ESSAYS ON NETWORK FORMATION

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Contents

Summary

The importance of networks has been highlighted in numerous economic studies. To address the key question of how networks emerge, several models have been developed to examine equilibrium networks and assess the efficiency of various network structures. This thesis presents our research on network formation, organized into three chapters.

In Chapter 1, we consider a non-cooperative model of network formations where agents decide on whom to form costly links to. Links are unilaterally formed and payoff flows one way to the active side. We study discontinuous information flows where agents only receive benefits from other agents at a distance of two in the network. For the static game, we show that the set of strict Nash equilibria encompasses a multiplicity of core-periphery network structures. We further study a noisy best response process to obtain long-run predictions. By doing so, we find that the set of stochastically stable states retains a multiplicity of network structures, many of which are not efficient.

Chapter 2 provides supportive simulation evidence for the theoretical model of evolutionary network formation, where agents form unilateral links and receive payoffs from others within distance two. We present a MatLab program to mimic both unperturbed and perturbed myopic best learning dynamics. The simulation of unperturbed dynamics shows that core-periphery networks are absorbing when agents never make mistakes. Further, when there is a small probability that agents make mistakes, the simulation of perturbed dynamics shows that core-periphery networks

are uniquely stochastically stable.

In Chapter 3, we present an evolutionary model of coordination and network formation where there are two groups of agents who face either high or low linking constraints on the number of links. We study the agents' choices of actions in the 2×2 coordination game and the set of agents to whom they link. For the static game, we show that both monomorphic states (all agents play the same action) and polymorphic states (agents play different actions) are Nash equilibria. We then study a noisy best response learning dynamics to select among multiple Nash equilibria in the static game. We find that if both low and high constraints are loose, the risk-dominant strategy is selected. In contrast, if both low and high constraints are tight, the payoff-dominant action arises. Moreover, we present that the co-existence of the risk- and payoff-dominant actions can be observed for some game parameters.

In summary, the thesis contributes to the literature on network formation in both theoretical and simulation respects by considering the constraints of information transmission distance. Additionally, Furthermore, the work also adds to the literature on the coevolution of social coordination and network formation by incorporating heterogeneous linking constraints.

Chapter 1

Network Formation with Local Benefits

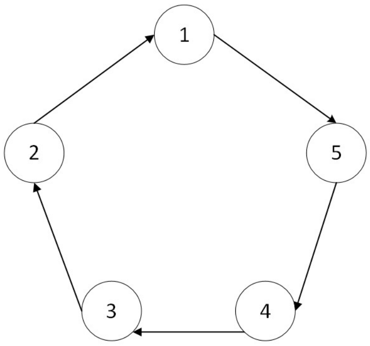
1.1 Introduction

The role of networks in various social and economic activities has attracted considerable attention in academic research. Networks have proven to be instrumental in explaining phenomena within fields such as economics, sociology, and related disciplines. Comprehending the impact of networks constitutes a pivotal area of inquiry across diverse contexts, ranging from R&D collaboration to labour markets.[1] Thus, it is important to know which network configurations will form and what drives the stability and efficiency of networks.

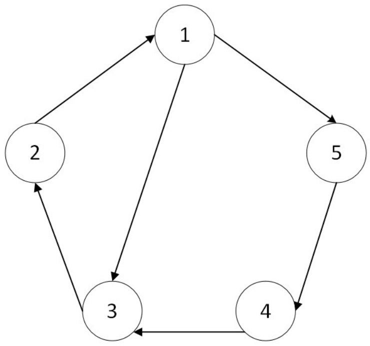
The motivation of this paper is based on the following two observations. First, many social networks exhibit some structures with small diameters.[2] The distance between any two nodes in a network is usually relatively small. Second, friction in the spread of information is inevitable in the real world due to noise or misinformation etc. This paper introduces a network formation model that effectively captures these characteristics, based on the key assumption that information transmission is constrained within two steps. Specifically, agents obtain information either directly

[1] Topics involve R&D networks (Goyal & Moraga-Gonzalez 2001; Goyal & Joshi 2003), networks in labour markets (Calvo-Armengol & Jackson 2004, 2007), public goods in networks (Bramoullé & Kranton 2007; Allouch 2015); social coordination (Goyal & Vega-Redondo 2005; Staudigl & Weidenholzer 2014 and Cui & Weidenholzer 2021).

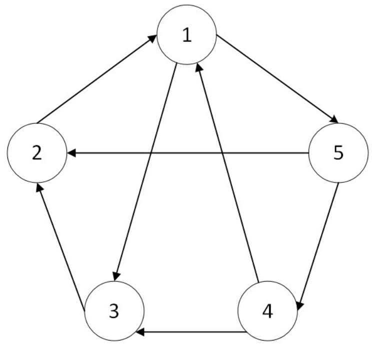
[2] Examples include the small-world phenomenon (Milgram 1967) and six degrees of separation (Guare 2016).



(a) *Wheel*



(b) *Wheel + one additional link*



(c) *Wheel + three additional links*

Figure 1.1. *Collapse of the wheel: consider an example of five agents, the wheel depicted in (a) is a strict Nash network in Bala & Goyal's one-way flow model without decay. When agents can only receive information from others within distance two, agent 1 has an incentive to form an additional link to agent 3 to get information from 2 and 3 as shown in (b). Similarly, agent 4 and agent 5 also have incentives to form additional links as shown in (c).*

from their immediate connections (referred to as "friends") or indirectly from the connections of their friends (commonly termed as "friends of friends").[3] A typical example is the network of citations within journal papers where researchers frequently cite the works of others, thus creating a network of scientific papers.[4] However, when analysing the citations of a specific paper, access is limited to the papers cited within that particular work. Additionally, it is unknown which papers are cited within the references cited in a given work. Consequently, to identify and access papers referenced within those citations, additional time and effort are required to extract information concerning the "citations of citations."

In this paper, we set up a non-cooperative model of network formation, where links are unilaterally formed and information flows one way. Under one-way flow, the payoff is only received by the agent who initiates the link.[5] The information carried by each agent is the source of payoffs. By forming costly links, agents receive payoffs from their friends and friends of friends.

Bala & Goyal (2000) have shown that in the absence of limits on information transmission,

[3] This setup is featured as 'truncated connections' in Jackson & Wolinsky (1996) or 'communication threshold' in Hojman & Szeidl (2008).

[4] See e.g. Price (1965) who analyses the growing citation networks and documents that they are scale-free networks.

[5] See e.g. Bala & Goyal 2000; Billand et al. 2008 and Cui et al. 2013. In contrast, two-way flow assumes that both sides of a link receive the payoff (see e.g. Bala & Goyal 2000; Feri 2007; Billand et al. 2011 and De Jaegher & Kamphorst 2015).

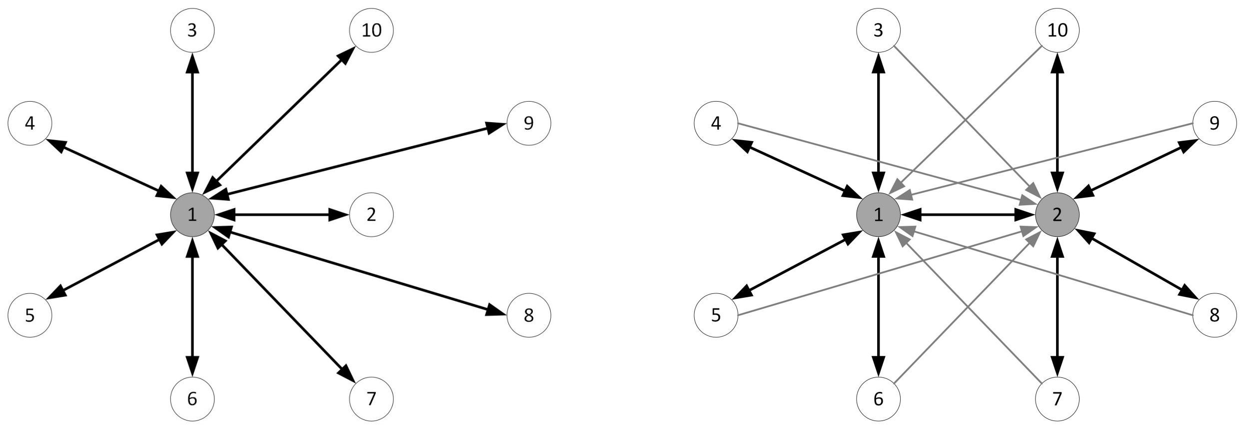
the wheel is the unique strict Nash equilibrium. On an intuitive level, when the information flows one way and the linking cost is low enough, the network will have to feature cycles, implying that everybody has to be connected so that there exists a path from any agent to any other agents while there also exists a path in the reverse direction. But it cannot be insistent where two paths cross in a strict Nash equilibrium because that would imply that some agents are indifferent in their linking choice; leaving the wheel as the only strict Nash equilibrium.

However, in the presence of constraint on distance, the wheel is not a strict Nash equilibrium since agents have incentives to form additional links to those beyond distance two. Figure 1.1 illustrates this point. The main result under our key assumption shows that the network formation game has multiple strict Nash network configurations, which we refer to *core-periphery networks*.[6] In these strict Nash equilibria, there is a small set of core agents who maintain some links to other core agents and periphery agents and a large set of periphery agents who maintain only links to core agents. Figure 1.2 shows two examples of such equilibrium networks. The logic is that due to the constraint imposed on information transmission, agents form links to keep others within a two-step distance. Conversely, each agent has the incentive to minimize the number of links required to fulfil this objective. In a core-periphery network, links formed by core agents ensure that any given agent can keep others within two steps by forming links to all core agents. Besides, the compact size of the core agent set enables agents to link to others with a minimal number of links.

Furthermore, due to the multiplicity of strict Nash equilibria, we study a noisy best response process to characterize stochastically stable states. This approach has various applications to the selection of multi-equilibria in the literature on network formations.[7] Each agent has a positive probability of receiving opportunities to update their linking strategies in a discrete time. Sometimes they make mistakes and fail to maximize their payoffs. We adopt the methodology developed

[6] Borgatti & Everett (2000) formalize the concept of core-periphery structure in the context of undirected networks. In this paper, our definition and notions of core-periphery networks are closely related to the definition of the directed core-periphery network in Elliott et al. (2020).

[7] See e.g. Jackson & Watts (2002*a*), Feri (2007) and Cui et al. (2013) for applications to network formation.



(a) *CP_1-star* **(b)** *CP_2 Network*

Figure 1.2. *Two core-periphery networks with 10 agents*

by Kandori et al. (1993) and Young (1993), allowing us to identify stochastically stable states, i.e. states in support of the invariant distribution of the Markov process as the probability of mistakes vanishes. Our results show that the set of stochastically stable states retains a multiplicity of network structures, which encompasses core-periphery networks. Additionally, we study the number of links that efficient networks have and show that core-periphery networks however are inefficient as they are payoff dominated by the star. Thus, with the constraint on the distance of information transmission, we may observe inefficient outcomes. This is in contrast to the case where there is no constraint and the unique strict Nash network, the wheel, is also efficient.

The paper is organized as follows. Section 1.2 discusses the relation between our paper and the existing literature. In section 1.3, we describe the details of the setups of the network formation game. In section 1.4, we present our results on strict Nash equilibria. Section 1.5 describes the learning dynamics and our analytical results of the long-run predictions. We also have a short discussion on the efficiency of stochastically stable states in section 1.5. Section 1.6 concludes. Table 1.1 provides the list of notions and their definitions in the paper and the Appendix A.1 contains the proofs of our key results.

Table 1.1. *List of parameters and their definitions*

Parameters	Definitions
n	Number of agents, which is larger than or equal to 4.
N	Set of all agents.
g_{ij}	Linking decision of agent i to j.
g_i	A N-tuple of agent i's linking strategy to each agent.
g	A network, i.e. a strategy profile of all agents.
$\mathscr{G}_i$	Strategy profile of agent i.
$\mathscr{G}$	Set of all strategy profiles.
g_{-i}	Network formed by agents other than i.
$g+ij$	Network obtained by adding the link from i to j.
$g-ij$	Network obtained by deleting the link from i to j.
$d(i,j;g)$	Distance from j to i in a given network g.
$N_i^d(g)$	d-neighbourhood of agent i, i.e. the set of agents who are at distance d to i.
$n_i^d(g)$	Number of agents in i's d-neighbourhood.
d_i^{out}	Out-degree of agent i, i.e. the number of links that agent i actively forms.
d_i^{in}	In-degree of agent i, i.e. the number of links that agent i passively receives.
C	Component of N.
g^i	Sub-network on component C_i.
c	Cost of forming a link.
$\mathscr{G}^\star$	Set of strict Nash equilibrium networks.
CP_ℓ	Core-periphery network with ℓ core agents.
$C(\ell;g)$	Set of core-agents given a core-periphery network g.
P	Set of periphery agents.
P_i	Set of agent i's periphery agents.
$\mathscr{C}\mathscr{P}_\ell$	Set of strict Nash CP_ℓ networks.
$\bar{\ell}$	Maximum number of core agents in any strict Nash core-periphery network.
$\mathscr{C}\mathscr{P}_{\bar{\ell}}$	Set of all strict Nash core-periphery networks.
$G^{\star\star}$	Absorbing set.
$\mathscr{G}^{\star\star}$	Set of all absorbing sets.
$\mathscr{G}^{\star\star\star}$	Set of stochastically stable states.
$r(g,g')$	Resistance of transition form network g to g'.
τ_i	A $G_i^{\star\star}$-tree, i.e. a spanning tree rooted in the absorbing set $G_i^{\star\star}$.
T_i	Set of all $G_i^{\star\star}$-trees.
$r(G_i^{\star\star})$	Resistance of a $G_i^{\star\star}$-tree.
$\gamma(G_i^{\star\star})$	Stochastic potential of the absorbing set $G_i^{\star\star}$.
$\boldsymbol{N}$	Number of all absorbing sets.
$W(g)$	Welfare generated by given network g.

1.2 Literature review

The present paper is closely related to the broad literature on network formation. Jackson & Wolinsky (1996) explore the truncated connection model with some bound D in the cooperative network formation model where forming a link requires mutual consent of both parties. They show that the pairwise stable networks exhibit a property that the maximum distance between any two players is $2D-1$. Further, Bala & Goyal (2000) presents a non-cooperative network formation model where Nash equilibrium can be used to characterise the stable network architectures. They have broad discussions on models of one-way flow and two-way flow in cases with decay (where there are frictions in information) and without decay (where there is no friction in information). They provide characterizations of Nash equilibrium networks and efficient networks, which exhibit some simple architectures, e.g. the wheel and the star. Based on Bala & Goyal (2000)'s two-way flow model with decay, Hojman & Szeidl (2008) study a model where links have decreasing returns and show that for some parameters, the unique non-empty Nash equilibrium network is the periphery-sponsored star at the presence of communication threshold. Our work differs from these models in one main direction. We introduce constraints on information transmission to Bala and Goyal's one-way flow model without decay and exhibit a different class of strict Nash equilibrium networks – core-periphery networks – which include the star.

This paper also adds to the literature on the dynamics of social networks. Watts (2001) presents a dynamic network formation with two-sided links through independent decisions and shows that the star is both stable and efficient for some parameters.[8] Jackson & Watts (2002*a*) study a stochastic evolution of network formation and find that a stochastically stable network is either pairwise stable or part of a closed cycle. Further, Feri (2007) considers the noisy best re-

[8]Two-sided link through independent decisions is that the formation of a link between two agents requires that both sides wish it (see e.g. Goyal & Vega-Redondo 2005 and Fosco & Mengel 2011).

sponse learning in Bala and Goyal's two-way flow model with decay and shows that the periphery-sponsored star is the unique stochastically stable network architecture. Cui et al. (2013) explore the evolutionary version of Bala and Goyal's one-way flow model with decay and find that either the empty network or the wheel is the stochastically stable state. Our work contributes to these results in two respects. First, we demonstrate that core-periphery networks exhibit stochastic stability, highlighting the potential for the emergence of networks other than simple architectures, e.g. the wheel and the star. Second, we showcase the possibility of obtaining inefficient network architectures in the long run.

Our paper also departs from the literature in modelling payoffs. Goyal & Vega-Redondo (2007) study a model of pairwise links where two parties of a link split the payoffs. They show that the star emerges in the absence of capacity constraints on links and the cycle network is stochastically stable when the capacity of links is relatively small to the population. In Galeotti & Goyal (2010), a player's payoff depends on how much information she and her neighbours acquire. They show that the equilibrium networks exhibit 'the Law of the Few' and have a core-periphery structure, i.e. few players in the core acquire information and many players in the periphery acquire no information. In contrast to these studies, in our model each agent carries information of value one and the payoff goes to the party who initiates the link.

A different branch in the literature analyses models of co-evolution of coordination games and network formations, where in addition to their linking choice, they also have to decide the action played in the coordination game. Jackson & Watts (2002*b*) study an evolutionary model in social coordination games where the network is bilaterally formed, which prescribes the use of the concept of pairwise stability in Jackson & Wolinsky (1996). They show that for some parameters the networks in stochastically stable states exhibit fully connected configurations. In Goyal & Vega-Redondo (2005), agents unilaterally decide on whom to link to. They show that the equilibrium network is either empty or complete. Staudigl & Weidenholzer (2014) set up a model with a restricted maximum number of links that each agent can support and show a variety of the equilibrium network structures in the long run. Cui & Weidenholzer (2021) consider a case where an agent is able to receive payoffs from links that other agents form to her. They find that the Nash equilibrium networks do not have to be fully connected and that architectures, where agents use different actions, may sometimes be stochastically stable.

1.3 Model

We consider a one-way flow model of network formations.[9] There is a population of n agents, denoted by $N = \{1, 2, \cdots, n\}$ with $n \geq 4$. Each agent $i \in N$ decides the set of agents to whom she forms links. A strategy used by agent i is given by a N-tuple $g_i = (g_{i1}, g_{i2}, \cdots, g_{in})$ where $g_{ij} \in \{0, 1\}$ is agent i's linking decision to agent j. We say i links up to j if $g_{ij} = 1$; otherwise, $g_{ij} = 0$. We assume that agents cannot link to themselves, i.e. $g_{ii} = 0, \forall i \in N$. Further, let $\mathscr{G}_i$ be the set of all possible link strategies that agent i can choose. One-way flow model implies that agent i's linking decision to j is independent with j's decision to i, i.e. g_{ji} and g_{ij} are not necessarily equal. A network $g = (g_i)_{i \in N} \in \mathscr{G}$ is the strategy profile of all agents, where $\mathscr{G} = \prod_{i \in N} \mathscr{G}_i$ is the set of all networks. Moreover, let $g_{-i} = g - g_i$ denote the network formed by agents other than i. Further, we denote by $g + ij$ the network obtained by adding the link from agent i to j to network g. Similarly, $g - ij$ denotes the network obtained by deleting the link from i to j in network g.

We say there exists a path from agent j to i if either $g_{ij} = 1$ or there is a set of agents $\{k_1, k_2, \cdots, k_m\}$, such that $g_{ik_1} = g_{k_1 k_2} = \cdots = g_{k_m j} = 1$. The distance from j to i, denoted by $d(i, j; g)$, is the number of links of the shortest path from agent j to i.[10] Further, we define agent i's d-neighbourhood as the set of agents who are at distance d to i, denoted by $N_i^d(g) =$

[9] We follow the notations introduced in Jackson & Wolinsky (1996) and Bala & Goyal (2000).

[10] In a directed network g, $d(i, j; g)$ and $d(j, i; g)$ can be different. We say $d(i, j; g) = \infty$ if j is not linked by i neither directly nor indirectly.

$\{j \in N : d(i,j;g) = d\}$, with $n_i^d(g) = |N_i^d(g)|$ the number of agent i's d-neighbours. We refer to $d_i^{out} := n_i^1(g) = \sum_{j \in N} g_{ij}$ as the out-degree of agent i, i.e. the number of active links that agent i forms. We also denote by $d_i^{in} = \sum_{j \in N} g_{ji}$ the in-degree of agent i, i.e. the number of passive links that agent i receives from others.

A subset $C \subseteq N$ is called a strongly connected component if $\forall i, j \in C$ with $i \neq j$, there exists a path from i to j as well as a path from j t i, and there is no strict superset, i.e. $C \subset C' \subseteq N$ for which this is true. A network g is strongly connected if it has a unique strongly connected component. Let g^i be the sub-network within agents in the strongly connected component C_i.

We now define the payoffs in our network formation game. As in Bala & Goyal (2000), each agent receives information from others by forming costly links and benefits from doing so. Without loss of generality, the value of information that each agent has is homogeneously normalized to one. The cost of each link is $c > 0$. We assume that the cost is only incurred to the party who initiates the link. We deviate from Bala & Goyal (2000) by studying the case where the distance that information can travel is constrained. Let D be the constraint. This implies that agents can receive information from others who are within a distance of D. As motivated in the introduction, we assume that agents can only observe their neighbours and the neighbours of neighbours, i.e. we focus on the case where $D = 2$.[11]

An agent's payoff is calculated as the sum of benefits derived from the information she receives from others, minus the total cost incurred from forming links. More formally, given a network $g = (g_i)_{i \in N}$, agent i's payoff is given by

$$U_i(g_i, g_{-i}) = 1 + n_i^1(g) + n_i^2(g) - c \cdot \sum_{j \in N} g_{ij} \tag{1.1}$$

[11] This is a special case of the communication threshold in Hojman & Szeidl (2008), which in contrast to our model focuses on two-way flow. The existence of limits in communication is also observed in the one-way flow case, which their model cannot characterize.

where the constant captures the value of information of agent i self.

1.4 Strict Nash Networks

In the first step, we characterize some important properties of strict Nash networks. We follow the definition of strict Nash networks in Bala & Goyal (2000), which is formalized as Definition 1.4.1.

Definition 1.4.1 (Strict Nash Networks). *A network $g = (g_i)_{i \in N}$ is a strict Nash network if and only if $U_i(g_i, g_{-i}) > U_i(g_i', g_{-i})$ for all $g_i' \in \mathscr{G}_i$ and $i \in N$.*

We denote by $\mathscr{G}^\star$ the set of all permissible strict Nash networks. The first two technical lemmas establish some useful properties of the natures of out-degrees and in-degrees in a strict Nash network.

Lemma 1.4.1. *For any non-empty strict Nash network $g \in \mathscr{G}^\star$, $d_i^{out} \geq 1$ for any $i \in N$.*

Lemma 1.4.1 provides that every agent supports at least one active link. The intuition is that if supporting an active link is profitable for some agents, then it must be the case that it is profitable for every agent. It is trivial for the case where $c < 1$, forming a link to another agent yields at least $1 - c > 0$. Note that in this case, any strict Nash network has to be non-empty. The next lemma establishes a similar insight regarding the in-degrees of agents.

Lemma 1.4.2. *For any non-empty strict Nash network $g \in \mathscr{G}^\star$, $d_i^{in} \geq 1$ for any $i \in N$.*

Lemma 1.4.2 implies that every agent receives at least one passive link. For the case $c < 1$, this result is trivial because if an agent i receives no passive links, another agent j will get an additional payoff $1 - c$ by forming a link to i. Consider the other case $c > 1$, the intuition is more complicated. The main idea in this case is based on the observation that agents without passive links turn out to be either indifferent between whom to form links to or have profitable deviation. This is incompatible with a strict Nash network.

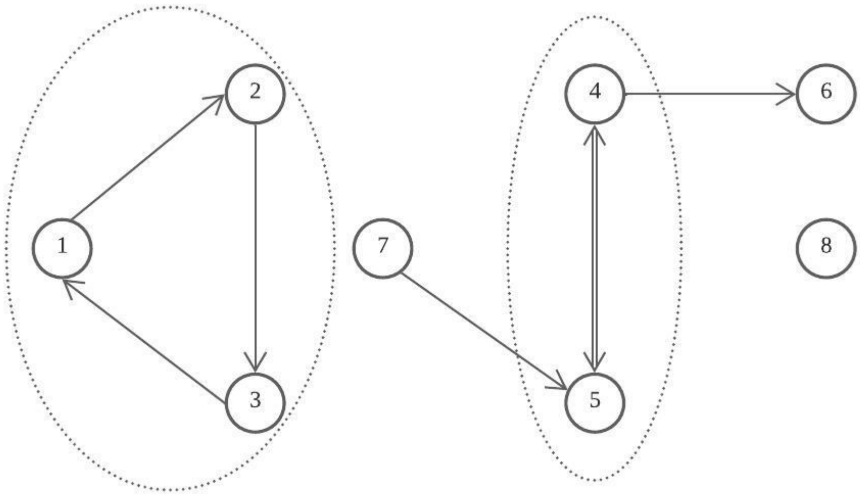


Figure 1.3. *A network which is not strongly connected*

We now move towards studying the implications of the two observations for the general structures of strict Nash networks. Recall the definition of strongly connected networks. Strong connectedness implies that there exists a unique strongly connected component in the network, which requires that for any two agents i and j, there is a path from i to j and a path from j to i. Any network that is not strongly connected, has to consist of multiple strongly connected components. These strongly connected components could be isolated from each other, or a path exists from one component to another, but not vice versa. The network depicted in Figure 1.3 illustrates such properties. In this network, there are five strongly connected components in the network: $\{1,2,3\}, \{4,5\}, \{6\}, \{7\}$ and $\{8\}$. Component $\{8\}$ is totally isolated from other components, while components $\{6\}$ and $\{4,5\}$ are not strongly connected since there is no path from agent 6 to either agent 4 or agent 5.

Lemma 1.4.3. *Any non-empty strict Nash network is strongly connected.*

Lemma 1.4.3 is trivial for the case $c < 1$. If a strict Nash network g is not strongly connected, then there exist two agents i and j such that there is no path from i to j. Therefore, there is a profitable deviation for agent i to form a link to j. For $c > 1$, the lemma follows from combining the previous two lemmas and pointing out that in any not strongly connected component some agents will necessarily have an incentive to link to agents in other components.

While we are able to obtain the previous results for the general case where c may be larger than one, for the following analysis we have to restrict ourselves to $c < 1$. Revisit networks depicted in Fig 1.2. In the star network shown in Fig 1.2a, the agent in the centre links to everyone else whilst receiving links from everyone. One can check that the star is a strict Nash network as the agent in the centre and agents in the periphery all give a unique best response.

In fact, there also exists another class of networks that possess these properties. We refer to these networks as core-periphery networks.[12] Less formally, a core-periphery network consists of two sets of agents: core and periphery. Each agent in the core (termed as *core agent*) links to every agent in the core and also links to a subset of the other agents in the periphery. Each agent in the periphery (termed as *periphery agent*) links to all core agents and forms no link to any other periphery agent. The more formal definition is given as follows.

Definition 1.4.2. *A network g is called a core-periphery network, denoted by CP_ℓ if*

1) each agent is either a core agent or a periphery agent, i.e. $N = C(\ell;g) \cup P(g)$ and $C(\ell;g) \cap P(g) = \emptyset$, where $C(\ell;g) = \{1,2,\cdots,\ell\}$ is the set of core agents and $P(g) = \{\ell+1,\ell+2,\cdots,n\}$ is the set of periphery agents;

2) core agents link to each other directly, i.e. $g_{ij} = 1, \forall i, j \in C(\ell;g)$ with $i \neq j$;

3) each core agent i links to a subset of periphery agents, i.e. $P_i(g) = \{j \in P(g) : g_{ij} = 1\}$;

4) each periphery agent is linked by a single core agent, i.e. $\bigcap_{i\in C(\ell;g)} P_i(g) = \emptyset$ and $P(g) = \bigcup_{i\in C(\ell;g)} P_i(g)$;

5) each periphery agent links to all core agents, i.e. $g_{ij} = 1, \forall i \in P(g), j \in C(\ell;g)$;

[12] Core-periphery networks are also featured previously in the literature (see in Borgatti & Everett 2000 and Elliott et al. 2020), but with slightly different definitions in our paper.

6) there is no link between periphery agents, i.e. $g_{jk} = g_{kj} = 0, \forall j,k \in P(g)$.

The network shown in Fig 1.2b depicts such a core-periphery network with two core agents. Agents 1 and 2 are core agents and all other agents link up to them. The other eight agents are periphery agents, each of whom is linked by either agent 1 or agent 2.

We classify the core-periphery networks by the number of core agents so that CP_ℓ denotes a core-periphery network with ℓ core agents. Note that for a given number of core agents ℓ, there is a multiplicity of different CP_ℓ networks, varying in the identities of core agents and their peripheries with similar structures. The network in Fig 1.2b is a CP_2 network and the star is the special case of CP_1 network. One can also check that the CP_2 network depicted in Fig 1.2b is a strict Nash network since core and periphery agents all give a unique best response.

Having defined core-periphery networks, we proceed to prove that the set of strict Nash networks includes CP_ℓ networks with a certain condition. The following proposition exhibits our main results on strict Nash networks.

