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# Entwining Yang–Baxter maps over Grassmann algebras

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ABSTRACT

In this work we construct novel solutions to the set-theoretical entwining Yang–Baxter equation. These solutions are birational maps involving non-commutative dynamical variables which are elements of the Grassmann algebra of order $n$. The maps arise from refactorisation problems of Lax supermatrices associated to a nonlinear Schrödinger equation. In this non-commutative setting, we construct a spectral curve associated to each of the obtained maps using the characteristic function of its monodromy supermatrix. We find generating functions of invariants for the entwining Yang–Baxter maps from the moduli of the spectral curves. Moreover, we show that a hierarchy of birational entwining Yang–Baxter maps with commutative variables can be obtained by fixing the order $n$ of the Grassmann algebra, and we present the cases $n = 1$ (dual numbers) and $n = 2$. Then we discuss the integrability properties, such as Lax matrices, invariants, and measure preservation, for the obtained discrete dynamical systems.

## 1. Introduction

The first appearances of the Yang–Baxter (YB) equation can be traced back to the study of quantum many-body systems and exactly solvable models in statistical mechanics [1,2]. After that, the YB equation appeared in a broad range of different fields, from quantum field theory and quantum inverse scattering method to gauge theory, and quantum groups. See for example [3,4], and references therein, for an introduction and a collection of the original papers in the field. Naturally, an intensive focus on finding and classifying solutions to the equation followed [5,6]. Originally, the focus was on finding solutions of the YB equation

$$\mathcal{R}^{12}\mathcal{R}^{13}\mathcal{R}^{23} = \mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12} \tag{1}$$

that are linear maps $\mathcal{R} : V \otimes V \to V \otimes V$, where $V$ is a $\mathbb{F}$-vector space. Here, $\mathcal{R}^{13}$, for example, denotes the action of $\mathcal{R}$ on the first and third copy of $V \otimes V \otimes V$. The study of another class of solutions to the YB equation was proposed by V. Drinfeld in [7], where now these solutions are maps $\mathcal{R} : A \times A \to A \times A$ and $A$ can be any set. Such solutions are often called set-theoretical solutions or Yang–Baxter maps, with the latter term introduced by Veselov in [8] following [9].

A generalisation of (1) which is relevant to the current work originates in the study of quantum integrable systems (see for example [10–12]) and is given by the following equation

$$\mathcal{S}^{12}\mathcal{R}^{13}\mathcal{T}^{23} = \mathcal{T}^{23}\mathcal{R}^{13}\mathcal{S}^{12} \tag{2}$$

which is known as entwining YB equation. A triplet of maps $\mathcal{S}$, $\mathcal{R}$, $\mathcal{T}$ satisfying (2) were first derived in [13], inspired by the work in [14], and other works on constructing such maps using e.g. classical star-triangle relations [15], symmetries of YB maps [16], or scattering of matrix solitons [17] followed.

A plethora of works on the YB equation and its generalisations has appeared in various physical applications in the past few decades such as in relation to collisions of relativistic particles [18,19], knot theory [20], geometric crystals [21], discrete dynamical systems and soliton theory [8,22]. See also [23–25] for related classifications of such maps. In particular, there are deep connections between the theory of nonlinear integrable partial differential and lattice equations and YB maps [26–31]. For example, interactions of solitons, of vector or matrix generalisations of known integrable PDEs, are described by maps for the internal degrees (polarisations) which satisfy the Yang–Baxter equation [32–36]. Moreover, higher dimensional analogues of the YB equation (and its entwining version), known as n-simplex equations, is an area of active research which has gained increased popularity, see [5,37–41] and references therein.

In recent years different types of solutions of the YB equation have been derived from various algebraic constructions. On one hand, combinatorial solutions of the YB and the associated braid equation have been produced using certain algebraic structures such as braces, racks and quandles, see for example [42,43]. On the other hand, birational

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solutions of the YB equation have been constructed using ideas from the theory of integrable systems. In particular, the connection between Darboux and Bäcklund transformations for integrable PDEs and YB maps has been explored recently (see e.g. [44,45]), and it is also relevant to the current work. Further, birational solutions to the YB equation in non-commutative settings having been studied in, for example, [46–52]. YB maps containing bosonic and fermionic dynamical variables, related to a super extended integrable NLS equation, were derived in [48] using the formalism of Grassmann algebras. Moreover, in [53] the authors derived an extension of a YB map over Grassmann algebras starting from the Darboux transformation associated to a super KdV hierarchy [54], and which is linked to the discrete potential KdV equation.

