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Entropic uncertainty principle, partition function and holographic principle derived from Liouville's Theorem

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ABSTRACT

An entropic version of Liouville's Theorem is defined in terms of the conjugate variables ("hyperbolic position" and "entropic momentum") of an *entropic* Hamiltonian. It is used to derive the Holographic Principle as applied to holomorphic structures that represent maximum entropy configurations. The Bekenstein-Hawking expression for black hole entropy is a consequence. Based on the entropic commutator derived from Liouville's Theorem and the same entropic conjugate variables, an entropic Uncertainty Principle (in units of Boltzmann's constant) isomorphic to the kinematic Uncertainty Principle (in units of Planck's constant) is also derived. These formal developments underpin the previous treatment of Quantitative Geometrical Thermodynamics (QGT) which has established (entirely on geometric entropy grounds) the stability of the double-helix, the double logarithmic spiral, and the sphere. Since in the QGT formalism the Boltzmann and Planck constants are quanta of quantities orthogonal to each other in Minkowski spacetime, a solution of the Schrödinger Equation is demonstrated isomorphic to a probability term of an *entropic* Partition Function, where both are defined by path integrals obeying the stationary principle: this isomorphism represents an important symmetry of the formalism. The geometry of a holomorphic structure must also exhibit at least C2 symmetry.

1. Introduction

The Holographic Principle has attracted considerable interest since it was first explicitly defined by 't Hooft [1] and Susskind [2] in the 1990s, having emerged from the quantum gravity and black hole physics developments of the 1970s. It was reviewed authoritatively by Bousso in 2002 [3] and also summarised more accessibly by Bekenstein [4]: "*the holographic principle holds that … a fully three-dimensional [image of the Universe] could be [written] on a … surface*". That is, the properties of a volume can be represented ***entirely*** by the properties of a surface. We show here how such a holographic theory emerges when considered from the viewpoint of Quantitative Geometrical Thermodynamics (**QGT**), as derived by Parker & Jeynes [5] (which for convenience we will refer to here as "**PJ2019**").

PJ2019 construct the QGT formalism from a rigorous restatement (in their Appendix A) of Parker & Walker (2010) [6], who use the standard definition of entropy as *differential information* (following standard literature and Parker & Walker, 2004 [7]) to build a geometric entropy theoretical framework. Integrating the differential information in a contour integral across 4-space (that is, Minkowski spacetime) such that the line integral consistently follows a path in the *positive* time direction ensures that that the geometric structures frequently observed in nature (such as the double-helix of DNA) explicitly obey the Second Law of Thermodynamics, even though the result of the integration is a static geometrical structure (independent of time).

It is necessary to appreciate that not one but two conceptual steps are made in this move. The first step is to show that the *information* emerges from a contour integral over *time* in 4-space, which therefore reduces the dimensionality of the structures under consideration to three and is the basis for the static description required for Maximum Entropy objects. The second step is then to explicitly derive expressions for the entropy of certain *holomorphic* geometries, which must have C2 symmetry in physical systems. This step privileges one of the three spatial axes (designated x_3 in the following analysis; its properties have been demonstrated by PJ2019). The key degrees of freedom of the system are therefore to be found on only the other two spatial axes: perhaps this informally explains how our approach yields holomorphic structures obeying the holographic principle with its reduced dimensionality.

Our QGT formalism employs a rigorously *geometric entropy* description for thermodynamic objects that (as quantitatively described in

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PJ2019) exhibit physical properties conforming to experimental observations over a very wide dynamic range, obtained analytically without any free parameters. It must be recognized that QGT is *complementary* to conventional thermodynamic and kinematic treatments, forming a different aspect of an overall unified thermodynamic theory that encompasses properties deriving from the geometry of a system as well as its statistical mechanical behaviour. PJ2019 is built on the Second Law in the usual way: QGT represents standard thermodynamics even though it is cast in an unfamiliar way.

PJ2019 establish an entropic version of the Hamiltonian equations of state, and show how there exists an entropic version of the *Principle of Least Action* (where the quantum of *action* is the reduced Planck's constant ħ): this is called the *Principle of Least Exertion* (where the quantum of *exertion* is Boltzmann's constant k_B). PJ2019 show that these two Principles are mathematically isomorphic, and specifically that the Planck and Boltzmann constants are quanta of quantities that can be considered to be mutually orthogonal in Minkowski spacetime. Conventionally, Liouville's Theorem is applied to the phase space description of the *temporal (kinematic) dynamics* of a system (to which the Principle of Least Action must apply): we will show here that the same Theorem is also valid in an entropic version that applies to the entropic phase space description of the *spatial (entropic) geometry* of a system (to which the Principle of Least Exertion must apply). Any system has both descriptions, which must therefore also be mutually consistent. Note that *action* and *exertion* are well-defined quantities referring, respectively, to *energy* and *entropy*. This work makes a sharp (and novel) distinction between these two sorts of description.

For example, conventional (kinematic or energy-based) descriptions of the thermodynamic potential invoke quantities such as the Gibbs free energy, which PJ2019 also employ to calculate the conformational energy difference between P-DNA and B-DNA. However, the *free entropy* is also a thermodynamic potential that Onsager [8] used to generate his reciprocal relations. Such an *entropic* potential (also known as the Massieu-Planck potential) was used recently in a discussion of the thermodynamics of field theories (Hongo, 2017 [9]). QGT employs an equivalent potential in the definition of the entropic Hamiltonian H_S, which is given by the sum of the 'kinetic entropy' T_S and the 'potential entropy' V_S, as also seen in the table of isomorphisms between kinematic and entropic quantities (PJ2019 Table 1).

We emphasise the distinction between the conventional (kinematic) Hamiltonian H of statistical mechanics, and our entropic Hamiltonian H_S as applied within QGT. In particular, our concept of an "entropic Hamiltonian" does *not* mean that we are applying the conventional kinematic Hamiltonian in an entropic context. Rather, whereas the kinematic Hamiltonian is of dimensionality energy [J] and is a conserved system quantity, our "entropic Hamiltonian" is of dimensionality "entropic momentum" [$JK^{-1}m^{-1}$] and is a conserved quantity (according to Nöther's theorem, see ref. [5]) in the hyperbolic space of QGT. Thus our entropic Hamiltonian H_S is functionally different to and incommensurate with the conventional kinematic Hamiltonian H.

This must have fundamental implications. For the first time we now apply here the entropic Hamiltonian formalism within an entropic Liouville Theorem to derive the general form of the entropy of a thermodynamically stable (that is, *Maximum Entropy* or "**MaxEnt**") system, and go on to show that the entropy of such a MaxEnt system must be *holographic* in nature, being proportional to the 2D surface surrounding the 3D system in question. Thus, we *prove* that the Holographic Principle is a real physical Principle with general (and very wide) application.

We then use this fundamental result to derive the specific case of the Bekenstein-Hawking entropy of a black hole (**BH**). Originally, the BH entropy was derived from heuristic argumentation (subsequently quantum field theory or string theory derivations were discovered). But we obtain this important result axiomatically from the entropic Liouville Theorem and its consequent general Holographic Principle as applied to MaxEnt structures.

It is interesting that Verlinde [10] employs both entropic and holographic principles to argue for emergent *kinematic* effects (that is, the law of gravitation) in a treatment that recognises their comparative properties, although we cannot digress here to engage with him. It is also interesting that Keppens [11] develops Verlinde's thought in certain ways, independently deriving the Bekenstein-Hawking equation.

Liouville's Theorem naturally employs the Poisson bracket, which is also the route to a description of a quantum commutator. Thus, we use our entropic Liouville Theorem to define the appropriate quantum commutator applicable in the entropic domain (with the conjugate variables of entropic momentum and hyperbolic position). This in effect describes an entropic system quantised through Boltzmann's constant; from this an *entropic* Uncertainty Principle is derived within an explicitly Hamiltonian/Lagrangian mathematical framework. This enables us to prove explicitly the major result that amplitude components of the *entropic* Partition Function are isomorphic to solutions of the Schrödinger equation, offering an entirely new slant on statistical mechanics. We should mention that Baldiotti et al. [12] also cover much of this ground but employing a fundamentally kinematic approach, so that their resulting thermodynamic (entropic) conclusions remain somewhat tentative.

