52}{9} & \text{otherwise.}\end{cases}$$

In particular, we see that $S(W^{\widehat{F}_0,C}_{\bullet,\bullet,\bullet}; Q) < A_{C,\Delta_C}(Q)$ in every possible case. The obtained contradiction completes the proof of Proposition 5.5.

5.2. Proof of Proposition 5.6. Let us use notation introduced earlier in this section before Proposition 5.6, and let P be a singular point of type $\mathbb{A}_2$ of the surface $B \in |2S^+|$. Then, up to a change of coordinates, we may assume that $P = (0,0,1,0,1)$ and

$$f_4(x_1, x_2, 1) = x_1^2 + x_2^3 + \text{higher order terms}.$$

Let $\rho\colon Y_0 \to Y$ be the blow up of the point P with weights $(3,2,3)$ with respect to variables (x_1, x_2, x_4); note that ρ is a toroidal log resolution of (Y, B). We may describe Y_0 as a toric variety given as $(\mathbb{C}^6 \setminus Z(I_0))/\mathbb{G}_m^3$, where the action is given by the matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 3 & 3 & 0 \end{pmatrix},$$

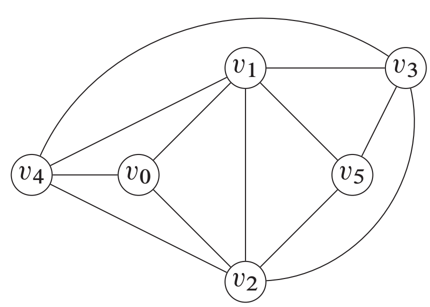
where the irrelevant ideal is

$$I_0 = \langle x_1, x_2, x_3 \rangle \cap \langle x_1, x_2, x_4 \rangle \cap \langle x_4, x_5 \rangle \cap \langle x_0, x_3 \rangle \cap \langle x_0, x_5 \rangle.$$

To describe the fan of the toric threefold Y_0, we denote by v_i the vector generating the ray corresponding to x_i. Then

$$\begin{aligned} v_0 &= (3,2,3), & v_1 &= (1,0,0), & v_2 &= (0,1,0), \\ v_3 &= (-1,-1,-2), & v_4 &= (0,0,1), & v_5 &= (0,0,-1), \end{aligned}$$

and the cone structure can be visualized with the following diagram:



Let $F_i = \{x_i = 0\} \subset Y_0$ and $C_{ij} = F_i \cap F_j$ for $i \neq j$ such that $\dim(F_i \cap F_j) = 1$. The geometric identifications of F_i and C_{ij} are the same as in previous section. Then

$$\overline{\mathrm{Eff}(Y_0)} = \langle F_0, F_1, F_5 \rangle \quad \text{and} \quad \overline{\mathrm{NE}(Y_0)} = \langle C_{12}, C_{15}, C_{01} \rangle.$$

Intersections of divisors F_0, F_1, F_5 are described in following table:

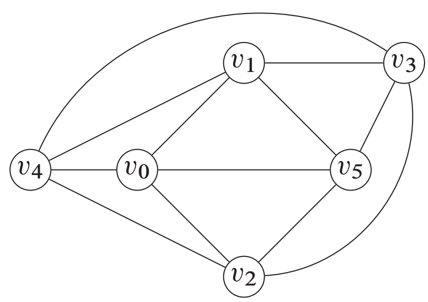
F_0^3	$F_0^2F_1$	$F_0^2F_5$	$F_0F_1^2$	$F_0F_1F_5$	$F_0F_5^2$	F_1^3	$F_1^2F_5$	$F_1F_5^2$	F_5^3
$\frac{1}{18}$	$-\frac{1}{6}$	0	$\frac{1}{2}$	0	0	$-\frac{3}{2}$	1	-2	4

This gives the following intersection table:

$\bullet$	F_0	F_1	F_5
C_{12}	$\frac{1}{3}$	-1	1
C_{15}	0	1	-2
C_{01}	$-\frac{1}{6}$	$\frac{1}{2}$	0

Now, we set $A = -(K_Y + \Delta)$. Take $u \in \mathbb{R}_{\geqslant 0}$. Set $L(u) = \rho^*(A) - uF_0$. Then we have $L(u) \sim_{\mathbb{R}} (9-u)F_0 + 3F_1 + F_5$, so $L(u)$ is pseudo-effective if and only if $u \leqslant 9$. Let us find the Zariski decomposition for $L(u)$.

Observe that $L(u)$ is nef for $u \in [0,3]$. Since $L(3) \cdot C_{12} = 0$ and C_{12} is unique in its numerical equivalence class, we consider a small $\mathbb{Q}$-factorial modification $Y_0 \dashrightarrow Y_1$ along the curve C_{12} such that $Y_1 = (\mathbb{C}^6 \setminus Z(I_1))/\mathbb{G}_m^3$, where the torus action is the same, and the irrelevant ideal is $I_1 = \langle x_1, x_2\rangle \cap \langle x_4, x_5\rangle \cap \langle x_0, x_3\rangle$. The fan of Y_1 is generated by the same vectors, but the cone structure is different:



Abusing our previous notation, we denote the divisor $\{x_i = 0\} \subset Y_1$ also by F_i, and we let $C_{ij} = F_i \cap F_j$ for $i \neq j$ such that $F_i \cap F_j$ is a curve. Then $\overline{\mathrm{NE}(Y_1)} = \langle C_{01}, C_{15}, C_{05}\rangle$, and intersections on Y_1 are described in the following two tables:

F_0^3	$F_0^2F_1$	$F_0^2F_5$	$F_0F_1^2$	$F_0F_1F_5$	$F_0F_5^2$	F_1^3	$F_1^2F_5$	$F_1F_5^2$	F_5^3
0	0	$-\frac{1}{6}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{5}{2}$

$\bullet$	F_0	F_1	F_5
C_{05}	$-\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{2}$
C_{15}	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
C_{01}	0	0	$\frac{1}{2}$

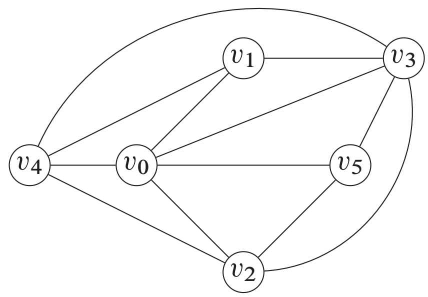
Thus, we see that the proper transform on Y_1 of the divisor $L(u)$ is nef for $u \in [3,5]$, and it intersects the curve C_{15} trivially for $u = 5$. Since C_{15} is unique in its numerical equivalence class, we consider another small $\mathbb{Q}$-factorial modification $Y_1 \dashrightarrow Y_2$ such that

$$Y_2 = (\mathbb{C}^6 \setminus Z(I_2))/\mathbb{G}_m^3,$$

where the torus action is again given by the matrix M and the irrelevant ideal

$$I_2 = \langle x_1, x_2\rangle \cap \langle x_4, x_5\rangle \cap \langle x_1, x_5\rangle \cap \langle x_0, x_2, x_3\rangle \cap \langle x_0, x_3, x_4\rangle.$$

Then the fan of Y_2 is generated by the same vectors, but the cone structure is different:



We abuse our notation again and denote the divisor $\{x_i = 0\} \subset Y_2$ also by F_i. Similarly, we let $C_{ij} = F_i \cap F_j$ for $i \neq j$ such that $F_i \cap F_j$ is a curve. Then $\overline{\mathrm{NE}}(Y_2) = \langle C_{01}, C_{03}, C_{05}\rangle$, and intersections on Y_2 are described in the following two tables:

F_0^3	$F_0^2F_1$	$F_0^2F_5$	$F_0F_1^2$	$F_0F_1F_5$	$F_0F_5^2$	F_1^3	$F_1^2F_5$	$F_1F_5^2$	F_5^3
$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{2}$	0	-1	$\frac{1}{2}$	0	0	3

$\bullet$	F_0	F_1	F_5
C_{05}	$\frac{1}{3}$	0	-1
C_{03}	$\frac{-2}{3}$	1	1
C_{01}	$\frac{1}{2}$	$-\frac{1}{2}$	0

The proper transform on Y_2 of the divisor $L(u)$ is nef for $u \in [5, 6]$, and it intersects both curves C_{01} and C_{05} trivially for $u = 6$. Furthermore, if $u \in [6, 9]$, then the negative part of the Zariski decomposition of the divisor $L(u)$ on the threefold Y_2 is

$$N(u) = (u-6)F_1 + \frac{u-6}{3}F_5,$$

while the positive part is $P(u) \sim_{\mathbb{R}} (9-u)(F_0 + F_1 + \frac{1}{3}F_5)$. This gives

$$\mathrm{vol}\big(L(u)\big) = \begin{cases} 13 - \frac{u^3}{18}, & u \in [0,3], \\ \frac{-u^2+3+23}{2}, & u \in [3,5], \\ \frac{1}{2}u^3 - 8u^2 + \frac{3}{2}u, & u \in [5,6], \\ -\frac{1}{9}u^3 + 3u^2 - 27u + 81, & u \in [6,9]. \end{cases}$$

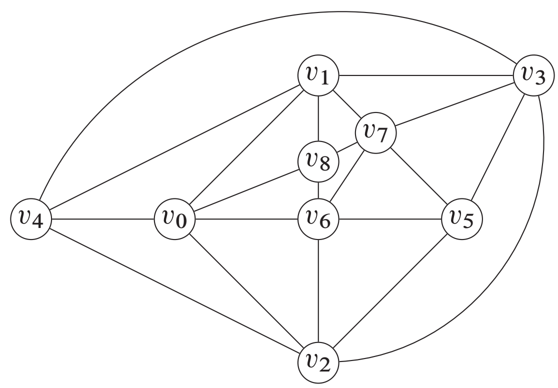
Integrating, we get $S_A(F_0) = \frac{127}{26}$. Since $A_{Y,\Delta}(F_0) = 5$, we get

$$\frac{A_{Y,\Delta}(F_0)}{S_A(F_0)} = \frac{130}{127} > 1.$$

Next we construct a partial common toric resolution for Y_0, Y_1, Y_2, which is easy to see from fan toric picture: we want to add the rays

$$\begin{aligned} v_6 &= (3,2,0) \in \langle v_1, v_2\rangle \cap \langle v_0, v_5\rangle, \\ v_7 &= (1,0,-1) \in \langle v_0, v_3\rangle \cap \langle v_0, v_3\rangle, \\ v_8 &= (3,1,0) \in \langle v_1, v_2\rangle \cap \langle v_0, v_3\rangle. \end{aligned}$$

Set $\widetilde{Y}$ be the toric variety corresponding to $v_0, \ldots, v_8$ with the following cone structure:



Then we have the following toric diagram:

$$\begin{tikzcd}
 & & \widetilde{Y} \arrow[dl, "\sigma_{12}"'] \arrow[dr, "\psi_{01}"] & & \\
 & Y'_{12} \arrow[d, "\sigma'"'] & & Y'_{01} \arrow[d, "\psi'"] & \\
 & Y_{12} \arrow[rr, dashed] \arrow[dl, "\sigma_2"'] \arrow[dr, "\sigma_1"] & & Y_{01} \arrow[dl, "\psi_1"'] \arrow[dr, "\psi_0"] & \\
Y_2 & & Y_1 \arrow[ll, dashed] & & Y_0, \arrow[ll, dashed]
\end{tikzcd}$$

where toric maps can be described as follows:

Map	Center	Weights	Exceptional divisor	Relation
ψ_0	$x_1 = x_2 = 0$	$(3,2)$	$\{x_6 = 0\}$	$3v_1 + 2v_2 = v_6$
ψ_1	$x_0 = x_5 = 0$	$(1,3)$	$\{x_6 = 0\}$	$v_0 + 3v_5 = v_6$
σ_1	$x_1 = x_5 = 0$	$(1,1)$	$\{x_7 = 0\}$	$v_1 + v_5 = v_7$
σ_2	$x_0 = x_3 = 0$	$(1,2)$	$\{x_7 = 0\}$	$v_0 + 2v_3 = v_7$
ψ'	$x_1 = x_5 = 0$	$(1,1)$	$\{x_7 = 0\}$	$v_1 + v_5 = v_7$
σ'	$x_0 = x_5 = 0$	$(1,3)$	$\{x_6 = 0\}$	$v_0 + 3v_5 = v_6$
ψ_{01}	$x_1 = x_6 = 0$	$\frac{1}{2}(3,1)$	$\{x_8 = 0\}$	$3v_1 + v_6 = 2v_8$
σ_{12}	$x_0 = x_7 = 0$	$\frac{1}{2}(1,3)$	$\{x_8 = 0\}$	$v_1 + 3v_7 = 2v_8$

Here, $\frac{1}{2}(a,b)$ indicates that the variety has an $\mathbb{A}_1$-singularity along the center of blow up.

Now, we set $\varphi_0 = \psi_{01} \circ \psi' \circ \psi_0$, $\varphi_1 = \psi_{01} \circ \psi' \circ \psi_1$, $\varphi_2 = \sigma_{12} \circ \sigma' \circ \sigma_2$. Let $\widetilde{F}_i$ be the toric divisor $\{x_i = 0\} \subset \widetilde{Y}$. Then

$$\varphi_0^*(F_0) \sim_{\mathbb{Q}} \widetilde{F}_0,$$

$$\varphi_0^*(F_1) \sim_{\mathbb{Q}} \widetilde{F}_1 + 3\widetilde{F}_6 + \widetilde{F}_7 + 3\widetilde{F}_8,$$

$$\varphi_0^*(F_5) \sim_{\mathbb{Q}} \widetilde{F}_5 + \widetilde{F}_7,$$

$$\varphi_1^*(F_0) \sim_{\mathbb{Q}} \widetilde{F}_0 + \widetilde{F}_6 + \frac{1}{2}\widetilde{F}_8,$$

$$\varphi_1^*(F_1) \sim_{\mathbb{Q}} \widetilde{F}_1 + \widetilde{F}_7 + \frac{3}{2}\widetilde{F}_8,$$

$$\varphi_1^*(F_5) \sim_{\mathbb{Q}} \widetilde{F}_5 + 3\widetilde{F}_6 + \widetilde{F}_7 + \frac{3}{2}\widetilde{F}_8,$$

$$\varphi_2^*(F_0) \sim_{\mathbb{Q}} \widetilde{F}_0 + \widetilde{F}_6 + \widetilde{F}_7 + 2\widetilde{F}_8,$$