Proposition 1.4.1. *Any CP_ℓ network g with $|P_i(g)| \geq 3, \forall i \in C(\ell;g)$ is a strict Nash equilibrium.*

The intuition of Proposition 1.4.1 is as follows. First, core agents link up to the other core agents and their respective peripheries such that they can observe everyone. Adding any other link yields no additional payoffs, and deleting any link means a reduction in payoffs. Further, periphery agents observe everybody by linking to all core agents. Thus, they have no incentives to form additional links. If they delete some links, they will lose access to the periphery agents of these core agents, leading to a decrease in payoffs. More importantly, the condition on the number of periphery agents ensures that the best response of each agent is unique, i.e. there is no alternative strategy that yields the same payoffs as in a CP_ℓ network. Figure 1.4 (a) depicts a CP_2 network where core agent 1 has only two periphery agents. In this network, the periphery agent 3 is indifferent between linking to agent 1 and agent 4 (see Figure 1.4b). Therefore, the CP_2 network

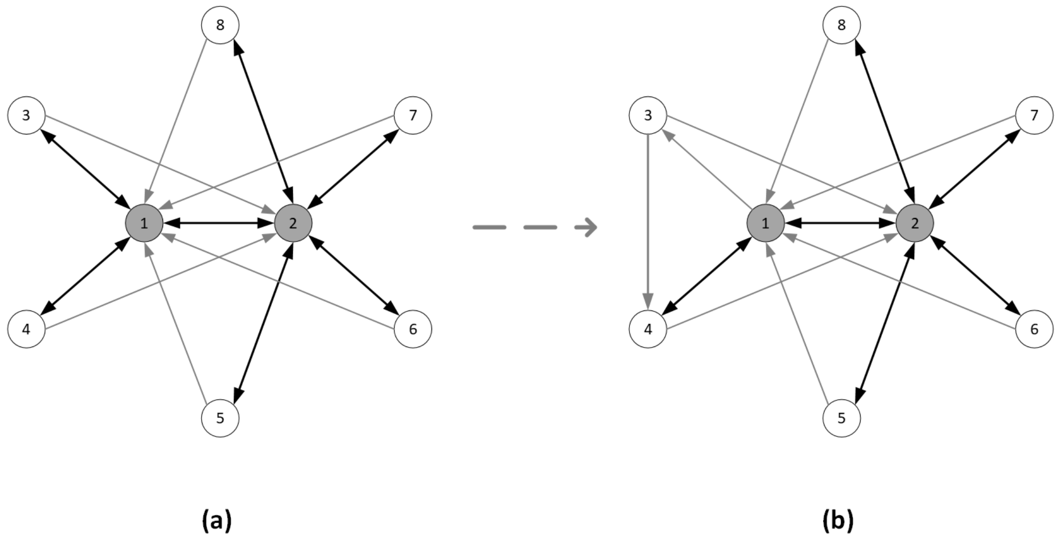


Figure 1.4. *A core-periphery network where the core agent 1 has only two periphery agents.*

in Figure 1.4 (a) is not a strict Nash equilibrium.

Proposition 1.4.1 shows that any core-periphery network in which each core agent links up to at least three periphery agents is a strict Nash network. Now, we denote by $\mathscr{CP}_\ell$ the set of strict Nash core-periphery networks with ℓ core agents. More formally, $\mathscr{CP}_\ell$ is defined by

$$\mathscr{CP}_\ell := \{g : g \text{ is a } CP_\ell \text{ network and } |P_i(g)| \geq 3, \text{for all } i \in C(\ell; g)\}.$$

Note that the condition on the size of each set of periphery agents imposes an upper bound on the number of core agents in a strict Nash core-periphery network. To be more precise, in a strict Nash CP_ℓ network, each core agent has to have at least three periphery agents, therefore the number of periphery agents is at least three times more than ℓ. Thus, given the number of all agents n, the number of core agents can never exceed $\lfloor \frac{n}{4} \rfloor := \bar{\ell}$. If a core-periphery network has more than L core agents, then it is not a strict Nash network. We denote by $\mathscr{CP}_{\bar{\ell}}$ the set of all strict Nash CP_ℓ networks in which the number of core agents is less or equal to $\bar{\ell}$. More formally, the set $\mathscr{CP}_{\bar{\ell}}$ at a given n is defined as

$$\mathscr{C}\mathscr{P}_{\bar{\ell}} := \bigcup_{1 \leq \ell \leq \bar{\ell}} \mathscr{C}\mathscr{P}_{\ell}.$$

Proposition 1.4.1 implies that there is a multiplicity of strict Nash network configurations. This result differs from Bala & Goyal (2000) and Hojman & Szeidl (2008). In Bala & Goyal (2000)'s one-way flow model without decay, the wheel is the unique strict Nash network for some parameters within certain ranges. With local benefits in our model, the wheel is not permissible as agents can never observe others who are more than distance two far away and thus have incentives to form additional links to those who are not observed by them. In Hojman & Szeidl's two-way flow model, the non-empty strict Nash network is unique and exhibits a structure of either periphery-sponsored stars or extended stars. Our model presents that the star, denoted by CP_1, is the unique strict Nash core-periphery network for any $n < 8$. For any $n \geq 8$, Proposition 1.4.1 shows that there are multiple strict Nash networks which present similar structures, i.e. the core-periphery networks (see also Figure 1.2 for an illustration).

Further, in the one-way flow case, the strict Nash networks have to feature cycles, implying that each pair of two agents has to be connected by two paths in both directions. In the absence of constraints on the distance of information transmission, the circumference of these cycles is not limited but it requires there exist no crossing cycles. However, in the presence of constraints, the circumference of the cycles is limited to the constraint but it allows the existence of crossing cycles.

1.5 Stochastically Stable States

Since there are multiple strict equilibria, we are interested in which kinds of network architectures are more likely to be selected in the long run. Previous literature (see e.g. Jackson & Watts 2002*a*, Feri 2007 and Cui et al. 2013) has established that the best response dynamics with random noise may select a subset of them. For this reason, we consider a best response learning dynamics due to Kandori et al. (1993) and Young (1993). An agent is randomly selected to renew her strategy at each period t in discrete time, i.e. $t = 0, 1, 2, \cdots$. The selected agent chooses a best response to the strategy profile of other agents at the previous period $t-1$, i.e.

$$g_i(t) \in arg \max_{g_i \in \mathscr{G}_i} U_i(g_i, g_{-i}(t-1))$$

where $g_i(t)$ refers to agent i's strategy at period t, and $g_{-i}(t-1)$ means the strategy profile of other agents except i at period $t-1$. In the case that there are multiple best responses, agents randomly choose one with equal probability.

Given the equation above, the new network configuration in period t only depends on the network in the previous period $t-1$. Technically, this revision process can be defined as a Markov chain on the strategy space $\mathscr{G} \equiv \mathscr{G}_1 \times \mathscr{G}_2 \times \cdots \mathscr{G}_n$. Each network g is a state in this space. An absorbing set is defined as a minimum subset of $\mathscr{G}$ with the property that the dynamics can never leave it once reached. We denote by $G^{\star\star}$ an absorbing set and $\mathscr{G}^{\star\star}$ denotes the set of all absorbing sets. First, we provide an important property of $\mathscr{G}^{\star\star}$.

Proposition 1.5.1. *All networks in $\mathscr{C}\mathscr{P}_{\bar{\ell}}$ are absorbing. Each of them forms a singleton absorbing set $G^{\star\star} \in \mathscr{G}^{\star\star}$.*

Proposition 1.5.1 implies that all strict Nash core-periphery networks are absorbing. The result derives from the fact that in a strict Nash equilibrium, no agent is indifferent between multiple strategies. Thus, all agents are playing their unique best response. As a consequence, they will remain at their current strategy whenever they receive the opportunity to revise. Thus, the revision dynamics can never leave a strict Nash equilibrium without any mistakes, implying that all networks in $\mathscr{C}\mathscr{P}_{\bar{\ell}}$ are singleton absorbing sets.

Note that Proposition 1.5.1 does not provide a full characterization of all absorbing sets. That is, other absorbing sets may exist, such as strict Nash networks different from core-periphery networks, and a collection of Nash networks among which the dynamics could circulate but never leave.[13] Let a denote the number of all absorbing sets given the number of the population n.

Now, we proceed to characterize the selection among multiple absorbing sets, by adopting the standard techniques developed by Kandori et al. (1993) and Young (1993). Consider that agents might fail to choose the optimal strategy during the revision process, which we call a mistake. The probability of agents making mistakes is positive, denoted by $\varepsilon > 0$. We assume that the agent who makes a mistake chooses randomly among all strategies. Given the positive ε, the revision process is ergodic and aperiodic. The Markov process is therefore irreducible and aperiodic, which means it has a unique stationary distribution $\mu(\varepsilon)$. As ε goes to zero, $\mu(\varepsilon)$ converges to a limited distribution $\mu^\star$, i.e. $\lim_{\varepsilon \to 0} \mu(\varepsilon) = \mu^\star$. A network g is called stochastically stable if $\mu^\star(g) > 0$. The set of stochastically stable states is defined as $\mathscr{G}^{\star\star\star} \equiv \{g \in \mathscr{G} : \mu^\star(g) > 0\}$.

The following algorithm introduced by Freidlin & Wentzell (1998) and Foster & Young (1990) is used to identify the set of stochastically stable states. Consider two states in different absorbing sets, $g \in G_i^{\star\star}$ and $g' \in G_j^{\star\star}$. Denote by $r(g,g') > 0$ the resistance of transition from g to g', which is the minimum number of mistakes required for this transition. Further, a $G_i^{\star\star}$-tree is defined as a spanning tree rooted in $G_i^{\star\star}$, such that there is a unique path from each other absorbing set to $G_i^{\star\star}$. Denote by τ_i a $G_i^{\star\star}$-tree and T_i denotes the set of all τ_i. The resistance

[13] Despite our best efforts, we have not identified such strict Nash equilibria or such cycles. Therefore. we have not been able to rule out the existence of such absorbing sets.

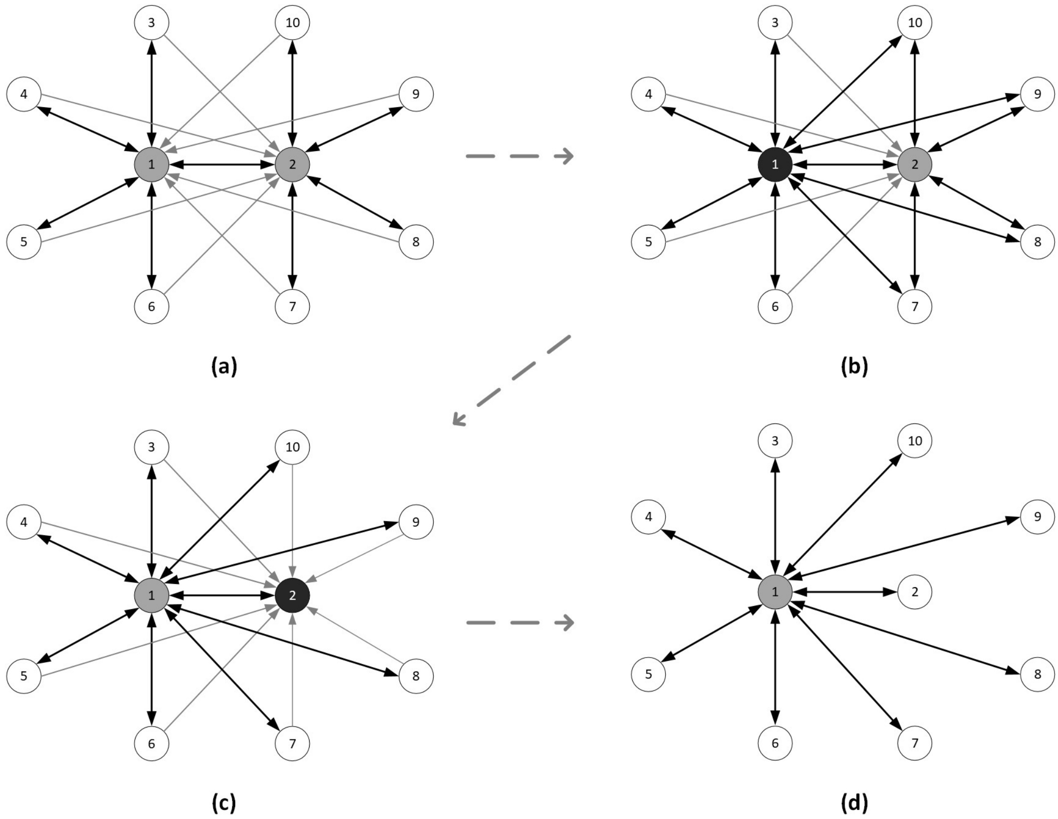


Figure 1.5. *The transition from a CP_2 network to a CP_1 star. The grey circles are core agents and the white circles are periphery agents. The black circles are the agents who are revising their strategies.*

of a $G_i^{\star\star}$-tree is defined as the sum of resistances of its edges, i.e. $r(\tau_i) = \sum_{(g,g')\in\tau_i} r(g',g)$. The stochastic potential of the absorbing set $G_i^{\star\star}$ is defined as the minimum resistance among all τ_i, i.e. $\gamma(G_i^{\star\star}) = arg \min_{\tau_i \in T_i} r(\tau_i)$. Finally, a state in the absorbing set $G_i^{\star\star}$ is stochastically stable if $G_i^{\star\star}$ has the minimum stochastic potential, i.e. $\gamma(G_i^{\star\star}) = \min_{G_j^{\star\star}\in\mathscr{G}^{\star\star}} \gamma(G_j^{\star\star})$. With this technique, we are able to identify a class of stochastically stable states by analysing the relative robustness of absorbing states to mistakes.

Before showing our main results, we exhibit two examples of transitions between the two core-periphery networks, which will play a key role in our analysis.

Example 1. Consider the CP_2 network in Figure 1.5 (a). Assume that agent 1 makes a mistake and forms additional links to agent 2's periphery agents as shown in Figure 1.5 (b). Following this, given a revision opportunity, agent 2 will delete the links to agents 7, 8, 9 and 10 as shown in Figure 1.5 (c). In the next steps, any periphery agent receiving revision opportunities will consequently

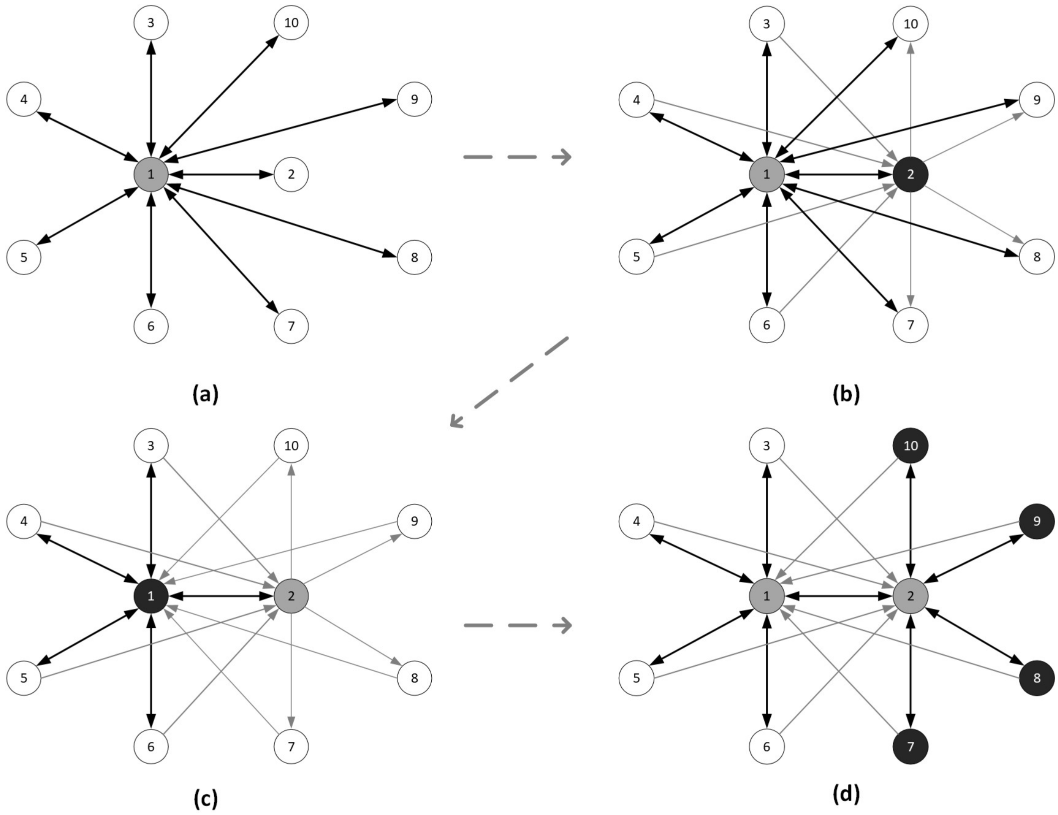


Figure 1.6. *The transition from a CP_1 star to a CP_2 network.*

delete the link to agent 2 as the network, see Figure 1.5 (d). Thus, with one mistake, we have reached another absorbing state which is a CP_1 network.

Example 2. Now consider a CP_1 network in Figure 1.6 (a). Assume that agent 2 makes a mistake and forms additional links to agents 7, 8, 9 and 10 as illustrated by Figure 1.6 (b). In the next step, agent 1 receives the opportunity to revise and find it optimal to delete the links to agents 7, 8, 9, and 10 as shown in Figure 1.6 (c). Following this, agents 7, 8, 9 and 10 receiving opportunities to revise will form a link to agent 2, see Figure 1.6 (d). Thus, with one mistake the dynamics has reached a CP_2 network where agents 1 and 2 are the two core agents.

To predict which kinds of network configurations are stochastically stable, we construct a sequence of absorbing sets where the transition between any two adjacent sets requires one mistake such as the two examples illustrated above. To do so, we first establish that the transition from any absorbing set to a CP_1 network requires one mistake. Second, we show that the transition from a

CP_ℓ network to a $CP_{\ell+1}$ network with ℓ common core agents is able at the cost of one mistake. By doing so, we argue that the stochastical potential of each absorbing set is equal to $N-1$, which is the minimum stochastic potential. Thus, all absorbing sets characterized in Proposition 1.5.1 are stochastically stable. The following proposition establishes this result.

Proposition 1.5.2. $\mathscr{C}\mathscr{P}_{\bar{\ell}} \subset \mathscr{G}^{\star\star\star}$.

The result that core-periphery networks are stochastically stable is significantly different to Feri (2007) who predicts the periphery sponsored star for the two-way flow model with decay and Cui et al. (2013) who predicts either the wheel or the empty network for the one-way flow model with decay. Our model implies that there are multiple networks that are stochastically stable.

In the last step, we are interested in the welfare properties of stochastically stable states. Welfare of a network g is defined as the sum of payoffs of individuals, i.e. $W(g) = \sum_{i \in N} U_i(g)$. A network g is said to be efficient if and only if $W(g) \geq W(g')$ for all $g' \in \mathscr{G}$. We derive one important property that efficient networks have to fulfil. The lemma proposes that the maximum number of links in any efficient network has to be at most $2 \cdot (n-1)$.

Lemma 1.5.1. *A network g is not efficient if the number of links in g exceeds $2 \cdot (n-1)$.*

This lemma follows from the following observation. Note that the welfare of a CP_1 network is given as

$$W(CP_1) = \underbrace{1 + (n-1) - c \cdot (n-1)}_{\text{payoff of agent in core}} + (n-1) \cdot \underbrace{[1 + (n-1) - c]}_{\text{payoff of agent in periphery}}$$
$$= n^2 - 2c \cdot (n-1)$$

Without considering linking costs, n^2 is the highest benefit that can be yielded by a network, where each agent receives benefits from all agents. Thus, if the number of links in a network g

is larger than $2 \cdot (n-1)$, the linking costs in g are larger than the linking cost in a CP_1 network, implying that g is not efficient since $W(g) < W(CP_1)$.

Lemma 1.5.1 implies that the set of stochastically stable states contains networks that are not efficient. To see this, consider a CP_ℓ network in $\mathscr{C}\mathscr{P}_{\bar{\ell}}$. By the definition of core-periphery networks, the number of links between core agents is $2 \cdot (\ell - 1)$. The number of links from core agents to periphery agents is $n - \ell$ as each periphery agent is linked by only one core agent. Further, the number of links from periphery agents to core agents is $(n-\ell) \cdot \ell$ as each periphery agent links up to all core agents. We thus have that the number of links in a CP_ℓ network is

$$f(\ell, n) = 2 \cdot (\ell - 1) + n - \ell + (n - \ell) \cdot \ell = -\ell^2 + (n+1) \cdot \ell + n - 2.$$

This function $f(\ell, n)$ is increasing provided that $\ell < \frac{n+1}{2}$. Note that any CP_ℓ network that is a strict Nash equilibrium and further stochastically stable state requires that each core agent forms links to at least three periphery agents. Thus given the number of agents n, the number of core agents fulfils that $\ell \leq \lfloor \frac{n}{4} \rfloor < \frac{n+1}{2}$. Hence, the number of links in any $g \in \mathscr{C}\mathscr{P}_{\bar{\ell}}$ increases with the number of core agents. Thus we have that $W(CP_1) > W(CP_2) > \cdots > W(CP_{\bar{\ell}})$.[14] Therefore, the set of stochastically stable states contains network configurations that are not efficient.

The implication is that when benefits are global as in Bala & Goyal (2000), the wheel network is efficient since the wheel exhibits a structure where every agent can use one link to get access to all other agents. In contrast with local benefits, to observe other agents who are located two steps away, forming additional links is necessary. This results in an increase in the number of links in the network, causing a reduction in welfare. Therefore, network structures may arise in the long run, which are dominated by others in terms of welfare.

[14]Whilst we have been able to use numerical calculations to show that CP_1 networks are efficient for $n = 5$ and 6, we have not been able to provide a general result showing that this for arbitrary n.

1.6 Conclusion

In this paper, we explored a non-cooperative framework for modelling the process of network formation, where agents unilaterally form costly links. The payoffs accrue to agents who initiate the links. We focus on investigating how constraints on information transmission affect stable networks in the long run.

In contrast to the conventional results of either the wheel or the periphery-sponsored star being the unique strict Nash network or stochastically stable, we reveal that core-periphery networks are strict Nash equilibria when agents can only receive information from their neighbours and the neighbours of their neighbours. This finding sheds light on the diverse range of equilibrium structures that can emerge in the context of network formation.

Additionally, because of the multiplicity of strict Nash networks, we study the selection between multiple equilibria in a perturbed best response learning dynamics, to find which kind of networks are stochastically stable. We show that the set of stochastically stable states encompasses multiple network configurations that exhibit the structure of the core-periphery networks. Surprisingly, our analysis reveals that the set of stochastically stable states includes network configurations that are inefficient from a welfare perspective.

Chapter 2

Network Formation with Local Benefits: Evidence from Simulation

2.1 Introduction

The significance of social networks has been emphasized in numerous studies within both economic and sociological literature. To address the question of which kinds of network configurations will be formed, several models have been developed to explore the theoretical foundations of how information networks emerge.[1] Additionally, many experimental analyses based on these theoretical models have been conducted, providing us with intuitive insights into the process of network formation. [2]

In Chapter 1, we have shown that constraints on the distance of information transmission significantly influence the formation of networks. Specifically, when agents can receive information only from their neighbours and neighbours of neighbours, core-periphery networks often emerge as Nash equilibria. The multiplicity of core-periphery network structures then raises a critical question of the stochastically stable states—those that persist in the long run within a model of noisy

[1] See e.g. Jackson & Wolinsky (1996), Jackson & Watts (2002*a*) for the cooperative model of network formation, and see e,g, Bala & Goyal (2000), Feri (2007) and Hojman & Szeidl (2008) for the non-cooperative model.

[2] See e.g. Callander & Plott (2005), Berninghaus et al. (2007) and Falk & Kosfeld (2012).

best-response learning dynamics. Results in Chapter 1 indicate that core-periphery networks are stochastically stable as the probability of mistakes approaches zero. However, the theoretical work offers only partial results of the set of stochastically stable states, leaving a full characterization unexplored.

The main contribution of this work is the simulation of the evolution of network formation. By simulating a model with constraints on information transmission, we analyze which types of network configurations emerge in the long run by varying some important parameters in the model. Our main aim is to confirm: first, whether core-periphery networks will emerge as predicted by the theoretical model; second, whether core-periphery networks are uniquely selected (which has not been proven by the analytical results).

There are two main findings in the paper. First, consistent with the analytical results, we provide that core-periphery networks emerge in both simulated unperturbed dynamics (where agents are assumed to never make mistakes during the revision process) and perturbed dynamics (where agents are allowed to make mistakes). Second, and most importantly, we also find that with sufficient large group sizes, the simulated processes consistently converge to core-periphery networks. This finding of the uniqueness of core-periphery networks complements the analytical results, affirming that core-periphery networks are uniquely absorbing and stochastically stable.

The paper is organized as follows. In Section 2.2 we present the related literature. Section 2.3 describes the details of the network formation game and the noisy best-response learning dynamics. In Section 2.4 we present in detail the simulation setups and results. Section 2.5 concludes.

2.2 Literature review

The present work adds to the theoretical literature on the evolution of social networks. Due to Jackson & Wolinsky (1996)'s concept of pairwise stability, Jackson & Watts (2002*a*) study the evolution of network formation and show that pairwise stable networks are stochastically stable. Feri (2007) studies an evolutionary version of Bala & Goyal (2000)'s two-way flow model with decay, which predicts the periphery-sponsored star in the long run. Further, based on Bala & Goyal (2000)'s one-way flow model with decay, Cui et al. (2013) consider a noisy best response learning dynamics where agents can update their links in discrete time and may mistakes. They find that the wheel is uniquely stochastically stable for some parameters. In our previous theoretical work, we explore Bala & Goyal (2000)'s one-way flow model without decay with a bound $D = 2$, where agents can only receive information from others within distance two.[3] We show that core-periphery networks are stochastically stable, highlighting the potential for the emergence of networks other than simple architectures, e.g. the wheel and the star. However, the analytical results have not been able to rule out other kinds of networks being stochastically stable. This paper contributes to these results in two respects. First, we simulate the theoretical model to gain some important insights into the network configurations selected in the long run. Second, our simulation results illustrate that core-periphery networks are unique stochastically stable states.

Our paper also contributes to the literature on empirical studies of social networks. Callander & Plott (2005) design a laboratory experiment to investigate the evolution of networks. Their results find that concepts of equilibria are the principle behind the convergence of network dynamics. Berninghaus et al. (2007)'s experimental results show that periphery-sponsored stars are the unique strict Nash equilibria, which is in line with the analytical results in Bala & Goyal (2000). In Goeree et al. (2009)' experiment, they extend Bala & Goyal (2000)'s two-way flow model with decay by considering heterogeneous agents with lower costs or higher benefits. They show that star networks are not formed with identical agents, while stars frequently occur with heterogeneous agents. Further, Falk & Kosfeld (2012) present an experiment based on Bala & Goyal (2000).

[3] See also in Jackson & Wolinsky (1996). They study the effect of such bound D in the cooperative network formation model where connection requires mutual consent.

Their results only support the prediction of the one-way flow model in Bala & Goyal (2000).

The contributions of the present paper are mainly in two directions. First, instead of designing a laboratory experiment, this paper uses a simulation method to illustrate the evolution of network formation. As the convergence of the evolutionary dynamics of networks usually takes a significantly long time to occur, the disadvantage of laboratory experiments is that there are always limits on the duration, which may not be long enough for convergence. However, we are able to set the evolutionary dynamics to last as long as possible till the convergence is observed by using simulation. Fosco & Mengel (2011) also present a simulation of the coevolution of networks and Prisoner's Dilemma games, where the results strongly support the model's prediction of core-periphery networks.

Second, the present work is based on a model where agents can only receive payoffs from others within distance two, which extends to Bala & Goyal (2000)'s one-way flow model without decay. Our results show that the simulated processes converge to core-periphery networks other than simple architectures, e.g. the wheel and the star. This supports the analytical results that core-periphery networks are stochastically stable. Further, this also complements that core-periphery networks are unique, which has not been proven by analytical methods.

2.3 Model

2.3.1 Network Formation Game

We consider a one-way flow model of network formations.[4] There is a population of n agents, denoted by $N = \{1,2,\cdots,n\}$ with $n \geq 4$. Each agent $i \in N$ decides the set of agents to whom she forms links. A strategy used by agent i is given by a N-tuple $g_i = (g_{i1}, g_{i2}, \cdots, g_{in})$ where $g_{ij} \in \{0,1\}$ is agent i's linking decision to agent j. We say i links up to j if $g_{ij} = 1$; otherwise,

[4] We follow the notations introduced in Jackson & Wolinsky (1996) and Bala & Goyal (2000).

$g_{ij} = 0$. We assume that agents cannot link to themselves, i.e. $g_{ii} = 0, \forall i \in N$. Further, let $\mathscr{G}_i$ be the set of all possible link strategies that agent i can choose. One-way flow model implies that agent i's linking decision to j is independent with j's decision to i, i.e. g_{ji} and g_{ij} are not necessarily equal. A network $g = (g_i)_{i \in N} \in \mathscr{G}$ is the strategy profile of all agents, where $\mathscr{G} = \prod_{i \in N} \mathscr{G}_i$ is the set of all networks.

We say there exists a path from agent j to i if either $g_{ij} = 1$ or there is a set of agents $\{k_1, k_2, \cdots, k_m\}$, such that $g_{ik_1} = g_{k_1 k_2} = \cdots = g_{k_m j} = 1$. The distance from j to i, denoted by $d(i, j; g)$, is the number of links of the shortest path from agent j to i. Further, we define agent i's d-neighbourhood as the set of agents who are at distance d to i, denoted by $N_i^d(g) = \{j \in N : d(i, j; g) = d\}$, with $n_i^d(g) = |N_i^d(g)|$ the number of agent i's d-neighbours.

We now define the payoffs in our network formation game. As in Bala & Goyal (2000), each agent receives information from others by forming costly links and benefits from doing so. Without loss of generality, the value of information that each agent has is homogeneously normalized to one. The cost of each link is $c > 0$. We assume that the cost is only incurred to the party who initiates the link. We deviate from Bala & Goyal (2000) by studying the case where the distance that information can travel is constrained. As motivated in the introduction, we assume that agents can only observe their neighbours and the neighbours of neighbours, i.e. we focus on the case where the constraint $D = 2$.[5]

Thus, an agent's payoff is calculated as the sum of benefits derived from the information she receives from others, minus the total cost incurred from forming links. More formally, given a network $g \in \mathscr{G}$, agent i's payoff is given by

$$U_i(g_i, g_{-i}) = 1 + n_i^1(g) + n_i^2(g) - c \cdot \sum_{j \in N} g_{ij} \tag{2.1}$$

[5]This is a special case of the communication threshold in Hojman & Szeidl (2008), which in contrast to our model focuses on two-way flow. Limits in communication are also observed in the one-way flow case, which their model cannot characterize.

where the constant captures the value of information of agent i self.