The integrability, in the sense of Liouville, of the YB maps and their corresponding transfer maps requires the existence of sufficient number of invariants, as well as a Poisson structure which is invariant under the action of the map and under which all invariants commute. The corresponding notion of integrability of maps over associative but not necessarily commutative algebras is a challenging open problem, see for example the review paper [55]. A step towards the understanding of the Liouville integrability for Grassmann extended YB maps was presented in [53] where the integrability of the Adler map over the complex dual numbers $\mathbb{C}[\theta]/\langle\theta^2\rangle$ was shown. The approach followed in [53] for extensions over the dual numbers can be used in the same way for integrable maps of other types. Indeed, recently in [56], the integrability properties of the Somos-4 sequences over the dual numbers were studied.

In what follows, we extend the investigation of integrability properties of maps with Grassmann variables, this time to solutions of the entwining YB equation (2). The structure of the paper is: Section 2 introduces notation and necessary background material in relation to the Grassmann algebra $\Gamma(n)$, the parametric entwining YB equation, and the Lax triple for entwining YB maps. In Section 3 we derive Grassmann entwining YB maps and their invariants. In Sections 4 and 5 we construct the first two members of a hierarchy of entwining YB maps with commutative variables, and discuss their integrability properties. Finally, in Section 6 we offer some concluding remarks in relation to this work and discuss some directions of future work. More specifically, the main outcomes of this work are described below:

- We derive novel birational parametric entwining YB maps starting from Lax matrices which in the commutative limit are Darboux transformations associated to the nonlinear Schrödinger equation.
- We construct the characteristic rational functions for the monodromy supermatrices associated to these maps, and we use them to derive invariants from the moduli of the associated spectral curves. To our knowledge, this is the first time that entwining YB maps of this type and their associated spectral curves appear in the literature.
- We obtain entwining YB maps with commutative variables in dimensions 8 and 16 associated to the Grassmann algebras $\Gamma(1)$ and $\Gamma(2)$, and we present their integrability properties such as Lax representation, invariants, and measure preservation. This new approach can be used to obtain entwining YB maps in dimensions $2^{n+2}$ and their corresponding Lax matrices and invariants.

## 2. Preliminaries

### 2.1. Grassmann algebras

We denote by $\Gamma(n)$ the Grassmann algebra of order $n$ over a field $\mathbb{F}$ of characteristic zero (such as $\mathbb{R}$ or $\mathbb{C}$). $\Gamma(n)$ is an associative algebra with unit 1 and $n$ generators $\theta_i$, $i = 1, \dots, n$, satisfying

$$\theta_i\theta_j + \theta_j\theta_i = 0\,. \tag{3}$$

The elements of $\Gamma(n)$ that contain sums of products of only even (resp. odd) number of $\theta_i$'s are called *even* (resp. *odd*) and are denoted by $\Gamma(n)_0$ (resp. $\Gamma(n)_1$). Even elements commute with all elements of $\Gamma(n)$, while the odd elements anticommute with each other. The Grassmann algebra $\Gamma(n)$, considered as a vector space, can be written as the direct sum of $\Gamma(n)_0$ and $\Gamma(n)_1$, namely $\Gamma(n) = \Gamma(n)_0 \oplus \Gamma(n)_1$. Moreover, we have that $\Gamma(n)$ has a natural $\mathbb{Z}_2$-grading, i.e. $\Gamma(n)_i\Gamma(n)_j \subseteq \Gamma(n)_{(i+j) \bmod 2}$ and therefore $\Gamma(n)_0$ is a subalgebra of $\Gamma(n)$. In what follows we denote elements of $\Gamma(n)_0$ by Latin letters, and elements of $\Gamma(n)_1$ by Greek letters, with the exception of $\lambda$ which plays the role of the spectral parameter and takes values in the field $\mathbb{F}$. A Grassmann algebra is an example of a superalgebra, i.e. a super vector space with a $\mathbb{Z}_2$-grading. For more details on superalgebras we direct the reader to [57–59].