As an entropic system, QGT operates in hyperbolic space; but such hyperbolic "velocities" (that is, spatial gradients calculated by taking the differential along a spatial axis) have the peculiar property of being dimensionless. In other words, there is no dimensional difference between a hyperbolic velocity and its inverse (that is, its reciprocal): they are not incommensurate as are the kinematical velocity and its inverse. PJ2019 simply used these properties without comment, but here we show some far-reaching implications. The product of the (entropic) hyperbolic velocity and its inverse, and the maximum ratio of the (kinematical) velocity of an object to the speed of light, are both necessarily unity: this intriguing relation is not accidental (being rooted in the hyperbolic and relativistic properties of 3+1 Minkowski spacetime), and enables a generalisation of the interpretation of *phase* and *group* velocities in wave mechanics.

It should be emphasised that in the present treatment we only consider *static* geometric structures in equilibrium; that is, structures that are in a stable, non-time-varying configuration (as discussed previously [13]) for which issues such as the kinematic velocity of the structures are not relevant, and relativistic issues play no direct part. However, QGT being defined within 4-space (Minkowski spacetime, as described in PJ2019) means that such geometric structures correctly obey the hyperbolic rotations prescribed by relativity. The issue of *non-equilibrium* structures that are evolving in time towards stability, whether near to or far from equilibrium (see Ilya Prigogine's concepts of minimum and maximum entropy production [14]), is not the subject of this paper. Note added in proof: QGT has also been used to determine the "entropy production of [idealised] galaxies" [15].

The ground-breaking non-equilibrium thermodynamics work of Prigogine's Brussels-Austin group has been helpfully reviewed by Robert Bishop (2004) [16]: in contrast to this group, our work concentrates on the properties of static, MaxEnt (equilibrium) geometries. Of course, this is a very severe restriction which, if unavoidable, would imply that the formalism could hardly apply to any real systems of interest. However, we here are only outlining elementary consequences of the formalism in the most symmetrical cases. We already know that the formalism is naturally extensible to non-equilibrium cases since spiral galaxies, idealised by the (holomorphic) double logarithmic spiral, are certainly highly dynamic systems in which we already have an expression for the *entropic force* (see PJ2019 Eq. (23)). But at this stage we are concerned not with motion in or evolution of systems, but with the entropy associated with certain static system geometries. We expect that the underlying hyperbolic (Minkowski spacetime) framework of QGT will easily extend to dynamic (relativistic) analyses (such as galaxies [15]).

PJ2019 rigorously established what has long been suspected, that there exists a thermodynamic isomorph of the Heisenberg uncertainty principle; in the present paper we show how the entropic Uncertainty

Principle is actually a consequence of the entropic Liouville Theorem. We have come to this understanding by considering the deep implications of *analytical continuation* in hyperbolic spacetime, whose mathematical consequences generate both holomorphic functions and an entropic Hamiltonian/Lagrangian framework based on spatial gradients. A fundamental feature of MaxEnt structures is *holomorphic pairing* (as already discussed in PJ2019), which creates complex-vector quantities in hyperbolic spacetime. The underlying fundamental applicability of Liouville's Theorem offers new insights into the complementary relationship between kinematic and entropic descriptions of any stable system. For example, we conjecture that the special axiom of analyticity required by Chruściel & Costa [17] to rigorously establish the "*no-hair*" Theorem will actually be provided by QGT.

PJ2019 have proved the applicability of the QGT formalism to the double helix (evidenced by DNA), and the logarithmic double spiral (the *spirae mirabilis* of Bernoulli; evidenced by spiral galaxies); and Parker & Jeynes [13] have also proved its applicability to spheres (evidenced by Buckminster-fullerene). Other entropic geometries of relevance in the universe include the large-scale filamental (dendritic) structure of galaxy groups and clusters with self-similar scaling relations investigated by Paul et al. (2017) [18]. The interaction of these structures with the high-entropy cosmic microwave background radiation is currently the object of intense investigation, for example by Hurier & Lacasa (2017) [19]. Structures such as neural networks (used for deep learning and artificial intelligence applications and frequently featuring intermediate 'hidden' layers) also have a strongly dendritic organisational structure (with feedforward and feedback elements), allowing complex and adaptable interconnections between the input and output planes (synapses) of the system. Such non-linear systems are important areas of current research, and entropic considerations are already being used to render them tractable to analysis [20], since branching topologies can also be studied using conservation of entropic momentum (that is, applying Nöther's Theorem). We would expect these and other developments (such as calculations by the computational chemists of fullerene stability) to be accelerated by systematic use of the QGT formalism amplified by the fundamental relations made explicit here.

In section §2 we show how Liouville's Theorem can be expressed in entropic terms, using the QGT formalism. Section §3 puts this in the context of a discussion of the Partition Function, maximum entropy systems and unitary objects. Then an expression for the entropy of a general system is obtained from the entropic Liouville Theorem (§4) with the entropy of specific systems as a consequence, including the Bekenstein-Hawking expression for the entropy of a black hole (§5). The holographic properties of Eqs. (11) and (12) emerge as a result of the holomorphism of the objects considered. The entropic Uncertainty Principle follows (§6) from the canonical commutation relation obtained from consideration of the Partition Function in terms of the Poisson bracket (which in quantum mechanics is given as an operator). Some rather technical, but useful and far-reaching, results (§7) are given for the hyperbolic velocity, and we finish (§§8,9) with Discussion and Conclusions.

This whole subject is intricately involved with the symmetry of the objects since holomorphic entities must have at least C2 symmetry: this was previously discussed at some length in the context of fullerene molecules [13]. Moreover, and more fundamentally, the whole formalism articulates the symmetry (isomorphism) between the complementary kinematic and entropic expressions: see previously Table 1 of PJ2019, with the centrally important isomorphism between solutions of the Schrödinger Equation and the probability amplitude of an entropic Partition Function shown here in Eq. (15).

2. The entropic Liouville Theorem

Consider a density of states distribution $\rho\,(x_1, x_2, \ldots x_N, p_1, p_2, \ldots p_N, t)$ of a system in $2N$-dimensional $(p, x)^N$ phase space, where p and x respectively are the conventional (kinematic) conjugate momentum and position variables of a system with $2N$ degrees of freedom. Then this density distribution ρ is the number of states per incremental volume $(\Delta p \Delta x)^N$, and Liouville's Theorem can be written in terms of the kinematic Hamiltonian H (following Wannier's standard treatment [21]):

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial\rho}{\partial t} + \left(\sum_{n=1}^{N} \dot{p}_n \frac{\partial\rho}{\partial p_n} + \dot{x}_n \frac{\partial\rho}{\partial x_n}\right) = \frac{\partial\rho}{\partial t} - \sum_{n=1}^{N}\left(\frac{\partial H}{\partial x_n}\frac{\partial}{\partial p_n} - \frac{\partial H}{\partial p_n}\frac{\partial}{\partial x_n}\right)\rho = 0 \tag{1a}$$

where the dot indicates the time derivative as usual. The quantity in the brackets on the RHS is known as the Liouville operator ("Liouvillian") $-\mathrm{i}\widehat{\boldsymbol{L}}$, and for the n^{th} instance is given by:-

$$-\mathrm{i}\widehat{\boldsymbol{L}}_n = \frac{\partial H}{\partial x_n}\frac{\partial}{\partial p_n} - \frac{\partial H}{\partial p_n}\frac{\partial}{\partial x_n} \tag{1b}$$

It is well-known that the kinematic equations of state are given by the identities:-

$$\dot{x}_n = \frac{\partial H}{\partial p_n} \quad \text{and} \quad \dot{p}_n = -\frac{\partial H}{\partial x_n} \tag{1c}$$

In the entropic domain the phase space now involves the *entropic momentum* (p) and the *hyperbolic space* (q) co-ordinates with the corresponding *entropic Hamiltonian* (H_S), all as derived by PJ2019. To avoid ambiguity, here (as in PJ2019) we use the symbol "x" for Euclidean space (as conventionally employed in kinematics, see Eqs. (1)) in contrast to the symbol "q" which we use to indicate *hyperbolic* space (needed for QGT). We also employ the subscript "S" that serves to remind us that a labelled quantity is *entropic*. The interpretation of the momentum "p" as entropic or kinematic is to be inferred, as appropriate, from context.