2.3.2 Learning Dynamics

We consider a noisy best response learning dynamics due to Kandori et al. (1993) and Young (1993). An agent is randomly selected to renew her strategy at each period t in discrete time, i.e. $t = 0, 1, 2, \cdots$. The selected agent chooses a best response to the strategy profile of other agents at the previous period $t-1$, i.e.

$$g_i(t) \in arg \max_{g_i \in \mathscr{G}_i} U_i(g_i, g_{-i}(t-1))$$

where $g_i(t)$ refers to agent i's strategy at period t, and $g_{-i}(t-1)$ means the strategy profile of other agents except i at period $t-1$. In the case that there are multiple best responses, agents randomly choose one with equal probability. Agents might fail to choose the optimal strategy during the revision process, which we call a mistake. The probability of agents making mistakes is positive, denoted by $\varepsilon > 0$. We assume the agent who makes a mistake chooses randomly among all strategies.

2.4 Simulation

In this section, we illustrate and complement the analytical results from previous literature through simulations. Two key features of the model are particularly important: the group size n and the probability of making mistakes ε. According to the analytical results, the probability of making mistakes differentiates between unperturbed and perturbed learning dynamics. The group size significantly influences the network configurations within the set of stochastically stable states. Therefore, we conduct twelve simulations varying these two factors. Specifically, we examine

Table 2.1. *Regression between the relative frequencies and the independent variables*

	Dependent variable:	
	Core-periphery Networks	Other Nash Networks
	(1)	(2)
Prob. Mistakes	-21.257^{**}	-0.607
	(8.221)	(2.149)
Group Size	0.024^{*}	-0.035^{***}
	(0.014)	(0.003)
Period	0.040	0.005
	(0.031)	(0.008)
Cons	-0.096	0.340^{***}
	(0.183)	(0.044)
Observations	51	54
R^2	0.186	0.673
Adjusted R^2	0.134	0.654
Residual Std. Error	0.264 (df = 47)	0.071 (df = 50)
F Statistic	3.586^{**} (df = 3; 47)	34.350^{***} (df = 3; 50)
Note:		*p<0.1; **p<0.05; ***p<0.01

twelve set-ups involving four different group sizes: 5, 6, 8, and 10, and for each group size, we consider three different probabilities of making mistakes: 0, 10^{-3} and 10^{-2}. This approach allows us to illustrate the effect of varying population sizes on network configurations in both unperturbed and perturbed dynamics, as well as the impact of mistake probabilities on convergence within a given group size.

Moreover, in all simulations, we focus on scenarios where the linking cost is lower than each agent's information value, specifically $c = 0.5$. Additionally, each simulation is conducted over three different periods: 10^3, 10^4 and 10^6 periods, and each setup is repeated 100 times. We monitor the network structures across various time periods of the dynamics and assess the frequency of core-periphery networks and other types of Nash equilibrium networks.

The main variable we look at is the relative frequencies of core-periphery networks and other permissible Nash networks. Table 2.1 presents the regression analyses examining the relationship between the relative frequencies of core-periphery networks, other Nash networks, and various influencing variables. e.g. probability of mistakes, group size and period.

We find that variables significantly associated with the emergence of core-periphery networks include the probability of mistakes and group size. Specifically, the probability of mistakes has a negative impact on the relative frequency of core-periphery networks at the 1% significance level, indicating that core-periphery networks are more likely to emerge as the probability of mistakes approaches zero. Additionally, group size has a positive effect at the 10%significance level, suggesting that larger group sizes are associated with a significantly higher relative frequency of core-periphery networks. Conversely, group size exerts a significantly negative effect on the relative frequency of other Nash networks at the 1% significance level. Furthermore, our regressions show that the effects of time periods are insignificant for both core-periphery and other Nash networks.

Following this, we proceed to analyze the effects of the probability of mistakes, group size, and periods individually, offering our explanations for the observed results.

We present our first results of simulations of the unperturbed dynamics (i.e. the scenarios where the probability of making mistakes ε is 0), concerning different group sizes.

Result 2.4.1. *Core-periphery networks always emerge for all group sizes. With a large enough group size, the simulated unperturbed process converges to core-periphery networks with approximately 100 percent.*

Support for Result 2.4.1 is provided by Figure 2.1. Figure 2.1 displays the relative frequencies of core-periphery networks and other types of Nash equilibrium networks in scenarios where the probability of making mistakes is zero, i.e. within the unperturbed best response learning dynamics. First, we look at the results for group sizes $n = 5$ and $n = 6$. We observe a high occurrence of non-Nash networks and a relatively high frequency of other Nash networks. Precisely, at $t = 10^6$, the emergence of Nash networks is 35 percent for $n = 5$, with 8 percent core-periphery networks, whilst for $n = 6$, the frequency of Nash networks is 11 percent, with only 3 percent corresponding

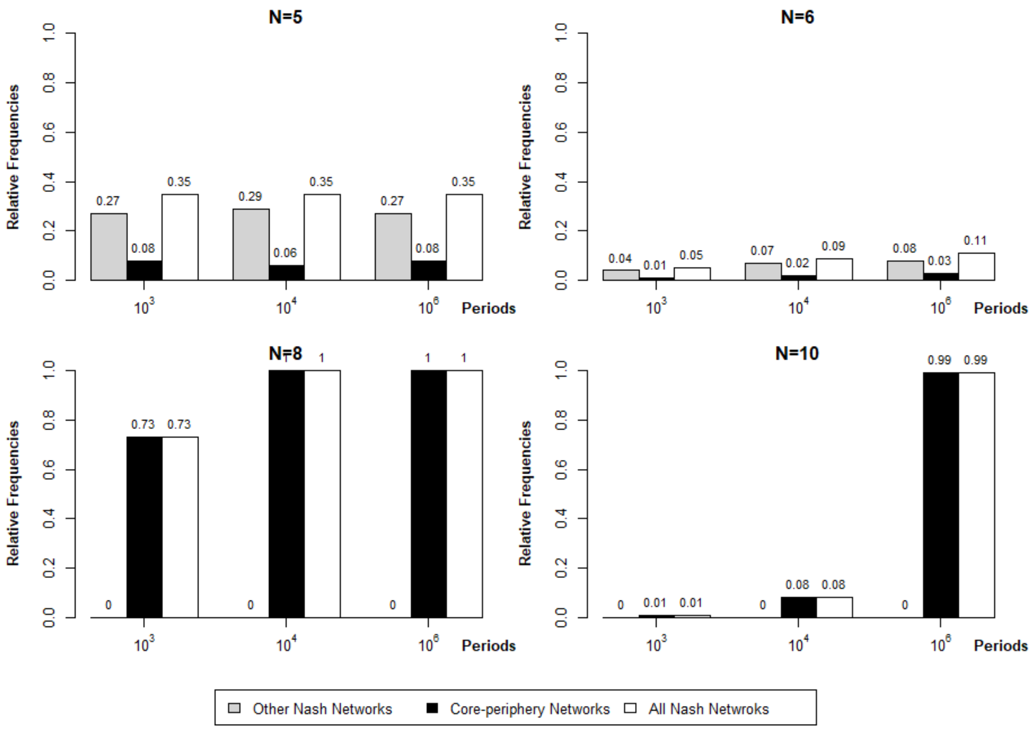


Figure 2.1. *Relative frequencies of core-periphery networks and other Nash networks in the unperturbed dynamics*

to core-periphery networks. In contrast to Nash networks, however the frequency is relatively low, we observe that core-periphery networks are the unique strict Nash networks.

One possible way to account for the low frequencies of core-periphery networks and other Nash networks is as follows. In smaller group sizes, there is a higher prevalence of Nash networks that are not strict Nash equilibria. Note that in a non-strict Nash network, agents may randomly choose among multiple best response strategies. Thus, a single agent switching to an alternative best response can cause the dynamics to reach toward a non-Nash network, as the transition in Figure 2.2. The higher prevalence of Nash networks prolongs the probability of such transitions, reducing the likelihood of the emergence of core-periphery networks within a given period t.

If we look at two graphs for group sizes $n = 8$ and $n = 10$ in Figure 2.1, the results are significantly different. The findings related to the convergence toward core-periphery networks are more compelling: the relative frequency of other Nash networks remains zero and the frequency of core-periphery networks approaches approximately 100 percent over time. Specifically, the

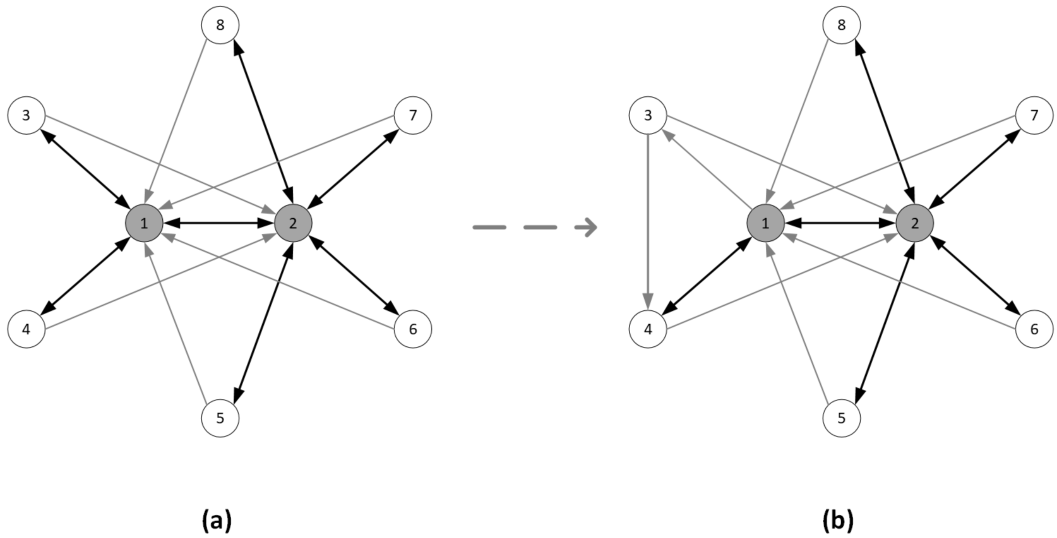


Figure 2.2. *Transition from a Nash network to a non-Nash network*

relative frequency of core-periphery networks for a group size of $n = 8$ is 73 percent at $t = 10^3$ and reaches 100 percent by $t = 10^4$. For a group size of $n = 10$, the frequency of core-periphery reaches 99 percent by $t = 10^6$. This reveals that with sufficiently large group sizes, the dynamics will inevitably converge to a core-periphery network, suggesting that core-periphery networks are uniquely absorbing states.

Our second result concerns the stochastical stability of networks, i.e. which kinds of networks that the dynamics coverage to if the probability of mistakes approaches zero.

Result 2.4.2. *The relative frequencies of core-periphery networks increase when the probability of mistakes approaches zero. With a large enough group size, the simulated stochastic process converges to core-periphery networks with approximately 100 percent.*

Support for Result 2.4.2 is provided by Figure 2.3. Figure 2.3 shows the relative frequencies of core-periphery networks and other Nash networks at period $t = 10^6$, across various group sizes and probabilities of mistakes.

First, note that stochastically stable states are the states in the support of the invariant distribution of the revision dynamics as introduced in the previous section. Figure 2.3 reveals that

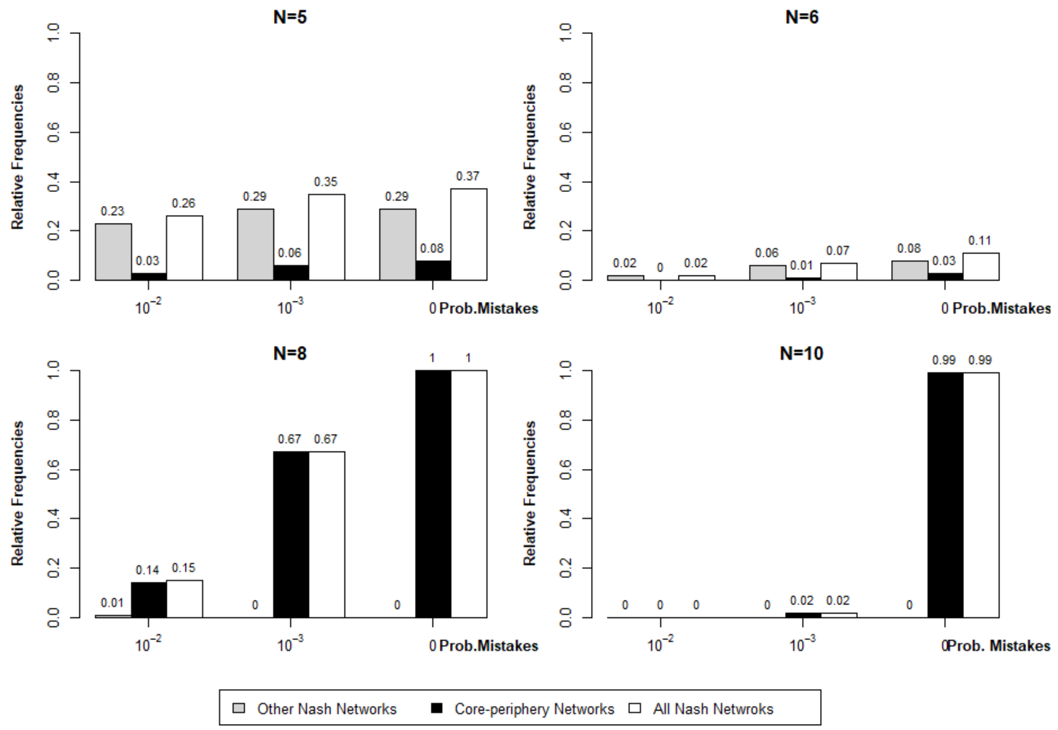


Figure 2.3. *Frequencies of core-periphery Networks and other Nash networks at* $t = 10^6$*, varying in probabilities of mistakes*

for group sizes $n = 5, 6, 8$ and 10, the set of stochastically stable states contains core-periphery networks. For instance, for group sizes $n = 5$ and $n = 6$, although the prevalence of core-periphery networks is relatively low, the frequency remains positive and increases as the probability of making mistakes approaches zero. As argued above, the low frequency of core-periphery networks may be attributed to the high prevalence of non-strict Nash networks. However, for these two smaller group sizes, the emergence of other network structures suggests that we cannot entirely rule out the possibility that other types of network configurations may also be stochastically stable.

Further, for sufficient large group sizes, such as $n = 8$ and $n = 10$, simulation results imply that core-periphery networks are the unique stochastically stable states. As shown in Figure 2.3, the relative frequency of core-periphery networks is 67 percent at $\varepsilon = 10^{-3}$ and reaches 100 percent at $\varepsilon = 0$ for $n = 8$. For $n = 10$, although the relative frequency of core-periphery networks remains low at $\varepsilon = 10^{-3}$, at just 2 percent. However, as the probability of mistakes approaches zero, this frequency increases to 99 percent. This provides empirical evidence supporting the uniqueness of the stochastical stability of core-periphery networks.

2.5 Conclusion

The present paper provides a simulation analysis of the evolution of non-cooperative network formation. Our simulation is based on a model extending to the one-way flow model of Bala & Goyal (2000), where agents can only receive information within a distance of two. The simulation results show that the prediction regarding absorbing sets and stochastically stable states of the model are consistent with the model. Core-periphery networks are observed across all different setups, indicating that they are not only absorbing but also stochastically stable. In addition, with sufficient large group sizes, the uniqueness of core-periphery networks is observed in the long run. Starting with any network, the noisy best-response learning dynamics will converge to core-periphery networks.

Chapter 3

The Effects of Heterogeneous Constraints on Social Coordination and Network Formation

3.1 Introduction

In various social and economic activities, people often benefit from adopting the same actions or adhering to some common standards (e.g. Latex vs. Microsoft, $C++$ vs. Python, Windows vs. MacOS, etc.). This can be characterized as coordination games, which have two pure Nash equilibria, i.e. payoff-dominant equilibrium and risk-dominant equilibrium (see in Harsanyi et al. 1988). Related literature points out that agents usually coordinate on the same action (see e.g. Kandori et al. 1993; Young 1993; Blume 1993, 1995; or Ellison 1993, 2000, etc.). However, some examples in our real world reveal that it is often the case people do not choose the same action as others, for example, both $C++$ and Python have positive market shares. Thus, it is important to know what drives people to choose different actions and which actions will be selected in the long run.

To solidify the idea, consider a group of students collaborating on a project. A student is better off if she forms a team with somebody using the same software, i.e. either $C++$ or Python. In addition to which software to use, her payoff from this joint project also depends on the choice of her teammates. Therefore, each student has to make two decisions: software and collaborators to maximize her payoff. This example gives rise to the co-evolutionary model of 2×2 coordination games and network formation (see e.g. Jackson & Watts 2002*b*, Goyal & Vega-Redondo 2005, Staudigl & Weidenholzer 2014). Moreover, although agents have the flexibility to choose whom they interact with, the number of interactions they can maintain is often limited due to constraints on socialising, through e.g. decreasing marginal payoff from socialising or increasing marginal cost of interaction (see e.g. Jackson & Watts 2002*b*, Staudigl & Weidenholzer 2014, Cui & Weidenholzer 2021, or Cui & Shi 2022). Previous work assumes that constraints on interactions are homogeneous for all agents. However, there is empirical evidence in real-life social networks like Twitter revealing that the number of links that agents can support is different (see e.g. Albert et al. 1999 and Kwak et al. 2010). Thus, it seems more realistic to assume that such constraints on interactions are heterogeneous across agents. Therefore, in this paper, we set up a co-evolutionary model of coordination game and network formation, to study how such heterogeneous constraints affect agents' action choices and linking decisions, and further, which action in the coordination game is selected in the long run.[1]

To be more specific, we follow the model of Staudigl & Weidenholzer (2014) but assume that agents face heterogeneous linking constraints. More precisely, we consider a 2×2 coordination game played among a finite number of agents. Every agent makes two choices simultaneously: the action played in the coordination game and a set of agents she plays the game with. Forming links is costly. The payoff of each agent is the sum of payoffs from the coordination game played with each agent she links to, minus the total cost of forming links. We assume that there are two groups of agents who face two different linking constraints: high and low. The size of the *low-constraint group* (i.e. the number of agents in the group) is assumed to be larger than the size of the

[1] See also Zeng (2019) for some static properties of Nash equilibria with heterogeneous constraints in 2×2 coordination games and Lu & Shi (2023) for a dynamics analysis of size-dependent minimum effort game.

high-constraint group.[2] Linking costs are assumed to be low enough so that in principle any link is beneficial. Thus, it is optimal for each agent to form as many links as her constraint. However, the optimal actions in the coordination game may be different for low- and high-constraint groups.

In fact, our model shows that *polymorphic states* (where agents with different constraints play different actions) may be Nash equilibria for some given game parameters. Specifically, the profiles where agents in the low-constraint group play the payoff-dominant action and agents in the high-constraint group play the risk-dominant action can be Nash equilibria if the low and high constraints are significantly different. To see this point, consider a polymorphic state as described above. Then it may be the case that agents in the low-constraint group focus all of their links on agents playing payoff-dominant action and thus get the highest possible payoff, which is the mechanism that drives the results in Staudigl & Weidenholzer (2014). However, agents in the high-constraint group may lack sufficient potential interaction partners with the payoff-dominant action. Instead, they face a distribution of mixed actions involving both risk-dominant and payoff-dominant actions, such that playing the risk-dominant action may yield a higher expected payoff. This is similar to the mechanism in Goyal & Vega-Redondo (2005) where the complete network is formed and the risk-dominant action does well. Thus, such a polymorphic state can be Nash equilibria for some given parameters. However, the other kind of polymorphic state where agents in the low-constraint group play the risk-dominant action and agents in the high-constraint group play the payoff-dominant action, can never be Nash equilibrium. The reason is that if agents with the higher constraint find there are sufficient potential interaction partners with the payoff-dominant action, it has to be also the case for agents with the lower constraint. Thus, it is always profitable for agents in the low-constraint group to deviate from the risk-dominant action. In addition, in line with Goyal & Vega-Redondo (2005) and Staudigl & Weidenholzer (2014), *monomorphic states*

[2] This assumption has support from the empirical literature (see e.g. Goyal et al. 2006 and Jackson & Rogers 2007) who find that a minority of agents support a large number of links.

(where all agents play the same action) are always Nash equilibria since when all other agents play the same action, an agent will always be better off if she chooses the same action as others.

Further, given the multiplicity of Nash equilibria, we study the co-evolution of the above static game in discrete time, to predict which kind of profiles will be selected in the long run. We assume that at each period agents may receive opportunities to revise their strategies based on a noisy myopic best-response rule. That is, agents choose actions and links that optimize their payoffs against the distribution of actions in the previous period. There is however a probability that agents make mistakes and choose a random strategy. We follow the standard methodology developed by Kandori et al. (1993), Young (1993) and Freidlin & Wentzell (1998) to identify the stochastically stable states as the long-run prediction, which are the states in the support of a unique invariant distribution when the probability of making mistakes approaches zero. Naturally, states that are more robust to mistakes are stochastically stable.

In the first step, we identify the absorbing sets, which are the sets of states once reached can never be left without mistakes. The literature considering homogeneous constraints (see e.g. Staudigl & Weidenholzer 2014 and Cui & Shi 2022) shows that the absorbing sets consist of only monomorphic states. In contrast, in our model, polymorphic states can also be absorbing if the two constraints on links are significantly different. Following this, we characterize the set of stochastically stable states by comparing the robustness of absorbing states to mistakes. In cases where the low and high constraints are close, the set of stochastically stable states contains only monomorphic states, which is in line with the model in Goyal & Vega-Redondo (2005) and Staudigl & Weidenholzer (2014). More precisely, the payoff-dominant action emerges in the long run if both constraints are small. In contrast, the risk-dominant action will be selected if both constraints are high. Surprisingly, we also find that if the low and the high constraints are significantly different from one another, the polymorphic states where agents in the low-constraint group play the payoff-dominant action and agents in the high-constraint group play the risk-dominant action, can also be stochastically stable.

The structure of this paper is as follows. In section 3.2, we review the related literature. Section 3.3 outlines our model. In section 3.4, we characterize the Nash equilibria of the static game. Section 3.5 presents our results on the set of stochastically stable states for different levels of linking constraints. Section 3.6 concludes. Formal proofs of our results are relegated to the Appendix A.2.

3.2 Literature review

This paper adds to the literature on the co-evolution of coordination and network formation games (see e.g. Jackson & Watts 2002*b*, Goyal & Vega-Redondo 2005, Staudigl & Weidenholzer 2014.). Jackson & Watts (2002*b*) consider a model where the network is bilaterally formed based on the concept of pairwise stability provided by Jackson & Wolinsky (1996) and point out that whether risk-dominant or payoff-dominant conventions are stochastically stable depends on the relationships between payoffs in the coordination games and linking costs. Goyal & Vega-Redondo (2005) consider the case where agents non-cooperatively form unilateral links and find that which convention will emerge also depends on the relative level of linking costs to payoffs. As the adjustment process in Goyal & Vega-Redondo (2005) is different to the one used in Jackson & Watts (2002*b*), the precise nature of the relationship between payoffs and linking costs differs too.[3] Goyal & Vega-Redondo (2005) show that agents will coordinate on the risk-dominant action if the linking cost is low, and they will select the payoff-dominant action if the linking cost is high. Further, Staudigl & Weidenholzer (2014) extend Goyal & Vega-Redondo's model by considering homogeneous constraints on the number of links agents can support. Their study shows that if the constraint is low

[3] As argued by Goyal & Vega-Redondo (2005), the main source of this discrepancy lies in the fact that in Goyal & Vega-Redondo (2005) actions and links are chosen simultaneously but follow independent process in Jackson & Watts (2002*b*).

compared to the population, the payoff-dominant action is selected, while the risk-dominant action will be selected in the long run if the constraint is high. Cui & Weidenholzer (2021) consider the effect of lock-in on the selection of conventions based on Staudigl & Weidenholzer (2014)'s model, where agents receive payoffs not only from links they form but also from the links they receive. They show that agents using different actions sometimes can also be stochastically stable. Our model differs from these studies in that agents face heterogeneous constraints on the number of links. Agents are distinguished by two different levels of linking constraints: high and low. We find that such heterogeneous linking constraints will lead to the co-existence of both risk-dominant and payoff-dominant actions.

The most closely related literature to the present work is the paper by Lu & Shi (2023). They also study a co-evolutionary model with heterogeneous constraints on links, featuring a minimum-effort game. They find that all agents will choose high effort levels if everybody faces low constraints, while low effort levels will be chosen if constraints are high for everybody. The coexistence of different effort levels happens if constraints are significantly different. We remark that at first sight the mechanism and results are similar. However, there are some important differences between their model and ours. While they focus on the size-dependent minimum-effort games, we study 2×2 coordination games. In minimum-effort games, agents always want to match the effort level of the lowest of their interaction partners. This implies that agents choosing the high effort level can never interact with agents choosing lower effort levels. In contrast, agents with payoff-dominant actions may interact with those with risk-dominant actions in equilibria of our model. Furthermore, our results emphasise that the coexistence is not driven by the particular best response structures of minimum-effort games, where it is always best to keep the effort level in line with the weakest link but carry over to games where the best response to a mixed profile depends on the exact distribution of actions. In addition, Bilancini & Boncinelli (2018) also study heterogeneous agents and build a model with two different types of agents, where interactions between different types result in additional costs. They show that when the costs of interactions with different types are high, one type will play the risk-dominant action and the other type will play the payoff-dominant action.

In addition to the literature on coordination and network formation games, there is also a strand of literature where agents can determine their interaction partners by moving among a set of locations or islands (see e.g. Oechssler 1997, Dieckmann 1999, Anwar et al. 2002, Bhaskar & Vega-Redondo 2004, and Pin et al. 2017). when there are restrictions on the mobility between locations, or constraints on the capacity of each location, the co-existence of conventions might be observed. However, the co-existence of conventions depends on the limited interaction between locations. In contrast, in our model agents have the flexibility to interact with anybody.

3.3 Model

We consider a 2×2 coordination game played among the population of n agents, denoted by $N = \{1, 2, \cdots, n\}$ ($n \geq 3$). Each agent i can choose an action a_i from the action set $\mathscr{A} = \{A, B\}$. The payoff of agent i is given by $u(a_i, a_j)$ when she plays this coordination game against agent j. An agent who chooses action A in the coordination game is called an A-agent. Similarly, B-agents are those who play action B. The payoff matrix of this coordination game is given in the following table.

	A	B
A	$(a,\ a)$	$(c,\ d)$
B	$(d,\ c)$	$(b,\ b)$

We assume that $b > c$ and $a > d$, so that strategy (A,A) and (B,B) are two pure strategy Nash-Equilibria. Further, we assume $b > a$ so that (B,B) is the payoff-dominant equilibrium that yields the highest payoff. Moreover, we assume that $a + c > b + d$ so that (A,A) is risk-dominant equilibrium according to Harsanyi et al. (1988), meaning that A is the best response against an agent who plays both actions with equal probability $(\frac{1}{2}, \frac{1}{2})$. Given all those assumptions, we can simply have $c > d$. Further, we assume $a > c$ such that A-agents prefer playing against A-agents over playing against B-agents. Combining all assumptions together we have the following order of payoffs $b > a > c > d$.

Apart from their actions in the coordination game, agents may also determine the set of agents that they link to. We denote by g_{ij} the link decision to agent j made by agent i, where $g_{ij} = 1$ denotes that agent i forms a link to agent j and otherwise $g_{ij} = 0$. We consider the case where links are unilaterally formed, i.e. agent i decides on the link g_{ij} and agent j does not have a say in this link.[4] We assume that agents cannot link to themselves, i.e. $g_{ii} = 0$. Agent i's linking strategy g_i can be defined as a n-dimensional vector, i.e. $g_i = (g_{i1}, g_{i2}, \cdots, g_{in}) \in \mathscr{G}_i = \{0,1\}^n$. The out-degree of agent i is denoted by $d_i^{out} = \sum_j g_{ij}$, i.e. the number of links that agent i forms. The network formed by all agents is denoted by $g = (g_i)_{i \in N}$. Agent i's pure strategy s_i includes her action choice $a_i \in \mathscr{A}$ and linking strategy $g_i \in \mathscr{G}_i$, i.e. $s_i = (a_i, g_i) \in \mathscr{A} \times \mathscr{G}_i = \mathscr{S}_i$. Further, a strategy profile is denoted by $s = (s_1, s_2, \cdots, s_n) \in \Pi_{i \in N} \mathscr{S}_i = \mathscr{S}$.

Agents play the coordination game only with those agents they link to. We assume that the payoff generated by the coordination game only goes to the agent who forms the link. The agent who is passively linked gets zero from the coordination game played.[5] Forming links is costly and the cost is denoted by γ. The total payoff of an agent is given by the sum of payoffs she receives by playing the coordination game with each agent she links to, minus the total cost incurred by forming those links. Thus, given a strategy profile $s = (s_i)_{i \in N}$, the total payoff for agent i is given

[4] There is also some literature considering bilateral links (e.g. Jackson & Wolinsky 1996, Dutta & Mutuswami 1997 or Jackson & Watts 2002*b*) where forming a link requires the consents of both parties.

