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Counting graphs induced by Gauss diagrams and families of mutant alternating knots

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ABSTRACT

The construction known as Gauss diagrams or Gauss words is one of the oldest in knot theory and has been studied extensively both in the context of knots and in the context of closed curves with self-intersections. When we studied graphs induced by Gauss diagrams, we produced all examples of these graphs of small sizes, and we published the number of these graphs as sequence A343358 in the OEIS. The aim of this article is to clarify several subtle theoretical points concerning A343358. Most importantly, we explain why our numbers, produced using graph-theoretical constructions, reflect the number of so-called mutant knots.

1. Introduction

An important concept in knot theory are objects called *Gauss diagrams* [1–4]. They can be studied using certain graphs induced by Gauss diagrams, which are usually called *interlacement graphs* [5,6]. As a part of our research of Gauss diagrams and their interlacement graphs [7–10], we produced all examples of these graphs of size up to 13. The number of these examples, published in the OEIS as A343358[1] provides experimental cross-validation for some recent theoretical and experimental results produced by other researchers using other methods.

The aim of this article is to clarify several subtle theoretical points underlying the numbers in A343358. On the one hand, we revisit the definitions of two other sequences in the OEIS, A002864[2] and A264759,[3] and explain why the numbers in our sequence A343358 are not the same as in A002864 and A264759 (see Table 1). On the other hand, we compare A343358 with the number of examples of knots in certain knot-theoretical datasets and explain why these numbers coincide.

Table 1
Values of sequences for sizes $n = 3, \ldots, 13$.

size n	A343358	A002864	A264759
3	1	1	1
4	1	1	1
5	2	2	2
6	3	3	3
7	7	7	10
8	18	18	27
9	41	41	101
10	123	123	364
11	361	367	1 610
12	1257	1288	7 202
13	4573	4878	34 659

2. Definitions

A broad theme of this field of research is classifying types of closed curves. Doing this, some researchers are interested primarily in closed curves, whereas for other researchers, closed curves represent simplified representations of knots, called *shadows* (or *projections*) of knots.

Naturally, if two closed curves on the plane can be transformed into one another by a smooth transformation of the ambient plane, these two curves are deemed equivalent. Now let us consider more complicated constructions. Out of the two curves in Figs. 1 and 2, one can be turned into the other by 'turning it inside out', informally speaking. The most widely used way of expressing 'turning inside out' mathematically is to say that these two curves are drawn not on a plane but on the surface of a sphere. On a sphere, you can transform one of

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1 https://oeis.org/A343358.
2 https://oeis.org/A002864.
3 https://oeis.org/A264759.

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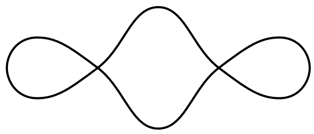

Fig. 1. A curve with two crossings, example 1.

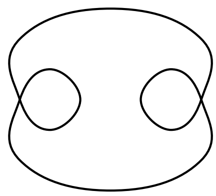

Fig. 2. A curve with two crossings, example 2.

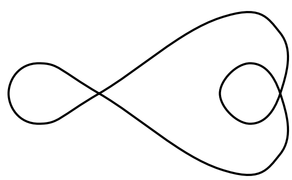

Fig. 3. A curve with two crossings, example 3.

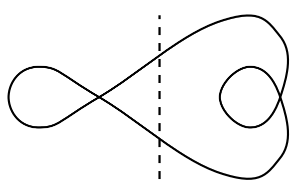

Fig. 4. Connected sum illustrated.

these curves into the other by a smooth transformation of the ambient sphere (and by changing the point from which you look at the sphere, as needed).

What if you turn one side of the curve in Fig. 1 'inside out', as shown in Fig. 3? One way to speak of this transformation is to say that this 'turning inside out' of one fragment of the curve has been made possible by the fact that the curve is so-called *connected sum*, that is, can be split into two fragments which are connected to each other only at two connecting points, as highlighted in Fig. 4.

Every closed curve can be decomposed into a connected sum of *indecomposable* curves, that is, curves which cannot be decomposed into a connected sum of simpler curves. Hence, one concentrates on studying indecomposable curves; in particular, numbers in Table 1 are produced for indecomposable curves.

Speaking of simplifying a curve, also it is considered 'not interesting' if a fragment of a curve goes in a loop from an intersection back to the same intersection without passing through other intersections on the way. Each of the curves in the three examples above consists of two instances of these 'not interesting' loops. A curve that does not contain such loops is called *irreducible*; the smallest irreducible curve, known in knot theory as the *trefoil*, is presented in Fig. 5. Numbers in Table 1 are produced for irreducible curves.