Parker & Jeynes considered the geometric thermodynamics of the double logarithmic spiral (of which the double helix is a special case) [5], and of the sphere (which can be represented by a linear combination of two identical double-helices; that is, thereby conforming to the stable holomorphic-pair description) [13]. These are demonstrated to be maximum entropy (MaxEnt) structures since the appropriate Euler-Lagrange equations are satisfied (PJ2019 Eq.13b), instanced in nature by (respectively) spiral galaxies, DNA, and Buckminster-fullerene (C_{60}). Note also that such MaxEnt structures conforming to the QGT formalism are necessarily holomorphic.

We can see that in these three cases (spiral galaxies, DNA, and C_{60}) there exist axes of symmetry, to which we assign the (spatial) x_3 co-ordinate. In these cases the condition for maximum entropy is given by the stationary behaviour not of the (conventional) integral of the kinematic Hamiltonian H along the special (temporal, t) axis:-

$$\delta \int H \mathrm{d}t = 0 \tag{2a}$$

but of the integral of the *entropic* Hamiltonian H_S along the special (spatial, x_3) axis:-

$$\delta \int H_s \mathrm{d}x_3 = 0 \tag{2b}$$

Clearly, the key controlling variable of Eq. (2b) is spatial (x_3) in contrast to Eq. (2a) where it is temporal (t). This isomorphism is explicit in the final row of the summary Table 1 of PJ2019, and can be understood in detail by considering the geometric (Clifford) algebra (including the quaternion subalgebra and resulting spinor isomorphism) of the QGT representation. This spinor representation sees a natural pairing of the four space-time variables (co-ordinates) into conjugate pairs, such that x_0 (i.e. time t) is conjugate to x_3 (see Eq.A.5 of Appendix A in PJ2019).

PJ2019 also showed the close quantitative relation of *information* and *entropy*, such that an *info-entropy* field can be defined. This field is obtained in general by an integral over time in complex (that is, Minkowski) 4-space. They also showed that the info-entropy expressions for

Maximum Entropy structures (that is, those structures such as double helices and logarithmic spirals that are frequently encountered in nature) are holomorphic functions where the key controlling variable (x_3) and its derivative (dx_3) are space-like; this is in contrast to the conventional dynamics of a system where the key variable (t) and its derivative (dt) determining the kinematics are time-like.

Then the *entropic* Liouville equation equivalent to Eq. (1a), with the appropriate entropic density of states ρ_S (q_1, q_2, $_{...}q_N$, p_1, p_2, $_{...}p_N$, x_3), is given by:

$$\frac{d\rho_S}{dx_3}=\frac{\partial\rho_S}{\partial x_3}+\sum_{n=1}^{N}\left(p_n^{'}\frac{\partial\rho_S}{\partial p_n}+q_n^{'}\frac{\partial\rho_S}{\partial q_n}\right)=\frac{\partial\rho_S}{\partial x_3}-\sum_{n=1}^{N}\left(\frac{\partial H_S}{\partial q_n}\frac{\partial}{\partial p_n}-\frac{\partial H_S}{\partial p_n}\frac{\partial}{\partial q_n}\right)\rho_S=0 \tag{3}$$

where $(p, q)_n$ are the conjugate variables for the n^{th} pair of degrees of freedom of the holomorphic structure, and x_3 is the coordinate along the symmetry axis of the structure. The appropriate entropic (hyperbolic space) equations of state corresponding to Eq. (1c) are now given by the identities proved by PJ2019 (Appendix B Eq.B17)

$$q_n^{'}=\frac{\partial H_S}{\partial p_n} \quad \text{and} \quad p_n^{'}=-\frac{\partial H_S}{\partial q_n} \tag{4a}$$

noting that the prime symbol indicates differentiation with respect to the x_3 co-ordinate:

$$q^{'}\equiv\frac{\partial q}{\partial x_3} \quad \text{and} \quad p^{'}\equiv\frac{\partial p}{\partial x_3} \tag{4b}$$

and where the appropriate entropic Hamiltonians will depend on the maximum entropy (MaxEnt) structures considered. So the fundamental structure is the logarithmic double-spiral, whose entropic Hamiltonian is given by PJ2019 (Appendix B Eq.B40b). The double helix is a special case of the logarithmic double-spiral: its entropic Hamiltonian is given by PJ2019 (Appendix B Eq.B19b). The sphere is a combination of double-helices, with an entropic Hamiltonian given by Parker & Jeynes 2020 [13] Eq. (8a).

3. Entropic granularity, MaxEnt, and unitary objects

An elemental $2N$-volume within the density of states ρ_S along the trajectory is given by $d\Omega_S = \Pi_n(dp_n{\cdot}dq_n)$, $n\in\{1...N\}$; however in the special case of simple MaxEnt systems which are specified by a *single* pair of entropic conjugate variables ($N = 1$), it is given by:

$$d\Omega s = dp \cdot dq \tag{5a}$$

Only unitary objects are specified by a single pair of conjugate variables since unitary objects have no choice about their specification: they have no additional degrees of freedom. The fact that the double-helix is an eigenvector of the entropic Hamiltonian in the QGT formalism of PJ2019 means that as an entropic structure it is specified by a single pair (p,q) of the entropic conjugate variables; that is, it is a *unitary* object. This looks odd, since DNA (the exemplar of the double-helix) encodes huge quantities of information, and therefore cannot be unitary! However, entropy is a scale-less quantity, whose hierarchical scale must be carefully chosen for any system so as to correctly represent that system at the appropriate scale. The *granularity* of the system (that is, precisely how the Partition Function is calculated) is a matter of choice: for a very general discussion of this issue of granularity see Penrose (2004 [22] §27; 2010 [23] §1.3). The basic backbone structure of DNA (the double-helix) is indeed unitary (holomorphic) in QGT, and Parker & Walker, 2004 [7] have proved that holomorphic functions cannot encode information. As a unitary object the double-helix of DNA cannot itself encode information: the encoded genetic information is at a different granularity.

The fact that a structure is MaxEnt means that it is a *most likely* geometry, that is, it has maximum stability. A MaxEnt structure cannot evolve any further (there are no more degrees of freedom available) at that level of granularity. Note that a MaxEnt structure may not be in thermodynamic equilibrium: for example, spiral galaxies may approximate the double logarithmic spiral geometry (also a unitary structure in hyperbolic space) but galaxies are manifestly evolving since the central supermassive BH must accrete material. Such a system will be described by a succession of local MaxEnt states, tending eventually (after cosmic times!) to the global MaxEnt state when all the mass resides in the BH.

Thus, a MaxEnt structure has minimised its currently available degrees of freedom (DoFs). Also, a MaxEnt structure requires the least possible additional information to fully specify it; a MaxEnt object has attained the simplest possible configuration (at that granularity) and is therefore unitary. Black holes (BHs) are the prime example of objects that have maximised their entropy in 3-space: it is already known that no object of that volume can have more entropy. But we can also note that a BH must be a MaxEnt unitary object since it can be fully specified by 4 parameters (conventionally: mass, charge and angular momentum; and a fourth parameter to specify the scale of the system, which for BHs is fixed as the Planck length).

In the general (not MaxEnt) case the entropic Liouville Theorem requires a multiplicity of degrees of freedom ($2N$) as indicated in Eq. (3). However, as a system evolves towards a MaxEnt state, then the available number of degrees of freedom reduces. Absolutely maximum entropy occurs when the system can no longer evolve or simplify, and the available number of degrees of freedom is minimised. The limiting case for such a MaxEnt configuration is $N = 1$, corresponding to the fundamental double-helix eigenvector of QGT which we will explicitly consider (in §5 below).

We mention the so-called *no-hair theorem* [24,25], which guarantees that 3 parameters of a black hole (mass, electric charge, and angular momentum) are sufficient to completely determine the relevant solutions to Einstein's equations of general relativity. It also guarantees that a black hole geometry must be MaxEnt: this is equivalently expressed by asserting the uniqueness of the solutions of the Liouville equation (Eq. (3)) as a partial differential equation (PDE) for $N = 1$. It is no accident that the same is true for Maxwell's equations for electromagnetism, which also describe a double-helical geometry for the holomorphic-pair description of the electric and magnetic fields (the simplest case is a circularly-polarised EM field) implied by the Riemann-Silberstein complex vector representation [26]. Courant & Hilbert [27] (Vol.II, ch.6 §6 *passim*) explicitly demonstrate in detail the existence and uniqueness of solutions of hyperbolic PDEs (such as Maxwell's equations) with particular regard to their kinematic properties.