[5] Goyal & Vega-Redondo (2005) and Cui & Weidenholzer (2021) also consider a model where agents receive benefits from passive links as well.

by

$$U_i(s_i, s_{-i}) = \sum_{j=1}^{n} g_{ij} \cdot u_i(a_i, a_j) - \gamma \cdot d_i^{out}. \tag{3.1}$$

We focus on a case where the number of links that agent i can support is limited by k_i, i.e. $d_i^{out} \leq k_i$ as in Staudigl & Weidenholzer (2014).[6] In addition, we are interested in a scenario where linking constraints are heterogeneous among agents. Particularly, we consider a case where there are two types of agents, one with a lower constraint k^ℓ and the other with a higher constraint k^h, i.e. $k^\ell < k^h$. We define the set of agents with the lower constraint as *low-constraint group*, denoted by N_ℓ with $n_\ell = |N_\ell|$ the number of agents. Similarly, *high-constraint group* is the set of agents with the higher constraint, denoted by N_h with $n_h = |N_h|$. We focus on the case $n_\ell > n_h$ where the low-constraint group is larger than the high-constraint group.

Consider a scenario where the linking cost is low, i.e. $\gamma < d$, so that in principle an agent wants to form links to any other agents regardless of their actions. In this case, agents will form the maximum number of links they can support, i.e. k^ℓ or k^h. Thus, the total payoff function above is equivalent to

$$U_i(s_i, s_{-i}) = \sum_{j=1}^{n} g_{ij} \cdot u_i(a_i, a_j) - \gamma \cdot k_i \tag{3.2}$$

where $k_i \in \{k^\ell, k^h\}$ is the linking constraint for agent i.

3.4 Nash Equilibrium

In our characterization of Nash equilibrium, two types of states play an important role. Firstly, we denote by $\overrightarrow{XX}$ the set of monomorphic states, where $X \in \{A, B\}$. In a monomorphic state, every

[6] Alternatively. we can think of this assumption as links becoming prohibitively expensive as a certain threshold is crossed.

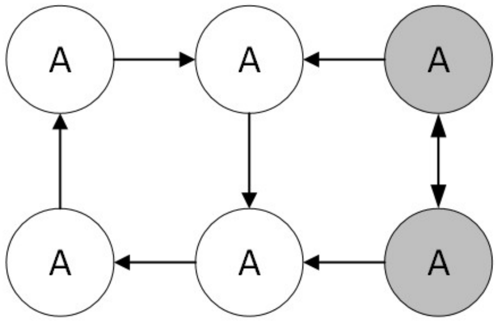


Figure 3.1. *A monomorphic state in* $\overrightarrow{AA}$ *when* $n_\ell = 4, n_h = 2, k^\ell = 1$ *and* $k^h = 2$.

agent chooses the same action and forms the maximum number of links. More formally, the set of monomorphic states is given by

$$\overrightarrow{XX} = \{s \in S | (a_i = a_j = X) \wedge (d_i^{out} = k^\ell, d_j^{out} = k^h), \forall i \in N_\ell, j \in N_h\}.$$

For example, in a monomorphic state $s \in \overrightarrow{AA}$, all agents play action A, agents in the low-constraint group support k^ℓ links, and agents in the high-constraint group support k^h links. Fig 3.1 depicts an example of a monomorphic state in $\overrightarrow{AA}$ when $n_\ell = 4, n_h = 2, k^\ell = 1$ and $k^h = 2$.

Secondly, we denote by $\overrightarrow{XY}$ the set of polymorphic states where $X, Y \in \{A, B\}$ and $X \neq Y$. Less formally, in a polymorphic state, all agents in the low-constraint group play one action and all agents in the high-constraint group play the other action. And in terms of linking strategy, all agents exhaust their constraints. Furthermore, agents will first form links to other agents within the same group, and then link to agents in the other group to fill up their remaining slots if any. More formally, $\overrightarrow{XY}$ is defined by

$$\overrightarrow{XY} = \{s \in S | (a_i = X, a_j = Y, a_j \neq a_i) \wedge (d_i^{out} = k^\ell, d_j^{out} = k^h) \wedge (\sum_{i' \in N_\ell} g_{ii'}$$

$$= \min\{k^\ell, n_\ell - 1\}, \sum_{j' \in N_h} g_{jj'} = \min\{k^h, n_h - 1\}), \forall i \in n_\ell, j \in N_h\}.$$

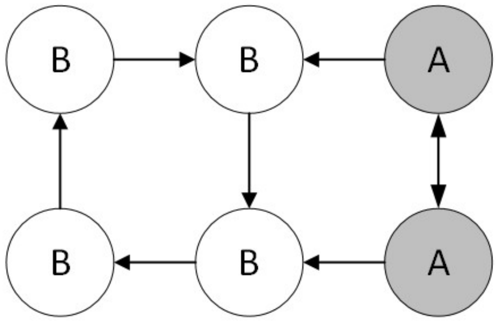


Figure 3.2. *A polymorphic state in* $\overrightarrow{BA}$ *when* $n_\ell = 4, n_h = 2, k^\ell = 1$ *and* $k^h = 2$.

For example, in a polymorphic state $s \in \overrightarrow{AB}$, all agents in the low-constraint group play the same action A and support k^ℓ links, whereas all agents in the high-constraint group N_h play action B and support k^h links. A- agents link to other A-agents first and then link to B-agents if they still have remaining slots, e.g. if $k^\ell > n_\ell - 1$. Similarly, B- agents link to other B-agents first and then link to A-agents if $k^h > n_h - 1$. Fig 3.2 depicts an example of a polymorphic state in $\overrightarrow{BA}$ when $n_\ell = 4, n_h = 2, k^\ell = 1$ and $k^h = 2$.

Following the same mechanism as in Staudigl & Weidenholzer (2014), finding the best response can be divided into two steps: First, for each of the two actions, determine the payoff optimizing linking strategy and calculate the payoffs associated with it. This is summarized by the link-optimized payoff functions (for short, the LOPs). And second, compare the LOPs across actions and choose the one with the highest payoff.

We denote by m the number of A-agents at a given strategy profile s. The number of B-agents is thus $n - m$. The LOPs are thus given by

$$v_i(a_i, m) = \max_{g_i \in \mathscr{G}_i} U_i((a_i, g_i), m), \quad \forall i \in N.$$

where $U_i((a_i, g_i), m)$ is agent i's payoff given her strategy $s_i = (a_i, g_i)$ and the number of A-agents m. Consider an agent i whose linking constraint is $k_i \in \{k^\ell, k^h\}$. Given the distribution of actions

$(m, n-m)$, her LOP of choosing action A is given by

$$v_i(A,m) = a \cdot \min\{k_i, m-1\} + c \cdot (k_i - \min\{k_i, m-1\}) - M \cdot k_i.$$

Intuitively, given the order of payoffs $a > c$, A-agents prefer playing against other A-agents over playing against B-agents. Thus, agent i will first link to other A-agents. Considering different levels between the constraint k_i and the number of other A-agents $m-1$, the maximum number of links to A agents that i could form is $\min\{k_i, m-1\}$. After forming links to A-agents, agent i will then fill her remaining slots $k_i - \min\{k_i, m-1\}$ by linking to B-agents if there are any remaining slots left.

Similarly, the order of payoffs $b > d$ implies that B-agents prefer forming links with other B-agents first. Then they will fill up their remaining slots by linking to A-agents. Note that a B-agents faces $n-(m-1)$ other B-agents if agent i chooses to play action B. Agent i's LOP of choosing action B is thus given by

$$v_i(B,m) = b \cdot \min\{k_i, n-m-1\} + d \cdot (k_i - \min\{k_i, n-m-1\}) - M \cdot k_i.$$

Given the LOPs, we now define the concept of Nash equilibrium in our game. Consider a strategy profile $s \in \mathscr{S}$ and the corresponding distribution of actions $(m, n-m)$. Strategy profile s is a Nash equilibrium if the following two conditions hold:

i) $v_i(A,m) \geq v_i(B, m-1)$ for all i with $a_i = A$;

ii) $v_j(B,m) \geq v_j(A, m+1)$ for all j with $a_j = B$.

We denote by $\mathscr{S}^\star$ the set of Nash equilibria. Given the previous observations, we are now able to state the following proposition which characterizes the set of Nash equilibria. In particular, Nash equilibria correspond to the monomorphic states and the polymorphic states.

Proposition 3.4.1. *There exist two thresholds* $\overline{k^\ell} = \frac{(b-d)(n_\ell-1)-(a-c)n_h}{(c-d)}$ *and* $\underline{k^h} = \frac{(b-d)n_\ell-(a-c)(n_h-1)}{(c-d)}$, *such that:*

i) if $k^\ell \leq \overline{k^\ell}$ *and* $k^h \geq \underline{k^h}$, *then* $\mathscr{S}^\star = \overrightarrow{AA} \cup \overrightarrow{BB} \cup \overrightarrow{BA}$;

ii) if $k^\ell > \overline{k^\ell}$ *or* $k^h < \underline{k^h}$, *then* $\mathscr{S}^\star = \overrightarrow{AA} \cup \overrightarrow{BB}$.

The proof to Proposition 3.4.1 proceeds using a series of lemmas. Note that each agent's strategy consists of two parts: action choice and linking choice. First, we prove that only monomorphic states and polymorphic states can potentially be a Nash equilibrium, i.e. $\mathscr{S}^\star \subseteq \overrightarrow{AA} \cup \overrightarrow{BB} \cup \overrightarrow{AB} \cup \overrightarrow{BA}$. Next, we show that any strategy profile in $\overrightarrow{AB}$ is not a Nash equilibrium. That is, a Nash equilibrium cannot be a state where agents in the low-constraint group play the risk-dominant action and agents in the high-constraint group play the payoff-dominant action. Then, we prove that for any k^ℓ and k^h, monomorphic states are always Nash equilibria. In the last step, we prove that a strategy profile in $\overrightarrow{BA}$ is Nash equilibrium if and only if $k^\ell \leq \overline{k^\ell}$ and $k^h \geq \underline{k^h}$.

Lemma 3.4.1. *If* $s \notin \overrightarrow{AA} \cup \overrightarrow{BB} \cup \overrightarrow{AB} \cup \overrightarrow{BA}$, *then* s *is not a Nash equilibrium.*

Intuitively, since agents in the same group have the same constraints, they face the same situation. This implies that whenever it is optimal for one agent to stay at her action, then it is also optimal for agents with the other action to switch. It follows that all agents in the same group have to choose the same action in a Nash equilibrium.

The next lemma establishes that all monomorphic states are Nash equilibria.

Lemma 3.4.2. $\overrightarrow{AA} \cup \overrightarrow{BB} \subset \mathscr{S}^\star$ *for any* k^ℓ, k^h *and* n.

The proof of Lemma 3.4.2 is straightforward. First, consider a strategy profile s in the monomorphic set $\overrightarrow{AA}$. The corresponding distribution of actions is $(n,0)$. Since there are only A-agents, none of them will deviate from playing action A as switching to B will lower their payoff per link by $a-d$. Additionally, no agent has incentives to form fewer links since each link yields

$a-\gamma$, which is strictly positive. Thus, no one wants to deviate either from her current action choice or from her linking choice. Therefore, s is a Nash equilibrium. Following the same argument as $s \in \overrightarrow{AA}$, we can also prove that any strategy profile $s \in \overrightarrow{BB}$ is also a Nash equilibrium.

Note that coordinating on the same action always yields a higher payoff than not coordinating. when all agents choose the same action, no one has incentives to switch to the other action. The next two lemmas extend the discussion on Nash equilibrium to the polymorphic states.

Lemma 3.4.3. *No state $s \in \overrightarrow{AB}$ is a Nash equilibrium.*

Intuitively, independent of the sizes of the two groups, a Nash equilibrium cannot be a state where agents in the low-constraint group choose the risk-dominant action and agents in the high-constraint group choose the payoff-dominant action. If agents in the high-constraint group choose the payoff-dominant action, it implies that there are sufficient B-agents around for agents with the higher constraint. Thus, for agents with the lower constraint, the number of B-agents is also sufficient. Their best response therefore is choosing the payoff-dominant action B.

The following lemma establishes that a polymorphic state $s \in \overrightarrow{BA}$ could be a Nash equilibrium for some constraints k^ℓ and k^h.

Lemma 3.4.4. *$\overrightarrow{BA} \subset \mathscr{S}^\star$ iff $k^\ell \leq \overline{k^\ell}$ and $k^h \geq \underline{k^h}$.*

Lemma 3.4.4 provides us with conditions for the co-existence of the risk-dominant action and the payoff-dominant action in a Nash equilibrium. Such a Nash equilibrium is characterized by agents in the low-constraint group choosing the payoff-dominant action, and agents in the high-constraint group choosing the risk-dominant action. We provide two examples to develop intuition for our findings.

Example 3.4.1. *Figure 3.2 depicts a polymorphic state in $\overrightarrow{BA}$ where $n_\ell = 4, n_h = 2, k^\ell = 1$ and $k^h =$ 2. For any payoffs (a,b,c,d) fulfilling our assumptions, one can check that $k^\ell \leq \frac{(b-d)\cdot 3-(a-c)\cdot 2}{c-d}$*

and $k^h < \frac{(b-d)\cdot 4-(a-c)}{c-d}$ hold, so that the second condition for polymorphic equilibrium is violated.[7] *Therefore, there exists no polymorphic equilibrium. To develop intuition, consider the strategy profile depicted in Figure 3.2. If agents in the high-constraint group choose action B, their optimal linking choice is to link to two B-agents and they will get $2\cdot b$ by doing so. If they choose action A, the highest payoff is $a+c$ by linking to one A-agent and one B-agent. Since $b>a>c>d$, we have $2\cdot b>a+c$, then agents in the high-constraint group will always switch to action B.*

Example 3.4.1 highlights that if the conditions identified in Lemma 3.4.4 fail, then there cannot be polymorphic equilibrium. Intuitively, while the low-constraint group has a small enough constraint so that choosing B is optimal, the constraint of agents in the high-constraint group is so small that choosing B would be optimal for them too. If they had a larger constraint, it may be optimal for them to stay with A as the following example shows.

Example 3.4.2. *Figure 3.3 depicts another example of a polymorphic state in $\overrightarrow{BA}$. Assume that $a+3c\geq 3b+d$.*[8] *Agents in the low-constraint group will stay with B provided $b>a$. Furthermore, agents in the high-constraint group will stay with A provided $a+3c\geq 3b+d$. To see this, note that each agent in N_h forms three links with B-agents and one link with the other A-agent. By playing action A, she gets $3c$ from playing against all B-agents and a from playing against the other A-agent. If she switches to B, she gets $3b$ from playing against all B-agents and d from playing against the other A-agent. Given that $a+3c\geq 3b+d$, she will stay with action A. Thus, the state depicted in Figure 3.3 is a Nash equilibrium.*[9]

[7]Note that $b>a>c>d$. Inequality $k^\ell \leq \frac{(b-d)\cdot 3-(a-c)\cdot 2}{c-d}$ holds since that $\frac{(b-d)\cdot 3-(a-c)\cdot 2}{c-d} = \frac{(b-d)+(b-a+c-d)\cdot 2}{c-d} > \frac{(b-d)}{c-d} > 1 = k^\ell$. Inequality $k^h < \frac{(b-d)\cdot 4-(a-c)}{c-d}$ holds since that $\frac{(b-d)\cdot 4-(a-c)}{c-d} = \frac{(b-d)\cdot 3+(b-a+c-d)}{c-d} > \frac{(b-d)\cdot 3}{c-d} > 3 > 2 = k^h$.

[8]One can check that $a+3c\geq 3b+d$ is plausible, e.g. $(a,b,c,d)=(10,11,9,1)$.

[9]If $a+3c\geq 3b+d$, then $k^\ell \leq \frac{(b-d)\cdot 2-(a-c)\cdot 2}{c-d}$ and $k^h \geq \frac{(b-d)\cdot 3-(a-c)}{c-d}$ hold so that the constraints fulfill the conditions identified in Lemma 3.4.4.

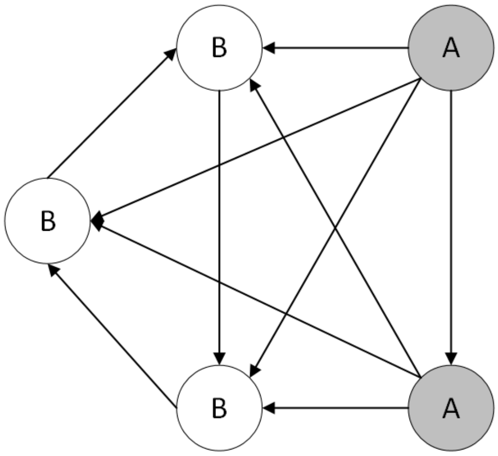


Figure 3.3. *A polymorphic state in $\overrightarrow{BA}$ for $n_\ell = 3, n_h = 2, k^\ell = 1$ and $k^h = 4$, which is a Nash equilibrium in a coordination game if $a + 3c \geq 3b + d$. White circles represent four agents in the low-constraint group. Grey circles represent two agents in the high-constraint group.*

3.5 Myopic best response learning

As we have seen in the previous section, our model may feature a multiplicity of equilibria. To assess which of these equilibria is most likely to arise in the long run, we consider a model of myopic best response learning where agents make occasional mistakes in the spirit of Kandori et al. (1993), Young (1993) and Ellison (1993, 2000). The unperturbed model is defined as follows. In discrete-time $t = 0, 1, 2, 3, \cdots$, each agent may receive the opportunity to revise her strategy (both action and links) with a positive probability $\lambda \in (0, 1)$. This probability is independent among all agents and periods. When getting the opportunity to revise in period t, each agent chooses a strategy that maximizes her payoff in the last period $t - 1$. More formally, agent i chooses a strategy in the period t as follows:

$$s_i^t \in arg \max_{s_i \in \mathscr{S}_i} U_i(s_i, s_{-i}^{t-1})$$

where s_{-i}^{t-1} is the strategy profile played by agents except i in the last period $t - 1$. If there are multiple best responses, agents choose one of them at random.

In light of our discussion in the previous section, this revision protocol can be analyzed in two steps: i) for each action, agents first determine the optimal linking strategy, and ii) given the optimal linking strategies for both actions, agents then determine which of two actions is optimal. This approach is captured by the LOPs. Formally, an agent i chooses her action in the following way:

i) when $a_i^{t-1} = A$, switch to B if $v_i(B, m^{t-1} - 1) > v_i(A, m^{t-1})$, remain with A if $v_i(B, m^{t-1} - 1) < v_i(A, m^{t-1})$, randomize between A and B if $v_i(B, m^{t-1} - 1) = v_i(A, m^{t-1})$;

ii) when $a_i^{t-1} = B$, switch to A if $v_i(A, m^{t-1} + 1) > v_i(B, m^{t-1})$, remain with B if $v_i(A, m^{t-1} + 1) < v_i(B, m^{t-1})$, randomize between A and B if $v_i(A, m^{t-1} + 1) = v_i(B, m^{t-1})$,

where a_i^{t-1} denotes i's action and m^{t-1} is the number of A-agents in the last period $t-1$.

The revision rule outlined above gives rise to a Markov chain on the state space $\mathscr{S} \equiv \mathscr{S}_1 \times \mathscr{S}_2 \times \cdots \times \mathscr{S}_n$. In this context, a state s in the space $\mathscr{S}$ is equivalent to a strategy profile $s = (s_i)_{i \in N}$.

We are interested in sets of states to which this process converges. These sets are known as absorbing sets (see e.g. Kandori et al. (1993), Young (1993), Freidlin & Wentzell (1998), and Ellison (2000)). An absorbing set, denoted by $S^{\star\star}$, is a minimum subset of $\mathscr{S}$ such that:

i) for any pair of states $s, s' \in S^{\star\star}$, the probability of a transition from s to s' is positive;

ii) for any two states $s \in S^{\star\star}$ and $s'' \notin S^{\star\star}$, the probability of a transition from s to s'' is zero.

We denote the set of all absorbing sets by $\mathscr{S}^{\star\star}$.

We now proceed to characterize those absorbing sets. By considering various ranges of the game parameters m, n, k^ℓ, and k^h, we have computed the switching thresholds for agents in both groups, that is, we provide conditions on the distribution of actions when agents find it optimal to switch actions and when they will remain at their current actions. These results are summarized in the Table 3.1. This allows us to have a full characterization of absorbing sets which is presented as the following proposition. [10]

Table 3.1. *Where "a.s" means that an agent always switches to the other action and "n.s" means that an agent never switches to the other action.*

Switching Thresholds for A-agents			
$v(B,m-1) \geq v(A,m)$	$k^h > k^\ell \geq m-1$	$k^h \geq m-1 > k^\ell$	$m-1 > k^h > k^\ell$
	$k^h > k^\ell \geq n-m$	$k^h > k^\ell \geq n-m$	$k^h > k^\ell \geq n-m$
$i \in N_\ell$	$m \leq \frac{(n-1)(b-d)-k^\ell(c-d)}{a+b-c-d}+1 := M_\ell^1$	$m \leq n-\frac{a-d}{b-d}k^\ell := M_\ell^2$	$m \leq n-\frac{a-d}{b-d}k^\ell$
$j \in N_h$	$m \leq \frac{(n-1)(b-d)-k^h(c-d)}{a+b-c-d}+1 := M_h^1$	$m \leq \frac{(n-1)(b-d)-k^h(c-d)}{a+b-c-d}+1$	$m \leq n-\frac{a-d}{b-d}k^h := M_h^2$
	$k^h > k^\ell \geq m-1$	$k^h \geq m-1 > k^\ell$	$m-1 > k^h > k^\ell$
	$k^h \geq n-m > k^\ell$	$k^h \geq n-m > k^\ell$	$k^h \geq n-m > k^\ell$
$i \in N_\ell$	$a.s.$	$a.s.$	$a.s.$
$j \in N_h$	$m \leq \frac{(n-1)(b-d)-k^h(c-d)}{a+b-c-d}+1$	$m \leq \frac{(n-1)(b-d)-k^h(c-d)}{a+b-c-d}+1$	$m \leq n-\frac{a-d}{b-d}k^h$
		$n-m > k^h > k^\ell$	
$i \in N_\ell$	$a.s.$	$a.s.$	$a.s.$
$j \in N_h$	$a.s.$	$a.s.$	$a.s.$
Switching Thresholds for B-agents			
$v(A,m+1) \geq v(B,m)$	$k^h > k^\ell > m$	$k^h > m \geq k^\ell$	$m \geq k^h > k^\ell$
	$k^h > k^\ell > n-m-1$	$k^h > k^\ell > n-m-1$	$k^h > k^\ell > n-m-1$
$i \in N_\ell$	$m \geq \frac{(n-1)(b-d)-k^\ell(c-d)}{a+b-c-d}$	$m \geq n-1-\frac{a-d}{b-d}k^\ell$	$m \geq n-1-\frac{a-d}{b-d}k^\ell$
$j \in N_h$	$m \geq \frac{(n-1)(b-d)-k^h(c-d)}{a+b-c-d}$	$m \geq \frac{(n-1)(b-d)-k^h(c-d)}{a+b-c-d}$	$m \geq n-1-\frac{a-d}{b-d}k^h$
	$k^h > k^\ell > m$	$k^h > m \geq k^\ell$	$m \geq k^h > k^\ell$
	$k^h > n-m-1 \geq k^\ell$	$k^h > n-m-1 \geq k^\ell$	$k^h > n-m-1 \geq k^\ell$
$i \in N_\ell$	$n.s.$	$n.s.$	$n.s.$
$j \in N_h$	$m \geq \frac{(n-1)(b-d)-k^h(c-d)}{a+b-c-d}$	$m \geq \frac{(n-1)(b-d)-k^h(c-d)}{a+b-c-d}$	$m \geq n-1-\frac{a-d}{b-d}k^h$
		$n-m-1 \geq k^h > k^\ell$	
$i \in N_\ell$	$n.s.$	$n.s.$	$n.s.$
$j \in N_h$	$n.s.$	$n.s.$	$n.s.$

Proposition 3.5.1. *There exist thresholds* $\overline{k^\ell} = \frac{(b-d)(n_\ell-1)-(a-c)n_h}{(c-d)}$ *and* $\underline{k^h} = \frac{(b-d)n_\ell-(a-c)(n_h-1)}{(c-d)}$, *such that:*

i) if $k^\ell < \overline{k^\ell}$ *and* $k^h > \underline{k^h}$, *then* $\mathscr{S}^{\star\star} = \overrightarrow{AA} \cup \overrightarrow{BB} \cup \overrightarrow{BA}$;

ii) if $k^\ell \geq \overline{k^\ell}$ *or* $k^h \leq \underline{k^h}$, *then* $\mathscr{S}^{\star\star} = \overrightarrow{AA} \cup \overrightarrow{BB}$.

[10] The existence of different classes of absorbing sets in this setting has already been characterized by Zeng (2019). This proposition goes beyond that result by identifying relevant thresholds.

Proposition 3.5.1 shows that when linking constraints k^ℓ and k^h sufficiently differ from one another, polymorphic states could be contained in $\mathscr{S}^{\star\star}$. This implies that the co-existence of the payoff-dominant action and the risk-dominant action, in the long run, could emerge. Intuitively, agents in the low-constraint group have a constraint low enough such that they can fill sufficiently many of their slots with B-agents. On the other hand, the constraint of agents in the high-constraint group is too large to do so and they will consequently find it optimal to choose A.

As we have seen in above, there may be a multiplicity of absorbing sets under the unperturbed myopic best response learning dynamics. To find which kind of profile is more likely to emerge in the long run we now move forward to characterize which absorbing sets are stochastically stable. In order to do this, we consider a case where agents may make occasional mistakes, which is also known as perturbed myopic best response learning.

Agents are assumed to make mistakes probability $\varepsilon \in (0,1)$, i.e. they choose a state different to the one prescribed by the unperturbed myopic best response learning dynamics. The probability ε is assumed to be independent across agents, periods, and payoffs. Foster & Young (1990) demonstrate that if the perturbed dynamics is ergodic, irreducible, and aperiodic, then it, which is captured by a Markov process, has a unique invariant distribution $\mu(\varepsilon)$ for each fixed ε. The limit of this invariant distribution exists and is $\mu^\star = \lim_{\varepsilon \to 0} \mu(\varepsilon)$. A state s such that $\mu^\star(s) > 0$ is a so-called stochastically stable state or a long-run equilibrium. We denote the set of all stochastically stable states by $\mathscr{S}^{\star\star\star} = \{s \in \mathscr{S} | \mu^\star(s) > 0\}$.

With this technique, we move forward to identify the set of stochastically stable states. Which profile turns out to be stochastically stable will depend on the level of linking constraints. The following propositions establish our main results for various ranges of linking constraints k^ℓ and k^h.

In the first step, we focus on the case where there are only two monomorphic absorbing sets $\overrightarrow{AA}$ and $\overrightarrow{BB}$. After that, we turn to the case where $\overrightarrow{BA}$ is also absorbing.

Proposition 3.5.2. *For any given $k^h \leq \underline{k^h}$, there exist two thresholds $\underline{\underline{k^\ell}} \leq \overline{\overline{k^\ell}}$, such that: i) if $k^\ell < \underline{\underline{k^\ell}}$, then $\mathscr{S}^{\star\star\star} = \overrightarrow{BB}$; ii) if $k^\ell \in [\underline{\underline{k^\ell}}, \overline{\overline{k^\ell}}]$, then $\mathscr{S}^{\star\star\star} = \overrightarrow{BB} \cup \overrightarrow{AA}$; iii) if $k^\ell > \overline{\overline{k^\ell}}$, then $\mathscr{S}^{\star\star\star} = \overrightarrow{AA}$.*

And for any k^h and k^ℓ such that $k^h > \underline{k^h}$ and $k^\ell \geq \overline{k^\ell}$, we have that $\mathscr{S}^{\star\star\star} = \overrightarrow{AA}$.

Now, we provide technical intuitions for the results by using the case when both constraints are less than half of the number of the other agents, i.e. $k^\ell < k^h < \frac{n-1}{2}$. First, consider the transition from $\overrightarrow{AA}$ to $\overrightarrow{BB}$. Assume that there are k^ℓ agents who mutate to action B and choose any linking strategy. Then, A-agents in N_ℓ will find it optimal to switch to B and link to k^ℓ B-agents. It follows that the number of B-agents now is at least n_ℓ. Since $k^h < \frac{n-1}{2} < n_\ell$, the number of B-agents is sufficient for A-agents in N_h to switch. Thus, k^ℓ mutations are sufficient for this transition, i.e. $r(\overrightarrow{AA}, \overrightarrow{BB}) \leq k^\ell$. Next, consider the transition from $\overrightarrow{BB}$ to $\overrightarrow{AA}$. Note that k_ℓ mutations are insufficient for this transition. To see this, assume that k_ℓ agents mutate to A. After this, there will still be $N - k_\ell$ agents playing B. Since $k^\ell < k^h \leq \frac{n-1}{2}$, we have $k^h \leq n - k^\ell - 1$. This implies that any revising agent (either in N_ℓ or N_h) will find it optimal to either stay with B or switch back to B. It follows that $r(\overrightarrow{BB}, \overrightarrow{AA}) > k^\ell$. We thus have that $r(\overrightarrow{BB}, \overrightarrow{AA}) > r(\overrightarrow{AA}, \overrightarrow{BB})$. Thus, $\overrightarrow{BB}$ is the unique stochastically stable set.

In the Appendix, we provide the proof with both necessary and sufficient conditions for the transitions to occur and thus provide a complete characterization of the set of stochastically stable states for the case where there are only two monomorphic absorbing sets.

Intuitively, when the constraints k^ℓ and k^h are both small, a small number of B-agents is enough for all agents to make choosing the payoff-dominant action optimal. With a logic similar to Staudigl & Weidenholzer (2014), the payoff-dominant convention thus will emerge in the long run. In contrast, if both constraints are sufficiently large, the payoff-dominant action being optimal requires more B-agents to show up. There is increased uncertainty concerning agents' actions with whom one forms links. Consequently, in the long run, agents tend to choose the risk-dominant action, which yields a higher expected payoff.