We denote by $\mathbb{F}_n^{k,l}$ the $(k,l)$-dimensional superspace consisting of tuples of $k$ even and $l$ odd variables of a Grassmann algebra of order $n$ over $\mathbb{F}$, namely

$$\mathbb{F}_n^{k,l} := \{(\boldsymbol{x}, \boldsymbol{\chi}) \mid \boldsymbol{x} \in \Gamma(n)_0^k,\ \boldsymbol{\chi} \in \Gamma(n)_1^l\}\,. \tag{4}$$

In Sections 4, 5 we present examples of maps over Grassmann algebras of order $n = 1$ and $n = 2$, respectively. Specifically, the $n = 1$ case is the algebra of dual numbers (over $\mathbb{F}$), where an element of the algebra is of the form $a + b\theta$ with $\theta^2 = 0$ and $a, b \in \mathbb{F}$. In the case $n = 2$, a generic element of the algebra can be written in the form $a + b\theta_1 + c\theta_2 + d\theta_1\theta_2$, with $\theta_1, \theta_2$ satisfying (3), and with $a + d\theta_1\theta_2 \in \Gamma(2)_0$, $b\theta_1 + c\theta_2 \in \Gamma(2)_1$ and $a, b, c, d \in \mathbb{F}$.

We will be working with square matrices with elements in $\Gamma(n)$ (such matrices are examples of supermatrices), of the block-form

$$M = \begin{pmatrix} P & \Pi \\ \Lambda & L \end{pmatrix},$$

where $P$, $L$ are $p \times p$ and $q \times q$ matrices with elements in $\Gamma(n)_0$, while $\Pi$, $\Lambda$ are $p \times q$ and $q \times p$ matrices with elements in $\Gamma(n)_1$. We say that, for example, $P$ is an element of $\mathrm{Mat}_p(\Gamma(n)_0)$ and $\Pi$ of $\mathrm{Mat}_{p,q}(\Gamma(n)_1)$. We also assume that $\det(L)$ and $\det(P)$ are non-zero. We denote the set of $(p+q)\times(p+q)$ supermatrices, such as $M$, by $\mathrm{M}_{p,q}$. The superdeterminant for $M \in \mathrm{M}_{p,q}$ is defined by:

$$\mathrm{sdet}(M) = \det(P - \Pi L^{-1}\Lambda)\det(L)^{-1} = \det(P)\det(L - \Lambda P^{-1}\Pi)^{-1}, \tag{5}$$

and is multiplicative, meaning

$$\mathrm{sdet}(M_1M_2) = \mathrm{sdet}(M_1)\,\mathrm{sdet}(M_2)\,, \tag{6}$$

for $M_1, M_2 \in \mathrm{M}_{p,q}$. It follows that the characteristic (rational) function

$$f_M(k) = \mathrm{sdet}(M - kI_{p,q}) \tag{7}$$

for a matrix $M \in \mathrm{M}_{p,q}$, with $I_{p,q}$ the unit supermatrix in $\mathrm{M}_{p,q}$, is invariant under similarity transformations $M \to UMU^{-1}$ for $U \in \mathrm{M}_{p,q}$, see [60], i.e.

$$f_{UMU^{-1}}(k) = f_M(k)\,, \tag{8}$$

which follows from (6) and the fact that $\mathrm{sdet}(U^{-1}) = \mathrm{sdet}(U)^{-1}$.

### 2.2. Parametric entwining Yang–Baxter equation

In this section we introduce a type of solutions of the entwining YB equation that depend on certain parameters in $\mathbb{F}$. For consistency we also assume that $A$ is $\mathbb{F}^d$ for a positive integer $d$. In many examples in the literature, the field $\mathbb{F}$ is $\mathbb{C}$ and the set $A$ is $\mathbb{C}^d$ or $\mathbb{CP}^d$ and the resulting (entwining) YB maps are birational isomorphisms of $A \times A$. For more exotic examples see [46,50].