$$\varphi_2^*(F_1) \sim_{\mathbb{Q}} \widetilde{F}_1,$$

$$\varphi_2^*(F_5) \sim_{\mathbb{Q}} \widetilde{F}_5 + 3\widetilde{F}_6.$$

Using this, we describe the Zariski decomposition of the divisor $\varphi_0^*(L(u))$ as follows:

$$\widetilde{P}(u) \sim_{\mathbb{R}} \begin{cases} (9-u)\widetilde{F}_0 + 3\widetilde{F}_1 + \widetilde{F}_5 + 9\widetilde{F}_6 + 4\widetilde{F}_7 + 9\widetilde{F}_8, & u \in [0,3], \\ (9-u)\widetilde{F}_0 + 3\widetilde{F}_1 + \widetilde{F}_5 + (12-u)\widetilde{F}_6 + 4\widetilde{F}_7 + \frac{21-u}{2}\widetilde{F}_8, & u \in [3,5], \\ (9-u)\widetilde{F}_0 + 3\widetilde{F}_1 + \widetilde{F}_5 + (12-u)\widetilde{F}_6 + (9-u)\widetilde{F}_7 + 2(9-u)\widetilde{F}_8, & u \in [5,6], \\ (9-u)(\widetilde{F}_0 + \widetilde{F}_1 + \frac{1}{3}\widetilde{F}_5 + 2\widetilde{F}_6 + \widetilde{F}_7 + 2\widetilde{F}_8), & u \in [6,9], \end{cases}$$

and

$$\widetilde{N}(u) = \begin{cases} 0, & u \in [0,3], \\ (u-3)\widetilde{F}_6 + \frac{u-3}{2}\widetilde{F}_8, & u \in [3,5], \\ (u-3)\widetilde{F}_6 + (u-5)\widetilde{F}_7 + (2u-9)\widetilde{F}_8, & u \in [5,6], \\ (u-6)\widetilde{F}_1 + \frac{u}{3}\widetilde{F}_5 + (2u-9)\widetilde{F}_6 + (u-5)\widetilde{F}_7 + (2u-9)\widetilde{F}_8, & u \in [6,9]. \end{cases}$$

where $\widetilde{P}(u)$ is the positive part, and $\widetilde{N}(u)$ is the negative part.

Now, we describe $\widetilde{P}(u)|_{\widetilde{F}_0}$ and $\widetilde{N}(u)|_{\widetilde{F}_0}$ for every $u \in [0,9]$. We have

$$\widetilde{Y} = (\mathbb{C}^9 \setminus \widetilde{I})/\mathbb{G}_m^6,$$

where the torus action is given by the matrix

$$\widetilde{M} = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ 0 & 1 & 1 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 3 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 6 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 3 & 6 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and the irrelevant ideal

$$\begin{aligned}\widetilde{I} = {} & \langle x_0, x_3\rangle \cap \langle x_0, x_5\rangle \cap \langle x_0, x_7\rangle \cap \langle x_1, x_2\rangle \cap \langle x_1, x_5\rangle \cap \langle x_1, x_6\rangle \cap \langle x_2, x_7\rangle \cap \langle x_2, x_8\rangle \\ & \cap \langle x_3, x_6\rangle \cap \langle x_3, x_8\rangle \cap \langle x_4, x_5\rangle \cap \langle x_4, x_6\rangle \cap \langle x_4, x_7\rangle \cap \langle x_4, x_8\rangle \cap \langle x_5, x_8\rangle.\end{aligned}$$

To obtain a similar description of the surface $\widetilde{F}_0$, set $x_0 = 0$, eliminate the first row in $\widetilde{M}$, and set $x_3 = x_5 = x_7 = 1$, since $\widetilde{I} \subset \langle x_0, x_3\rangle \cap \langle x_0, x_5\rangle \cap \langle x_0, x_7\rangle$. The resulting matrix is

$$\begin{pmatrix} x_1 & x_2 & x_4 & x_6 & x_8 \\ 3 & 2 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 1 & 3 & 0 & 1 \end{pmatrix}.$$

Using this, we see that $\widetilde{F}_0 = (\mathbb{C}^5 \setminus Z(I_{\widetilde{F}_0}))/\mathbb{G}_m^3$, where the torus action is given by

$$\begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix},$$

and $I_{\widetilde{F}_0} = \langle z_1, z_3\rangle \cap \langle z_1, z_4\rangle \cap \langle z_2, z_4\rangle \cap \langle z_2, z_5\rangle \cap \langle z_3, z_5\rangle$. We can see from the matrices that

$$x_1|_{\widetilde{F}_0} = z_1, \quad x_2^3|_{\widetilde{F}_0} = z_3, \quad x_4|_{\widetilde{F}_0} = z_2, \quad x_6^3|_{\widetilde{F}_0} = z_4, \quad x_8^3|_{\widetilde{F}_0} = z_5.$$

The fan of the toric surface $\widetilde{F}_0$ is given by

$$w_1 = (1,0), \quad w_2 = (-1,-2), \quad w_3 = (0,1), \quad w_4 = (1,2), \quad w_5 = (1,1)$$

with obvious cone structure. Note that we can also recover this structure by noticing that $F_0 \cong \mathbb{P}(1,1,2)$ is the exceptional divisor of the weighted blow up ρ and that the maps ψ_0 and ψ_{01} restrict to F_0 as weighted blow ups. For $i \in \{1,2,3,4,5\}$, let C_i be the curve in $\widetilde{F}_0$ given $z_i = 0$. The cone of effective divisors of the surface $\widetilde{F}_0$ is generated by the curves C_1, C_4, C_5, and their intersection form is given in the following table:

$\bullet$	C_1	C_4	C_5
C_1	$-\frac{1}{2}$	0	1
C_4	0	-1	1
C_5	1	1	-2

Further, we compute

$$\widetilde{P}(u)|_{\widetilde{F}_0} \sim_{\mathbb{R}} \begin{cases} \frac{u}{3}C_1 + \frac{u}{3}C_4 + \frac{u}{3}C_5, & u \in [0,3], \\ \frac{u}{3}C_1 + C_4 + (\frac{1}{2} + \frac{u}{6})C_5, & u \in [3,5], \\ \frac{u}{3}C_1 + C_4 + (3 - \frac{u}{3})C_5, & u \in [5,6], \\ (6 - \frac{2u}{3})C_1 + (3 - \frac{u}{3})C_4 + (3 - \frac{u}{3})C_5, & u \in [6,9], \end{cases}$$

and

$$\widetilde{N}(u)|_{\widetilde{F}_0} = \begin{cases} 0, & u \in [3,5], \\ \frac{u-3}{6}(2C_4 + C_5), & u \in [3,5], \\ \frac{u-3}{3}C_4 + \frac{2u-9}{3}C_5, & u \in [5,6], \\ (u-6)C_1 + \frac{2u-9}{3}(2C_4 + C_5), & u \in [6,9]. \end{cases}$$