Instead of turning curves or their fragments 'inside out', a more general and powerful approach is to consider the order of crossings on the curve instead of the curve itself. Consider travelling along the curve; in mathematical terms, if the curve on the plane is represented in the parametric way by a smooth function $t \mapsto (x, y)$, consider increasing t and recording the crossings that you encounter, until you have visited the whole length of the curve. For this purpose, denote each crossing

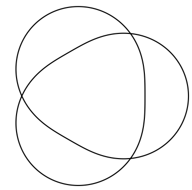

Fig. 5. An irreducible curve.

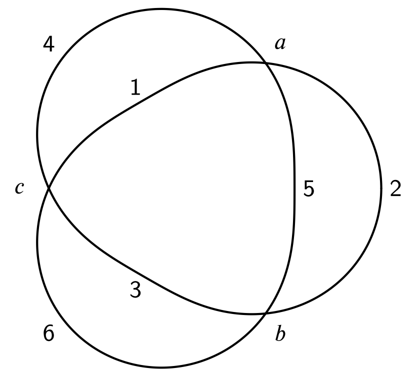

Fig. 6. The trefoil labelled up for producing its Gauss word.

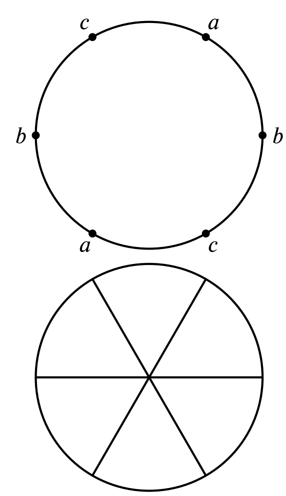

Fig. 7. Producing the Gauss diagram of the trefoil.

by a letter. For example, suppose we travel along the trefoil curve as shown in Fig. 6, along arcs 1, 2, 3, 4, 5, 6, in this order. If we denote the three crossings of the curve by a, b, c, we visit them in the order $abcabc$.

The sequence of letters we have produced is called a *Gauss word*. Obviously, the actual word depends on the point from which you start on the curve, and the choice of direction in which you travel along the curve, and the choice of notation for letters; one does not distinguish between Gauss words which differ from one another by these details. To stress that these details are not important, one writes the letters of the Gauss word consecutively around the circle (as shown in the first half of Fig. 7), and then replaces each pair of identical letters with a chord connecting these positions on the circle (see the second half of Fig. 7). The produced chord diagram is called a *Gauss diagram*.

The Gauss diagram in the example above happens to be symmetric relative to the mirror reflection. Not all of them are; as discussed above, in the context of studying closed curves, we do not distinguish between a Gauss diagram and its mirror reflection, since this difference is an artefact of the choice of the direction of travel along the curve.

To every chord diagram, a graph corresponds, in which the vertices correspond to the chords of the chord diagram, and two vertices are connected with an edge if and only if the two chords intersect in the chord diagram. In graph theory, this type of graph is called a *circle graph*, but in knot theory, the circle graph of a Gauss diagram is usually called the *interlacement graph* of the Gauss diagram.

3. Crunching the numbers

A Gauss diagram is called *indecomposable* if the set of chords cannot be split into two subsets C and D such that none of the chords in C intersects any of the chords in D. Equivalently, a Gauss diagram is indecomposable if and only if its interlacement graph is connected.

A chord diagram is called *realizable* if it is a Gauss diagram of a closed curve drawn on the plane. Not every chord diagram is realizable.

It is known [11,12] that there is a one-to-one correspondence between indecomposable irreducible closed curves and indecomposable realizable Gauss diagrams. Sequence A264759 in Table 1 presents the number of indecomposable realizable Gauss diagrams with n chords or, equivalently, the number of indecomposable irreducible closed curves with n crossings. Further discussion and pictures of all indecomposable realizable Gauss diagrams up to the size of 7 chords can be found in [13].

It is known [6,10,14,15] that a Gauss diagram is realizable if and only if its interlacement graph satisfies certain conditions. Thus, one can meaningfully speak of *realizable* interlacement graphs, that is, those that correspond to realizable Gauss diagrams.

Our sequence A343358, quoted in Table 1, presents the number of non-isomorphic connected realizable interlacement graphs with n vertices.

Several chord diagrams can share the same interlacement graph. Most importantly to us, several (indecomposable) realizable Gauss diagrams can share the same (connected) realizable interlacement graph. In this sense, there are more indecomposable realizable Gauss diagrams of size n than realizable interlacement graphs of size n. This is why the numbers in the column A343358 in Table 1 are smaller than the numbers in the column A264759, starting from size $n = 7$.