We consider the maps

$$\mathcal{R}_{a,b},\ \mathcal{S}_{a,b},\ \mathcal{T}_{a,b}\ :\ A \times A \to A \times A\,,$$

which depend on parameters $a, b \in \mathbb{F}$. Given for example map $\mathcal{R}_{a,b}$, we denote by $\mathcal{R}_{a,b}^{i,j}$ with $i \neq j \in \{1,2,3\}$ the extended map which acts as $\mathcal{R}_{a,b}$ on the $i$ and $j$ copies of the triple Cartesian product of $A$ with itself, and identically on the remaining copy of $A$. More precisely, we have

$$\mathcal{R}_{a,b}^{12} = \mathcal{R}_{a,b} \times id\,, \quad \mathcal{R}_{a,b}^{23} = id \times \mathcal{R}_{a,b}\,, \quad \mathcal{R}_{a,b}^{13} = \pi^{12} \circ \mathcal{R}_{a,b}^{23} \circ \pi^{12}\,,$$

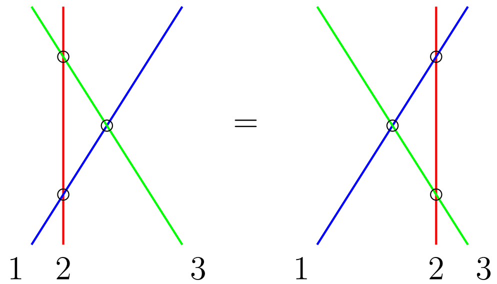


**Fig. 1.** Diagrammatic representation of entwining YB equation (9).

with $\pi^{12}$ the extension of the permutation (flip) map $\pi : (x, y) \to (y, x)$ on $A \times A$, and $id$ the identity map on $A$. In this paper we will be concerned with parametric ordered triplets of maps $(\mathcal{S}_{a,b}, \mathcal{R}_{a,b}, \mathcal{T}_{a,b})$ which satisfy the parametric entwining YB equation

$$\mathcal{S}^{12}_{a,b} \circ \mathcal{R}^{13}_{a,c} \circ \mathcal{T}^{23}_{b,c} = \mathcal{T}^{23}_{b,c} \circ \mathcal{R}^{13}_{a,c} \circ \mathcal{S}^{12}_{a,b} \,. \tag{9}$$

We call such maps entwining YB maps. Eq. (9) is to be understood as equality of compositions of maps over the triple product $A \times A \times A$ (see Fig. 1).

The entwining YB equation (9) can be represented by the following diagram, where the lines are coloured to indicate that each crossing corresponds to a different map, e.g. the red-blue crossing corresponds to $\mathcal{S}^{12}_{a,b}$, etc. When all lines have the same colour, i.e. when $\mathcal{S}_{a,b} = \mathcal{R}_{a,b} = \mathcal{T}_{a,b}$, then Eq. (9) reduces to the parametric YB equation

$$\mathcal{R}^{12}_{a,b} \circ \mathcal{R}^{13}_{a,c} \circ \mathcal{R}^{23}_{b,c} = \mathcal{R}^{23}_{b,c} \circ \mathcal{R}^{13}_{a,c} \circ \mathcal{R}^{12}_{a,b} \,. \tag{10}$$

In general, a parametric YB map $R_{a,b}(x, y) = (u_{a,b}(x, y), v_{a,b}(x, y))$ is called non-degenerate if the maps $u_{a,b}(\cdot, y) : A \to A$ and $v_{a,b}(x, \cdot) : A \to A$ are bijective [21,25]. More recently, non-degenerate YB maps which are also birational have been referred to as quadrirational YB maps [23,24]. We use the same terminology for maps that satisfy the entwining YB equation (9).

Following [13], we define a strong Lax triple for maps $\mathcal{S}_{a,b}, \mathcal{R}_{a,b}, \mathcal{T}_{a,b}$ to be a triple of matrices $\mathcal{L}_a, \mathcal{M}_a, \mathcal{N}_a$, each depending on a point $x \in A$, a parameter $a \in \mathbb{F}$ and a spectral parameter $\lambda \in \mathbb{F}$, such that the matrix refactorisation problems

$$\mathcal{L}_a(u)\mathcal{M}_b(v) = \mathcal{M}_b(y)\mathcal{L}_a(x)\,, \tag{11a}$$

$$\mathcal{L}_a(u)\mathcal{N}_b(v) = \mathcal{N}_b(y)\mathcal{L}_a(x)\,, \tag{11b}$$

$$\mathcal{M}_a(u)\mathcal{N}_b(v) = \mathcal{N}_b(y)\mathcal{M}_a(x)\,, \tag{11c}$$