We now turn to the case where the polymorphic set $\overrightarrow{BA}$ is also absorbing. The following proposition shows our main results of the stochastically stable set when two constraints k^ℓ and k^h are significantly various.

Proposition 3.5.3. *If $k^h > \underline{k^h}$ and $k^\ell < \overline{k^\ell}$, there exist two thresholds $k^{\ell\star} < \overline{k^\ell}$ and $k^{h\star} > \underline{k^h}$, such that whenever $k^\ell \leq k^{\ell\star}$ and $k^h \geq k^{h\star}$, $\overrightarrow{BA} \subseteq \mathscr{S}^{\star\star\star}$. Further, for $k^{\ell\star\star} < k^{\ell\star}$ and $k^{h\star\star} > k^{h\star}$, such that whenever $k^\ell < k^{\ell\star\star}$ and $k^h > k^{h\star\star}$, $\mathscr{S}^{\star\star\star} = \overrightarrow{BA}$.*

Thus we have identified a region of parameters such that co-existence occurs.[11] Proposition 3.5.3 shows that if constraints are significantly heterogeneous, the risk-dominant profile and payoff-dominant profile can co-exist. To be more specific, the polymorphic states that agents in the low-constraint group play the payoff-dominant action and agents in the high-constraint group play the risk-dominant action can be stochastically stable if the lower constraint is tighter and the higher constraint is looser.

We now revisit Example 3.4.2 for the intuition of Proposition 3.5.3.

Example 3.4.2 revisited. Recall that when the parameters are $n_\ell = 3, n_h = 2, k^\ell = 1$ and $k^h = 4$, and the payoffs in the coordination game fulfils that $a + 3c \geq 3b + d$, $\overrightarrow{BA}$ is an absorbing set. Figure 3.4 depicts transitions from monomorphic states to polymorphic states and the other way around, with which we can determine the robustness of these profiles to mistakes. Note that white circles are the agents who play the risk-dominant action B and grey circles are agents who play the payoff-dominant action A.

First, we study the transition from $\overrightarrow{BB}$ to $\overrightarrow{BA}$ as Figure 3.4a shows. Agents 1 and 2 can support four links, while agents 3, 4, and 5 can only support one link. Now assume that agent 1 makes a mistake and switches to A. In the next step, agent 2 will also switch since switching to A yields

[11] In the cases not covered by the parameter ranges of the Proposition 3.5.3, i.e. $k^\ell \geq k^{\ell\star}$ or $k^h \leq k^{h\star}$, either the risk-dominant convention $\overrightarrow{AA}$ or the payoff-dominant convention $\overrightarrow{BB}$ arises as stochastically stable states. While we have been able to obtain partial results, unfortunately, a complete characterization has eluded us.

$a+2c$, which is larger than $3b+d$ from reaming at B. Following this, agents in the low-constraint group will remain at B and link with other B-agents. Thus, with one mistake we have reached a state in $\overrightarrow{BA}$.

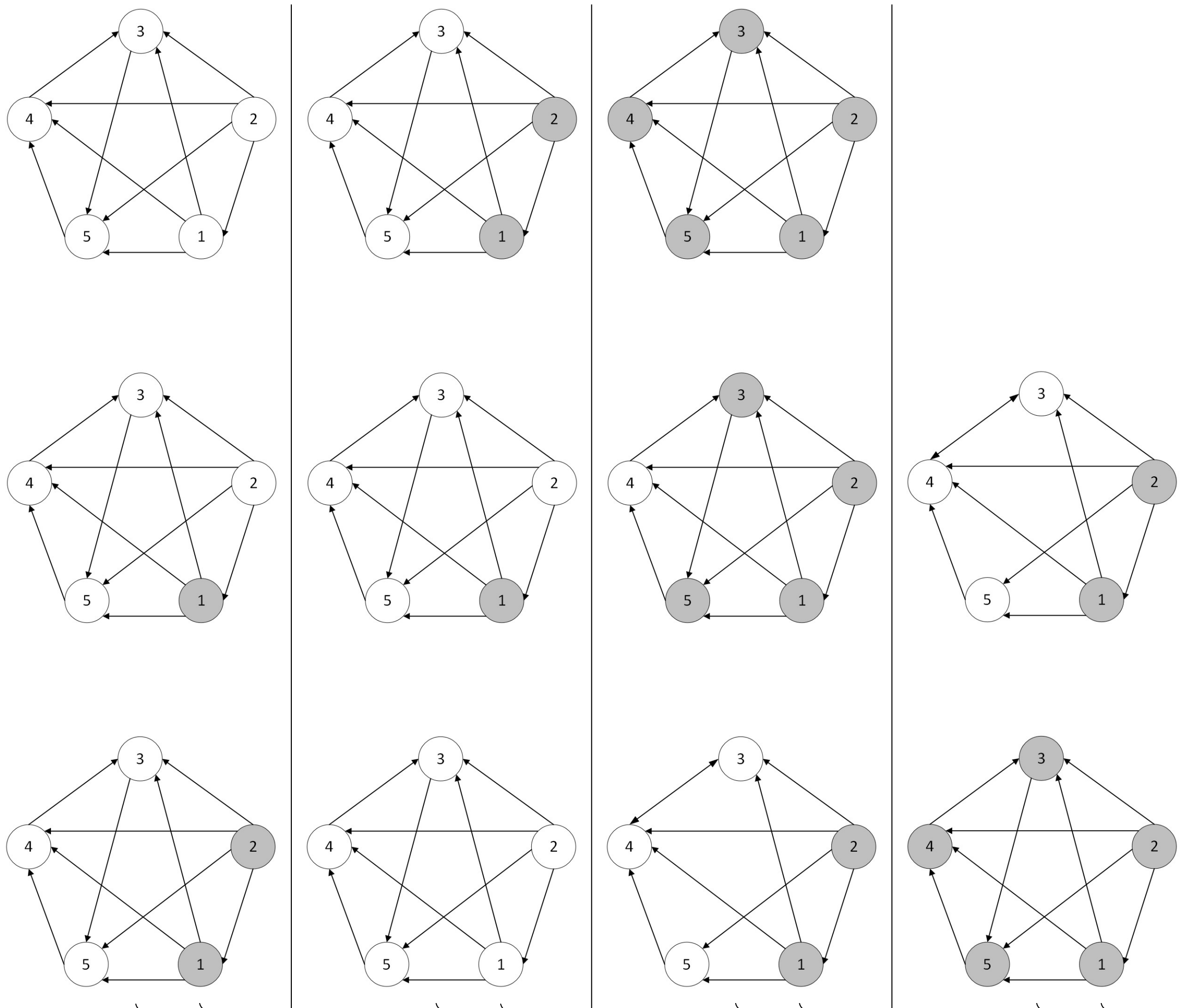
Then, consider the transition from $\overrightarrow{BA}$ to $\overrightarrow{BB}$ as Figure 3.4b. Assume that agent 2 makes a mistake and switches to B. Following this, agent 1 will switch since there are no other A-agents. B-agents in the low-constraint group will remain since there are sufficient B-agents around. Hence, we have reached a state in $\overrightarrow{BB}$ with one mistake.

The transition from $\overrightarrow{AA}$ to $\overrightarrow{BA}$ is similar to the above (see Figure 3.4c). One mistake is sufficient. To see this, assume that agent 4 makes a mistake and switches to B. In the next step, one B-agent is enough for other agents in the low-constraint group to switch. However, in the present setting, i.e. $a+3c \geq 3b+d$, agents 1 and 2 may choose to remain. We thus have reached a state in $\overrightarrow{BA}$ with one mutation.

Now, consider the transition from $\overrightarrow{BA}$ to $\overrightarrow{AA}$. For agents in the low-constraint group to switch requires there are no B-agents around. Hence, three mistakes are required for this transition.

Thus, $\overrightarrow{BB}$ and $\overrightarrow{BA}$ can be reached from each absorbing set via a sequence of one mistake for parameters in the present example. Consequently, both $\overrightarrow{BB}$ and $\overrightarrow{BA}$ are stochastically stable. However, $\overrightarrow{AA}$ cannot be reached via such a sequence, implying that more mistakes are required and thus $\overrightarrow{AA}$ is not stochastically stable.

Transitions among absorbing sets are similar to the spread of actions. Transitions between two monomorphic absorbing sets $\overrightarrow{AA}$ and $\overrightarrow{BB}$ can be split into two steps: transitions into and out of the intermediate state $\overrightarrow{BA}$. Thus, it can be the case that from $\overrightarrow{AA}$ and $\overrightarrow{BB}$, transitions into $\overrightarrow{BA}$ is easier than out of it. To see this point, if constraints are tight, transition into B is easier, while if constraints are loose, transition into A is easier instead, which is the mechanism that drives the results in Staudigl & Weidenholzer (2014). Note that significantly different constraints in our model imply that the lower constraint is tight and the higher constraint is loose. Thus, for the



(a) $\overrightarrow{BB}$ *to* $\overrightarrow{BA}$ **(b)** $\overrightarrow{BA}$ *to* $\overrightarrow{BB}$ **(c)** $\overrightarrow{AA}$ *to* $\overrightarrow{BA}$ **(d)** $\overrightarrow{BA}$ *to* $\overrightarrow{AA}$

Figure 3.4. *Tansitions among absorbing sets. Grey circles are agents who are playing action A, and white circles represent agents who are playing action B.*

low-constraint group, the payoff-dominant action B is more robust to mistakes, i.e. transition from $\overrightarrow{AA}$ into $\overrightarrow{BA}$ requires fewer mistakes than the other way around. Similarly, for the high-constraint group, the risk-dominant action A is more robust to mistakes, i.e. transition from $\overrightarrow{BB}$ into $\overrightarrow{BA}$ requires fewer mistakes than the other way around, which is similar to the mechanism in Goyal & Vega-Redondo (2005). Hence, transition into $\overrightarrow{BA}$ from each other may require the fewest mistakes than transitions into the other two absorbing sets $\overrightarrow{AA}$ and $\overrightarrow{BB}$. Consequently, $\overrightarrow{BA}$ is stochastically stable.

3.6 Conclusion

In this paper, we present an evolutionary model of coordination and network formation where there are two groups of agents who face either high or low linking constraints on the number of links they can form. We show that the heterogeneous constraints significantly affect the selection of conventions.

First, the present work reinforces the results of homogeneous constraints where the payoff-dominated action is selected if agents face tight constraints while the risk-dominant action is favoured if the constraints are loose. Moreover, in contrast to the conventional results of only monomorphic states being stochastically stable, we reveal that the co-existence of conventions can be observed when the constraints are significantly different. In this paper, we provide both necessary and sufficient conditions such that the risk-dominant convention and the payoff-dominant convention may co-exist. Specifically, if a large portion of the population has a very tight constraint, with the other having a very loose constraint, the larger group tends to choose the payoff-dominant action and the smaller group is more likely to choose the risk-dominant action.

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Appendix A

Mathematical Proofs

A.1 Appendix A

A.1.1 Proofs in Section 1.4

Proof of Lemma 1.4.1. Suppose g is a non-empty strict Nash network. Thus, there exists one agent i who maintains at least one link. Suppose now there exists one agent j who supports no links. Consider payoffs received by agents i and j. If $U_i(g) > 1$, agent j is strictly better off by replicating i's links; if $U_i(g) < 1$, agent i is better off by deleting all her links; if $U_i(g) = 1$, agent j is indifferent between maintaining no links and replicating i's links. All cases contradict g being a strict Nash network. This implies that there exists no agent who supports no links. Thus, every agent supports at least one link in a non-empty strict Nash network. □

Proof of Lemma 1.4.2. Given a non-empty strict Nash network g, note that there can exist at most one agent without any incoming links. To see this, assume that there exist two agents without incoming links, denoted by i and j. If $U_i(g) < U_j(g)$, i is strictly better off by replicating j's links. Analogously, if $U_i(g) > U_j(g)$, j is strictly better off by replicating i's links. Otherwise, if $U_i(g) = U_j(g)$, either agent i or j is indifferent between maintaining her current links and replicating the other's links. All cases contradict g being a strict Nash network. Thus, there cannot exist more than one agent without any incoming links.

In the next step, we show that in fact there cannot exist a single agent without any incoming links. To see this, assume an agent i has no incoming links. First, consider the case where there is an agent k, such that $U_k(g) > U_i(g)$. Note that payoffs of all other agents are independent of i's linking strategy. So by deleting her current links and forming the links that k supports, agent i can assure herself the same payoff as k gets, contradicting the assumption that g is a strict Nash equilibrium.

Second, we consider the case where there exists an agent k whose payoff is equal to i's i.e. $U_k(g) = U_i(g)$. In the first sub-case $g_i \neq g_k$, i is indifferent between replicating k's links and maintaining her current links, contradicting g being a strict Nash network. Then consider the second sub-case $g_i = g_k$. According to Lemma 1.4.1, k has at least one incoming link. Suppose that agent ℓ is the agent who forms a link to k. Since i and k have the same linking strategies, ℓ is indifferent between linking to k and to i, which also contradicts g being a strict Nash equilibrium. Thus, it is impossible to have an agent k whose payoff is equal to i's.

Third, we consider the case where agent i's payoff is the highest among all agents, i.e. $U_i(g) > U_k(g), \forall k \in N$. In the first sub-case, there are some agents from whom the distance to agent i is larger than two, i.e. $\exists j \in N, d(i, j; g) > 2$. As a result of the restriction on information transition, agent i's payoff is independent of j's linking strategy. Agent j is strictly better off by replicating i's links, which contradicts g being a strict Nash equilibrium. In the second sub-case, the furthest distance from any agent to i is two. There exists an agent ℓ with $d(i, \ell; g) = 2$. Sort i's $1-$ neighbours by the number of links they form. Without loss generality, rename them $i_1, i_2, \cdots, i_{n_i^1}$, where the number of their links weakly increases with subscripts, i.e. $n_{i_1}^1 \leq n_{i_2}^1 \leq \cdots \leq n_{i_{n_i^1}}^1$. We can rewrite i's payoff function in the following way:

$$U_i(g_i, g_{-i}) = 1 + \sum_{x=1}^{n_x^1} (1 + n_{i_x}^1 - c) \tag{A.1}$$

Analogously, sort ℓ's $1-$ neighbours by the numbers of links they form and without loss generality, rename them $\ell_1, \ell_2, \cdots, \ell_{n_\ell^1}$, where the number of their links weakly increases with subscripts, i.e. $n_{\ell_1}^1 \leq n_{\ell_2}^1 \leq \cdots \leq n_{\ell_{n_\ell^1}}^1$. Agent ℓ's payoff function can be written as:

$$U_l(g_\ell, g_{-\ell}) = 1 + \sum_{x=1}^{n_\ell^1} (1 + n_{\ell_x}^1 - c) \tag{A.2}$$

There are two sub-subcases:

i) The number of ℓ's links is at least as large as i's, i.e. $n_\ell^1 \geq n_i^1$. Since $U_i(g) > U_\ell(g)$, the largest number of links that agents in $N_i^1(g)$ form must be larger than the smallest number of links that agents in $N_\ell^1(g)$ form, i.e. $n_{i_{n_i^1}}^1 > n_{\ell_1}^1$.[1] Then, agent ℓ is strictly better off by deleting the link to ℓ_1 and replicating i's link to $i_{n_i^1}$, which contradicts g being a strict Nash equilibrium.[2]

ii) Agent ℓ forms less links than i does, i.e. $n_\ell^1 < n_i^1$. Again there are two sub-cases.

a) If $n_{\ell_{n_\ell^1}}^1 \geq n_{i_1}^1$, then i is strictly better off by deleting the link to i_1 and replicating ℓ's link to $\ell_{n_\ell^1}$.[3]

b) If $n_{\ell_{n_\ell^1}}^1 < n_{i_1}^1$, then ℓ is better off by deleting the link to ℓ_1 and replicating i's link to

[1] Consider an inequality $a_1 + a_2 + \cdots + a_n > b_1 + b_2 + \cdots + b_{n'}$, with $a_1 \leq a_2 \leq \cdots \leq a_n$, $b_1 \leq b_2 \leq \cdots \leq b_{n'}$ and $n \leq n'$. Since $n \cdot a_n > a_1 + a_2 + \cdots + a_n > b_1 + b_2 + \cdots + b_{n'} > n' \cdot b_1 > n \cdot b_1$, it must be true that $a_n > b_1$ must be true.

[2] Even though agents i and ℓ may have some common neighbours, we can always find an agent i_x who is not in $N_\ell^1(g)$ and forms more links than ℓ_1. To see this, consider the inequality $a_1 + a_2 + \cdots + a_n > b_1 + b_2 + \cdots + b_{n'}$. Agents i and ℓ have common neighbours implying that there exists at least one pair of a_i and b_j, such that $a_i = b_j$. We can eliminate a_i and b_j on both sides and still apply the property above.

[3] Analogously, even if i and ℓ have common neighbours, following an argument similar to the one in the footnote 2, we can always find an agent ℓ_x that $n_{\ell_x}^1 \geq n_{i_1}^1$.

$i_{n_i^1}$.

Thus, if there exists a single agent i such that $U_i(g) > U_k(g), \forall k \in N$, it follows that the distance from any agent to agent i is less than one, i.e. $d(i,k;g) \leq 1, \forall k \neq i$. In other words, agent i links up to all other agents, i.e. $k \in N_i^1, \forall k \neq i$. According to Lemma 1.4.1, note that every agent must have at least one outgoing link, and by assumption, there exists no agent linking to i. It follows that $\exists m,n \in N_i^1, m \neq n$, such that m links up to n, or vice versa. This contradicts g being a strict Nash network since i is strictly better off by deleting the link to n.

Thus, it follows that it is impossible to have a single agent without incoming links. Consequently, every agent has at least one incoming link. □

Proof of Lemma 1.4.3. The proof proceeds by contradiction. Consider a non-empty strict Nash network g that is not strongly connected. Assume that there exists a non-empty network $g \in \mathscr{G}$ which is a strict Nash network but not strongly connected. So there are multiple strongly connected components $\{C_1, C_2, \cdots, C_m\}$ with $m \geq 2$. Without loss of generality, we consider the case where $m = 2$. [4]. Let g^1 and g^2 be the two sets of strategies used by agents in C_1 and C_2 respectively Note that g consists of several strongly connected components. Without loss of generality, consider any two strongly connected components C_1 and C_2. Lemma 1.4.1 and Lemma 1.4.2 imply that both C_1 and C_2 contain multiple agents. Then there are two cases:

i) C_1 and C_2 are separated. Consider two agents $i,m \in C_1$ and two agents $j,k \in C_2$. We assume that m forms a link to i, i.e. $i \in N_m^1(g)$ and k forms a link to j, i.e. $j \in N_k^1(g)$. Since g is a strict Nash network, the payoff of m's link to i is positive and no larger than $1 + n_i^1 - c$. [5] Therefore, it must be true that $1 + n_i^1 - c > 0$. Then, consider agent k in C_2. Since k does not receive benefit from any agent in C_1, she is strictly better off by forming a link to i since $1 + n_i^1 - c > 0$. It contradicts the assumption that g is a strict Nash network. Now consider the link from k to j. By the same logic,

[4]For any $m \geq 3$, we can start with any two of the components and iteratively apply the same logic as with $m = 2$.
[5]It is possible that m links to agents in $N_i^1(g)$ directly or via other paths

m is strictly better off by forming a link to agent j.

ii) There exist paths from agents in C_1 to agents in C_2, but not vice versa. Following a similar argument in the case above, an agent in C_2 is strictly better off by forming a link to an agent in C_1.

Therefore, C_1 and C_2 are strongly connected and contained in one strongly connected component. Consequently, all strongly connected components are strongly connected. The network g is strongly connected. □

Proof of Proposition 1.4.1. Consider a core-periphery network CP_ℓ in which $|P_i| \geq 3$ for any $i \in C(\ell; g)$. A core agent i's payoff is given by

$$U_i(g_i, g_{-i}) = N - c \cdot \sum_{j \in N} g_{ij} = N - c \cdot (\ell + |P_i|)$$

Notice that agent i

i) has no incentives to form more links. Since choosing g_i already allows i to receive benefits from every other agent, adding more links increases i's cost but her benefit remains the same.

ii) has no incentives to delete any link. If i deletes one link to her periphery j, she would reduce her cost by c but lose the benefit from j. Since linking costs $c < 1$, i's payoff would decrease by $1 - c$. If i deletes one link to another core agent k, she would at least lose the benefits from k's periphery agents. Agent i's payoff would decrease by at least $|P_k| - c$. In both cases, player i is worse off by deleting a link.

Now consider a periphery agent j, her payoff is given by

$$U_j(g_j, g_{-j}) = N - c \cdot \sum_{k \in N} g_{jk} = N - c \cdot \ell$$

we argue in the following that agent j

i) has no incentives to delete any link to core agents. If j deletes one link to a core agent i, she would lose the benefit from i's periphery agents. The payoff of j would decrease by $|P_i| - c$. Note that $|P_i| \geq 3$ and $c < 1$. Agent j is worse off by deleting a link to the core agent.

ii) has no incentives to form any link to other periphery agents. Since by linking to all core agents, j receives benefits from every agent, adding more links only increases j's cost.

Moreover, we consider a case where no agent has incentives to replace any one of her links with another. To see this, replacing one link can be divided into two steps: deleting a link and forming a new link. Note that any agent already links to all core agents. An agent is unable to choose to form a new link to any other core agent since she has already linked to all core agents. Thus, there are two sub-cases for a core agent i to discuss.

a) Deleting a link to a core agent k and forming a new link to a periphery agent j. The first step reduces i's payoff by at least $|P_k| - c$. The second step can increase i's benefit by at most $1 - c$.[6] Overall, i's payoff is reduced by $|P_k| - 1$. Since $|P_k| \geq 3$, agent i is worse off by doing so.

b) Deleting a link to a periphery agent and forming a new link to another periphery agent. The first step reduces i's payoff by $1 - c$. The second step reduces i's payoff by c since she can observe every other periphery agent via other core agents. Thus, i's payoff decreases by $1 - c + c = 1$.

In both sub-cases, i is worse off. Thus, i doesn't have any incentives to replace any of her links. Note that periphery agents form no links to other periphery agents. The discussion of link replacement of a periphery agent can focus only on the sub-case a).

To conclude, neither core agents nor periphery agents have incentives to change their links. □

[6] If j is k's periphery agent, forming the link to j yields $1 - c$. If j is the periphery agent of another core agent, this link only costs c since she already receives the benefit from j via another core agent.

A.1.2 Proofs in Section 1.5

Proof of Proposition 1.5.1. Consider a network $g \in \mathscr{CP}_{\bar{\ell}}$. Note that all networks in $\mathscr{CP}_{\bar{\ell}}$ are strict Nash networks. Any agent i revising her strategy will remain at her current strategy since $U_i(g_i, g_{-i}) > U_i(g_i', g_{-i})$ for all g_i'. Thus, without any mistakes, the network will remain unchanged. □

Proof of Proposition 1.5.2. First, consider the transition from any state $g \in \mathscr{CP}_{\bar{\ell}}$ to a state $g' \in \mathscr{CP}_1$ with agent i as the unique core agent. One mistake is sufficient for this transition, i.e. $r(g, g') = 1$. To see this, assume that agent i makes a mistake and forms links to all other agents. Now, consider other agents who get the chance to revise. Forming a single link to agent i is a best response for them. Consequently, the dynamics reaches the CP_1 network with i as the unique core agent.

Then, consider the transition from $g \in \mathscr{CP}_1$ to $g' \in \mathscr{CP}_2$ where the core agent in g is still a core agent in g'. One mistake is sufficient for this transition. To see this, consider the case where a periphery agent j makes a mistake and forms three extra links to other periphery agents of core agent i. Now, given the revision opportunity the core agent i will find it optimal to delete links to those periphery agents of j. Following this, give the revision opportunity to other periphery agents. Their best response is forming one additional link to agent j to get access to the periphery agents of j. Consequently, the dynamics reaches a CP_2 network with i and j as two core agents.

Now, consider two states $g \in \mathscr{CP}_\ell$ and $g' \in \mathscr{CP}_{\ell+1}$ where they have ℓ common core agent, i.e. $C(\ell; g) \subset C(\ell+1; g')$. The resistance of the transition from g to g' is also one. Following a similar argument as above, a periphery agent j makes a mistake and links to three periphery agents. Consider core agents who link to the three periphery agents. They are indifferent or have a profitable deviation by deleting the links to these periphery agents and forming a link to j. Following this, consider other core agents and periphery agents. Their best response is to form an link to j. Consequently, the dynamics reaches a state in $\mathscr{CP}_{\ell+1}$ with j as the $\ell+1$ core agent. The cost of this transition is one, i.e. $r(g,g')=1$.

Now, consider the transition from a state g in any $\mathscr{CP}_{\ell}$ to g' in any $\mathscr{CP}_{\ell'}$. Note that g and g' are different in two dimensions. First, the numbers of core agents are different, i.e. $\ell \neq \ell'$. Second, the identities of core agents vary, i.e. $C(\ell;g) \neq C(\ell';g')$. To identify the resistance of this transition, we conduct a path of states $\{s_0,s_1,s_2,\cdots,s_{\ell'-1},s_{\ell'}\}$ with $s_0 = g$ and $s_{\ell'} = g'$, which are characterized by the following properties: $s_k \in \mathscr{CP}_k$ such that $C(k-1;s_{k-1}) \subset C(k;s_k) \subseteq C(\ell';g')$, and $P_j(s_i) = P_j(g')$ for each j in the core, for all $k = 2,3,\cdots,\ell'$.

(i) $s_1 \in \mathscr{CP}_1$ such that $C(1;s_1) \subseteq C(\ell';g')$, i.e. the core agent in s_1 is also a core agent in g';

(ii) $s_k \in \mathscr{CP}_k$ such that $C(k-1;s_{k-1}) \subset C(k;s_k) \subseteq C(\ell';g')$, and $P_j(s_i) = P_j(g')$ for each j in the core, for all $k = 2,3,\cdots,\ell'$.

As argued above, the resistance of transition from s_i to s_{i+1} is one. Thus, the resistance of this path is equal to the sum of resistances, i.e. $r(g,g') = \sum_{i=1}^{\ell'} r(s_{i-1},s_i) = \ell'$.

Now we move on to characterize the set of stochastically stable states. Note that the number of all absorbing sets is N. Recall that a $G_i^{\star\star}$-tree consists of edges that connect every absorbing set. Thus, the number of edges of every permissible $G_i^{\star\star}$-tree is $N-1$. Note that the stochastic potential of an absorbing set $G_i^{\star\star}$ is defined as the minimum sum of resistances of edges of the $G_i^{\star\star}$-tree. The minimum is obtained when the resistance of every edge of the $G_i^{\star\star}$-tree is one.

Following this, we move forward to calculate the stochastical potential of each absorbing set characterized by Proposition 1.5.1. First, consider the stochastic potential of absorbing set $G_i^{\star\star}$ formed by a $g \in \mathscr{CP}_1$. Consider the $G_i^{\star\star}$-tree where the transition from each absorbing set to $G_i^{\star\star}$ is direct as the structure depicted in Figure A.1. Note that as we argued above, the resistance from any state to a CP_1 network is one. The stochastic potential of this $G_i^{\star\star}$-tree is given by the sum of resistances of all edges, i.e. $\gamma(G_i^{\star\star}) = N-1$.