In kinematics the time dimension is not considered a degree of freedom (DoF), so also in the entropic geometry the x_3 axis (conjugate to time t) does not represent a DoF of the system. Thus in QGT a double-helix (as a unitary object) is categorised only by its radius R and pitch λ (with $N = 1$) and not additionally by the number of turns along the axis of the helix.

The entropic Hamiltonian H_S is constant along the x_3 axis, as was proved by PJ2019 (Appendix B Eq.B.40b). Therefore the sum of the kinetic entropy T_S and the potential entropy V_S is also constant: $H_S = T_S + V_S$. But since the kinetic entropy (KEnt) T_S is dependent on the hyperbolic velocity q', and the potential entropy (PEnt) V_S is dependent on the actual geometry of the system, it is clear that the available DoFs are given by the PEnt (not the KEnt) terms. Therefore QGT requires that as a system evolves and the number of DoFs decreases, V_S must also decrease (with a consequent rise in T_S). PJ2019 have proved (Appendix B, Eqs.B18,B42) that the PEnt of the double-helix vanishes but the PEnt of the double logarithmic spiral does not vanish. Therefore the double-helix cannot evolve to any higher entropy state (it must have the minimum possible DoFs), but the double logarithmic spiral when considered in a wider context must have extra DoFs available due to the non-zero PEnt remaining. Galaxies are geometrically stable structures (the double logarithmic spiral is a holomorphic or unitary object) but they must evolve over time such that their total entropy increases (their central black holes must grow).

Clearly, for a system which is still evolving towards a MaxEnt state, additional data (in the form of relevant boundary conditions) are required to specify a solution for the entropic Liouville PDE (Eq. (3) with $N>1$). Fortunately, exploiting the comprehensive isomorphism between the kinematic (energy) and entropic quantities outlined in PJ2019 and discussed further here, we can see that many of the PDE solutions directly applicable in this entropic context are explicitly treated by Courant & Hilbert [27]. The details remain outside our present scope, but we conjecture that an appropriate QGT treatment of the entropic Liouville Theorem will lead to an alternative proof of the no-hair theorem. We suspect that such a proof would be related to the result proved by Birmingham et al. [28] using "*a theorem of hyperbolic geometry*."

4. The general entropy of systems from Liouville

Standard results in conventional dynamics for the invariance properties of phase space (see for example Wannier's treatment of Liouville's Theorem [21], Eq.3.07) show that the logarithmic derivative of the phase space element $d\Omega$ at any point along the t (time) co-ordinate trajectory is zero. All that is required for this to carry over into the entropic domain is for the phase space trajectory to be spatial rather than temporal, and to be properly expressed in an entropic Hamiltonian-Lagrangian representation: these were demonstrated by PJ2019. The *entropic* phase space element $d\Omega_S$ is therefore also constant along the *entropic* x_3 trajectory:

$$\frac{1}{d\Omega_s}\frac{\mathrm{d}}{\mathrm{d}x_3}(d\Omega_s) = \frac{\mathrm{d}}{\mathrm{d}x_3}(\ln dx_3) = 0 \tag{5b}$$

This is an alternative expression of Liouville's Theorem, that is, Eq. (5b) is equivalent to Eq. (3). In Eq. (5b) we make the subtle distinction between the operator "d" and the infinitesimal "d" notation, which is also seen in the progression of equations Eqs. (6), such that Eq. (6b) is in terms of the infinitesimal "$d\Omega_S$" whereas Eq. (6c) uses the operator notation of the integral ("$\mathrm{d}\Omega_S$").

It should be noted that in contrast to the conventional kinematic phase space description, here we employ for the first time a phase space Ω_S that describes the geometric entropy aspects of a system. In addition, our approach can be seen to have the additional novelty of applying the entropic Liouville Theorem in the context of the Principle of Least Exertion where the quantum of exertion is given by Boltzmann's constant.

The microcanonical ensemble of the system can be assumed to have uniform probability P in the entropic phase space (i.e. thereby conforming to Liouville's Theorem) subject to the overall system entropy S and the system's number of degrees of freedom $2N$, such that $\int P\mathrm{d}\Omega_S = 1$, formally noting $\int \mathrm{d}\Omega_S = \Omega_S$, with $P = 1/\Omega_S$ being constant. However, at a MaxEnt equilibrium the system has a higher (but still constant) probability distribution (i.e. density of states distribution ρ_S) along the trajectory in the entropic phase space. In defining the local density ρ_S we divide the phase space volume Ω_S into cells of macroscopic volume $(\Delta p\Delta q)^N$, where Δp, Δq must physically be regarded as quanta (rather than the mathematically convenient infinitesimals dp, dq); a convenient scale size for Δp and Δq being chosen such that each macroscopic volume $(\Delta p\Delta q)^N$ is expected to contain an average M_Δ states.

This may appear curious since conventionally one might expect Δp and Δq to be chosen such that there is an expectation for a cell to contain only a *single* state. However, entropy is scale-less (in the same sense that the Partition Function can assume any granularity), and these quanta (Δp, Δq) can be taken at any scale (although there must be a physical scale minimum which we will treat as the expression of the appropriate Uncertainty Principle, see §6 below). Rather, we take here the *general* case (as enabled by the intrinsic scale-lessness of entropy) to allow a cell to contain an average M_Δ states; this allows us to associate a subsystem entropy S_Δ with each trajectory.

Using the (appropriately modified) formalism of Swendsen & Wang (2015) [29], the local density ρ_S in the vicinity of the subsystem trajectory in the entropic phase-space is then approximately given by:

$$\rho_S = \frac{M_\Delta}{N!(\Delta p\Delta q)^N} \tag{6a}$$

The factor $N!$ arises from the fact that identical eigenvectors (as associated with each DoF) occupying the same cell location in phase space are indistinguishable, and also ensures entropic extensivity. We note here Jaynes' [30] careful analysis of whether entropy is *extensive,* where (in the context of a deep discussion of the Gibbs 'paradox') he writes that "*entropy is just as much, and just as little, extensive in classical statistics as in quantum statistics*," and indeed (in the context of examples of processes in which the energy does not increase whereas the entropy does) that "*entropy stands strongly contrasted to energy*". Thus, entropy is not intrinsically an extensive quantity (although for many simple systems it is approximately extensive): in any case it is obvious that the *holographic principle* cannot of itself be *extensive*!

The overall number of available macroscopic states conforming to a subsystem entropy S_Δ is then found by integrating the density ρ_S over the complete entropic phase space Ω_S. Note that Ω_S is *not* a function of $\Delta p\Delta q$. The number of states available within an elemental volume $d\Omega_S$ along the phase space trajectory for the subsystem is then given by $dM_\Delta = \rho_S\, d\Omega_S$, such that:

$$dM_\Delta = \rho_S d\Omega_S = \frac{M_\Delta}{N!(\Delta p\Delta q)^N} d\Omega_S \tag{6b}$$

We rearrange and integrate both sides:

$$\int \frac{\mathrm{d}M_\Delta}{M_\Delta} = \int \frac{\mathrm{d}\Omega_S}{N!(\Delta p\Delta q)^N} \tag{6c}$$

where the integration is performed across the entire entropic phase-space (Ω_S), and where for simplicity (and without loss of generality) we will ignore the constant of integration. For completeness we note parenthetically that Jaynes insists (see [30] Eq. (9)) that this is "*not an arbitrary constant but an arbitrary function*". Again, we do not have to go into this further here. Then:

$$\ln M_\Delta = \frac{\Omega_S}{N!(\Delta p\Delta q)^N} \tag{6d}$$

where M_Δ represents the total number of states across the entire phase space Ω_S for a trajectory conforming to a subsystem entropy S_Δ.

Note here that Liouville's Theorem, being fundamental, is also rather subtle: the formalism aids a correct manipulation. Even though ρ_S (as per Eq. (6a)) is constant along the trajectory, $1/M_\Delta$ (being a function of $\Delta p\Delta q$, that is, considered variable in the formalism) integrates in Eq. (6c) to $\ln M_\Delta$, whereas the RHS of Eq. (6c) integrates simply to Ω_S, since the magnitude of Ω_S is *not* a function of $\Delta p\Delta q$.