Let $\theta\colon \widetilde{F}_0 \to F_0$ be the morphism induced by φ_0. Then θ is a birational morphism that contracts C_4 and C_5. Set $\overline{C}_1 = \theta(C_1)$, $\overline{C}_2 = \theta(C_2)$, $\overline{C}_3 = \theta(C_3)$, identify $F_0 = \mathbb{P}(1,1,2)$ with coordinates $\bar{z}_1, \bar{z}_2, \bar{z}_3$ such that $\overline{C}_1 = \{\bar{z}_1 = 0\}$, $\overline{C}_2 = \{\bar{z}_2 = 0\}$, $\overline{C}_3 = \{\bar{z}_3 = 0\}$, where $\bar{z}_1$ and $\bar{z}_2$ are coordinates of weight 1, and $\bar{z}_3$ is a coordinate of weight 2. Then

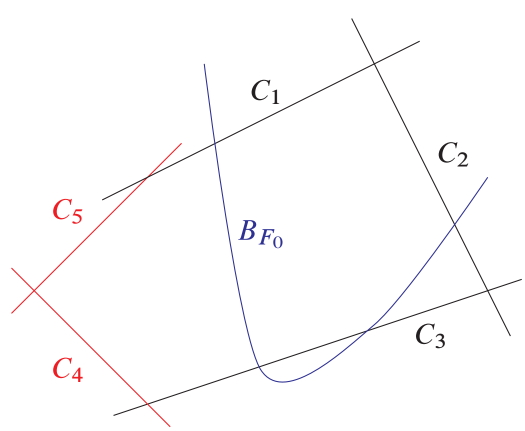
$$\theta(C_4) = \theta(C_5) = \overline{C}_1 \cap \overline{C}_3 = [0:1:0],$$

and θ is a composition of the ordinary blow up at the point $[0:1:0]$ with the consecutive blow up at the point on the proper transform of the curve $\overline{C}_3$. Note that C_5 is the proper transform of the exceptional curve for the first blow up and C_4 is the exceptional curve for the second blow up.

Let B_0 be the proper transform on Y_0 of the surface B. Set $\Delta_0 = \frac{1}{2}B_0$ and $B_{F_0} = B_0|_{F_0}$. Then, changing the coordinates $\bar{z}_1, \bar{z}_2, \bar{z}_3$, we may also assume that

$$B_{F_0} = \{\bar{z}_1^2 + \bar{z}_2^2 = \bar{z}_3\} \subset F_0.$$

This curve is smooth, it does not contain the singular point of F_0, and $[0:1:0] \notin B_{F_0}$. The geometry of the surface F_0 can be illustrated by the following picture:



Note that the surface Y_0 is singular along the curve $\overline{C}_3$. We set

$$\Delta_{F_0} = \frac{1}{2}B_{F_0} + \frac{2}{3}\overline{C}_3.$$

Then

$$K_{F_0} + \Delta_{F_0} \sim_{\mathbb{Q}} (K_{Y_0} + \Delta_0)|_{F_0},$$

and Δ_{F_0} is the corresponding different [40].

Now, we are ready to apply [2,7,22]. Let Q be a point in F_0, let C be a smooth curve in the surface F_0 that contains Q, let $\widetilde{C}$ be its proper transform on $\widetilde{F}_0$. For $u \in [0,9]$, let

$$t(u) = \inf\{v \in \mathbb{R}_{\geqslant 0} \mid \text{the divisor } \widetilde{P}(u)|_{\widetilde{F}_0} - v\widetilde{C} \text{ is pseudo-effective}\}.$$

For a real number $v \in [0, t(u)]$, let $P(u,v)$ and $N(u,v)$ be the positive part and the negative part of the Zariski decomposition of the divisor $\widetilde{P}(u)|_{\widetilde{F}_0} - v\widetilde{C}$, respectively. Set

$$S_L(W^{F_0}_{\bullet,\bullet}; C) = \frac{3}{A^3}\int_0^9 (\widetilde{P}(u)|_{\widetilde{F}_0})^2 \operatorname{ord}_{\widetilde{C}}(\widetilde{N}(u)|_{\widetilde{F}_0})\, du + \frac{3}{A^3}\int_0^9\int_0^{t(u)} (P(u,v))^2\, dv\, du.$$

Write $\theta^*(C) = \widetilde{C} + \Sigma$ for an effective divisor Σ on the surface $\widetilde{F}_0$. For $u \in [0, 9]$, write

$$\widetilde{N}(u)|_{\widetilde{F}_0} = d(u)\widetilde{C} + N'(u),$$

where $d(u) = \operatorname{ord}_{\widetilde{C}}(\widetilde{N}(u)|_{\widetilde{F}_0})$, and $N'(u)$ is an effective divisor on $\widetilde{F}_0$. Set

$$S(W^{F_0,C}_{\bullet,\bullet,\bullet}; Q) = \frac{3}{A^3}\int_0^9\int_0^{t(u)} (P(u,v)\cdot\widetilde{C})^2\, dv\, du + F_Q(W^{F_0,C}_{\bullet,\bullet,\bullet})$$

for

$$F_Q(W^{F_0,C}_{\bullet,\bullet,\bullet}) = \frac{6}{A^3}\int_0^9\int_0^{t(u)} (P(u,v)\cdot\widetilde{C}) \cdot \operatorname{ord}_Q\big((N'(u) + N(u,v) - (v + d(u))\Sigma)|_{\widetilde{C}}\big)\, dv\, du,$$

where we consider Q as a point in $\widetilde{C}$ using the isomorphism $\widetilde{C} \cong C$ induced by θ.

We will choose C such that the pair $(F_0, C + \Delta_{F_0} - \operatorname{ord}_C(\Delta_{F_0})C)$ has purely log terminal singularities. In this case, the curve C is equipped with an effective divisor Δ_C such that

$$K_C + \Delta_C \sim_{\mathbb{Q}} (K_{F_0} + C + \Delta_{F_0} - \operatorname{ord}_C(\Delta_{F_0})C)|_C,$$

and the pair (C, Δ_C) has Kawamata log terminal singularities. The $\mathbb{Q}$-divisor Δ_C is known as the different, and it can be computed locally near any point in C; see [40] for details.

Let $\mathbf{F}$ be a prime divisor over Y such that $P = C_Y(\mathbf{F})$. Recall that

$$\beta_{Y,\Delta}(\mathbf{F}) = A_{Y,\Delta}(\mathbf{F}) - S_A(\mathbf{F}) = A_{Y,\Delta}(\mathbf{F}) - \frac{1}{A^3}\int_0^\infty \operatorname{vol}(A - u\mathbf{F})\, du.$$

Suppose $\beta_{Y,\Delta}(\mathbf{F}) \leqslant 0$. Then, using [22, Corollary 4.18], we obtain

$$1 \geqslant \frac{A_{Y,\Delta}(\mathbf{F})}{S_A(\mathbf{F})} \geqslant \delta_P(Y,\Delta) \geqslant \min\left\{\frac{A_{Y,\Delta}(F_0)}{S_A(F_0)}, \inf_{Q\in F_0}\min\left\{\frac{A_{F_0,\Delta_{F_0}}(C)}{S_A(W^{F_0}_{\bullet,\bullet}; C)}, \frac{A_{C,\Delta_C}(Q)}{S(W^{F_0,C}_{\bullet,\bullet,\bullet}; Q)}\right\}\right\},$$

where the choice of C in the infimum depends on Q. Thus, since

$$\frac{A_{Y,\Delta}(F_0)}{S_A(F_0)} \geqslant 1,$$

we have

$$\inf_{Q\in F_0}\min\left\{\frac{A_{F_0,\Delta_{F_0}}(C)}{S_A(W^{F_0}_{\bullet,\bullet}; C)}, \frac{A_{C,\Delta_C}(Q)}{S(W^{F_0,C}_{\bullet,\bullet,\bullet}; Q)}\right\} \leqslant 1.$$

In fact, since

$$\frac{A_{Y,\Delta}(F_0)}{S_A(F_0)} = \frac{130}{127} > 1,$$

it follows from [22, Corollary 4.18] and [2, Theorem 3.3] that we have a strict inequality

$$\inf_{Q\in F_0}\min\left\{\frac{A_{F_0,\Delta_{F_0}}(C)}{S_A(W^{F_0}_{\bullet,\bullet};C)},\frac{A_{C,\Delta_C}(Q)}{S(W^{F_0,C}_{\bullet,\bullet,\bullet};Q)}\right\}<1.$$

Let us use this to obtain a contradiction, which would finish the proof of Proposition 5.6.