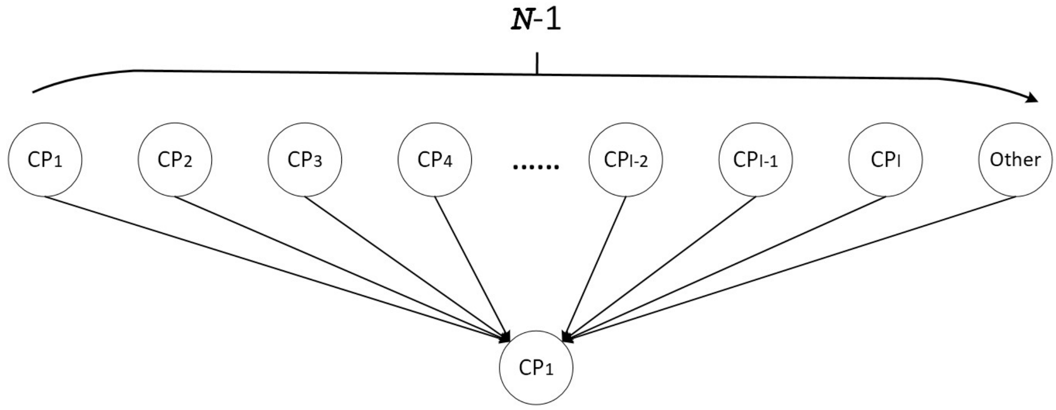


Figure A.1. *The structure of a $G_i^{\star\star}$-tree formed by $g \in \mathscr{C}\mathscr{P}_1$.*

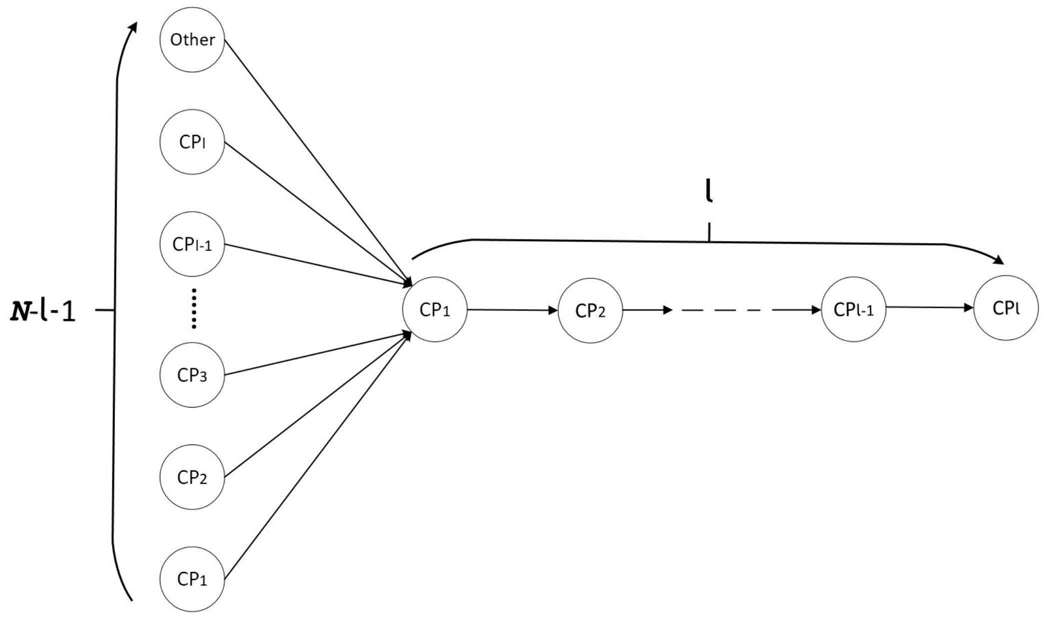


Figure A.2. *An sketch structure of a $G_j^{\star\star}$-tree with any $g \in \mathscr{C}\mathscr{P}_\ell$.*

Next, consider the absorbing sets $G_{\ell}^{\star\star}$ formed by a state $g \in \mathscr{C}\mathscr{P}_{\ell}$. We first conduct a sequence of absorbing sets $\{G_1^{\star\star}, G_2^{\star\star}, \cdots, G_{\ell-1}^{\star\star}, G_{\ell}^{\star\star}\}$ with $G_j^{\star\star}$ formed by a state s_i in $\mathscr{C}\mathscr{P}_j$, $j = 1, 2, \cdots, \ell$, where s_i holds the same property as above. Then, for those absorbing sets not included in this sequence, consider the transition from them to $G_1^{\star\star}$ directly. Figure A.2 depicts such a $G_{\ell}^{\star\star}$-tree. According to the above construction, the resistance of transition from $G_i^{\star\star}$ to $G_{i+1}^{\star\star}$ is one, for any $i = 1, 2, \cdots, \ell-1$. Further, from other $N-\ell-1$ absorbing sets, one mistake is sufficient from each absorbing set to $G_1^{\star\star}$. Thus, the sum of resistance of this $G_j^{\star\star}$-tree is $\ell + N - \ell - 1$, i.e. the stochastic potential of this $G_j^{\star\star}$-tree is $\gamma(G_j^{\star\star}) = N - 1$.

Since $N-1$ is the minimum stochastic potential, for any absorbing state $g \in \mathscr{C}\mathscr{P}_{\bar{\ell}}$ is thus stochastically stable according to Kandori et al. (1993) and Young (1993). □

A.2 Appendix B

A.2.1 Proofs of Section 3.4

Proof of Lemma 3.4.1. First, consider the case where some agents do not form the maximum number of links they can support. Given $\gamma < d < c < a < b$, those agents will fill their remaining slots as any extra link yields at least $d - \gamma$. As a result, every agent will form the maximum number of links.

Next, consider the case where agents within the same group play different actions, e.g. some agents in the low-constraint group play action A and the other agents play action B. Note that the LOPs for any agent in the same group are identical given any strategy profile. First, consider agents in the low-constraint group. Their LOPs are given by

$$v(A,m) = a \cdot \min\{k^{\ell}, m-1\} + c \cdot (k^{\ell} - \min\{k^{\ell}, m-1\}) - \gamma \cdot k^{\ell} \tag{A.3}$$

and

$$v(B,m) = b \cdot \min\{k^{\ell}, n-m-1\} + d \cdot (k^{\ell} - \min\{k^{\ell}, n-m-1\}) - \gamma \cdot k^{\ell} \tag{A.4}$$

Consider two agents i and j in N_{ℓ}. Assume that agent i plays action A and agent j plays action B. If agent i behaves optimally, then it must be the case $v(A,m) \geq v(B,m-1)$. By equations (A.3) and (A.4), we have

$$a \cdot \min\{k^{\ell}, m-1\} + c \cdot (k^{\ell} - \min\{k^{\ell}, m-1\}) \geq b \cdot \min\{k^{\ell}, n-m\} + d \cdot (k^{\ell} - \min\{k^{\ell}, n-m\}). \tag{A.5}$$

Similarly, agent j behaves optimally if $v(B,m) \geq v(A,m+1)$. We thus have

$$b \cdot \min\{k^{\ell}, n-m-1\} + d \cdot (k^{\ell} - \min\{k^{\ell}, n-m-1\}) \geq a \cdot \min\{k^{\ell}, m\} + c \cdot (k^{\ell} - \min\{k^{\ell}, m\}). \tag{A.6}$$

Note that for agents in the low-constraint group to choose different actions as best responses, inequalities (A.5) and (A.6) have to hold simultaneously. We solve inequalities (A.5) and (A.6) independently by discussing different levels of k^{ℓ}. Table A.1 presents the solutions for these two inequalities for various relevant ranges of thresholds.

One can check that inequalities (A.5) and (A.6) never have a common solution for various levels of k^{ℓ}. This implies that both agents can't behave optimally simultaneously. Thus, a strategy profile where agents in the low-constraint group play different actions is not a Nash equilibrium.

The argument for agents in the high-constraint group follows the same logic and is omitted. Therefore, for any $s \notin \overrightarrow{AA} \cup \overrightarrow{BB} \cup \overrightarrow{AB} \cup \overrightarrow{BA}$, s is not a Nash equilibrium. □

Proof of Lemma 3.4.3. The proof proceeds by contradiction. Consider two agents $i \in N_{\ell}$ and $j \in N_h$. Note that the distribution of actions in a strategy profile $s \in \overrightarrow{AB}$ is (n_{ℓ}, n_h). Then agent i is now playing action A and agent j is playing action B. We assume that s is a Nash equilibrium,

Table A.1. *Conditions on m such that i and j behave optimally at different levels of* k^ℓ.

	$k^\ell > m-1$ and $k^\ell > n-m$	$k^\ell \leq m-1$ and $k^\ell > n-m$	$k^\ell \leq n-m$
$v(A,m) \geq v(B,m-1)$	$m \geq \frac{(n-1)(b-d)-k^\ell(c-d)}{a+b-c-d}+1$	$m \geq n - \frac{a-d}{b-d}k^\ell$	never
	$k^\ell > m$ and $k^\ell > n-m-1$	$k^\ell \leq m$ and $k^\ell > n-m-1$	$k^\ell \leq n-m-1$
$v(B,m) \geq v(A,m+1)$	$m \leq \frac{(n-1)(b-d)-k^\ell(c-d)}{a+b-c-d}$	$m \leq n-1-\frac{a-d}{b-d}k^\ell$	always

then neither i nor j will deviate.

Consider agent i. Agent i will stay with A if

$$\begin{aligned} v_i(A,n_\ell) &= a\cdot\min\{k^\ell,n_\ell-1\}+c\cdot(k^\ell-\min\{k^\ell,n_\ell-1\})-M\cdot k^\ell \\ &\geq b\cdot\min\{k^\ell,n_h\}+d\cdot(k^\ell-\min\{k^\ell,n_h\})-M\cdot k^\ell = v_i(B,n_\ell-1). \end{aligned} \tag{A.7}$$

Similarly, agent j will stay with B if

$$\begin{aligned} v_j(B,n_\ell) &= b\cdot\min\{k^h,n_h-1\}+d\cdot(k^h-\min\{k^h,n_h-1\})-M\cdot k^h \\ &\geq a\cdot\min\{k^h,n_\ell\}+c\cdot(k^h-\min\{k^h,n_\ell\})-M\cdot k^h = v_j(A,n_\ell+1). \end{aligned} \tag{A.8}$$

Note that in a Nash equilibrium, inequalities (A.7) and (A.8) have to hold simultaneously. First, we solve inequality (A.7) to find the switching threshold for agent i. We have three sub-cases by considering the order of k^ℓ, $n_\ell-1$, and n_h.

i) if $k^\ell \leq n_h \leq n_\ell - 1$, we have

$$a\cdot k^\ell + c\cdot(k^\ell-k^\ell) \geq b\cdot k^\ell + d\cdot(k^\ell-k^\ell) \Rightarrow a\cdot k^\ell \geq b\cdot k^\ell \Rightarrow a \geq b$$

which is in contradiction to the order of payoffs in the coordination game. Thus, agent i will deviate.

ii) if $n_h < k^\ell \leq n_\ell - 1$, we have

$$a \cdot k^\ell + c \cdot (k^\ell - k^\ell) \geq b \cdot n_h + d \cdot (k^\ell - n_h) \Rightarrow k^\ell \geq \frac{b-d}{a-d} \cdot n_h. \tag{A.9}$$

One can check that inequality (A.9) holds iff $\frac{b-d}{a-d} \cdot n_h \leq n_\ell - 1$.[7] Note that $\frac{b-d}{a-d} \cdot n_h > n_h$ holds as $b - d > a - d$. The solution to inequality (A.9) is $\frac{b-d}{a-d} \cdot n_h \leq k^\ell \leq n_\ell - 1$. Moreover, if $\frac{b-d}{a-d} \cdot n_h \leq n_\ell - 1$, then we have that $n_h \leq \frac{a-d}{b-d} \cdot n_\ell - (a-d)$. Thus, agent i will stay with A if $n_h \leq \frac{a-d}{b-d} \cdot n_\ell - (a-d)$.

iii) if $n_h \leq n_\ell - 1 < k^\ell$, we have

$$a \cdot (n_\ell - 1) + c \cdot (k^\ell - (n_\ell - 1)) \geq b \cdot n_h + d \cdot (k^\ell - n_h) \Rightarrow k^\ell \geq \frac{(b-d)n_h - (a-c)(n_\ell - 1)}{c-d}.$$

Agent i will stay with A if $k^\ell \geq \max\{\frac{(b-d)n_h-(a-c)(n_\ell-1)}{c-d}, n_\ell - 1\}$. Furthermore, one can check that $n_h \leq \frac{a-d}{b-d} \cdot n_\ell - (a-d)$ if $\frac{(b-d)n_h-(a-c)(n_\ell-1)}{c-d} \leq n_\ell - 1$. Thus, we have that $k^\ell > n_\ell - 1$ if $n_h \leq \frac{a-d}{b-d} \cdot n_\ell - (a-d)$ and $k^\ell \geq \frac{(b-d)n_h-(a-c)(n_\ell-1)}{c-d}$ if $n_h > \frac{a-d}{b-d} \cdot n_\ell - (a-d)$.

Summary up, agent i will stay with A if

$$k^\ell \geq \begin{cases} \frac{b-d}{a-d} \cdot n_h, & \text{if } n_h \leq \frac{a-d}{b-d} \cdot n_\ell - (a-d). \\ \frac{(b-d)n_h-(a-c)(n_\ell-1)}{c-d}, & \text{if } n_h > \frac{a-d}{b-d} \cdot n_\ell - (a-d). \end{cases} \tag{A.10}$$

Similarly, we solve inequality (A.8) by considering various ranges of k^h.

i) if $k^h \leq n_h - 1 < n_\ell$, we have

$$b \cdot k^h + d \cdot (k^h - k^h) \geq a \cdot k^h + c \cdot (k^h - k^h) \Rightarrow b \cdot k^h \geq a \cdot k^h \Rightarrow b \geq a$$

which is consistent with the order of payoffs in the coordination game. Thus, agent j will stay with

[7] If $\frac{b-d}{a-d} \cdot n_h > n_\ell - 1$, then $n_h > \frac{a-d}{b-d} \cdot n_\ell - (a-d)$. Thus, we have that $k^\ell \leq n_\ell - 1$ which contradicts $k^\ell \geq \frac{b-d}{a-d} \cdot n_h$. In this case, inequality (A.9) does not hold. This implies that agent i will deviate.

B.

ii) if $n_h - 1 < k^h < n_\ell$, we have

$$b \cdot (n_h - 1) + d \cdot (k^h - (n_h - 1)) \geq a \cdot k^h + c \cdot (k^h - k^h) \Rightarrow k^h \leq \frac{b-d}{a-d} \cdot (n_h - 1).$$

Agent j will stay with B if $k^h \leq \min\{\frac{b-d}{a-d} \cdot (n_h - 1), n_\ell\}$. Moreover, we obtain that $n_h < \frac{a-d}{b-d} \cdot n_\ell + (b-d)$ from $\frac{b-d}{a-d} \cdot (n_h - 1) < n_\ell$. Thus, we have that $k^h \leq \frac{b-d}{a-d} \cdot (n_h - 1)$ if $n_h < \frac{a-d}{b-d} \cdot n_\ell + (b-d)$ and $k^h < n_\ell$ if $n_h \geq \frac{a-d}{b-d} \cdot n_\ell + (b-d)$.

iii) if $n_h - 1 < n_\ell \leq k^h$, we have

$$b \cdot (n_h - 1) + d \cdot (k^h - (n_h - 1)) \geq a \cdot n_\ell + c \cdot (k^h - n_\ell) \Rightarrow k^h \leq \frac{(b-d)(n_h - 1) - (a-c)n_\ell}{c-d}. \quad \text{(A.11)}$$

Inequality (A.11) has solution if and only if $\frac{(b-d)(n_h-1)-(a-c)n_\ell}{c-d} \geq n_\ell$.[8] Furthermore, we have that $n_h \geq \frac{a-d}{b-d} \cdot n_\ell + (b-d)$ if $\frac{(b-d)(n_h-1)-(a-c)n_\ell}{c-d} \geq n_\ell$. Thus, agent j will stay with B if $n_\ell \leq k^h \leq \frac{(b-d)(n_h-1)-(a-c)n_\ell}{c-d}$, in the case where $n_h \geq \frac{a-d}{b-d} \cdot n_\ell + (b-d)$.

Summary up, agent j will stay with B if

$$k^h \leq \begin{cases} \frac{b-d}{a-d} \cdot (n_h - 1), & \text{if } n_h < \frac{a-d}{b-d} \cdot n_\ell + (b-d). \\ \frac{(b-d)(n_h-1)-(a-c)n_\ell}{c-d}, & \text{if } n_h \geq \frac{a-d}{b-d} \cdot n_\ell + (b-d). \end{cases} \quad \text{(A.12)}$$

Note that equations (A.10)) and (A.12) have to hold simultaneously in a Nash equilibrium. To find the solution to these two equations, there are three sub-cases for the various ranges of n_ℓ and N_h.

First, consider the sub-case where $n_h \leq \frac{a-d}{b-d} \cdot n_\ell - (a-d)$. We have that agent i will stay with A if $k^\ell \geq \frac{b-d}{a-d} \cdot n_h$. As $\frac{a-d}{b-d} \cdot n_\ell - (a-d) < \frac{a-d}{b-d} \cdot n_\ell + (b-d)$, we have agent j will stay with B if

[8] If $\frac{(b-d)(n_h-1)-(a-c)n_\ell}{c-d} < n_\ell$, then $n_h < \frac{a-d}{b-d} \cdot n_\ell + (b-d)$. Thus, we have that $k^h \leq \frac{(b-d)(n_h-1)-(a-c)n_\ell}{c-d}$ which contradicts $k^h \geq n_\ell$. In this case, inequality (A.11) does not have a solution, and agent j will deviate.

and only if $k^h \leq \frac{b-d}{a-d} \cdot (n_h - 1)$. Therefore, the condition for both i and j staying in their action is $k^h \leq \frac{b-d}{a-d} \cdot (n_h - 1) < \frac{b-d}{a-d} \cdot n_h \leq k^\ell$, which contradicts our assumption $k^\ell < k^h$. Thus, either agent i or j will switch.

Next, consider the sub-case where $\frac{a-d}{b-d} \cdot n_\ell - (a-d) < n_h < \frac{a-d}{b-d} \cdot n_\ell + (b-d)$. We have that agent i will stay with A if $k^\ell \leq \frac{(b-d)n_h-(a-c)(n_\ell-1)}{c-d}$ and agent j will stay with B if $k^h \leq \frac{b-d}{a-d} \cdot (n_h - 1)$. Since $n_h > \frac{a-d}{b-d} \cdot n_\ell - (a-d)$ and $k^\ell \geq \frac{(b-d)n_h-(a-c)(n_\ell-1)}{c-d}$, we obtain that $k^\ell > n_\ell - 1$. And since $n_h < \frac{a-d}{b-d} \cdot n_\ell + (b-d)$ and $k^h \leq \frac{b-d}{a-d} \cdot (n_h - 1)$, we have that $k^h < n_\ell$. Moreover, since n_ℓ is an integer, $k^\ell \geq n_\ell - 1$ implies that $k^\ell \geq n_\ell$, and $k^h < n_\ell$ implies that $k^h \leq n_\ell - 1$. Thus, we have that $k^h \leq n_\ell - 1 < n_\ell \leq k^\ell$, which contradicts $k^\ell < k^h$. Therefore, either agent i or j will deviate.

Finally, consider the sub-case where $n_h \geq \frac{a-d}{b-d} \cdot n_\ell + (b-d)$. We have that agent j will stay with B if $k^h \leq \frac{(b-d)(n_h-1)-(a-c)n_\ell}{c-d}$. As $\frac{a-d}{b-d} \cdot n_\ell + (b-d) > \frac{a-d}{b-d} \cdot n_\ell - (a-d)$, from equation (A.10) we find that agent i will stay with A if and only if $k^\ell \geq \frac{(b-d)n_h-(a-c)(n_\ell-1)}{c-d}$. Thus, the condition for both agents staying in their actions is $k^\ell \geq \frac{(b-d)n_h-(a-c)(n_\ell-1)}{c-d} > \frac{(b-d)(n_h-1)-(a-c)n_\ell}{c-d} \geq k^h$, which contradicts $k^\ell < k^h$. Therefore, either agent i or j will switch in this sub-case.

Consequently, there does not exist any n_ℓ and n_h such that both agents i and j stay with their actions, i.e. either i or j will deviate. Thus, $s \in \overrightarrow{AB}$ is not a Nash equilibrium. □

Proof of Lemma 3.4.4. Note that the distribution of actions in a strategy profile $s \in \overrightarrow{BA}$ is (n_h, n_ℓ). Consider two agents $i \in N_\ell$ and $j \in N_h$. Note that agent i is playing action B and agent j is playing action A. As s is a Nash equilibrium, neither i nor j will deviate from their current actions.

First, consider agent i. She will stay with B if and only if

$$\begin{aligned} v_i(B, n_h) &= b \cdot \min\{k^\ell, n_\ell - 1\} + d \cdot (k^\ell - \min\{k^\ell, n_\ell - 1\}) - M \cdot k^\ell \\ &\geq a \cdot \min\{k^\ell, n_h\} + c \cdot (k^\ell - \min\{k^\ell, n_h\}) - M \cdot k^\ell = v_i(A, n_h + 1). \end{aligned} \tag{A.13}$$

Similarly, agent j will stay with A if and only if

$$\begin{aligned} v_j(A,n_h) &= a\cdot\min\{k^h,n_h-1\}+c\cdot(k^h-\min\{k^h,n_h-1\})-M\cdot k^h \\ &\geq b\cdot\min\{k^h,n_\ell\}+d\cdot(k^h-\min\{k^h,n_\ell\})-M\cdot k^h = v_j(B,n_h-1). \end{aligned} \tag{A.14}$$

Note that in a Nash equilibrium, inequalities (A.13) and (A.14) have to hold simultaneously. First, we solve inequalities (A.13) to obtain switching thresholds for agent i. There are three sub-cases by considering various orders of k^ℓ, $n_\ell-1$, and n_h.

i) if $k^\ell \leq n_h < n_\ell - 1$, we have

$$b\cdot k^\ell + d\cdot(k^\ell-k^\ell) \geq a\cdot k^\ell + c\cdot(k^\ell-k^\ell) \Rightarrow b\cdot k^\ell \geq a\cdot k^\ell \Rightarrow b\geq a$$

which coincides with the order of payoffs in the coordination game. Thus, agent i will stay.

ii) if $n_h < k^\ell \leq n_\ell - 1$, we have

$$b\cdot k^\ell + d\cdot(k^\ell-k^\ell) \geq a\cdot n_h + c\cdot(k^\ell-n_h) \Rightarrow k^\ell \geq \frac{a-c}{b-c}\cdot n_h.$$

Note that $\frac{a-c}{b-c}\cdot n_h < n_h$ holds as $a-c<b-c$. Following that, we have $k^\ell > \frac{a-c}{b-c}\cdot n_h$ whenever $n_h < k^\ell \leq n_\ell - 1$. Thus, inequality (A.13) holds in the relevant range of k^ℓ. This implies that agent i will stay with action B whenever $n_h < k^\ell \leq n_\ell - 1$.

iii) if $n_h \leq n_\ell - 1 < k^\ell$, we have

$$b\cdot(n_\ell-1) + d\cdot(k^\ell-(n_\ell-1)) \geq a\cdot n_h + c\cdot(k^\ell-n_h) \Rightarrow k^\ell \leq \frac{(b-d)(n_\ell-1)-(a-c)n_h}{c-d} = \overline{k^\ell}.$$

One can check that $\overline{k^\ell} > (n_\ell - 1)$.[9] Thus, inequality (A.13) holds if and only if $n_\ell - 1 < k^\ell \leq \overline{k^\ell}$

[9] This is obtained by considering $\frac{(b-d)(n_\ell-1)-(a-c)n_h}{c-d} - (n_\ell-1) = \frac{(b-d)(n_\ell-1)-(a-c)n_h-(c-d)(n_\ell-1)}{c-d} = \frac{(b-c)(n_\ell-1)-(a-c)n_h}{c-d} \geq \frac{(b-c)n_h-(a-c)n_h}{c-d} > 0$.

and agent i will stay with B in the relevant range. As we have seen in cases i) and ii), agents will stay with B if $k^\ell \leq n_h < n_\ell - 1$ and $n_h < k^\ell \leq n_\ell - 1$. Combining these results with the condition obtained in case iii) yields that agent i will stay with action B if and only if $k^\ell \leq \overline{k^\ell}$.

Now, consider agent j. Similarly, we solve inequality (A.14) by considering different orders of k^h, $n_h - 1$ and n_ℓ.

i) if $k^h \leq n_h - 1 < n_\ell$, we have

$$a \cdot k^h + c \cdot (k^h - k^h) \geq b \cdot k^h + d \cdot (k^h - k^h) \Rightarrow a \cdot k^h \geq b \cdot k^h \Rightarrow a \geq b$$

which contradicts the order of payoffs $b > a$. Thus, agent j will deviate.

ii) if $n_h - 1 < k^h < n_\ell$, we have

$$a \cdot (n_h - 1) + c \cdot (k^h - (n_h - 1)) \geq b \cdot k^h + d \cdot (k^h - k^h) \Rightarrow k^h \leq \frac{a-c}{b-c} \cdot (n_h - 1).$$

Note that $\frac{a-c}{b-c} \cdot (n_h - 1) < n_h - 1$ holds since $b > a$. There is a contradiction between $k^h > n_h - 1$ and $k^h \leq \frac{a-c}{b-c} \cdot (n_h - 1)$. This implies that inequality (A.14) does not hold and agent j will deviate in this range.

iii) if $n_h - 1 < n_\ell \leq k^h$, we have

$$a \cdot (n_h - 1) + c \cdot (k^h - (n_h - 1)) \geq b \cdot n_\ell + d \cdot (k^h - n_\ell) \Rightarrow k^h \geq \frac{(b-d)n_\ell - (a-c)(n_h - 1)}{c-d} = \underline{k^h}.$$

Note that $\underline{k^h} > n_\ell$.[10] Thus, agent j will stay with action A if and only if $k^h \geq \underline{k^h}$. Attending the results in cases i) and ii), agents will stay with B if $k^h \leq n_h - 1 < n_\ell$ and $n_h - 1 < k^h \leq n_\ell$. Combining these results with the condition obtained in case iii) yields that agent j will stay with

[10] This is obtained by considering $\frac{(b-d)n_\ell-(a-c)(n_h-1)}{c-d} - n_\ell = \frac{(b-d)n_\ell-(a-c)(n_h-1)-(c-d)n_\ell}{c-d} = \frac{(b-c)n_\ell-(a-c)(n_h-1)}{c-d} > \frac{(b-c)n_h-(a-c)n_h+(a-c)}{c-d} > 0$.

action A if and only if $k^h \geq \underline{k^h}$.

Consequently, for both agents i and j to stay with their current actions requires that $k^\ell \leq \overline{k^\ell}$ and $k^h \geq \underline{k^h}$. Furthermore, if both $k^\ell \leq \overline{k^\ell}$ and $k^h \geq \underline{k^h}$ hold, then both agents i and j with stay with their current actions. □

A.2.2 Proofs of Section 3.5

Proof of Proposition 3.5.1. This proof proceeds in two steps. In the first step, we prove that from any state s, the dynamics leads to a monomorphic or polymorphic state, i.e. a state $s' \in \overrightarrow{AA} \cup \overrightarrow{BB} \cup \overrightarrow{AB} \cup \overrightarrow{BA}$, with a positive probability. In the second step, we show that from each state s', this process converges to a Nash equilibrium.

We prove the first step by constructing a sequence of revisions leading to a monomorphic or polymorphic state from any state s. This sequence of revisions consists of multiple rounds where in each round, one of the two groups is selected and all agents in this selected group can revise their strategies.[11] Moreover, we assume that if agents are indifferent between two actions, they will remain with their current actions.[12]

Consider an initial state s with distribution of actions $(m, n-m)$. In the first round, give the revision opportunity to agents in N_ℓ. Consider the case where A-agents in N_ℓ remain. This implies that $v(A,m) \geq v(B,m-1)$, i.e. $m \geq M^1_\ell$ (or $m \geq M^2_\ell$). Table 3.1 reveals that for any B-agents in N_ℓ, the optimal choice is to switch. Thus, we have reached a state where all agents in N_ℓ play action A. Now, consider the case where A-agents in N_ℓ switch. This implies that $v(B,m-1) > v(A,m)$, i.e. $m < M^1_\ell$ (or $m < M^2_\ell$). This implies that $m \leq M^1_\ell - 1$ (or $m \leq M^2_\ell - 1$), and furthermore, $v(A,m+1) \leq v(B,m)$. Thus, B-agents will remain and we have reached a state where all agents in

[11] This sequence occurs with positive probability since the probability of each agent receiving the revision opportunity is positive.

[12] Note that since agents randomize between two actions when they are indifferent, the probability of their remains is positive.

N_ℓ play action B.

In the second round, give the revision opportunity to agents in N_h. Assume that the distribution of actions is $(m', n-m')$ after agents in N_ℓ have revised. Agents in N_h decide whether to remain or to switch based on this new distribution of actions. Following similar arguments as for those agents in N_ℓ, we will arrive at a state where all agents in N_h play the same action A or B.

Consequently, after two rounds of revisions, we have reached a state where agents in the same group play the same action, i.e. a state $s' \in \overrightarrow{AA} \cup \overrightarrow{BB} \cup \overrightarrow{AB} \cup \overrightarrow{BA}$.

In the second step, we show this process will converge to a Nash equilibrium from any state s'. First, consider any state $s' \in \overrightarrow{AA} \cup \overrightarrow{BB}$. If all agents play action A (and also B), no one will switch since $v_i(A,n) > v_i(B,n-1), \forall i \in N$ (and since $v_i(B,0) > v_i(A,1)$).

Now consider a state $s' \in \overrightarrow{AB}$. If agents in N_ℓ find it optimal to play action A, then we have that $v_i(A,n_\ell) > v_i(B,n_\ell - 1), \forall i \in N_\ell$, which implies that $n_\ell \geq M_\ell^1$ (or $n_\ell \geq M_\ell^2$). Note that $M_\ell^2 > M_h^2$ and $M_\ell^1 > M_h^1$ holds (see in Table 3.1). We thus obtain that $n_\ell > M_h^1$ (or $n_\ell > M_h^2$). Table 3.1 reveals that it is optimal for B-agents in N_h to switch to A. Similarly, one can check that if agents in N_h find it optimal to play action B, then it is optimal for A-agents in N_ℓ to switch to B. Thus, we will arrive at a state $s \in \overrightarrow{AA} \cup \overrightarrow{BB}$.

Then, consider a state $s' \in \overrightarrow{BA}$. In the proof of Lemma 3.4.4 we have argued that it is optimal for agents in N_ℓ to play B and for agents in N_h to play A iff $k^\ell \leq \overline{k^\ell}$ and $k^h \geq \underline{k^h}$. This implies that whenever $k^\ell < \overline{k^\ell}$ and $k^h > \underline{k^h}$, agents will strictly prefer to remain at their actions when they receive the revision opportunity. It follows that if $k^\ell \geq \overline{k^\ell}$, agents in N_ℓ will find it optimal to play A and switch. Similarly, if $k^h \leq \underline{k^h}$, agents in N_h will find it optimal to play B and switch. Then we will reach a state $s \in \overrightarrow{AA} \cup \overrightarrow{BB}$.

Consequently, this process will finally converge to a state $s \in \overrightarrow{AA} \cup \overrightarrow{BB} \cup \overrightarrow{BA}$ if $k^\ell < \overline{k^\ell}$ and $k^h > \underline{k^h}$, and will converge to a state $s \in \overrightarrow{AA} \cup \overrightarrow{BB}$ otherwise. According to Proposition 3.4.1, s is a Nash equilibrium for the relevant ranges of k^ℓ and k^h.