There are many knot diagrams corresponding to any given knot shadow. However, if we consider only so-called alternating knots, and do not distinguish between a knot and its mirror image, diagrams of so-called alternating knots behave 'almost' like closed curves, with only some subtle differences. The number of alternating knots with n crossings is presented in Table 1 in column A002864. It is not surprising that starting from size $n = 7$ the number of alternating knots is smaller than the number of shadows, because one knot can have many shadows. However, it is surprising that starting from size $n = 11$ the number of interlacement graphs is becoming just slightly smaller than the number of alternating knots. What is the reason for this small difference?

A knot-theoretical construction of *mutation* is defined as follows. Suppose you have a knot diagram which consists of two fragments having four free ends each (such fragments are usually called *tangles*) and connected as shown in the first half of Fig. 8. A mutation is the operation of rotating one of the tangles half a turn, as shown in the second half of Fig. 8. In most cases, the result of a mutation is the same knot, but sometimes, it is a different knot; in this case, the knots that can be produced from one another using mutations are called *mutant knots*.

In [16] an operation on chord diagrams is described, called the *flip of shares*, which expresses the operation of knot mutation in the language of chord diagrams. Then in [16], Theorem in Section 4.8.5 is proved, see also [17,18], which states that two chord diagrams share the same interlacement graph if and only if they can be produced from one another using flips of shares.

Hence, the difference between A343358 and A002864 is explained by the fact that starting from size $n = 11$, families of alternating mutant knots start occurring. Let us demonstrate how this conclusion is cross-validated by experimental data from other sources.

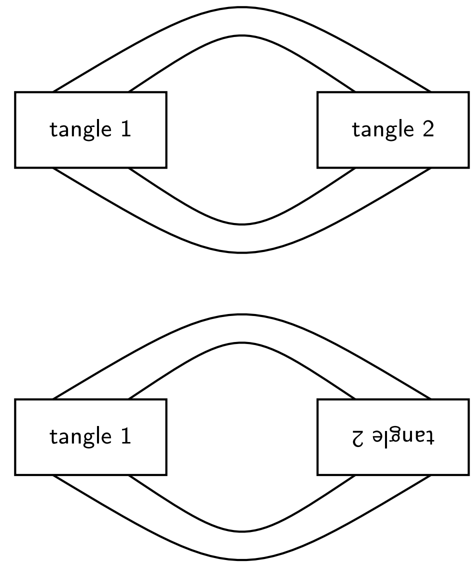

Fig. 8. Mutation of knots.

It is well known that no alternating mutant knots exist up to the size 10; that explains why A343358 and A002864 coincide up to $n = 10$. Stoimenov's Knot Data Tables[4] present explicit lists of mutant knots (both alternating and non alternating) for $n = 11, \ldots, 15$. For $n = 11$ there exist 6 pairs of mutant alternating knots, and that explains the difference when $n = 11$, that is, $367 = 361 + 6$. Independently, this conclusion for $n = 11$ is also corroborated by Section 4 in [19] which lists 6 pairs of alternating mutant knots of size 11.

For $n = 12$, Knot Data Tables list 27 pairs of alternating mutant knots and 2 triples of alternating mutant knots; this is again consistent with our results, namely, $1288 = 1257 + 27 + 2 \times 2$. For size $n = 13$, manual comparison becomes difficult, so we wrote a script which counts mutant alternating knots in Knot Data Tables; it is known that there are 4878 alternating knots of size 13, and Stoimenov's list of mutant knots of size 13 includes 574 alternating knots in 269 families of mutant alternating knots; hence, there are 305 more alternating knots than their families, and 4878 – 305 is our entry 4573.

4. Remarks on implementation

To produce A343358, we found all non-equivalent Gauss diagrams using our permutation-based algorithm from [7] (up to $n = 11$) and incremental algorithm from [9] (for $n = 12$), then for the case $n = 13$ we have used Tait Curves program by J. Betrema.[5] Checking if graphs are isomorphic was done by calling a NetworkX library[6] function in Python code.

5. Conclusion

We presented a detailed definition of what examples have been counted in our sequence A343358 in the OEIS; namely, A343358 counts non-isomorphic connected realizable interlacement graphs.

We clarified why the numbers in A343358 differ from those in A002864 (the number of alternating knots) and A264759 (the number of curves). We explained how the numbers in A343358 cross-validate a theorem in [16] and experimental data in Stoimenov's Knot Data Tables and in [19].

4 http://stoimenov.net/stoimeno/homepage/ptab/.
5 https://github.com/j2b2/TaitCurves.
6 https://networkx.org/.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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