imply uniquely the maps $\mathcal{S}_{a,b}, \mathcal{R}_{a,b}, \mathcal{T}_{a,b} : (x, y) \to (u, v)$, respectively. If Eqs. (11a)–(11c) are satisfied for given $\mathcal{S}_{a,b}, \mathcal{R}_{a,b}, \mathcal{T}_{a,b}$ maps, then the triple of matrices is called simply a Lax triple. In general we omit the dependence of the Lax matrices on the spectral parameter $\lambda$ for convenience. It was proved in [13] that if $\mathcal{L}_a, \mathcal{M}_a, \mathcal{N}_a$ is a strong Lax triple for maps $\mathcal{S}_{a,b}, \mathcal{R}_{a,b}, \mathcal{T}_{a,b}$ and the following equality

$$\mathcal{L}_a(x)\mathcal{M}_b(y)\mathcal{N}_c(z) = \mathcal{L}_a(x')\mathcal{M}_b(y')\mathcal{N}_c(z') \tag{12}$$

implies that $x = x'$, $y = y'$ and $z = z'$ then the maps are entwining YB maps. If $\mathcal{L}_a = \mathcal{M}_a = \mathcal{N}_a$, then the refactorisation problems (11a)–(11c) coincide and $\mathcal{L}_a$ is a strong Lax matrix for the parametric YB map $\mathcal{R}_{a,b}$.

In Section 3 we derive maps $\mathcal{S}_{a,b}, \mathcal{R}_{a,b}, \mathcal{T}_{a,b}$ satisfying the parametric entwining YB equation (9) over the Grassmann algebra $\Gamma(n)$. In this case, the set $A$ is the $(k, l)$-dimensional superspace $\mathbb{F}^{k,l}_n$ for given positive integers $k, l$, and the obtained entwining YB maps are birational maps of $\mathbb{F}^{k,l}_n \times \mathbb{F}^{k,l}_n$. We will consider the reduction $\mathcal{N}_a \equiv \mathcal{L}_a$, hence the Lax triple will be $\mathcal{L}_a, \mathcal{M}_a, \mathcal{L}_a$ and each matrix will depend on a point $(\boldsymbol{x}, \chi) \in \mathbb{F}^{k,l}_n$. The maps we construct are Grassmann extensions of the following entwining YB maps

$$\begin{aligned}
\mathrm{S}_{a,b}(x_1, x_2, y_1, y_2) &= \left( \frac{y_1^2}{bx_1} + \frac{y_1}{b}(y_2 - a), \frac{b}{y_1}, x_1, a + x_1x_2 - \frac{y_1}{x_1} \right), \\
\mathrm{R}_{a,b}(x_1, x_2, y_1, y_2) &= \left( y_1 - \frac{a-b}{1+x_1y_2}x_1, y_2, x_1, x_2 + \frac{a-b}{1+x_1y_2}y_2 \right), \\
\mathrm{T}_{a,b}(x_1, x_2, y_1, y_2) &= \left( \frac{a}{y_2}, b + y_1y_2 - \frac{a}{x_1y_2}, x_1, \frac{a}{x_1^2y_2} + \frac{x_2 - b}{x_1} \right),
\end{aligned} \tag{13}$$

which admit a strong Lax triple and are Liouville integrable having polynomial and rational invariants that Poisson-commute [61].

## 3. Grassmann entwining YB maps

In this section we derive birational, parametric, entwining YB maps over the Grassmann algebra $\Gamma(n)$, starting from the refactorisation problems of certain Lax supermatrices. These Lax matrices are Darboux matrices associated to a Grassmann generalisation of the NLS equation [62]. Refactorisation problems of certain Darboux matrices over Grassmann algebras resulting to YB maps were also considered in [48]. Moreover, in [61] the parametric entwining YB maps given in (13) were derived from the refactorisation problems of the Darboux matrices which were presented in [63]. The resulting entwining YB maps of this section are generalisations of (13) involving non-commutative (Grassmann) variables. We note here that while the maps that we obtain in this paper are birational, they are degenerate or non-quadrirational.