Now, we proceed to show that this process moves between any pair of states s and s' in $\overrightarrow{AA}$ (and also for any pair in $\overrightarrow{BB}$ and $\overrightarrow{BA}$) with positive probability. Note that s and s' only differ in the linking strategies of agents. As agents are indifferent between linking to any of those agents with the same action, this process will move between any such two strategies with a positive probability. Thus, all states in $\overrightarrow{AA}$ (also in $\overrightarrow{BB}$ and $\overrightarrow{BA}$) form an absorbing set.[13] □

Proof of Proposition 3.5.2. First, note that if $k^\ell \geq \underline{k^\ell}$ or $k^h \leq \overline{k^h}$, $\overrightarrow{AA}$ and $\overrightarrow{BB}$ are the only two absorbing sets.

First, consider the transition from $\overrightarrow{AA}$ to $\overrightarrow{BB}$.

Note that A-agents with the lower constraint k^ℓ require fewer mutations to switch than agents with the higher constraint k^h. To find the minimum number of mutations required for this transition, we thus start with agents in N_ℓ. Denote by $n - m_\ell^{AB}$ the minimum number of B-agents required for the successful transition of agents in N_ℓ. Consequently, m_ℓ^{AB} is the maximum number of remaining A-agents.

First, note that A-agents with the lower constraint will always switch if $m_\ell^{AB} \leq n - k^\ell$. Since that $m_\ell^{AB} \leq n - k^\ell$ implies $k^\ell \leq n - m_\ell^{AB}$, the number of B-agents is sufficient such that A-agents can fill all their slots with B-agents. Thus, we now turn to the case where $m_\ell^{AB} > n - k^\ell$, i.e. the number of B-agents is insufficient such that A-agents cannot fill all their slots with B-agents. We now need to determine the payoff A-agents get when they stay with A. Hence we need to distinguish two sub-cases: i) A-agents can fill all their slots with other A-agents, i.e. $m_\ell^{AB} \geq k^\ell + 1$, and ii) A-agents have to link to both A- and B-agents, i.e. $m_\ell^{AB} < k^\ell + 1$.

Consider sub-case i). According to Table 3.1, the switching threshold for A-agents is $m_\ell^{AB} = \lfloor M_\ell^2 \rfloor$. Observe now that this sub-case will happen if indeed $m_\ell^{AB} = \lfloor M_\ell^2 \rfloor \geq k^\ell + 1$. Attending to the definition of M_ℓ^2 and solving for k^ℓ, we have that $k^\ell \leq \frac{(n-1)(b-d)}{a+b-2d}$.

[13] The discussion regarding states in $\overrightarrow{BA}$ is consistently established on the condition $k^\ell < \overline{k^\ell}$ and $k^h > \underline{k^h}$.

Now, consider sub-case ii). According to Table 3.1, the switching threshold for A-agents in this sub-case is given by $m_\ell^{AB} = \lfloor M_\ell^1 \rfloor$. Solving for k^ℓ yields $k^\ell > \frac{(n-1)(b-d)}{a+b-2d}$.

Recall that if there are $n - k^\ell$ or less A-agents, all agents in N_ℓ will switch to B. Thus, the maximum number of A-agents for the transition to occur is characterized by

$$m_\ell^{AB} = \begin{cases} \max\{\lfloor M_\ell^2 \rfloor, n-k^\ell\}, & \text{if } k^\ell \leq \frac{(n-1)(b-d)}{a+b-2d}. \\ \max\{\lfloor M_\ell^1 \rfloor, n-k^\ell\}, & \text{if } k^\ell > \frac{(n-1)(b-d)}{a+b-2d}. \end{cases}$$

One can check that $\lfloor M_\ell^2 \rfloor > n - k^\ell$ always holds for any k^ℓ, and $\lfloor M_\ell^1 \rfloor > n - k^\ell$ holds whenever $k^\ell > \frac{(n-1)(a-c)}{a+b-2c}$. Since $\frac{(n-1)(a-c)}{a+b-2c} < \frac{(n-1)(b-d)}{a+b-2d}$, we thus have

$$m_\ell^{AB} = \begin{cases} \lfloor M_\ell^2 \rfloor, & \text{if } k^\ell \leq \frac{(n-1)(b-d)}{a+b-2d}. \\ \lfloor M_\ell^1 \rfloor, & \text{if } k^\ell > \frac{(n-1)(b-d)}{a+b-2d}. \end{cases}$$

Therefore, the minimum number of mutations required for the transition of A-agents in N_ℓ is

$$n - m_\ell^{AB} = \begin{cases} n - \lfloor M_\ell^2 \rfloor, & \text{if } k^\ell \leq \frac{(n-1)(b-d)}{a+b-2d}. \\ n - \lfloor M_\ell^1 \rfloor, & \text{if } k^\ell > \frac{(n-1)(b-d)}{a+b-2d}. \end{cases}$$

Now, we assess the largest number of B-agents after the mutations and switches (excluding switches among agents in N_h for now), i.e. agents who have mutated and agents in N_ℓ who have switched to B. For this, assume all mutations occur in N_h. Thus, the number of B-agents is $n_\ell + n - m_\ell^{AB}$. It follows that the number of remaining A-agents in N_h is $m_\ell^{AB} - n_\ell$. We have that $m_\ell^{AB} - n_\ell < n_h$ holds for any relevant range of k^ℓ since the mutations occur among N_h and every agent in N_ℓ switched.

Now consider agents in N_h. First, denote by m_h^{AB} the number of A-agents required for agents in N_h to switch. Following the same argument as for agents in N_ℓ, attending to Table 3.1 reveals that m_h^{AB} is given by

$$m_h^{AB} = \begin{cases} \lfloor M_h^2 \rfloor, & \text{if } k^h \leq \frac{(n-1)(b-d)}{a+b-2d}. \\ \lfloor M_h^1 \rfloor, & \text{if } k^h > \frac{(n-1)(b-d)}{a+b-2d}. \end{cases}$$

If the number of A-agents $m_\ell^{AB} - n_\ell$ is less than m_h^{AB}, then agents in N_h switch without requiring more mutations. Otherwise, extra mutations are needed for their transition.

First, consider the case where $k^h \leq \frac{(n-1)(b-d)}{a+b-2d}$. One can check that $\lfloor M_h^2 \rfloor > n_h$ holds. Thus, we have that $m_\ell^{AB} - n_\ell < \lfloor M_h^2 \rfloor$, i.e. the number of existing A-agents is smaller than the number of A-agents required for the transition. Therefore, the number of B-agents is sufficient for A-agents in N_h to switch.

Next, consider the case where $\frac{(n-1)(b-d)}{a+b-2d} < k^h \leq \underline{k^h}$. We have that $\lfloor M_h^1 \rfloor \geq n_h$ holds. This implies that no extra mutation is required for A-agents in N_h to switch.

Now, consider the case where $k^h > \underline{k^h}$. We have that $\lfloor M_h^1 \rfloor < n_h$. Recall that we are now focusing on the case where there are only two absorbing sets. Thus, the range of k^ℓ is restricted on $k^\ell \geq \overline{k^\ell}$ whenever $k^h > \underline{k^h}$. One can check that $m_\ell^{AB} - n_\ell \leq 0$ if $k^\ell \geq \overline{k^\ell}$. Note that if $m_\ell^{AB} - n_\ell \leq 0$, all agents are now playing B and we have reached $\overrightarrow{BB}$.

Combining the results of all three cases, the number of B-agents after mutations and switches among N_ℓ is sufficient for agents in N_h to switch. Denote by $n - m^{AB}$ the minimum number of B-agents for the transition among N_ℓ and N_h. In summary, we have that

$$n - m^{AB} = \begin{cases} n - \lfloor M_\ell^2 \rfloor, & \text{if } k^\ell \leq \frac{(n-1)(b-d)}{a+b-2d}, k^h \leq \frac{(b-d)n_\ell-(a-c)(n_h-1)}{(c-d)}. \\ n - \lfloor M_\ell^1 \rfloor, & \text{if } k^\ell > \frac{(n-1)(b-d)}{a+b-2d}, k^h \leq \frac{(b-d)n_\ell-(a-c)(n_h-1)}{(c-d)}. \\ n - \lfloor M_\ell^1 \rfloor, & \text{if } k^\ell \geq \frac{(b-d)(n_\ell-1)-(a-c)n_h}{(c-d)}, k^h > \frac{(b-d)n_\ell-(a-c)(n_h-1)}{(c-d)}. \end{cases} \tag{A.15}$$

Since there are only two absorbing sets, we have that the stochastic potential of $\overrightarrow{BB}$ is given by

$$r(\overrightarrow{AA},\overrightarrow{BB}) = n - m^{AB}.$$

Second, consider the transition from $\overrightarrow{BB}$ to $\overrightarrow{AA}$.

Note that fewer mutations are required for B-agents with the higher constraint to switch. Thus, to obtain the minimum number of mutations required for this transition, we start with agents in the high-constraint group. Denote by m_h^{BA} the minimum number of mutations required for B-agents in N_h to switch.

Whenever $m_h^{BA} \leq n - k^h - 1$, B-agents in N_h will always stay since there are sufficient other B-agents for them to link to. Thus, we turn to the case where $m_h^{BA} > n - k^h - 1$, i.e. B-agents have to link to both A- and B-agents. Now we have to determine the payoff B-agents get when they switch to A. Thus, we have to consider two sub-cases: i) A-agents have to link to both A- and B-agents after they switch, i.e. $m_h^{BA} < k^h$, and ii) A-agents can fill all their slots with other A-agents, i.e. $m_h^{BA} \geq k^h$.

In sub-case i), the switching threshold for B-agents is given by $m_h^{BA} = \lceil M_h^1 \rceil - 1$ according to Table 3.1. Solving $\lceil M_h^1 \rceil - 1 < k^h$ yields that $k^h \geq \frac{(b-d)n}{a+b-2d} + \frac{a-c}{a+b-2d}$.

In sub-case ii), the switching threshold for B-agents is $m_h^{BA} = \lceil M_h^2 \rceil - 1$ according to Table 3.1. Then by solving $\lceil M_h^2 \rceil - 1 \geq k^h$, we obtain that $k^h < \frac{(b-d)n}{a+b-2d}$.[14]

It remains to be classified what happens in the range $k^h \in \left[\frac{(b-d)n}{a+b-2d}, \frac{(b-d)n}{a+b-2d} + \frac{a-c}{a+b-2d}\right)$. Assume that $m_h^{BA} < k^h$. Since the number of A-agents is less than the constraint, A-agents will have to link to both A- and B-agents. Attending Table 3.1 reveals that for B-agents to switch requires that $m_h^{BA} \geq \lceil M_h^1 \rceil - 1$, which can in turn be written as $k^h \geq \frac{(b-d)n}{a+b-2d} + \frac{a-c}{a+b-2d}$. This lies out of our interval, yielding a contradiction. Thus, we consider $m_h^{BA} \geq k^h$. Now observe that for

[14]Note that k^h is a positive integer. If $k^h > \lceil M_h^1 \rceil - 1$, then $k^h \geq M_h^1$. Similarly, if $k^h \leq \lceil M_h^2 \rceil - 1$, then $k^h < M_h^2$.

$m_h^{BA} = k^h$, we have that $v_i(A, k^h+1) \geq v_i(B, k^h)$ holds, provided that $k^h \geq \frac{(n-1)(b-d)}{a+b-2d}$. Because $k^h \in \left[\frac{(b-d)n}{a+b-2d}, \frac{(b-d)n}{a+b-2d} + \frac{a-c}{a+b-2d}\right)$, exactly k^h mutation are sufficient for the transition in this range. In summary, we have that

$$m_h^{BA} = \begin{cases} \lceil M_h^2 \rceil - 1 & \text{if } k^h < \frac{n(b-d)}{a+b-2d}. \\ k^h & \text{if } k^h \in \left[\frac{n(b-d)}{a+b-2d}, \frac{n(b-d)+(a-c)}{a+b-2d}\right). \\ \lceil M_h^1 \rceil - 1 & \text{if } k^h \geq \frac{n(b-d)+(a-c)}{a+b-2d}. \end{cases} \tag{A.16}$$

Now, observe that $k^h = \lceil M_h^2 \rceil - 1$ if $k^h < M_h^2 \leq k^h + 1$. This holds for $\frac{(n-1)(b-d)}{a+b-2d} \leq k^h < \frac{n(b-d)}{a+b-2d}$. Similarly, we find that $k^h = \lceil M_h^1 \rceil - 1$ if $k^h < M_h^1 \leq k^h + 1$ which can in turn be written as $k^h \in [\frac{(n-1)(b-d)}{a+b-2d}, \frac{n(b-d)}{a+b-2d} + \frac{(a-c)}{a+b-2d})$. Thus, the equation (A.16) is equivalent to

$$m_h^{BA} = \begin{cases} \lceil M_h^2 \rceil - 1 & \text{if } k^h < \frac{(n-1)(b-d)}{a+b-2d}. \\ \lceil M_h^1 \rceil - 1 & \text{if } k^h \geq \frac{(n-1)(b-d)}{a+b-2d}. \end{cases} \tag{A.17}$$

Moreover, we find that $\lceil M_h^2 \rceil - 1 > n - k^h - 1$ holds for any k^h, and $\lceil M_h^1 \rceil - 1 > n - k^h - 1$ Whenever $k^h > \frac{(n-1)(a-c)}{a+b-2c}$. As $\frac{(n-1)(a-c)}{a+b-2c} < \frac{(n-1)(b-d)}{a+b-2d}$, equation (A.17) is true in the relevant range of k^h.

Now, denote by m^{BA} the minimum number of mutations for agents among both N_ℓ and N_h to switch. To maximize the impact of the mutations, assume that all mutations occur in the low-constraint group N_ℓ. Thus, after all B-agents in N_h have switched, the maximum number of A-agents is $\min\{n, m_h^{BA} + n_h\}$. It follows that the minimum number of B-agents now is $\max\{0, n_\ell - m_h^{BA}\}$. If $n_\ell - m_h^{BA} \leq 0$, i.e. if there are no B-agents, then we have reached $\overrightarrow{AA}$ and no extra mutations are required. Thus, we have $m^{BA} = m_h^{BA}$ for the relevant range of k^h.

Consider the case where there are still $m_h^{BA} + n_h$ A-agents left after the mutation and switch, i.e.

$n_\ell - m_h^{BA} > 0$. Now, we have to determine whether the number of A-agents is enough for B-agents in N_ℓ to switch. Following the same argument as above, the switching threshold for B-agents in N_ℓ is given by

$$m_\ell^{BA} = \begin{cases} \lceil M_\ell^2 \rceil - 1 & \text{if } k^\ell < \frac{(n-1)(b-d)}{a+b-2d}. \\ \lceil M_\ell^1 \rceil - 1 & \text{if } k^\ell \geq \frac{(n-1)(b-d)}{a+b-2d}. \end{cases}$$

where m_ℓ^{BA} is the minimum number of A-agents required for agents in N_ℓ to switch to A. It follows that if $m_h^{BA} + n_h > m_\ell^{BA}$, then no extra mutation are required for this transition. If $m_h^{BA} + n_h \leq m_\ell^{BA}$, an additional $m_\ell^{BA} - (m_h^{BA} + n_h)$ mutations are needed. Then total number of mutations is $m_\ell^{BA} - n_h$. In summary, the minimum number of mutations required is

$$m^{BA} = \max\{m_h^{BA}, m_\ell^{BA} - n_h\} \tag{A.18}$$

Since there are only two absorbing sets, we have that the stochastic potential of $\overrightarrow{BB}$ for the relevant ranges of k^ℓ and k^h is given by

$$r(\overrightarrow{BB}, \overrightarrow{AA}) = m^{BA}.$$

Having characterized the stochastic potentials of the absorbing sets, we now proceed to identify the set of stochastically stable states $\mathscr{S}^{\star\star\star}$ for the various ranges of k^ℓ and k^h. Denote by $\Delta(k^\ell, k^h)$ the difference between the stochastic potentials of $\overrightarrow{BB}$ and $\overrightarrow{AA}$, i.e. $\Delta(k^\ell, k^h) = r(\overrightarrow{AA}, \overrightarrow{BB}) - r(\overrightarrow{BB}, \overrightarrow{AA})$. If $\Delta(k^\ell, k^h) > 0$, then $\mathscr{S}^{\star\star\star} = \overrightarrow{AA}$; if $\Delta(k^\ell, k^h) = 0$, then $\mathscr{S}^{\star\star\star} = \overrightarrow{AA} \cup \overrightarrow{BB}$, and if $\Delta(k^\ell, k^h) < 0$, then $\mathscr{S}^{\star\star\star} = \overrightarrow{BB}$.

First, consider the case where $k^\ell < k^h < \frac{(n-1)(b-d)}{a+b-2d}$. We have obtained above that $r(\overrightarrow{BB}, \overrightarrow{AA}) = \max\{\lceil M_h^2 \rceil - 1, \lceil M_\ell^2 \rceil - 1 - n_h\}$ and $r(\overrightarrow{AA}, \overrightarrow{BB}) = n - \lfloor M_\ell^2 \rfloor$ for the relevant ranges of k^ℓ and k^h.

Thus, we have that

$$\Delta(k^\ell,k^h) = \min\{n-\lfloor M_\ell^2\rfloor-\lceil M_h^2\rceil+1, n+n_h+1-\lfloor M_\ell^2\rfloor-\lceil M_\ell^2\rceil\}$$
$$= \min\{\left\lceil \frac{a-d}{b-d}\cdot k^\ell\right\rceil+\left\lfloor \frac{a-d}{b-d}\cdot k^h\right\rfloor-n+1, \left\lceil \frac{a-d}{b-d}\cdot k^\ell\right\rceil+\left\lfloor \frac{a-d}{b-d}\cdot k^\ell\right\rfloor-n_\ell+1\}.$$

One can check that $n-\lfloor M_\ell^2\rfloor-\lfloor M_h^2\rfloor+1<0$ holds whenever $k^\ell<k^h<\frac{(n-1)(b-d)}{a+b-2d}$, which implies that $\Delta(k^\ell,k^h)<0$. Thus, $\mathscr{S}^{\star\star\star}=\overrightarrow{BB}$. In this case, the two thresholds in the proposition are given by $\overline{\overline{k^\ell}}=\underline{\underline{k^\ell}}=\frac{(n-1)(b-d)}{a+b-2d}$.

Second, consider the case where $k^\ell\leq\frac{(n-1)(b-d)}{a+b-2d}\leq k^h\leq\underline{k^h}$. We have that $r(\overrightarrow{BB},\overrightarrow{AA})=\max\{\lceil M_h^1\rceil-1,\lceil M_\ell^2\rceil-1-n_h\}$, and $r(\overrightarrow{AA},\overrightarrow{BB})=n-\lfloor M_\ell^2\rfloor$. Thus,

$$\Delta(k^\ell,k^h) = \min\{n-\lfloor M_\ell^2\rfloor-\lceil M_h^1\rceil+1, n+n_h+1-\lfloor M_\ell^2\rfloor-\lceil M_\ell^2\rceil\}$$
$$= \min\{\left\lceil \frac{a-d}{b-d}\cdot k^\ell\right\rceil+\left\lfloor \frac{k^h(c-d)-(n-1)(b-d)}{a+b-c-d}\right\rfloor, \left\lceil \frac{a-d}{b-d}\cdot k^\ell\right\rceil+\left\lfloor \frac{a-d}{b-d}\cdot k^\ell\right\rfloor-n_\ell+1\}$$
$$:= \min\{\phi(k^\ell,k^h),\psi(k^\ell,k^h)\}.$$

Given that $b>a>c>d$, $\Delta(k^\ell,k^h)$ is weakly increasing in both k^ℓ and k^h. Thus, $\Delta(k^\ell,k^h)$ obtains its minimum at the boundary where $k^\ell=1$ and $k^h=\frac{(n-1)(b-d)}{a+b-2d}$. At this point, we have that

$$\phi(1,\frac{(n-1)(b-d)}{a+b-2d}) = \left\lceil\frac{a-d}{b-d}\right\rceil-\left\lfloor\frac{(n-1)(b-d)}{a+b-2d}\right\rfloor\leq 0;$$
$$\psi(1,\frac{(n-1)(b-d)}{a+b-2d}) = 2-n_\ell\leq 0.$$

Thus, we have that $\Delta(1,\frac{(n-1)(b-d)}{a+b-2d})\leq 0$. [15] When n is sufficiently large, we have that $\Delta(1,\frac{(n-1)(b-d)}{a+b-2d})$ is strictly negative.

We now assess the maximum of $\Delta(k^\ell,k^h)$, which is obtained at the boundary where $k^\ell=$

[15] Notice that $\Delta(1,\frac{(n-1)(b-d)}{a+b-2d})=0$ hold if and only if $n=3$ and $n_\ell=2$, otherwise, $\Delta(1,\frac{(n-1)(b-d)}{a+b-2d})<0$. Furthermore, note that when $n=3$ and $n_\ell=2$, we have that $k_\ell=1$ and $k_h=2$. The transition from $\overrightarrow{AA}$ to $\overrightarrow{BB}$ requires one mutation and the transition from $\overrightarrow{BB}$ to $\overrightarrow{AA}$ requires two mutations. Thus, $\overrightarrow{BB}$ is the unique set of stochastically stable states. Thus, in the main context, we only discuss the case when n is sufficiently large.

$\frac{(n-1)(b-d)}{a+b-2d}$ and $k^h = \underline{k^h}$. We have that

$$\phi(\frac{(n-1)(b-d)}{a+b-2d}, \underline{k^h}) = \left\lceil n_\ell - \frac{(n-1)(b-d)}{a+b-2d} \right\rceil;$$
$$\psi(\frac{(n-1)(b-d)}{a+b-2d}, \underline{k^h}) = \left\lceil \frac{(n-1)(a-d)}{a+b-2d} \right\rceil + \left\lfloor \frac{(n-1)(a-d)}{a+b-2d} \right\rfloor - n_\ell + 1.$$

We find that $\phi(\frac{(n-1)(b-d)}{a+b-2d}, \underline{k^h}) = 0$ if $\frac{(n-1)(b-d)}{a+b-2d} - 1 < n_\ell \leq \frac{(n-1)(b-d)}{a+b-2d}$. Thus, we have that $\phi(\frac{(n-1)(b-d)}{a+b-2d}, \underline{k^h}) < 0$ holds whenever $n_\ell \leq \frac{(n-1)(b-d)}{a+b-2d} - 1$ and $\phi(\frac{(n-1)(b-d)}{a+b-2d}, \underline{k^h}) > 0$ holds whenever $n_\ell > \frac{(n-1)(b-d)}{a+b-2d}$. Similarly, we find that $\psi(\frac{(n-1)(b-d)}{a+b-2d}, \underline{k^h}) = 0$ whenever $\frac{2(n-1)(a-d)}{a+b-2d} \leq n_\ell < \frac{2(n-1)(a-d)}{a+b-2d} + 2$. Thus, we have that $\psi(\frac{(n-1)(b-d)}{a+b-2d}, \underline{k^h}) < 0$ holds whenever $n_\ell \geq \frac{2(n-1)(b-d)}{a+b-2d} + 2$ and $\psi(\frac{(n-1)(b-d)}{a+b-2d}, \underline{k^h}) > 0$ holds whenever $n_\ell < \frac{2(n-1)(b-d)}{a+b-2d}$. In summary, we have that

$$\begin{cases} \Delta(\frac{(n-1)(b-d)}{a+b-2d}, \underline{k^h}) < 0 & \text{if } n_\ell \in (2, \frac{(n-1)(b-d)}{a+b-2d} - 1] \cup [\frac{2(n-1)(a-d)}{a+b-2d} + 2, n-1); \\ \Delta(\frac{(n-1)(b-d)}{a+b-2d}, \underline{k^h}) = 0 & \text{if } n_\ell \in (\frac{(n-1)(b-d)}{a+b-2d} - 1, \frac{(n-1)(b-d)}{a+b-2d}] \cup [\frac{2(n-1)(a-d)}{a+b-2d}, \frac{2(n-1)(a-d)}{a+b-2d} + 2); \\ \Delta(\frac{(n-1)(b-d)}{a+b-2d}, \underline{k^h}) > 0 & \text{if } n_\ell \in (\frac{(n-1)(b-d)}{a+b-2d}, \frac{2(n-1)(a-d)}{a+b-2d}). \end{cases}$$

Consequently, whenever $n_\ell \in (2, \frac{(n-1)(b-d)}{a+b-2d} - 1] \cup [\frac{2(n-1)(a-d)}{a+b-2d} + 2, n-1)$, we have that $\Delta(k^\ell, k^h) < 0$ for any k^ℓ and k^h in the relevant ranges , which implies that $\mathscr{S}^{\star\star\star} = \overrightarrow{BB}$. In this case, the two thresholds in the proposition are given by $\overline{\overline{k^\ell}} = \underline{\underline{k^\ell}} = \frac{(n-1)(b-d)}{a+b-2d}$.

It follows that if $n_\ell \in (\frac{(n-1)(b-d)}{a+b-2d} - 1, \frac{(n-1)(b-d)}{a+b-2d}] \cup [\frac{2(n-1)(a-d)}{a+b-2d}, \frac{2(n-1)(a-d)}{a+b-2d} + 2)$, we have that $\Delta(k^\ell, k^h) = 0$ holds if and only if $k^\ell = \frac{(n-1)(b-d)}{a+b-2d}$ and $k^h = \underline{k^h}$, which implies $\mathscr{S}^{\star\star\star} = \overrightarrow{BB} \cup \overrightarrow{AA}$. Furthermore, we have that $\Delta(k^\ell, k^h) < 0$ for any pair of k^ℓ and k^h such that $k^\ell < \frac{(n-1)(b-d)}{a+b-2d} \leq k^h < \underline{k^h}$, which implies that $\mathscr{S}^{\star\star\star} = \overrightarrow{BB}$. The two thresholds in this case are thus given by $\overline{\overline{k^\ell}} = \underline{\underline{k^\ell}} = \frac{(n-1)(b-d)}{a+b-2d}$.

Moreover, if $n_\ell \in (\frac{(n-1)(b-d)}{a+b-2d}, \frac{2(n-1)(a-d)}{a+b-2d})$, we have that the maximum of $\Delta(k^\ell, k^h)$ is positive, i.e. $\Delta(\frac{(n-1)(b-d)}{a+b-2d}, \underline{k^h}) > 0$ and the minimum is negative, i.e. $\Delta(1, \frac{(n-1)(b-d)}{a+b-2d}) < 0$. Thus, for each

$k^h \in [\frac{(n-1)(b-d)}{a+b-2d}, \underline{k^h})$, there exists a corresponding interval of k^ℓ, such that for any k^ℓ in this interval we have that $\Delta(k^\ell, k^h) = 0$. Note that $\Delta(k^\ell, k^h)$ is weakly increasing in both k^ℓ and k^h. We have that $\Delta(k^\ell, k^h) < 0$ if k^ℓ falls below this interval and $\Delta(k^\ell, k^h) > 0$ if k^ℓ falls above. Therefore, for each $k^h \in [\frac{(n-1)(b-d)}{a+b-2d}, \underline{k^h})$, we have that

$$\mathscr{S}^{\star\star\star} = \begin{cases} \overrightarrow{BB}, & \text{if } k^\ell < \underline{\underline{k^\ell}}. \\ \overrightarrow{BB} \cup \overrightarrow{AA}, & \text{if } k^\ell \in [\underline{\underline{k^\ell}}, \overline{\overline{k^\ell}}]. \\ \overrightarrow{AA}, & \text{if } k^\ell > \overline{\overline{k^\ell}}. \end{cases}$$

where $\overline{\overline{k^\ell}}$ and $\underline{\underline{k^\ell}}$ are the two thresholds which are given by the upper and lower boundaries of this interval respectively.

Now, consider the case where $\frac{(n-1)(b-d)}{a+b-2d} < k^\ell < k^h \leq \underline{k^h}$. we have that $r(\overrightarrow{BB}, \overrightarrow{AA}) = \max\{\lceil M_h^1 \rceil - 1, \lceil M_\ell^1 \rceil - 1 - n_h\}$, and $r(\overrightarrow{AA}, \overrightarrow{BB}) = n - \lfloor M_\ell^1 \rfloor$. Thus,

$$\begin{aligned}
\Delta(k^\ell, k^h) &= \min\{n - \lfloor M_\ell^1 \rfloor - \lceil M_h^1 \rceil + 1, n + n_h + 1 - \lfloor M_\ell^1 \rfloor - \lceil M_\ell^1 \rceil\} \\
&= \min\{n + \left\lceil \frac{k^\ell(c-d) - (n-1)(b-d)}{a+b-c-d} \right\rceil + \left\lfloor \frac{k^h(c-d) - (n-1)(b-d)}{a+b-c-d} \right\rfloor - 1, \\
&\quad n + n_h + \left\lceil \frac{k^\ell(c-d) - (n-1)(b-d)}{a+b-c-d} \right\rceil + \left\lfloor \frac{k^\ell(c-d) - (n-1)(b-d)}{a+b-c-d} \right\rfloor - 1\} \\
&:= \min\{\phi(k^\ell, k^h), \psi(k^\ell, k^h)\}.
\end{aligned}$$

As above, $\Delta(k^\ell, k^h)$ is weakly increasing in both k^ℓ and k^h. Thus, $\Delta(\frac{(n-1)(b-d)}{a+b-2d}, \frac{(n-1)(b-d)}{a+b-2d}) < \Delta(k^\ell, k^h) < \Delta(\underline{k^h}, \underline{k^h})$. One can check that $\Delta(\underline{k^h}, \underline{k^h}) > 0$ hold since both $\phi(\underline{k^h}, \underline{k^h})$ and $\psi(\underline{k^h}, \underline{k^h})$ are strictly positive. We now assess the sign of $\Delta(\frac{(n-1)(b-d)}{a+b-2d}, \frac{(n-1)(b-d)}{a+b-2d})$. Note that $\Delta(\frac{(n-1)(b-d)}{a+b-2d}, \frac{(n-1)(b-d)}{a+b-2d}) = \min\{\phi(\frac{(n-1)(b-d)}{a+b-2d}, \frac{(n-1)(b-d)}{a+b-2d}), \psi(\frac{(n-1)(b-d)}{a+b-2d}, \frac{(n-1)(b-d)}{a+b-2d})\}$ where

$$\phi(\frac{(n-1)(b-d)}{a+b-2d}, \frac{(n-1)(b-d)}{a+b-2d}) = n - \left\lceil \frac{(n-1)(b-d)}{a+b-2d} \right\rceil - \left\lfloor \frac{(n-1)(b-d)}{a+b-2d} \right\rfloor - 1 < 0.$$

and

$$\psi(\frac{(n-1)(b-d)}{a+b-2d},\frac{(n-1)(b-d)}{a+b-2d}) = n+n_h-\left\lceil\frac{(n-1)(b-d)}{a+b-2d}\right\rceil-\left\lfloor\frac{(n-1)(b-d)}{a+b-2d}\right\rfloor-1.$$

Since $\phi(\frac{(n-1)(b-d)}{a+b-2d},\frac{(n-1)(b-d)}{a+b-2d})<0$ holds, we have that $\Delta(\frac{(n-1)(b-d)}{a+b-2d},\frac{(n-1)(b-d)}{a+b-2d})<0$. Thus, for each $k^h \in (\frac{(n-1)(b-d)}{a+b-2d},\underline{k^h}]$, there exists a corresponding interval of k^ℓ, such that for any k^ℓ in this interval we have $\Delta(k^\ell,k^h)=0$. As above, We have that $\Delta(k^\ell,k^h)<0$ if k^ℓ falls below this interval and $\Delta(k^\ell,k^h)>0$ if k^ℓ falls above. Therefore, for each $k^h \in (\frac{(n-1)(b-d)}{a+b-2d},\underline{k^h}]$, we have that

$$\mathscr{S}^{\star\star\star}=\begin{cases}\overrightarrow{BB}, & \text{if } k^\ell<\underline{\underline{k^\ell}}.\\ \overrightarrow{BB}\cup\overrightarrow{AA}, & \text{if } k^\ell\in[\underline{\underline{k^\ell}},\overline{\overline{k^\ell}}].\\ \overrightarrow{AA}, & \text{if } k^\ell>\overline{\overline{k^\ell}}.\end{cases}$$

where $\overline{\overline{k^\ell}}$ and $\underline{\underline{k^\ell}}$ are the two thresholds which are given by the upper and lower boundaries of this interval respectively.