Namely, we will show that, for every point $Q\in F_0$, there exists a smooth irreducible curve $C\subset F_0$ such that $Q\in C$, the log pair $(F_0, C+\Delta_{F_0}-\mathrm{ord}_C(\Delta_{F_0})C)$ has purely log terminal singularities, and the following two inequalities hold:

$$S_A(W^{F_0}_{\bullet,\bullet};C)\leqslant A_{F_0,\Delta_{F_0}}(C), \tag{5.6}$$

$$S(W^{F_0,C}_{\bullet,\bullet,\bullet};Q)\leqslant A_{C,\Delta_C}(Q). \tag{5.7}$$

To be precise, we will choose the curve C as follows:

- if $Q\in\overline{C}_1$, we let $C=\overline{C}_1$;
- if $Q\notin\overline{C}_1$ and $Q\in\overline{C}_3$, we let $C=\overline{C}_3$;
- if $Q\notin\overline{C}_1\cup\overline{C}_3$, we let C be the unique curve in $|\overline{C}_1|$ such that $Q\in C$.

Lemma 5.14. *Let Q be a point in $\overline{C}_1$. Set $C=\overline{C}_1$. Then* (5.6) *and* (5.7) *hold.*

Proof. Note that $A_{F_0,\Delta_{F_0}}(C)=1$ and $\Sigma=\overline{C}_4+\overline{C}_5$. We have

$$d(u)=\begin{cases}0, & u\in[0,6]\\ u-6, & u\in[6,9],\end{cases}\quad\text{and}\quad t(u)=\begin{cases}\frac{u}{3}, & u\in[0,6],\\ 6-\frac{2u}{3}, & u\in[6,9].\end{cases}$$

Moreover, we have

$$N(u,v)=\begin{cases}v(C_4+C_5), & u\in[0,3],\ v\in[0,\frac{u}{3}],\\ \frac{v}{2}C_5, & u\in[3,5],\ v\in[0,\frac{u}{3}-1],\\ \frac{3v+3-u}{3}C_4+\frac{6v+3-u}{6}C_5, & u\in[3,5],\ v\in[\frac{u}{3}-1,\frac{u}{3}],\\ 0, & u\in[5,6],\ v\in[0,u-5],\\ \frac{v+5-u}{2}C_5, & u\in[5,6],\ v\in[u-5,\frac{u}{3}-1],\\ \frac{3v+3-u}{3}C_4+\frac{3v+9-2u}{3}C_5, & u\in[5,6],\ v\in[\frac{u}{3}-1,\frac{u}{3}],\\ 0, & u\in[6,9],\ v\in[0,3-\frac{u}{3}],\\ \frac{3v+u-9}{3}(C_4+C_5), & u\in[6,9],\ v\in[3-\frac{u}{3},6-\frac{2u}{3}],\end{cases}$$

and

$$P(u,v)\sim_{\mathbb{R}}\begin{cases}\frac{u-3v}{3}(C_1+C_4+C_5), & u\in[0,3],\ v\in[0,\frac{u}{3}],\\ \frac{u-3v}{3}C_1+C_4+\frac{3+u-3v}{6}C_5, & u\in[3,5],\ v\in[0,\frac{u}{3}-1],\\ \frac{u-3v}{3}(C_1+C_4+C_5), & u\in[3,5],\ v\in[\frac{u}{3}-1,\frac{u}{3}],\\ \frac{u-3v}{3}C_1+C_4+\frac{9-u}{3}C_5, & u\in[5,6],\ v\in[0,u-5],\\ \frac{u-3v}{3}C_1+C_4+\frac{3+u-3v}{6}C_5, & u\in[5,6],\ v\in[u-5,\frac{u}{3}-1],\\ \frac{u-3v}{3}(C_1+C_4+C_5), & u\in[5,6],\ v\in[\frac{u}{3}-1,\frac{u}{3}],\\ (\frac{18-2u-3v}{3}C_1+\frac{9-u}{3}(C_4+C_5), & u\in[6,9],\ v\in[0,3-\frac{u}{3}],\\ \frac{18-2u-3v}{3}(C_1+C_4+C_5), & u\in[6,9],\ v\in[3-\frac{u}{3},6-\frac{2u}{3}],\end{cases}$$

which gives

$$(P(u,v))^2=\begin{cases}\frac{(u-3v)^2}{18}, & u\in[0,3],\ v\in[0,\frac{u}{3}],\\ \frac{u}{3}-v-\frac{1}{2}, & u\in[3,5],\ v\in[0,\frac{u}{3}-1],\\ \frac{(u-3v)^2}{18}, & u\in[3,5],\ v\in[\frac{u}{3}-1,\frac{u}{3}],\\ -\frac{u^2}{2}+uv-\frac{v^2}{2}-13+\frac{16}{3}u-6v, & u\in[5,6],\ v\in[0,u-5],\\ \frac{u}{3}-v-\frac{1}{2}, & u\in[5,6],\ v\in[u-5,\frac{u}{3}-1],\\ \frac{(u-3v)^2}{18}, & u\in[5,6],\ v\in[\frac{u}{3}-1,\frac{u}{3}],\\ -2u+9+\frac{u^2}{9}-\frac{v^2}{2}, & u\in[6,9],\ v\in[0,3-\frac{u}{3}],\\ \frac{(18-2u-3v)^2}{18}, & u\in[6,9],\ v\in[3-\frac{u}{3},6-\frac{2u}{3}],\end{cases}$$

and

$$P(u,v)\cdot C=\begin{cases}\frac{u-3v}{6}, & u\in[0,3],\ v\in[0,\frac{u}{3}],\\ \frac{1}{2}, & u\in[3,5],\ v\in[0,\frac{u}{3}-1],\\ \frac{u-3v}{6}, & u\in[3,5],\ v\in[\frac{u}{3}-1,\frac{u}{3}],\\ \frac{6-u+v}{2}, & u\in[5,6],\ v\in[0,u-5],\\ \frac{1}{2}, & u\in[5,6],\ v\in[u-5,\frac{u}{3}-1],\\ \frac{u-3v}{6}, & u\in[5,6],\ v\in[\frac{u}{3}-1,\frac{u}{3}],\\ \frac{v}{2}, & u\in[6,9],\ v\in[0,3-\frac{u}{3}],\\ \frac{18-2u-3v}{6}, & u\in[6,9],\ v\in[3-\frac{u}{3},6-\frac{2u}{3}].\end{cases}$$

Integrating, we get

$$S(W^{F_0}_{\bullet,\bullet};C)=\frac{10}{13}<1=A_{F_0,\Delta_{F_0}}(C),$$

so (5.6) holds.