We consider the following supermatrices in $\mathrm{M}_{2,1}$

$$\mathcal{L}_a(\boldsymbol{x}, \chi) = \begin{pmatrix} x_1x_2 + \chi_1\chi_2 + a + \lambda & x_1 & \chi_1 \\ x_2 & 1 & 0 \\ \chi_2 & 0 & 1 \end{pmatrix},$$

$$\mathcal{M}_a(\boldsymbol{x}, \chi) = \begin{pmatrix} x_2 + \lambda & x_1 & \chi_1 \\ \frac{a}{x_1} & 0 & 0 \\ \chi_2 & 0 & 1 \end{pmatrix}, \tag{14}$$

with $(\boldsymbol{x}, \chi) = (x_1, x_2, \chi_1, \chi_2) \in \mathbb{F}^{2,2}_n$, $a \in \mathbb{F}$ a parameter, and $\lambda \in \mathbb{F}$ a spectral parameter. The $2 \times 2$ blocks of $\mathcal{L}_a$ and $\mathcal{M}_a$ with entries in $\Gamma(n)_0$ constitute the Darboux matrices for NLS derived in [63]. The refactorisation problems (11a)–(11c) for matrices (14), with $\mathcal{N}_a \equiv \mathcal{L}_a$, have unique solutions for $\big((\boldsymbol{u}, \xi), (\boldsymbol{v}, \boldsymbol{\eta})\big)$ in terms of $\big((\boldsymbol{x}, \chi), (\boldsymbol{y}, \boldsymbol{\psi})\big)$. These give rise to eight-dimensional birational maps $\mathcal{R}_{a,b}$, $\mathcal{S}_{a,b}$, $\mathcal{T}_{a,b}$ with even–odd Grassmann variables which act as

$$\big((x_1, x_2, \chi_1, \chi_2), (y_1, y_2, \psi_1, \psi_2)\big) \mapsto \big((u_1, u_2, \xi_1, \xi_2), (v_1, v_2, \eta_1, \eta_2)\big)\ .$$

In particular, map $\mathcal{R}_{a,b}$ is defined by the following expressions

$$\mathcal{R}_{a,b} : \begin{cases} u_1 = y_1 - \frac{(a-b)(1+x_1y_2-\chi_1\psi_2)}{(1+x_1y_2)^2}x_1\,, & v_1 = x_1\,, \\ u_2 = y_2\,, & v_2 = x_2 + \frac{(a-b)(1+x_1y_2-\chi_1\psi_2)}{(1+x_1y_2)^2}y_2\,, \\ \xi_1 = \psi_1 - \frac{a-b}{1+x_1y_2}\chi_1\,, & \eta_1 = \chi_1\,, \\ \xi_2 = \psi_2\,, & \eta_2 = \chi_2 + \frac{a-b}{1+x_1y_2}\psi_2\,, \end{cases} \tag{15}$$

and map $\mathcal{S}_{a,b}$ is given by

$$\mathcal{S}_{a,b} : \begin{cases} u_1 = \frac{y_1^2(1+\chi_1\psi_2)}{bx_1} + \frac{y_1(y_2-a-\psi_1\psi_2)}{b}\,, & v_1 = x_1\,, \\ u_2 = \frac{b}{y_1}\,, & v_2 = a + x_1x_2 - \frac{y_1}{x_1} + \chi_1\chi_2\,, \\ \xi_1 = \psi_1 - \frac{y_1}{x_1}\chi_1\,, & \eta_1 = \chi_1\,, \\ \xi_2 = \psi_2\,, & \eta_2 = \chi_2 + \frac{y_1}{x_1}\psi_2\,. \end{cases} \tag{16}$$

Finally, below we prove that map $\mathcal{T}_{a,b}$ is related to $\mathcal{S}_{a,b}$ by

$$\mathcal{T}_{a,b} = \pi \circ \mathcal{S}^{-1}_{b,a} \circ \pi, \tag{17}$$

where now $\pi$ is the flip map in $\mathbb{F}_n^{2,2} \times \mathbb{F}_n^{2,2}$, acting as $\pi((\boldsymbol{x},\chi),(\boldsymbol{y},\boldsymbol{\psi})) = ((\boldsymbol{y},\boldsymbol{\psi}),(\boldsymbol{x},\chi))$. This is a direct consequence of the reduction $\mathcal{N}_a \equiv \mathcal{L}_a$.