Moreover, although Ω_S is defined over $2N$ phase space, for a given system with $2N$ degrees of freedom (DoFs) these DoFs are not a variable of the system when it has attained a MaxEnt state and cannot further evolve at that level of granularity. Rather, the $2N$ DoFs are a given (constant) parameter of that MaxEnt system, making Ω_S essentially independent of them: varying N is equivalent to configuring a *qualitatively different* geometric system which also exhibits a quantitatively different entropy requiring a separate treatment.

The entropy S_Δ of the subsystem is then as usual simply given by:

$$S_\Delta = k_B \ln M_\Delta = \frac{\Omega_S}{N!(\Delta p\Delta q)^N} k_B \tag{7a}$$

Subsequently, the overall probability distribution M for the system is given by the product of the factors associated with each j^{th} subsystem:

$$M = \frac{1}{M_T}\prod_j M_{\Delta,j} \tag{7b}$$

with an appropriate normalizing factor M_T, which looks like a "constant offset" in the overall entropy. Because entropy is not an absolute quantity (due to its scale-less character, with the entropic scale defined by the adopted granularity) M_T also includes the constant of integration (from Eq. (6c)) that represents the background contribution to the overall entropy. Note also that Eq. (7b) is entirely comparable to Swendsen & Wang's [29] Eq. (7).

Then the general entropy for a geometric system featuring $2N$ degrees of freedom with a quantised cell size whose scale is undetermined, is given by:

$$S = k_B \ln M = k_B \left(\sum_j \ln M_{\Delta,j} - \ln M_T \right) = \sum_j S_{\Delta,j} - k_B \ln M_T \tag{7c}$$

However, we are interested in MaxEnt cases where the degrees of freedom are minimised; in particular, the "*maximal MaxEnt*" case occurring for $N = 1$. Such a system will have no subsystems at the entropic granularity of interest (that is, it is *unitary*), and will have an entropy simply given by:

$$S = k_B \ln M = \frac{\Omega_S}{\Delta p \Delta q} k_B \tag{7d}$$

That is, the entropy of such a unitary system is the number of ways the incremental area of phase space sub-volume $\Delta p \cdot \Delta q$ divides the total system phase space volume Ω_S, quantised through the Boltzmann constant. This novel result clearly depends ***a)*** on p and q being properly conjugate quantities, which is established by them satisfying the canonical relations (Eq. (4a)), and ***b)*** Liouville's Theorem (Eq. (3)) guaranteeing that the phase space volume is given by the product of conjugate variables (Eq. (5a)).

We will use Eq. (7d) to determine a general expression for the entropy of the double-helix (together with its property of Maximum Entropy, Eq. (11)) and thence immediately to derive the Bekenstein-Hawking expression for the entropy of a black hole (Eq. (12a)), and consequently to derive an explicit expression for the entropic uncertainty relation (§6).

5. Holographic double-spiral entropy

The double-helix, characterised by its radius R and pitch λ (both defined in Euclidean space) was discussed at length by PJ2019. It is the fundamental eigenvector of an entropic system, and it is holomorphic. They show that the entropic momentum for the double helix is given by $p = k_B/R$, and then the smallest change Δp in entropic momentum (found using $\partial p/\partial R = -k_B/R^2$) is given by:

$$\Delta p = \left(k_B / R^2\right) \Delta R \tag{8a}$$

We do the same for the hyperbolic position, defined as $q \equiv R \ln(x/R)$ (see PJ2019 Eq. (9a), where x is the associated Euclidean space co-ordinate, with x_3 being in the axial direction of the double helix). In particular, using

$$q' \equiv \partial q / \partial x3 = Rx'/x = iR\kappa \tag{8b}$$

where for a double helix described as a complex-vector (isomorphic to the Riemann-Silberstein vector description of an electromagnetic wave) we can assume $x = R \exp(i\kappa x_3)$. We see that the smallest change Δq in the hyperbolic position (q) is:

$$\Delta q = R\kappa \Delta x \tag{8c}$$

where the coupling parameter is given by $\kappa \equiv 2\pi/\lambda$. Alternatively, noting the fact that q' is an imaginary quantity (as indicated by Eq. (8b)), we can also express the smallest change in Δq as:

$$\Delta q = |q'| \Delta x \tag{8d}$$

To obtain an expression for the maximum entropy of this system (the double-helix) we need the volume Ω_S spanned by the maximum allowable ranges of p and q in the entropic phase-space. Considering the spatial co-ordinate (that is, the hyperbolic position q) it is clear that the maximum value q_{max} (that is, the maximal extent of q) is of the order of the helix radius R: $q_{max} \sim R$. To obtain the corresponding value of p_{max} for the double-helix we consider the entropic momentum p defined in terms of the entropic mass m_S (given by $m_S \equiv i\kappa k_B$) and the hyperbolic "velocity" q' (see Eq. (8b) and PJ2019 Eq. (9b)):

$$p \equiv m_S / q' \tag{9a}$$

Due to the stable and non-time varying nature of the systems under discussion in this paper, a scalar description of the QGT geometry is sufficient (in contrast to the fully general analysis including temporal behaviour which requires a vectorial analysis). This means that consideration of only one of the single helices of the double-helix is adequate. For the unwrapped single helical geometry as shown in Fig. 1, and exploiting Eq. (8b) we have

$$\tan \alpha = 2\pi R/\lambda = \kappa R = |q'| = \text{constant} \tag{9b}$$

We assume that the acute angle α in Fig. 1 has a minimum value given by $\alpha \geq 17.7°$, which corresponds to a minimum hyperbolic velocity for a helical geometry given by $|q'|_{min} \equiv 1/\pi$. This means that from Eq. (9a) we have $p_{max} = \pi\kappa k_B$, giving the overall phase-space volume for the double-helix spanning the maximum allowable ranges of p and q as:

$$\Omega_s = p_{max} q_{max} = \pi\kappa R k_B \tag{10}$$

Exploiting the geometry of Fig. 1 and substituting these expressions into Eq. (7d) allows us to write the entropy of the fundamental double-helical system as:

$$S = \frac{\Omega_s}{\Delta p \Delta q} k_B = \frac{\pi\kappa R k_B}{(k_B/R^2)\Delta R \cdot R\kappa\Delta x} k_B = \frac{1}{4} \frac{4\pi R^2}{\Delta R \Delta x} k_B = \frac{1}{4} \frac{A}{\Delta R \Delta x} k_B \tag{11}$$

where $A \equiv 4\pi R^2$.

We immediately see here that the geometric entropy of the double-helix is holographic in the sense that the entropy S depends on the *surface area* A of a sphere of radius R, and the number of small tiles of area $\Delta R \Delta x$ (defining the granularity of the system) covering the spherical surface. We have therefore demonstrated for the first time that a holographic system, described by a lower number of degrees of freedom (associated with a 2D surface) compared to the number as given by the associated (3-space) volume one might anticipate, therefore also conforms to the MaxEnt interpretation of the entropic Liouville Theorem. Thus we see that the RHS of Eq. (11) (as derived from the entropic Liouville Theorem) now offers a geometric interpretation for the QGT eigenvector (a double-helix) entropy that is based upon the properties of a sphere.

If we define the smallest possible increment of spatial granularity as being the Planck length l_P, such that $\Delta R = \Delta x = l_P$, with the radius R set as the Schwarzschild radius r_S, then we recover the celebrated Bekenstein-Hawking entropy of a black hole (note that this is therefore

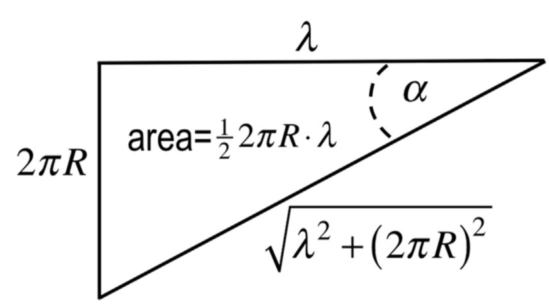

Fig. 1. Length of one period of an unwrapped helix showing its geometric properties.

also a consequence of the novel application of Liouville's Theorem in the entropic domain):

$$S_{BH} = \frac{1}{4}\frac{4\pi r_s^2}{l_p^2}k_B \tag{12a}$$

However, Eq. (11) has actually been derived assuming the geometry of a double-helix; whereas the geometry of a (vacuum) black hole must be that of a sphere. Fortunately, Parker & Jeynes [13] have shown that the complex-vector geometry of a sphere is equivalent to the sum of *two identical* double-helices. That is, for the radius of the sphere r_S, the radii of the two double-helices composing it are therefore both given by $R \equiv r_S/2$. Had we set $\Delta x = \Delta R = 1/k$ this would be equivalent to an entropy S for each double-helix of $S = 2\pi\kappa^2 R^2 k_B$ (PJ2019 Appendix D Eq.D6b). In which case the black hole entropy is also correctly given by $S_{BH} \equiv 2S$.