Similarly, we compute

$$S(W^{F_0,C}_{\bullet,\bullet,\bullet};Q)=\frac{9}{52}+F_Q(W^{F_0,C}_{\bullet,\bullet,\bullet}),$$

where

$$F_Q(W^{F_0,C}_{\bullet,\bullet,\bullet})=\begin{cases}\frac{1}{12}, & Q=\overline{C}_1\cap\overline{C}_3,\\ 0 & \text{otherwise.}\end{cases}$$

Observe that

$$A_{C,\Delta_C}(Q)=\begin{cases}\frac{1}{2}, & Q=\overline{C}_1\cap B_{F_0},\\ \frac{1}{2}, & Q=\overline{C}_1\cap\overline{C}_2,\\ \frac{1}{3}, & Q=\overline{C}_1\cap\overline{C}_3,\\ 1 & \text{otherwise.}\end{cases}$$

Thus, we have

$$\frac{A_{C,\Delta_C}(Q)}{S(W^{F_0,C}_{\bullet,\bullet,\bullet};Q)}=\begin{cases}\frac{13}{10} & Q=\overline{C}_1\cap\overline{C}_3,\\ \frac{26}{9} & Q=\overline{C}_1\cap\overline{C}_2,\\ \frac{26}{9}, & Q=\overline{C}_1\cap B_{F_0},\\ \frac{52}{9} & \text{otherwise,}\end{cases}$$

which implies (5.7). □

Lemma 5.15. *Let Q be a point in $\overline{C}_3 \setminus \overline{C}_1$. Set $C = \overline{C}_3$. Then* (5.6) *and* (5.7) *hold.*

Proof. For $u \in [0,9]$, we have $d(u) = 0$, $N'(u) = \widetilde{N}(u)|_{\tilde{F}_0}$. As $\widetilde{C} \sim C_3 + 2C_4 + C_5$, we have

$$t(u) = \begin{cases} \frac{u}{6}, & u \in [0,6], \\ \frac{9-u}{3}, & u \in [6,9]. \end{cases}$$

We compute

$$N(u,v) = \begin{cases} 2vC_4 + vC_5, & u \in [0,3],\ v \in [0, \frac{u}{6}], \\ 0, & u \in [3,5],\ v \in [0, \frac{u-3}{6}], \\ \frac{u-3}{6}(2C_4 + C_5), & u \in [3,5],\ v \in [\frac{u-3}{6}, \frac{u}{6}], \\ 0, & u \in [5,6],\ v \in [0, \frac{6-u}{3}], \\ \frac{3v+u-6}{3}(C_4), & u \in [5,6],\ v \in [\frac{6-u}{3}, \frac{2u-9}{3}] \\ \frac{6v+3-u}{3}(C_4) + \frac{v+9-2u}{3}(C_4), & u \in [5,6],\ v \in [\frac{2u-9}{3}, \frac{u}{6}], \\ \frac{2u-9}{3}(C_4 + C_5), & u \in [6,9],\ v \in [0, \frac{9-u}{3}], \end{cases}$$

and

$$P(u,v) \sim \begin{cases} \frac{u-6v}{3}(C_1 + C_4 + C_5), & u \in [0,3],\ v \in [0, \frac{u}{6}], \\ \frac{u-6v}{3}C_1 + \frac{3+u-6v}{6}C_5 + C_4, & u \in [3,5],\ v \in [0, \frac{u-3}{6}], \\ \frac{u-6v}{3}(C_1 + C_4 + C_5), & u \in [3,5],\ v \in [\frac{u-3}{6}, \frac{u}{6}], \\ \frac{u-6v}{3}C_1 + \frac{9-u-3v}{3}C_5 + C_4, & u \in [5,6],\ v \in [0, \frac{6-u}{3}], \\ \frac{u-6v}{3}C_1 + \frac{9-u-3v}{3}(C_5 + C_4), & u \in [5,6],\ v \in [\frac{6-u}{3}, \frac{2u-9}{3}], \\ \frac{u-6v}{3}(C_1 + C_4 + C_5), & u \in [5,6],\ v \in [\frac{2u-9}{3}, \frac{u}{6}], \\ \frac{9-u-3v}{3}(2C_1 + C_4 + C_5), & u \in [6,9],\ v \in [0, \frac{9-u}{3}], \end{cases}$$

which gives

$$(P(u,v))^2 = \begin{cases} \frac{u^2}{18} + 2v^2 - \frac{2}{3}uv, & u \in [0,3],\ v \in [0, \frac{u}{6}], \\ \frac{u}{3} - 2v - \frac{1}{2}, & u \in [3,5],\ v \in [0, \frac{u-3}{6}], \\ \frac{u^2}{18} + 2v^2 - \frac{2}{3}uv, & u \in [3,5],\ v \in [\frac{u-3}{6}, \frac{u}{6}], \\ \frac{16}{3}u - 2v - \frac{13}{2}u^2, & u \in [5,6],\ v \in [0, \frac{6-u}{3}], \\ 4u - 6v - 9 - \frac{7}{18}u^2 + v^2 + \frac{2}{3}uv, & u \in [5,6],\ v \in [\frac{6-u}{3}, \frac{2u-9}{3}] \\ 9 - 6v - 2u + v^2 + \frac{u^2}{9} + \frac{2}{3}uv, & u \in [5,6],\ v \in [\frac{2u-9}{3}, \frac{u}{6}], \\ \frac{2u-9}{3}(C_4 + C_5), & u \in [6,9],\ v \in [0, \frac{9-u}{3}], \end{cases}$$

and

$$P(u) \cdot C = \begin{cases} \frac{u}{3} - 2v, & u \in [0,3],\ v \in [0, \frac{u}{6}], \\ 1, & u \in [3,5],\ v \in [0, \frac{u-3}{6}], \\ \frac{u}{3} - 2v, & u \in [3,5],\ v \in [\frac{u-3}{6}, \frac{u}{6}], \\ 1, & u \in [5,6],\ v \in [0, \frac{6-u}{3}], \\ 3 - v - \frac{u}{3}, & u \in [5,6],\ v \in [\frac{6-u}{3}, \frac{2u-9}{3}], \\ \frac{u}{3} - 2v, & u \in [5,6],\ v \in [\frac{2u-9}{3}, \frac{u}{6}], \\ 3 - \frac{u}{3} - v, & u \in [6,9],\ v \in [0, \frac{9-u}{3}]. \end{cases}$$

Thus, integrating we get

$$S(W^{F_0}_{\bullet,\bullet}; C) = \frac{10}{39} < \frac{1}{3} = A_{F_0,\Delta_{F_0}}(C),$$

so (5.6) holds.