**Theorem 3.1.** *The maps $\mathcal{S}_{a,b}$, $\mathcal{R}_{a,b}$, $\mathcal{T}_{a,b}$ : $\mathbb{F}_n^{2,2}\times\mathbb{F}_n^{2,2} \to \mathbb{F}_n^{2,2}\times\mathbb{F}_n^{2,2}$ defined in (15)–(17) admit a strong Lax triple $\mathcal{L}_a, \mathcal{M}_a, \mathcal{L}_a$, with $\mathcal{L}_a, \mathcal{M}_a$ given in (14), and they satisfy the parametric entwining Yang–Baxter equation (9).*

**Proof.** The first part of the proof can be shown directly by solving the following refactorisation problems

$$\mathcal{L}_a(\boldsymbol{u},\xi)\mathcal{M}_b(\boldsymbol{v},\boldsymbol{\eta}) = \mathcal{M}_b(\boldsymbol{y},\boldsymbol{\psi})\mathcal{L}_a(\boldsymbol{x},\chi)\,, \tag{18a}$$

$$\mathcal{L}_a(\boldsymbol{u},\xi)\mathcal{L}_b(\boldsymbol{v},\boldsymbol{\eta}) = \mathcal{L}_b(\boldsymbol{y},\boldsymbol{\psi})\mathcal{L}_a(\boldsymbol{x},\chi)\,, \tag{18b}$$

$$\mathcal{M}_a(\boldsymbol{u},\xi)\mathcal{L}_b(\boldsymbol{v},\boldsymbol{\eta}) = \mathcal{L}_b(\boldsymbol{y},\boldsymbol{\psi})\mathcal{M}_a(\boldsymbol{x},\chi)\,, \tag{18c}$$

and showing that they admit a unique solution for $(\boldsymbol{u},\xi,\boldsymbol{v},\boldsymbol{\eta})$ in terms of $(\boldsymbol{x},\chi,\boldsymbol{y},\boldsymbol{\psi})$. The proof that the obtained maps satisfy the entwining YB equation is given in Appendix. □

**Remark 3.2.** The relation (17) between maps $\mathcal{S}_{a,b}$ and $\mathcal{T}_{a,b}$ can be readily deduced by observing that the refactorisation problems (18a) and (18c) are related by the transformation $a \leftrightarrow b$, $(\boldsymbol{u},\xi) \leftrightarrow (\boldsymbol{y},\boldsymbol{\psi})$, $(\boldsymbol{x},\chi) \leftrightarrow (\boldsymbol{v},\boldsymbol{\eta})$. This shows the birationality of maps $\mathcal{S}_{a,b}$ and $\mathcal{T}_{a,b}$. Moreover, the invariance of (18b) under the above transformation shows that $\mathcal{R}_{a,b}$ is also a birational map. In what follows we will only consider the maps $\mathcal{R}_{a,b}$ and $\mathcal{S}_{a,b}$.

Map $\mathcal{R}_{a,b}$ is an extension over Grassmann algebras of the Adler–Yamilov map [64]. This map and its corresponding matrix refactorisation problem (18b) were studied in [48], where it was shown that $\mathcal{R}_{a,b}$ is a birational Yang–Baxter map, and also reversible i.e. it satisfies the relation $\mathcal{R}_{a,b}^{-1} = \pi \circ \mathcal{R}_{b,a} \circ \pi$. Taking the commutative limit in maps $\mathcal{S}_{a,b}$ and $\mathcal{T}_{a,b}$, i.e. sending all the odd variables to zero, we obtain the birational maps which were derived in [61]. Therefore, the maps $\mathcal{S}_{a,b}, \mathcal{R}_{a,b}, \mathcal{T}_{a,b}$ in Theorem 3.1 form a generalisation over Grassmann algebras of the entwining Yang–Baxter maps given in (13). We show that invariants of these maps can be obtained using the invariance of the superdeterminant under similarity transformations, see (8) in Section 2. By invariant of a map, say $\mathcal{R}_{a,b}$, we mean a function $I$ such that $I \circ \mathcal{R}_{a,b} = I$. Moreover, an anti-invariant $I$ is a function such that $I \circ \mathcal{R}_{a,b} = -I$. It follows that the product of two different anti-invariants or the square of anti-invariants are all invariants of the given map.