The geometric entropy of both the sphere and the double-helix are clearly holographic in nature, since they are proportional to the surface areas of enclosed volumes. We note that spiral galaxies usually have at least *two* pairs of spirals: this may be a consequence of a sphere being composed of *two* double-helices in geometric entropy terms.

So the entropy of the DNA molecule is given (PJ2019 Eq.D5) by the surface area $A = 2\pi RL$ of the cylinder surrounding the double helix of radius R and length L (again taking $\Delta x = \Delta R = 1/k$ with the pitch of the DNA molecule λ_0 given by $\kappa = 2\pi/\lambda_0$). In this case, the factor (¼ in Eq. (11)) is now determined by the helical angle, $1/(2\sin\alpha)$:

$$S_{DNA} = \frac{1}{2\sin\alpha}\frac{A}{(\lambda_0/2\pi)^2}k_B \tag{12b}$$

The holographic principle can be seen again in the expression for the entropy of the entire galaxy (see PJ2019 Eq.25):

$$S_{Galaxy} = 2 \times \frac{1}{4}\frac{A}{(\lambda_G/2\pi)^2}k_B \tag{12c}$$

In this case, the surface area of an ellipsoid (radii $L/2$ and R_G) surrounding the galaxy is given by $A \approx 2\pi R_G L$, the factor two arises due to the overall geometry being composed of two double-helical eigenvectors, and the granularity of the galactic entropy is defined by the galactic wavelength λ_G.

It is clear that consideration of the geometric entropy of systems ranging in scale from the molecular (DNA), to the macroscopic (Schwarzschild black hole), through to the cosmic (galaxy) scales yields a common holographic interpretation of the resulting entropy. There appears to be an ambiguity as to the correct dimensionless factor to use in the holographic expression: we believe that, from a fundamental perspective, ¼ is correct. The use of $1/(2\sin\alpha)$ in Eq. (12b) arises as a result of different assumptions on the most appropriate system granularity. We know that the Planck length l_P and the physics of a black hole represent together a physical extreme with the system granularity therefore intrinsically defined. However, the other examples used here (DNA and a galaxy) each represent more intermediate-scale systems, where the intrinsic granularity is less self-evident; hence the slight variation in holographic factor. But the holographic principle itself remains clear, and is a consequence of the holomorphism (and MaxEnt state) of the objects considered.

6. Entropic uncertainty principle

Consideration in a Fourier or harmonic analysis of functions that are finite in the spatial domain and that are also causal (therefore also being finite in the temporal domain) implies consideration of the limits in accuracy for derived quantities. In particular, Fourier analysis requires the conjugate variables (of space and time, and their inverse counterparts) to be reciprocally related, such that a greater level of accuracy in the measurement of a quantity in the one domain necessitates a reduction in the accuracy available for the conjugate quantity. This is the mathematical viewpoint of the uncertainty principle: the best-known physical example is the Heisenberg uncertainty principle.

It is clear that the entropic conjugate variables p and q must also obey such an uncertainty principle. PJ2019 demonstrate a comprehensive isomorphism between the entropic geometry (based upon Boltzmann's constant k_B) and conventional kinematics (in which Heisenberg's uncertainty relation states that the uncertainty between the conjugate momentum and position variables is limited by the Planck constant: $\Delta p\Delta x \geq \hbar/2$); that is, the energy quantum is Planck's constant, and the entropy quantum is Boltzmann's constant.

This isomorphism between k_B and ħ was discussed at length by PJ2019 (see their Table 1), and is *Discussed* further below. We merely observe here that their close relationship has been long suspected, including by Heisenberg and Bohr: see a brief review and an illuminating discussion of the issues by Velazquez & Curilef (2009) [31]; Frieden (1992) [32] has also obtained a similar result in an entirely different and highly suggestive treatment. Beretta (2019) [33] has explicitly derived complementary expressions for the time/energy and time/entropy uncertainty relations involving k_B and ħ. We are concerned here with the geometric entropy in a treatment that does not involve time (that is, we do not consider the temporal evolution of the systems that are under consideration).

We therefore state the equivalent *entropic* Uncertainty Principle which we establish by the following argument that culminates in Eq. (18a):

$$\Delta p\Delta q \geq k_B \tag{13}$$

We note here that this is certainly plausible since the entropic momentum Δp and hyperbolic position Δq of Eq. (13) have already been proved by PJ2019 to be conjugate variables, obeying the canonical Hamiltonian (as well as the Lagrangian) relations.

There have been various previous attempts to define an equivalent set of entropic uncertainty relations based on the dimensionality of Eq. (13) (as discussed in Velazquez, 2012 [34]); however these have been empirical and dimensional best-guesses, without an underpinning Hamiltonian-Lagrangian theoretical framework. So Acosta et al. (2011) [35] consider the uncertainty in the entropy itself ΔS and the uncertainty in the number of particles composing the system under consideration ΔN; while Ruuge (2013) [36] assumes it is the uncertainty of the energy ΔU and the uncertainty of the inverse temperature $\Delta\beta$ that form the conjugate variables. But neither of these alternative representations form the basis of a consistent Hamiltonian-Lagrangian framework for the equations of state.

However, we justify the proposed entropic uncertainty principle of Eq. (13) by consideration of the Liouville Theorem. In particular, considering the MaxEnt geometry of unitary objects (such as the double-helix) where $N = 1$, we can re-write Eq. (3) (dropping the subscripts n) as:

$$\frac{d\rho_S}{dx_3} = \frac{\partial\rho_S}{\partial x_3} + p'\frac{\partial\rho_S}{\partial p} + q'\frac{\partial\rho_S}{\partial q} = \frac{\partial\rho_S}{\partial x_3} - \left(\frac{\partial H_S}{\partial q}\frac{\partial}{\partial p} - \frac{\partial H_S}{\partial p}\frac{\partial}{\partial q}\right)\rho_S = 0 \tag{14a}$$

It is clear that Eq. (14a) represents a Poisson bracket operation in the entropic p,q phase-space,

$$\frac{d\rho_s}{dx_3} = \left(\frac{\partial}{\partial x_3} - \{H_s, \cdot\}\right)\rho_s = 0 \tag{14b}$$

so that the *entropic* Liouvillian $\widehat{L}_S$ is given by:-

$$\widehat{L}_S = \{H_S, \cdot\} = \frac{\partial H_S}{\partial q}\frac{\partial}{\partial p} - \frac{\partial H_S}{\partial p}\frac{\partial}{\partial q} \tag{14c}$$

This is also consistent with Eq. (4a) so that we can write $q' = \partial H_s/\partial p = \{q, H_s\}$ and $p' = -\partial H_s/\partial q = \{p, H_s\}$. A classical canonical Hamiltonian system as described here can be transformed into a quantum system by replacing the Poisson brackets with the commutation operation (denoted by square brackets [ ]) and re-interpreting variables as quantum operators (denoted by the hats in Eq. (18)): then given

$d\rho_S/dx_3 = 0$, we write a quantum version of Eq. (14b) (noting that the dimensionality is correct):

$$\frac{\partial \rho_S}{\partial x_3} = \frac{-1}{2k_B}[H_S, \rho_S] \tag{14d}$$

where of course the equivalent quantum Liouville equation for kinematics is $\partial\rho/\partial t = (1/\mathrm{i}\hbar)[H,\rho]$ and the kinematical $\mathrm{i}\hbar$ is isomorphic to the entropic $-2k_B$ (as also seen below in Eq. (15)). The novel use of Boltzmann's constant in Eq. (14d) is the key difference from conventional quantum (kinematic) formalisms based upon Planck's constant. In other words, according to the Minkowski spacetime formalism, k_B (associated with space-like quantities) and ħ (identified with time-like quantities) are orthogonal to each other, that is, they transform to each other by a $\pi/2$ rotation in the complex plane (hence the imaginary factor i) as already noted previously by multiple authors (Acosta et al. [35], Córdoba et al. [37]; Baez & Pollard [38]; Velazquez [34]). Thus in Eq. (14d) k_B appears *without* the imaginary factor, and for the same reason, in Eq. (14c) the entropic Liouvillian $\widehat{\boldsymbol{L}}_S$ (being fundamentally based on Boltzmann's constant k_B) also appears *without* the imaginary factor, in contrast to the conventional kinematic Liouvillian of Eq. (1b).