Since $Q \neq \overline{C}_1 \cap \overline{C}_3$, we have $F_Q(W^{F_0,C}_{\bullet,\bullet,\bullet}) = 0$, which gives $S(W^{F_0,\overline{C}_3}_{\bullet,\bullet,\bullet}; Q) = \frac{9}{26}$. But

$$A_{C,\Delta_C}(Q) = \begin{cases} \frac{1}{2}, & Q \in B_{F_0}, \\ 1, & Q \notin B_{F_0}. \end{cases}$$

Thus, we have

$$\frac{A_{C,\Delta_C}(Q)}{S(W^{F_0}_{\bullet,\bullet}; C)} = \begin{cases} \frac{13}{10}, & Q \in B_{F_0}, \\ \frac{26}{9}, & Q \notin B_{F_0}, \end{cases}$$

which implies (5.6). □

Lemma 5.16. *Let Q be a point in F_0 such that $Q \notin \overline{C}_1 \cup \overline{C}_3$, and let C be the unique curve in the pencil $|\overline{C}_1|$ that contains Q. Then* (5.6) *and* (5.7) *hold.*

Proof. Note that $A_{F_0,\Delta_{F_0}}(C) = 1$, and $\widetilde{C} \sim C_1 + C_4 + C_5$. We have

$$t(u) = \begin{cases} \frac{u}{3}, & u \in [0,3], \\ 1, & u \in [3,6], \\ \frac{9-u}{3}, & u \in [6,9]. \end{cases}$$

For every $u \in [0,9]$, we have $d(u) = 0$ and $N'(u) = \widetilde{N}(u)|_{\widetilde{F}_0}$. We compute

$$N(u,v) = \begin{cases} 0, & u \in [0,3],\ v \in [0, \frac{u}{3}], \\ 0, & u \in [3,5],\ v \in [0,1], \\ 0, & u \in [5,6],\ v \in [0, 6-u], \\ (v+u-6)C_1, & u \in [5,6],\ v \in [6-u, 1], \\ vC_1, & u \in [6,9],\ v \in [0, 3-\frac{u}{3}], \end{cases}$$

and

$$P(u,v) \sim \begin{cases} \frac{u-3v}{3}(C_1 + C_4 + C_5), & u \in [0,3],\ v \in [0, \frac{u}{3}], \\ \frac{u-3v}{3}C_1 + (1-v)C_4 + \frac{3+u-6v}{6}C_5, & u \in [3,5],\ v \in [0,1], \\ \frac{u-3v}{3}C_1 + (1-v)C_4 + \frac{9-u-3v}{3}C_5, & u \in [5,6],\ v \in [0, 6-u], \\ \frac{18-2u-6v}{3}C_1 + (1-v)C_4 + \frac{9-u-3v}{3}C_5, & u \in [5,6],\ v \in [6-u, 1], \\ \frac{9-u-3v}{3}(2C_1 + C_4 + C_5), & u \in [6,9],\ v \in [0, 3-\frac{u}{3}], \end{cases}$$

which gives

$$(P(u,v))^2 = \begin{cases} \frac{(u-3v)^2}{18}, & u \in [0,3],\ v \in [0, \frac{u}{3}], \\ -\frac{1}{2} + \frac{u}{3} - \frac{1}{3}uv + \frac{1}{2}v^2, & u \in [3,5],\ v \in [0,1], \\ -\frac{u^2}{2} - \frac{uv}{3} + \frac{v^2}{2} - 13 + \frac{16}{3}u, & u \in [5,6],\ v \in [0, 6-u], \\ 5 + \frac{2uv}{3} + v^2 - \frac{2u}{3} - 6v, & u \in [5,6],\ v \in [6-u, 1], \\ \frac{(3-\frac{u}{3}-v)^2}{2}, & u \in [6,9],\ v \in [0, 3-\frac{u}{3}], \end{cases}$$

and

$$P(u)\cdot\widetilde{C}=\begin{cases}\frac{u-3v}{6}, & u\in[0,3],\ v\in[0,\frac{u}{3}],\\ \frac{u-3v}{6}, & u\in[3,5],\ v\in[0,1],\\ \frac{u-3v}{6}, & u\in[5,6],\ v\in[0,6-u],\\ \frac{9-u-3v}{3}, & u\in[5,6],\ v\in[6-u,1],\\ \frac{9-u-3v}{3}, & u\in[6,9],\ v\in[0,3-\frac{u}{3}].\end{cases}$$

Thus, integrating, we get

$$S(W^{F_0}_{\bullet,\bullet};C)=\frac{9}{26}<1=A_{F_0,\Delta_{F_0}}(C),$$

so (5.6) holds.

Since $Q\notin\overline{C}_1\cup\overline{C}_3$, we have $F_Q(W^{F_0,C}_{\bullet,\bullet,\bullet})=0$ and

$$A_{C,\Delta_C}(Q)=\begin{cases}\frac{1}{2}, & Q\in B_{F_0},\\ 1, & Q\notin B_{F_0}.\end{cases}$$

Integrating, we get $S(W^{F_0,C}_{\bullet,\bullet,\bullet};Q)=\frac{10}{39}$, so that

$$\frac{A_{C,\Delta_C}(Q)}{S(W^{F_0}_{\bullet,\bullet};C)}=\begin{cases}\frac{39}{20}, & Q\in B_{F_0},\\ \frac{39}{10}, & Q\notin B_{F_0},\end{cases}$$

which implies (5.6). □

Lemmas 5.14, 5.14, 5.16 complete the proof of Proposition 5.6.

6. On the K-moduli spaces

In this section, we prove Corollary 1.13. The proof of Corollary 1.14 is almost identical, so we omit it. To start with, let us present the following well-known assertion.

Lemma 6.1. *Let X be a smooth Fano threefold. Then*

$$h^0(X,T_X)-h^1(X,T_X)=\chi(X,T_X)=\frac{-K_X^3}{2}-18+b_2(X)-\frac{b_3(X)}{2},$$

where $b_2(X)$ and $b_3(X)$ are the second and the third Betti numbers of X, respectively.

Proof. The required assertion immediately follows from the Akizuki–Nakano vanishing theorem and the Hirzebruch–Riemann–Roch theorem, since $-K_X\cdot c_2(X)=24$. □

Now, let us use notation and assumptions introduced in Corollary 1.13.

Lemma 6.2. *Let $f\in T$ and let X_f be the Casagrande–Druel* 3*-fold constructed from* $\{f=0\}$*. Suppose that f is GIT semistable with respect to the Γ-action. Then X_f is K-semistable.*

Proof. There exists a one-parameter subgroup $\lambda\colon\mathbb{G}_m\to\Gamma$ such that

$$[f_0]=\lim_{t\to 0}\lambda(t)\cdot[f]$$

is a GIT polystable point in T. Let X_0 be the corresponding Casagrande–Druel threefold constructed from $\{f_0 = 0\}$. Then it follows from Theorem 1.11 that X_0 is K-polystable. On the other hand, the subgroup λ gives isotrivial flat degeneration of X_f to X_0, which implies that X_f is K-semistable, because K-semistability is an open condition. □

Now, we are ready to prove Corollary 1.13.

Proof of Corollary 1.13. Since the construction of Casagrande–Druel 3-folds is functorial, there exists a Γ-equivariant flat morphism $\pi_T\colon X_T \to T$ such that $\pi_T^{-1}([f]) \cong X_f$. We set $X_{T^{\mathrm{ss}}} = \pi_T^{-1}(T^{\mathrm{ss}})$. Then the restriction morphism $X_{T^{\mathrm{ss}}} \to T^{\mathrm{ss}}$ is a Γ-equivariant flat family of K-semistable Fano 3-folds by Lemma 6.2.