Finally, consider the case where $k^\ell\geq\overline{k^\ell}$ and $k^h>\underline{k^h}$. In this case, we also have that $r(\overrightarrow{BB},\overrightarrow{AA})=\max\{\lceil M_h^1\rceil-1,\lceil M_\ell^1\rceil-1-n_h\}$, and $r(\overrightarrow{AA},\overrightarrow{BB})=n-\lfloor M_\ell^1\rfloor$. Thus,

$$\begin{aligned}\Delta(k^\ell,k^h)&=\min\{n-\lfloor M_\ell^1\rfloor-\lceil M_h^1\rceil+1,n+n_h+1-\lfloor M_\ell^1\rfloor-\lceil M_\ell^1\rceil\}\\&=\min\{n+\left\lceil\frac{k^\ell(c-d)-(n-1)(b-d)}{a+b-c-d}\right\rceil+\left\lfloor\frac{k^h(c-d)-(n-1)(b-d)}{a+b-c-d}\right\rfloor-1,\\&\qquad n+n_h+\left\lceil\frac{k^\ell(c-d)-(n-1)(b-d)}{a+b-c-d}\right\rceil+\left\lfloor\frac{k^\ell(c-d)-(n-1)(b-d)}{a+b-c-d}\right\rfloor-1\}\\&:=\min\{\phi(k^\ell,k^h),\psi(k^\ell,k^h)\}.\end{aligned}$$

One can check that both $\phi(k^\ell,k^h)$ and $\psi(k^\ell,k^h)$ are strictly positive if $k^\ell\geq\overline{k^\ell}$ and $k^h>\underline{k^h}$. Therefore, we have that $\Delta(k^\ell,k^h)>0$ and consequently, $\mathscr{S}^{\star\star\star}=\overrightarrow{AA}$ for any k^ℓ and k^h with $k^\ell\geq\overline{k^\ell}$ and

$k^h > \underline{k^h}$. In this case, the two thresholds in the proposition are given by $\overline{\overline{k^\ell}} = \underline{\underline{k^\ell}} = \overline{k^\ell}$. □

Proof of Proposition 3.5.3. First, note that according to Proposition 3.5.1, for any k^ℓ and k^h with $k^\ell < \overline{k^\ell}$ and $k^h > \underline{k^h}$ there are three absorbing sets $\overrightarrow{AA}$, $\overrightarrow{BB}$ and $\overrightarrow{BA}$. The proof proceeds by using techniques by Young (1993) and Kandori et al. (1993), which include three steps: i) calculate the resistance of transition from one absorbing set to another; ii) calculate the stochastic potential of each absorbing set, and iii) compare the stochastic potentials and find the smallest one.

i) Calculate the resistances of transitions.

First, we consider the transition from $\overrightarrow{AA}$ to $\overrightarrow{BA}$. Following the same argument as in the proof of Proposition 3.5.2, the minimum number of mutations for the transitions of agents in N_ℓ is given by

$$n - m_\ell^{AB} = \begin{cases} n - \lfloor M_\ell^2 \rfloor, & \text{if } k^\ell \leq \frac{(n-1)(b-d)}{a+b-2d}. \\ n - \lfloor M_\ell^1 \rfloor, & \text{if } \frac{(n-1)(b-d)}{a+b-2d} < k^\ell < \overline{k^\ell}. \end{cases}$$

Assume that all mutations occur in N_ℓ. After the mutations and consequent switches have occured, we have reached a state in $\overrightarrow{BA}$. Note that $\overrightarrow{BA}$ is absorbing. Thus, no agent will switch without further mutations. Therefore, the resistance of the transition from $\overrightarrow{AA}$ to $\overrightarrow{BA}$ is given by

$$r(\overrightarrow{AA}, \overrightarrow{BA}) = \begin{cases} n - \lfloor M_\ell^2 \rfloor, & \text{if } k^\ell \leq \frac{(n-1)(b-d)}{a+b-2d}. \\ n - \lfloor M_\ell^1 \rfloor, & \text{if } \frac{(n-1)(b-d)}{a+b-2d} < k^\ell < \overline{k^\ell}. \end{cases} \tag{A.19}$$

Second, we consider the transition from $\overrightarrow{BA}$ to $\overrightarrow{AA}$. Following the same argument as in the proof of Proposition 3.5.2, we have that the minimum number of A-agents required for agents in

N_ℓ to switch from B to A is given by

$$m_\ell^{BA} = \begin{cases} \lceil M_\ell^2 \rceil - 1, & \text{if } k^\ell < \frac{(n-1)(b-d)}{a+b-2d}. \\ \lceil M_\ell^1 \rceil - 1, & \text{if } \frac{(n-1)(b-d)}{a+b-2d} \leq k^\ell < \overline{k^\ell}. \end{cases}$$

Since there are already n_h A-agents before the mutations, the number of mutations required for agents in N_ℓ to switch is $m_\ell^{BA} - n_h$. Thus, for any $k^h > \underline{k^h}$, the resistance of the transition from $\overrightarrow{BA}$ to $\overrightarrow{AA}$ is given by

$$r(\overrightarrow{BA}, \overrightarrow{AA}) = \begin{cases} \lceil M_\ell^2 \rceil - n_h - 1, & \text{if } k^\ell < \frac{(n-1)(b-d)}{a+b-2d}. \\ \lceil M_\ell^1 \rceil - n_h - 1, & \text{if } \frac{(n-1)(b-d)}{a+b-2d} \leq k^\ell < \overline{k^\ell}. \end{cases} \tag{A.20}$$

Third, consider the transition from $\overrightarrow{BB}$ to $\overrightarrow{BA}$. Consider agents in N_h. Attending to Table 3.1 reveals that the minimum number of mutations required for agents in N_h to switch from B to A is given by $m_h^{BA} = \lceil M_h^1 \rceil - 1$. Assume that all mutations occur in N_h. After the mutations and consequent switches have occured, we have reached a state in $\overrightarrow{BA}$. Since $\overrightarrow{BA}$ is absorbing, no agent will switch without further mutations. Thus, the resistance of the transition from $\overrightarrow{BB}$ to $\overrightarrow{BA}$ is given by

$$r(\overrightarrow{BB}, \overrightarrow{BA}) = \lceil M_h^1 \rceil - 1. \tag{A.21}$$

Next, consider the transition from $\overrightarrow{BA}$ to $\overrightarrow{BB}$. Denote by $n - m_h^{AB}$ the minimum number of B-agents required for agents in N_h to switch form action A to B. Thus, m_h^{AB} is the maximum number of A-agents allowed for this transition. According to Table 3.1, we have that $m_h^{AB} = \lfloor M_h^1 \rfloor$. Since there are already n_ℓ B-agents before the mutations, the minimum number of mutations required is thus $n - m_h^{AB} - n_\ell$. Hence, the resistance of the transition from $\overrightarrow{BA}$ to $\overrightarrow{BB}$ is given by

$$r(\overrightarrow{BA}, \overrightarrow{BB}) = n_h - \lfloor M_h^1 \rfloor . \tag{A.22}$$

Now, consider the transition from $\overrightarrow{AA}$ to $\overrightarrow{BB}$. Following the same argument as in the proof of Proposition 3.5.2, the minimum number of mutations for the transitions of agents in N_ℓ is given by

$$n - m_\ell^{AB} = \begin{cases} n - \lfloor M_\ell^2 \rfloor , & \text{if } k^\ell \leq \frac{(n-1)(b-d)}{a+b-2d}. \\ n - \lfloor M_\ell^1 \rfloor , & \text{if } \frac{(n-1)(b-d)}{a+b-2d} < k^\ell < \overline{k^\ell}. \end{cases}$$

We now assess the largest number of B-agents after the mutations and switches (excluding switches among agents in N_h for now), i.e. agents who have mutated and agents in N_ℓ who have switched to B. For this, assume all mutations occur in N_h. Thus, the largest number of B-agents is $n_\ell + n - m_\ell^{AB}$. It follows that the minimum number of remaining A-agents in N_h is $m_\ell^{AB} - n_\ell$. Consider the transitions of agents in N_h now. As above, the maximum number of A-agents allowed for the transitions of agents in N_h is $m_h^{AB} = \lfloor M_h^1 \rfloor$ whenever $k^h > \underline{k^h}$. If the number of A-agents $m_\ell^{AB} - n_\ell$ is less than m_h^{AB}, then agents in N_h will switch without further mutations. In this case, the resistance of the transition is given by

$$r(\overrightarrow{AA}, \overrightarrow{BB}) = \begin{cases} n - \lfloor M_\ell^2 \rfloor , & \text{if } k^\ell \leq \frac{(n-1)(b-d)}{a+b-2d}. \\ n - \lfloor M_\ell^1 \rfloor , & \text{if } \frac{(n-1)(b-d)}{a+b-2d} < k^\ell < \overline{k^\ell}. \end{cases}$$

Otherwise, if $m_\ell^{AB} - n_\ell \geq m_h^{AB}$, additional mutations are needed. In this case, assume that the minimum number of mutations required is x. Then, we have that $n - (n_\ell + x) < m_h^{AB}$ must hold, i.e. the number of A-agents left after the mutations and switched in N_ℓ is less than the switching threshold. Thus, we have that $x > n_h - m_h^{AB}$. Since x is the minimum number, $x = n_h - m_h^{AB} + 1$.

One can check that $n_h - m_h^{AB} + 1 > n - m_\ell^{AB}$. Therefore, we have that

$$r(\overrightarrow{AA}, \overrightarrow{BB}) \geq n - m_\ell^{AB} = r(\overrightarrow{AA}, \overrightarrow{BA}). \tag{A.23}$$

Finally, consider the transition from $\overrightarrow{BB}$ to $\overrightarrow{AA}$. As above, the minimum number of mutations required for agents in N_h to switch from B to A is given by $m_h^{BA} = \lceil M_h^1 \rceil - 1$.

To maximize the impact of the mutations, assume that all mutations occur in the low-constraint group N_ℓ. Thus, after all B-agents in N_h have switched, the maximum number of A-agents is $\min\{n, m_h^{BA} + n_h\}$. It follows that the minimum number of B-agents now is $\max\{0, n_\ell - m_h^{BA}\}$. If $n_\ell - m_h^{BA} \leq 0$, i.e. if there are no B-agents, then we have reached $\overrightarrow{AA}$ and no extra mutations are required. Thus, we have $m^{BA} = m_h^{BA}$ for the relevant range of k^h.

Consider the case where there are still $m_h^{BA} + n_h$ A-agents left after the mutation and switch, i.e. $n_\ell - m_h^{BA} > 0$. Now, we have to determine whether the number of A-agents is enough for B-agents in N_ℓ to switch. Following the same argument as above, the switching threshold for B-agents in N_ℓ is given by

$$m_\ell^{BA} = \begin{cases} \lceil M_\ell^2 \rceil - 1 & \text{if } k^\ell < \frac{(n-1)(b-d)}{a+b-2d}. \\ \lceil M_\ell^1 \rceil - 1 & \text{if } k^\ell \geq \frac{(n-1)(b-d)}{a+b-2d}. \end{cases}$$

where m_ℓ^{BA} is the minimum number of A-agents required for agents in N_ℓ to switch to A. It follows that if $m_h^{BA} + n_h > m_\ell^{BA}$, then no extra mutation are required for this transition. In this case, we have that

$$r(\overrightarrow{BB}, \overrightarrow{AA}) = \lceil M_h^1 \rceil - 1.$$

If $m_h^{BA} + n_h \leq m_\ell^{BA}$, an additional $m_\ell^{BA} - (m_h^{BA} + n_h)$ mutations are needed. Then total number of mutations is $m_\ell^{BA} - n_h$. In this case, we have that

$$r(\overrightarrow{BB},\overrightarrow{AA}) = \begin{cases} \lceil M_\ell^2 \rceil - n_h - 1 & \text{if } k^\ell < \frac{(n-1)(b-d)}{a+b-2d}. \\ \lceil M_\ell^1 \rceil - n_h - 1 & \text{if } k^\ell \geq \frac{(n-1)(b-d)}{a+b-2d}. \end{cases}$$

One can check that in both cases, we have that

$$r(\overrightarrow{BB},\overrightarrow{AA}) \geq \lceil M_h^1 \rceil - 1 = r(\overrightarrow{BB},\overrightarrow{BA}). \tag{A.24}$$

ii) Calculate the stochastic potential of each absorbing set.

Having obtained the resistances of transitions, we are now able to compute the stochastic potentials of each absorbing set. We denote by $r_j(S_i^{\star\star})$ the resistance of the j-th $S_i^{\star\star}$-tree. Figure A.3 depicts all possible $\overrightarrow{AA}$, $\overrightarrow{BB}$ and $\overrightarrow{BA}$-trees.

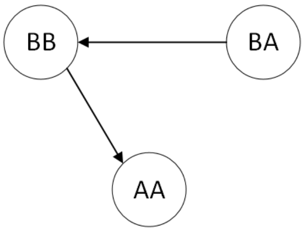
First, consider all $\overrightarrow{AA}$-trees depicted as sub-figures A.3a to A.3c in Figure A.3. The resistances of these trees are given by

$$r_1(\overrightarrow{AA}) = r(\overrightarrow{BA},\overrightarrow{BB}) + r(\overrightarrow{BB},\overrightarrow{AA}),$$
$$r_2(\overrightarrow{AA}) = r(\overrightarrow{BB},\overrightarrow{BA}) + r(\overrightarrow{BA},\overrightarrow{AA}),$$
$$r_3(\overrightarrow{AA}) = r(\overrightarrow{BB},\overrightarrow{AA}) + r(\overrightarrow{BA},\overrightarrow{AA}).$$

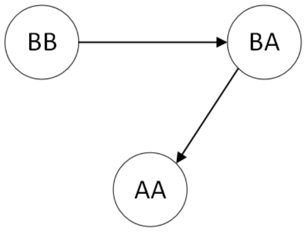
Given inequality (A.24), we have that $r_3(\overrightarrow{AA}) \geq r_2(\overrightarrow{AA})$. Thus, the stochastic potential of $\overrightarrow{AA}$ is given by

$$\gamma(\overrightarrow{AA}) = \min\{r_1(\overrightarrow{AA}), r_2(\overrightarrow{AA})\}. \tag{A.25}$$

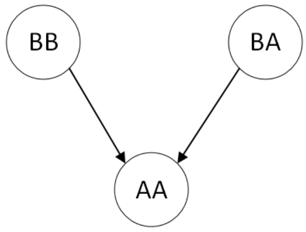
Now, consider all $\overrightarrow{BB}$-trees depicted as sub-figures A.3d to A.3f in Figure A.3. The resistances



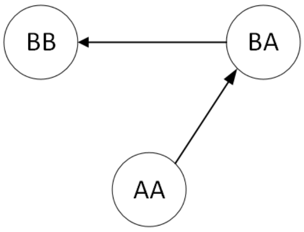
(a) *1st $\overrightarrow{AA}$-tree*



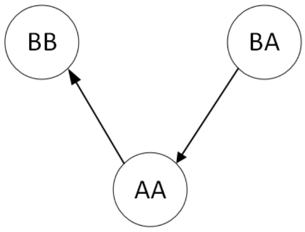
(b) *2nd $\overrightarrow{AA}$-tree*



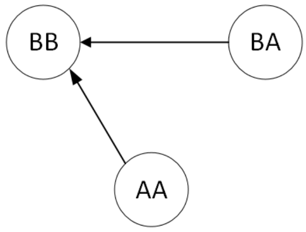
(c) *3rd $\overrightarrow{AA}$-tree*



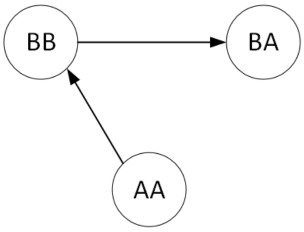
(d) *1st $\overrightarrow{BB}$-tree*



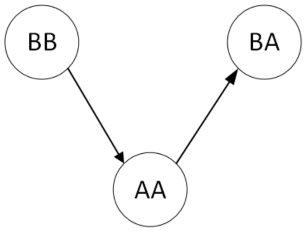
(e) *2nd $\overrightarrow{BB}$-tree*



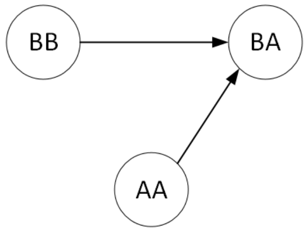
(f) *3rd $\overrightarrow{BB}$-tree*



(g) *1st $\overrightarrow{BA}$-tree*



(h) *2nd $\overrightarrow{BA}$-tree*



(i) *3rd $\overrightarrow{BA}$-tree*

Figure A.3. *All $S_i^{\star\star}$-trees*

of these trees are given by

$$r_1(\overrightarrow{BB}) = r(\overrightarrow{AA}, \overrightarrow{BA}) + r(\overrightarrow{BA}, \overrightarrow{BB}),$$

$$r_2(\overrightarrow{BB}) = r(\overrightarrow{BA}, \overrightarrow{AA}) + r(\overrightarrow{AA}, \overrightarrow{BB}),$$

$$r_3(\overrightarrow{BB}) = r(\overrightarrow{BA}, \overrightarrow{BB}) + r(\overrightarrow{AA}, \overrightarrow{BB}).$$

Given inequality (A.23), we have that $r_3(\overrightarrow{BB}) \geq r_1(\overrightarrow{BB})$. Thus, the stochastic potential of $\overrightarrow{BB}$ is given by

$$\gamma(\overrightarrow{BB}) = \min\{r_1(\overrightarrow{BB}), r_2(\overrightarrow{BB})\}. \tag{A.26}$$

Finally, consider all $\overrightarrow{BB}$-trees depicted as sub-figures A.3g to A.3i in Figure A.3. The resistances of these trees are given by

$$r_1(\overrightarrow{BA}) = r(\overrightarrow{AA}, \overrightarrow{BB}) + r(\overrightarrow{BB}, \overrightarrow{BA}),$$

$$r_2(\overrightarrow{BA}) = r(\overrightarrow{BB}, \overrightarrow{AA}) + r(\overrightarrow{AA}, \overrightarrow{BA}),$$

$$r_3(\overrightarrow{BA}) = r(\overrightarrow{BB}, \overrightarrow{BA}) + r(\overrightarrow{AA}, \overrightarrow{BA}).$$

Given the two inequalities (A.23) and (A.24) , we have that $r_1(\overrightarrow{BA}) \geq r_3(\overrightarrow{BA})$ and $r_2(\overrightarrow{BA}) \geq r_3(\overrightarrow{BA})$. Thus, the stochastic potential of $\overrightarrow{BA}$ is given by

$$\gamma(\overrightarrow{BA}) = r_3(\overrightarrow{BA}) = \begin{cases} n - \lfloor M_\ell^1 \rfloor + \lceil M_h^1 \rceil - 1 & \text{if } k^\ell \geq \frac{(n-1)(b-d)}{a+b-2d}. \\ n - \lfloor M_\ell^2 \rfloor + \lceil M_h^1 \rceil - 1 & \text{if } k^\ell < \frac{(n-1)(b-d)}{a+b-2d}. \end{cases} \tag{A.27}$$

iii) Find the region of k^ℓ and k^h such that the stochastic potential of $\overrightarrow{BA}$ is the smallest.

Having obtained the stochastic potentials, we now move on to find the regions of k^ℓ and k^h where the stochastic potential of $\overrightarrow{BA}$ is the smallest. To do so, both $\gamma(\overrightarrow{BA}) \leq \gamma(\overrightarrow{AA})$ and $\gamma(\overrightarrow{BA}) \leq \gamma(\overrightarrow{BB})$ must hold. Given equations (A.25), (A.26) and (A.27), we thus have that

$$\gamma(\overrightarrow{AA}) - \gamma(\overrightarrow{BA}) = \min\{r_1(\overrightarrow{AA}) - r_3(\overrightarrow{BA}), r_2(\overrightarrow{AA}) - r_3(\overrightarrow{BA})\} \geq 0. \tag{A.28}$$

and

$$\gamma(\overrightarrow{BB}) - \gamma(\overrightarrow{BA}) = \min\{r_1(\overrightarrow{BB}) - r_3(\overrightarrow{BA}), r_2(\overrightarrow{BB}) - r_3(\overrightarrow{BA})\} \geq 0. \tag{A.29}$$

Note that the above two inequalities (A.28) and (A.29) hold if and only if the following four inequalities hold

$$r_1(\overrightarrow{AA}) \geq r_3(\overrightarrow{BA}), \quad r_2(\overrightarrow{AA}) \geq r_3(\overrightarrow{BA});$$
$$r_1(\overrightarrow{BB}) \geq r_3(\overrightarrow{BA}), \quad r_2(\overrightarrow{BB}) \geq r_3(\overrightarrow{BA}). \tag{A.30}$$

By substituting the above equations of the resistances, we rewrite inequalities in (A.30) as follows:

$$r(\overrightarrow{BA},\overrightarrow{BB}) + r(\overrightarrow{BB},\overrightarrow{AA}) \geq r(\overrightarrow{BB},\overrightarrow{BA}) + r(\overrightarrow{AA},\overrightarrow{BA}), \tag{A.31a}$$

$$r(\overrightarrow{BA},\overrightarrow{AA}) - r(\overrightarrow{AA},\overrightarrow{BA}) \geq 0, \tag{A.31b}$$

$$r(\overrightarrow{BA},\overrightarrow{BB}) - r(\overrightarrow{BB},\overrightarrow{BA}) \geq 0, \tag{A.31c}$$

$$r(\overrightarrow{BA},\overrightarrow{AA}) + r(\overrightarrow{AA},\overrightarrow{BB}) \geq r(\overrightarrow{BB},\overrightarrow{BA}) + r(\overrightarrow{AA},\overrightarrow{BA}). \tag{A.31d}$$

Now, we substitute our results of resistances of transitions in the above four inequalities (A.31a) to (A.31d), we have that

$$n_h + 1 - \lfloor M_h^1 \rfloor - \lceil M_h^1 \rceil \geq 0, \tag{A.32a}$$

$$\lfloor M_\ell \rfloor + \lceil M_\ell \rceil - n - n_h - 1 \geq 0, \tag{A.32b}$$

$$\lfloor M_\ell \rfloor - \lfloor M_h^1 \rfloor - n_\ell \geq 0, \tag{A.32c}$$

$$\lceil M_\ell \rceil - \lceil M_h^1 \rceil - n_h \geq 0. \tag{A.32d}$$

where

$$M_\ell = \begin{cases} M_\ell^2, & \text{if } k^\ell \leq \frac{(n-1)(b-d)}{a+b-2d}. \\ M_\ell^1, & \text{if } \frac{(n-1)(b-d)}{a+b-2d} < k^\ell < \overline{k^\ell}. \end{cases}$$

Let $\Phi(k^h) = n_h + 1 - \lfloor M_h^1 \rfloor - \lceil M_h^1 \rceil$ and $\Psi(k^\ell) = \lfloor M_\ell \rfloor + \lceil M_\ell \rceil - n - n_h - 1$. We have that

$$\Phi(k^h) = n_h - 1 - \left\lfloor \frac{(n-1)(b-d) - k^h(c-d)}{a+b-c-d} \right\rfloor - \left\lceil \frac{(n-1)(b-d) - k^h(c-d)}{a+b-c-d} \right\rceil. \tag{A.33}$$

and

$$\Psi(k^\ell) = \begin{cases} \left\lfloor \frac{(n-1)(b-d)-k^\ell(c-d)}{a+b-c-d} \right\rfloor + \left\lceil \frac{(n-1)(b-d)-k^\ell(c-d)}{a+b-c-d} \right\rceil - n - n_h + 1 & \text{if } k^\ell \geq \frac{(n-1)(b-d)}{a+b-2d}. \\ \left\lfloor n - \frac{a-d}{b-d} \cdot k^\ell \right\rfloor + \left\lceil n - \frac{a-d}{b-d} \cdot k^\ell \right\rceil - n - n_h - 1 & \text{if } k^\ell < \frac{(n-1)(b-d)}{a+b-2d}. \end{cases} \tag{A.34}$$

We find that $\Phi(k^h) = 0$ whenever $k^h \in [\frac{2(n-1)(b-d)-n_h(a+b-c-d)}{2(c-d)} - \frac{a+b-c-d}{c-d}, \frac{2(n-1)(b-d)-n_h(a+b-c-d)}{2(c-d)})$. Let $k^{h\star} \equiv \frac{2(n-1)(b-d)-n_h(a+b-c-d)}{2(c-d)} - \frac{a+b-c-d}{c-d}$. Note that $\Phi(k^h)$ is weakly increasing in k^h. Thus, for any $k^h \geq k^{h\star}$, we have that $\Phi(k^h) \geq 0$.

Moreover, $\Psi(k^\ell)$ is weakly decreasing in k^ℓ with $k^\ell < \overline{k^\ell}$. Thus, we have that

$$\Psi_{\min}(k^\ell) > \Phi(\overline{k^\ell}) = 1 - n_\ell < 0. \tag{A.35}$$

and

$$\Psi_{\max}(k^\ell) = \Phi(1) = n_\ell - 2 \geq 0. \tag{A.36}$$

Thus, there exists an interval of k^ℓ, such that for any k^ℓ in this interval we have $\Psi(k^\ell) = 0$. Since $\Psi(k^\ell)$ is weakly decreasing in k^ℓ, we have that $\Psi(k^\ell) > 0$ if k^ℓ falls below this interval and $\Psi(k^\ell) < 0$ if k^ℓ falls above. Let $k^{\ell\star}$ equal to the upper bound of this interval. Thus, for any $k^\ell \leq k^{\ell\star}$ we have that $\Psi(k^\ell) \geq 0$.

Whenever $k^\ell \leq k^{\ell\star}$ and $k^h \geq k^{h\star}$, one can check that $\lfloor M_\ell \rfloor - \lfloor M_h^1 \rfloor - n_\ell \geq 0$ and $\lceil M_\ell \rceil - \lceil M_h^1 \rceil - n_h \geq 0$ hold. Thus, for any k^ℓ and k^h with $k^\ell \leq k^{\ell\star}$ and $k^h \geq k^{h\star}$, we have that $\gamma(\overrightarrow{BA}) \leq \gamma(\overrightarrow{AA})$ and

$\gamma(\overrightarrow{BA}) \leq \gamma(\overrightarrow{BB})$. Consequently, $\overrightarrow{BA} \subseteq \mathscr{S}^{\star\star\star}$.

Now, we proceed to identify thresholds for k^ℓ and k^h such that $\overrightarrow{BA}$ is the unique set of stochastically stable states. The proof is almost the same as the argument above. The only difference is in inequalities (A.32a) to (A.32d), which now are required to be strictly positive. Then, instead of solving $\Phi(k^h) \geq 0$ and $\Psi(k^\ell) \geq 0$ as above, we now solve $\Phi(k^h) > 0$ and $\Psi(k^\ell) > 0$ for the ranges of k^ℓ and k^h. In this case, $k^{h^{\star\star}}$ is the upper bound of the solution such that $\Phi(k^h) = 0$. Similarly, $k^{\ell^{\star\star}}$ is now the lower bound of the solution such that $\Psi(k^\ell) \geq 0$. Consequently, for any k^ℓ and k^h with $k^\ell < k^{\ell^{\star\star}}$ and $k^h > k^{h^{\star\star}}$, we have that $\gamma(\overrightarrow{BA}) < \gamma(\overrightarrow{AA})$ and $\gamma(\overrightarrow{BA}) < \gamma(\overrightarrow{BB})$. Thus, $\mathscr{S}^{\star\star\star} = \overrightarrow{BA}$.

□