In PJ2019 (Appendix A, Eq.A.4), we found that mutually orthogonal quantities such as the electric and magnetic fields, as well as the information h and entropy s vectors, are Hodge duals of each other. It is equally clear that the (reduced) Planck constant ħ (being time-like) and the Boltzmann constant k_B (space-like) are mutually orthogonal to each other in 4-space: we conjecture that in some sense ħ (representing kinematic quantities) and k_B (representing entropic quantities) also represent a Hodge duality.

The factor 2 that appears in Eq. (14d) is also a reflection of an explicitly *entropic* version of the Partition Function being composed of probabilities that are analogous to the *modulus-squared* of the Schrödinger Equation, as described by Córdoba et al. (2013) [37] (their Eqs.29,30); that is, we now observe for the first time the following isomorphism, distinguished by its precise line-integral expression:

$$\sqrt{|\Psi|^2}\exp\left(\frac{-\mathrm{i}}{\hbar}\int H\mathrm{d}t\right) \leftrightarrow \frac{1}{\sqrt{Z_S}}\exp\left(\frac{-1}{2k_B}\int H_S\mathrm{d}x_3\right) \tag{15}$$

For convenience we will refer to this isomorphism as "Eq. (15)". The left-hand side of Eq. (15) is a solution to the Schrödinger Equation, and the right-hand side is its isomorph representing an amplitude component of the entropic Partition Function (see Eq. (16a), below). The Schrödinger Equation is normalised as is conventional by the probability factor $|\Psi|^2$; similarly, the entropic Partition Function defines the normalisation factor Z_S (see below, Eqs. (16) and (17)). The negative sign is also consistent with the entropic Partition Function itself being in a maximum entropy configuration described by a negative exponential distribution.

It is noteworthy that the RHS of Eq. (15) contains within it the entropic Hamiltonian H_S, in contrast to the conventional Partition Function of statistical mechanics which contains the kinematic Hamiltonian H. We will show that the RHS of Eq. (15) provides a consistent measure of the system entropy, alternative (and complementary) to the conventional one.

As in Eq. (2) above, both sides of Eq. (15) have a path integral. The Schrödinger Equation entails the *principle of stationary phase* (Eq. (2a)), since its stable solutions are characterised by the integral $\varphi = \int H\mathrm{d}t$ being stationary. Similarly, each amplitude component of the entropic Partition Function entails the MaxEnt principle (Eq. (2b)), since stable solutions are characterised by the integral $S = \int H_S\mathrm{d}x_3$ being stationary.

The conventional functional operations on the series of amplitude components (Eq. (15)) forming the entropic Partition Function Z_S (Eq. (16a)) yield the appropriate analogous quantities. For a general, overall geometric structure composed of multiple subsystems each representing an entropy component (that is, the overall geometry is composed of N unitary sub-structures, each representing two DoFs of the overall structure, and each with a sub-system (component) entropy S_n according to Eq. (7a)), then we can arrange each subsystem in ascending order of entropy, where we use the subscript n to denote the position of the n^{th} subsystem in the ordered array. In which case we form the following entropic Partition Function, based on the geometric entropy properties of the overall structure:

$$Z_S = \sum_n e^{-S_n/k_B} \equiv \sum_n e^{-\beta_S S_n} \tag{16a}$$

$$P_n = \frac{e^{-S_n/k_B}}{Z_S} \equiv \frac{e^{-\beta_S S_n}}{Z_S} \tag{16b}$$

where we assume the entropic parameter $\beta_S \equiv 1/k_B$ (the entropic equivalent to the conventional inverse temperature parameter $\beta \equiv 1/k_B T$), and the associated probability term P_n of Eq. (16b). Note that in this representation β_S is treated as a *functional* (which happens to be a constant). We then immediately write:

$$\langle S\rangle = -\frac{\partial \ln Z_S}{\partial \beta_S} \tag{17a}$$

$$S = \langle S\rangle + k_B \ln Z_S \tag{17b}$$

in analogy to the conventional statistical mechanical quantities for the expected energy $<E>$ and the entropy S, where $\langle E\rangle \equiv -\partial \ln Z/\partial\beta$ and $S \equiv k_B(\beta\langle E\rangle + \ln Z)$, respectively, for the conventional partition function Z.

It is clear that the entropic Partition Function of Eq. (16a) can be used to generate alternative, yet physically consistent and valid expressions for thermodynamic quantities such as the expected entropy $<S>$ of Eq. (17a), which represents the average entropy over the available eigenstate entropies S_n, and the overall entropy S of Eq. (17b) which represents the expected entropy $<S>$ plus the entropy determined by the granularity of the entropic Partition Function, with the MaxEnt condition $\partial S/\partial\beta_S = 0$ applying. Note that the summation in Eq. (16a) is over the N subsystems (corresponding to the $2N$ degrees of freedom) of the system; that is, the "canonical ensemble" in view here is the appropriate entropic one.

For entropic systems (based on the conjugate variables p and q) quantised using the Boltzmann constant, it is immediately apparent that the following entropic commutator result therefore also holds:

$$\left[\widehat{q}_S, \widehat{p}_S\right] = -2k_B \tag{18a}$$

showing that this representation is consistent since Eq. (18a) is the *canonical commutation relation* from which the entropic uncertainty principle of Eq. (13) immediately follows.

It is also clear that in progressing from the classical Poisson brackets to the quantum entropic commutator, the entropic conjugate variable p can be identified for the first time with the following *entropic* quantum operator:

$$\widehat{p}_S \equiv 2k_B\frac{\partial}{\partial q} \tag{18b}$$

Likewise, it is clear that the entropic Hamiltonian H_S also represents the appropriate operator in our QGT formalism with the double-helix acting as its associated eigenvector. Note that Eq. (18b) is based in hyperbolic space q, whereas the conventional kinematic momentum operator is based on Euclidean space x ($\widehat{p} \equiv -\mathrm{i}\hbar\partial/\partial x$) as is the conventional Heisenberg uncertainty principle. However, the entropic uncertainty principle in Euclidean space is also obtained by substituting Eq. (8d) into Eq. (13):

$$\Delta p \cdot |q'| \Delta x \geq k_B \tag{19}$$

From Eq. (8a) we substitute for Δp as follows:

$$\frac{\Delta R}{R^2} k_B \cdot |q'| \Delta x \geq k_B \tag{20a}$$

so that we can also write the entropic uncertainty principle in terms of the hyperbolic “velocity” q' (Eq. (4b)):

$$|q'| \geq \frac{R^2}{\Delta R \Delta x} \tag{20b}$$

7. Hyperbolic “velocity”

Eq. (20b) makes it clear that the entropic uncertainty principle is naturally dimensionless only in hyperbolic space. Given that $\Delta x = \Delta R \equiv 1/\kappa$, as previously noted for the fundamental double-helix geometry, then we can express the magnitude of the hyperbolic velocity as (recasting Eq. (20b)):

$$|q'| \geq \kappa^2 R^2 \tag{21a}$$

However, from Eq. (8b) we have for a double helix $|q'| = \kappa R$, such that Eq. (21a) can therefore also be expressed as:

$$|q'| \geq |q'|^2 \tag{21b}$$

with the novel implication that

$$|q'| \leq 1 \tag{22}$$

Kinematics makes a distinction between the *group velocity* $v_g \equiv \partial\omega/\partial k$ (equivalent to $v_g = \partial H/\partial p$ as in Eq. (1c)) and the *phase velocity* $v_\varphi \equiv \omega/k$; the group velocity represents the speed of a particle and corresponds to the speed of information transfer, where ω is the frequency and $k = 2\pi/\lambda$ is the propagation constant or wavenumber of the wave of wavelength λ. The phase velocity is not constrained by c since matter and information can still only travel at the group velocity.