Let $\{T^{\mathrm{ss}}/\Gamma\}$ be the fibered category over $(\mathrm{Sch}/\mathbb{C})_{\mathrm{fppf}}$ in the sense of [36, Example 4.6.7]. Then the family $X_{T^{\mathrm{ss}}} \to T^{\mathrm{ss}}$ gives a morphism $\{T^{\mathrm{ss}}/\Gamma\} \to \mathcal{M}^{\mathrm{Kss}}_{3,28}$ of fibered categories. This induces the morphism $[T^{\mathrm{ss}}/\Gamma] \to \mathcal{M}^{\mathrm{Kss}}_{3,28}$ between Artin stacks, since $[T^{\mathrm{ss}}/\Gamma]$ is the stackification of $\{T^{\mathrm{ss}}/\Gamma\}$ (see [36, Remark 4.6.8]).

Since M is the good moduli space of $[T^{\mathrm{ss}}/\Gamma]$, it follows from [4, Theorem 6.6] that there exists a natural morphism $\Phi\colon M \to M^{\mathrm{Kps}}_{3,28}$ that maps $[f]$ to $[X_f]$. We claim that Φ is injective. Since M is of Picard rank 1, it is enough to show this on the open subset of M parametrizing $[f]$ such that $(f = 0)$ is non-singular, so that the corresponding 3-fold X_f is smooth. Suppose that f_1 and f_2 are points in T and the corresponding Casagrande–Druel 3-folds X_{f_1} and X_{f_2} are both smooth and isomorphic. Let $\chi\colon X_{f_1} \to X_{f_2}$ be the isomorphism. Then χ maps any exceptional locus of the contraction of an extremal ray of X_{f_1} to an exceptional locus of the contraction of an extremal ray of X_{f_2}. There are exactly 4 such exceptional loci: S_1, S_2, E_1 and E_2. Since S_j and E_k are not isomorphic to each other, the image of $S_1 \subset X_{f_1}$ via χ must be either $S_1 \subset X_{f_2}$ or $S_2 \subset X_{f_2}$. In each case, the restriction of χ to S_1 gives a projective isomorphism between $(f_1 = 0)$ and $(f_2 = 0)$. Thus, the points f_1 and f_2 are contained in one Γ-orbit. Hence, the morphism Φ is injective.

Observe that M is normal. Take $[f] \in M$. Since the deformations of the 3-fold X_f are unobstructed by Proposition 2.6, the variety $M^{\mathrm{Kps}}_{3,28}$ is also normal at $[X_f]$ by Luna's étale slice theorem [6, Theorem 1.2]. Moreover, if X_f is smooth, then

$$\dim_{[X_f]}(M^{\mathrm{Kps}}_{3,28}) \leqslant h^1(X_f, T_{X_f}) = \dim(M)$$

by Lemma 6.1, since $h^0(X, T_X) = \dim(\mathrm{Aut}(X)) = 1$. Therefore, using the injectivity of Φ, we see that the image $\Phi(M) \subset M^{\mathrm{Kps}}_{3,28}$ is a connected component, and Φ is an isomorphism onto this connected component by Zariski's main theorem. □

The variety $M^{\mathrm{Kps}}_{(3.9)}$ is well-studied [26]. Let us describe $M^{\mathrm{Kps}}_{(4.2)} \cong T^{\mathrm{ss}} /\!\!/ \Gamma$. Recall that

$$T = \mathbb{P}\big(H^0(V, \mathcal{O}_V(2,2))^\vee\big)$$

and $\Gamma = (\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})) \rtimes \boldsymbol{\mu}_2$, where $V = \mathbb{P}^1 \times \mathbb{P}^1$. Set $\Gamma_0 = \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$.

Proposition 6.3 (Noam Elkies). *One has $T^{\mathrm{ss}} /\!\!/ \Gamma_0 \cong T^{\mathrm{ss}} /\!\!/ \Gamma \cong \mathbb{P}(1,2,3)$.*

Proof. Let $W = H^0(V, \mathcal{O}_V(2,2))$, let S be the symmetric algebra of $W^\vee$, let S^{Γ_0} be its subalgebra of invariants for the natural Γ_0-action, and let $H(t)$ be its Hilbert series

$$H(t) = \sum_{k \geqslant 0} \dim\big((\mathrm{Sym}^k(W^\vee))^{\Gamma_0}\big) t^k.$$

Then it follows from [39, §11.9] or [18, §4.6] that

$$H(t) = \int_0^1 \int_0^1 \frac{2 - z_1^2 - z_1^{-2}}{2} \cdot \frac{2 - z_2^2 - z_2^{-2}}{2} \cdot \prod_{j_1, j_2 \in \{-1,0,1\}} \frac{1}{1 - t \cdot z_1^{2j_1} z_2^{2j_2}} \, d\phi_1 \, d\phi_2$$

with $|t| < 1$, where $z_1 = e^{2\pi\sqrt{-1}\phi_1}$ and $z_2 = e^{2\pi\sqrt{-1}\phi_2}$. This gives

$$H(t) = \frac{1}{(1-t^2)(1-t^3)(1-t^4)}.$$

Let us find generators of S^{Γ_0}. Consider the standard basis

$$x_0^2y_0^2, x_0^2y_0y_1, x_0^2y_1^2, x_0x_1y_0^2, x_0x_1y_0y_1, x_0x_1y_1^2, x_1^2y_0^2, x_1^2y_0y_1, x_1^2y_1^2$$

of the space W, let $a_{00}, a_{01}, a_{02}, a_{10}, a_{11}, a_{12}, a_{20}, a_{21}, a_{22}$ be the dual basis of the space $W^\vee$, and let J_2, J_3, J_4 be the coefficients of the characteristic polynomial of the matrix

$$\begin{pmatrix} \frac{1}{2}a_{11} & -a_{10} & -a_{01} & 2a_{00} \\ a_{12} & -\frac{1}{2}a_{11} & -2a_{02} & a_{01} \\ a_{21} & -2a_{20} & -\frac{1}{2}a_{11} & a_{10} \\ 2a_{22} & -a_{21} & -a_{12} & \frac{1}{2}a_{11} \end{pmatrix}$$

such that $J_k \in \mathrm{Sym}^k(W^\vee)$ for $k \in \{2, 3, 4\}$. Then J_2, J_3, J_4 are Γ_0-invariant, and these polynomials are algebraically independent, which gives $S^{\Gamma_0} = \mathbb{C}[J_2, J_3, J_4]$, so that

$$T^{\mathrm{ss}} /\!\!/ \Gamma_0 \cong \mathbb{P}(2,3,4) \cong \mathbb{P}(1,2,3).$$

Since the polynomials J_2, J_3, J_4 are also Γ-invariant, we also get $T^{\mathrm{ss}} /\!\!/ \Gamma_0 \cong T^{\mathrm{ss}} /\!\!/ \Gamma$. □

Remark 6.4. In fact, Proposition 6.3 is a classical result – Peano [38] and Turnbull [44] showed that S^{Γ_0} is generated by J_2, J_3, J_4; see [44, §12] and [37, pages 242–246].

The surface $M^{\mathrm{Kps}}_{(4.2)}$ is a component of the K-moduli space of smoothable Fano threefolds. Another two-dimensional component of this K-moduli space has been described in [13], and all its one-dimensional components have been described in [1].

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