We underline the standard result of classical electrodynamics (see Jackson [39] §8.5) that in natural units (that is, $c = 1$) the product of group (v_g) and phase (v_φ) velocities is unity:

$$v_g v_\varphi = c^2 \tag{23a}$$

and note that Maxwell’s equations are a *hyperbolic* version of the Cauchy-Riemann equations (see Courant & Hilbert [27] vol.II p.178). Therefore, exploiting the comprehensive isomorphism between the kinematic and hyperbolic entropic velocities, we define for the first time a ‘phase’ hyperbolic velocity q'_φ, such that the product of q' (the ‘group’ hyperbolic velocity) and q'_φ is unity:

$$q' \equiv 1/q'_\varphi \tag{23b}$$

Any hyperbolic velocity greater than unity must be considered to be the *phase* hyperbolic velocity q'_φ, so that we therefore have a lower bound for the phase hyperbolic velocity:

$$|q'_\varphi| \geq 1 \tag{23c}$$

which is consistent with Eq. (22) and therefore also consistent with the entropic uncertainty principle Eq. (13). It should be noted that calculations for the radius and wavelength of lengths of double-helical B-DNA and P-DNA in PJ2019 with $R = \{1.0, 0.6\}$ nm and $\lambda = \{3.32, 1.28\}$ nm, respectively, indicate hyperbolic velocities of $|q'_{B-DNA}| = 1.89$ and $|q'_{P-DNA}| = 2.95$; these must therefore be understood to represent the relevant *phase* hyperbolic velocities.

This highlights an essential ambiguity in hyperbolic space between the hyperbolic velocity and its inverse, since being dimensionless they both yield the same magnitude (but of different sign) for the kinetic entropy $T_S \equiv -m_S \ln q'$ (PJ2019, Eq.10a). In which case, as indicated by Eq. (23), the *group* hyperbolic velocity is axiomatically *defined* to be less than or equal to unity (in conformance to the entropic uncertainty principle). In contrast, its reciprocal (the *phase* hyperbolic velocity) must be more than or equal to unity. Both hyperbolic velocities are calculated using the entropic Hamiltonian equation of state identity: $q' \equiv \partial H_S/\partial p$ from Eq. (4a).

8. Discussion

Section §5 considers explicitly the geometry of the double-helix, this being the fundamental eigenvector of the QGT formalism. In particular, the double-helix is mathematically described as a double plane-wave (Eq. (8) of PJ2019), and a general treatment would involve a complex vector representation (as per the Riemann-Silberstein description of Maxwell’s equations in optics, for example). §5 and consequently §§6,7 deal only with non-time-varying (stable) geometries, for which a scalar description is sufficient. A fully general analysis where the full temporal behaviour is described, would require a fully vectorial analysis; but we do not yet consider this, for now treating only static geometric structures in equilibrium. The study of non-equilibrium systems is now emerging as important (witness the work of Prigogine’s Brussels-Austin school [14,16]). This is also seen in other important recent developments in non-equilibrium thermodynamics, such as the fluctuation-dissipation theorem where the Jarzynski equality [40] is a relatively recent development. In the future it will clearly be desirable to achieve a fully-vectorial (relativistic) analysis that includes the time-varying aspects of systems within a dynamic setting: this will enable generalisation of this work to other non-equilibrium (maximum and minimum entropy production) physical phenomena (such as galaxies [15]).

We emphasise that the entropic Partition Function of Eq. (16a) (as exemplified by the entropic parameter $\beta_S \equiv 1/k_B$) is intrinsically independent of temperature T; that is, as a MaxEnt description, Eq. (16a) is valid for all values of T. In contrast, for the conventional Partition Function employing the kinematic Hamiltonian H, the principle of Maximum Entropy only applies for a particular temperature as determined by the value of the Lagrangian multiplier β. Temperature only starts playing a role in any entropic analysis when the connection to a kinematic description (that is, related to energy) is required. Therefore, temperature can be seen as the coupling parameter (coefficient) connecting the kinematic and entropic domains.

The isomorph between the Schrödinger Equation and entropic Partition Function probability amplitude component of Eq. (15) also has similarities to the *Wick rotation* which effects a transformation between the time evolution operator of quantum mechanics and the density operator of statistical mechanics. For the Wick rotation (see Penrose [22] §28.9) this transformation maps imaginary time onto “thermodynamic time” τ (see Eq. (18) of Córdoba et al., 2013 [37], and see also Hongo, 2017 [9]): $it \leftrightarrow \tau$. However, whereas the Wick rotation maintains the same kinematic Hamiltonian H as well as the reduced Planck constant $\hbar$ across the transformation into the statistical mechanical domain, it is clear in our analysis that we employ a different (that is to say, an entropic) Hamiltonian H_S in our entropic Partition Function as well as the Boltzmann constant k_B. Thus, our entropic analysis here shares intriguing features with the Wick rotation: compare our use of the Euclidean Pauli conjugate variable x_3, corresponding to a real space co-ordinate, $it \leftrightarrow x_3$.

Also compare the requirement for information-bearing (entropic) functions such as the entropic Hamiltonian H_S to satisfy the Hurwitz polynomial criterion for stability (see Parker & Walker, 2004 [7]); this is equivalent to the Wick rotation requirement that the imaginary component of complex *t* must be less than or equal to zero. It is also worth noting that when applying the Wick rotation to the path-integral of quantum-mechanical propagators, this transforms the conventional kinematic Lagrangean *L* into what is called the Euclidean Lagrangean L_E, since it is now defined in Euclidean (as opposed to Minkowski) space. However, following Wick-transformation the kinematic Euclidean Lagrangean also frequently becomes identical to the kinematic Hamiltonian where the kinetic and potential energy terms sum. From this perspective, another interesting common feature becomes evident that our entropic Partition Function with its entropic Hamiltonian shares with the Wick-rotated path integral; yet our entropic Partition Function is still resolutely within (hyperbolic) Minkowski 4-space.

In addition, the Wick rotation is also generally associated with a periodic boundary condition for its 'thermodynamic' time dimension; this also has a strong resonance with the periodicity λ seen along the x_3 axis of the MaxEnt double-helix eigenvector geometry. Overall, the Wick rotation represents an elegant connection between quantum mechanics and statistical mechanics; fully understanding the distinction between it and our entropic approach is the subject of further research.

We can also comment briefly on how this formalism accounts for the difference between bosons and fermions. Further work [41] on the alpha particle naturally incorporates the antisymmetrisation requirement of fermions. We have not so far treated photons (bosons) directly, except in so far as the field-free electromagnetic wave is well-known to be holomorphic.

9. Concluding remarks

The close relationship between quantum mechanics (as expressed by the Schrödinger Equation) and statistical mechanics (as defined by the Partition Function) is well known (via the Wick rotation). However, using geometric entropy and the entropic version of Liouville's Theorem, we have been able to describe a comprehensive isomorphism between the Schrödinger Equation and the amplitude components of an entropic form of the Partition Function which intrinsically conform to the principle of Maximum Entropy. In so doing, we have shown not only how the entropy of a MaxEnt system is holographic in nature, but also that there exists an associated entropic version of the uncertainty principle, based on the Boltzmann constant as the appropriate entropic counterpart to the Planck constant. The entropic uncertainty principle places bounds on the (dimensionless) hyperbolic velocity q' such that its upper bound is unity, which is isomorphic to the kinematic speed of light in vacuum *c*, and which also represents the hyperbolic boundary between the group and phase hyperbolic velocities.

Finally, the phenomenon of "dimensional reduction" observed in black hole physics has been characterised by Carlip [42] as "*a hint that spacetime at very short distances becomes effectively two dimensional*". We think that, whatever the merits of this view, the well-known holographic effects observed in black holes are not confined to black holes or Planck distances but are actually ubiquitous in nature, as partially demonstrated here. We have interpreted this as a consequence of an entropic application of Liouville's Theorem to maximum entropy systems in hyperbolic space and shown that it is essential to a quantitative treatment of geometric thermodynamics.

CRediT authorship contribution statement

M.C. Parker: Conceptualization, Formal analysis, Methodology, Validation, Visualization, Writing – original draft, Writing – review & editing. **C. Jeynes:** Methodology, Validation, Visualization, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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