**Theorem 3.3.** *The maps $\mathcal{R}_{a,b}, \mathcal{S}_{a,b}$ : $\mathbb{F}_n^{2,2}\times\mathbb{F}_n^{2,2} \to \mathbb{F}_n^{2,2}\times\mathbb{F}_n^{2,2}$, given in (15)–(16), admit the following $\mathcal{I}, \mathcal{J}$-sets of invariants, respectively:*

$$\begin{aligned}
&\mathcal{I}_1 = x_1x_2 + y_1y_2\,, \quad \mathcal{I}_2 = \chi_1\chi_2 + \psi_1\psi_2\,, \quad \mathcal{I}_3 = (x_1\psi_1 - y_1\chi_1)(x_2\psi_2 - y_2\chi_2)\,,\\
&\mathcal{I}_4 = b(x_1x_2 + \chi_1\chi_2) + a(y_1y_2 + \psi_1\psi_2) + y_1y_2(x_1x_2 + \chi_1\chi_2) + x_1x_2\psi_1\psi_2\\
&\quad + x_1y_2 + x_2y_1 + \chi_1\psi_2 + \psi_1\chi_2\,, \quad \mathcal{I}_5 = \chi_1\chi_2\psi_1\psi_2\,,
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\mathcal{J}_1 = y_2 + x_1x_2 + \chi_1\chi_2\,, \quad \mathcal{J}_2 = \chi_1\chi_2 + \psi_1\psi_2\,,\\
&\mathcal{J}_3 = (a + x_1x_2)\psi_1\psi_2 + (b + y_2)\chi_1\chi_2 + (1 - x_2y_1)\chi_1\psi_2 + \left(1 - \frac{bx_1}{y_1}\right)\psi_1\chi_2\,,\\
&\mathcal{J}_4 = b\frac{x_1}{y_1} + y_2(a + x_1x_2 + \chi_1\chi_2) + x_2y_1 + \chi_1\psi_2 + \psi_1\chi_2\,, \quad \mathcal{J}_5 = \chi_1\chi_2\psi_1\psi_2\,.
\end{aligned} \tag{20}$$

**Proof.** The invariants of map $\mathcal{S}_{a,b}$ are obtained using the monodromy supermatrix $P_S(\boldsymbol{x},\chi,\boldsymbol{y},\boldsymbol{\psi}) = \mathcal{M}_b(\boldsymbol{y},\boldsymbol{\psi})\mathcal{L}_a(\boldsymbol{x},\chi)$, with $\mathcal{L}_a, \mathcal{M}_b$ given in (14). From the refactorisation property (18a) we obtain the isospectrality property of the monodromy under the action of the map

$$P_S(\boldsymbol{u},\xi,\boldsymbol{v},\boldsymbol{\eta}) = \mathcal{M}_b(\boldsymbol{v},\boldsymbol{\eta})P_S(\boldsymbol{x},\chi,\boldsymbol{y},\boldsymbol{\psi})\mathcal{M}_b^{-1}(\boldsymbol{v},\boldsymbol{\eta})\,, \tag{21}$$

similarly to the commutative setting. It follows that the characteristic (rational) function of the monodromy supermatrix $f_{P_S}(k) = \mathrm{sdet}(P_S - kI_{2,1})$ generates invariants of the map $\mathcal{S}_{a,b}$.

The supermatrix $P_S - kI_{2,1}$ can be written in the form

$$P_S - kI_{2,1} = \begin{pmatrix} \mathrm{A} - kI & \mathrm{B} \\ \mathrm{C} & \mathrm{D}(k) \end{pmatrix}, \tag{22}$$

with $\mathrm{D}(k) = \psi_2\chi_1 + 1 - k$ and A, B, C functions of $\boldsymbol{x}, \chi, \boldsymbol{y}, \boldsymbol{\psi}$ and $\lambda$. Hence, from (5) we have that

$$\begin{aligned}
f_{P_S}(k,\lambda) &= \frac{\det(\mathrm{A} - kI - \mathrm{B}\mathrm{D}(k)^{-1}\mathrm{C})}{\det \mathrm{D}(k)}\\
&= \frac{1 - k - \psi_2\chi_1}{(1-k)^2}\det(\mathrm{A} - kI - \mathrm{B}\mathrm{D}(k)^{-1}\mathrm{C})\,.
\end{aligned}$$

Setting $\mu = 1 - k$, and factoring $\mu^{-2}$ outside the determinant, the characteristic function takes the form

$$f_{P_S}(\mu,\lambda) = \frac{\mu + \chi_1\psi_2}{\mu^6}\det(\mu^3 I + \mu^2(A - I) - \mu\mathrm{BC} - \chi_1\psi_2\mathrm{BC})\,. \tag{23}$$

The equation $\mu^3 f_{P_S}(\mu,\lambda) = 0$ defines the spectral curve associated to map $\mathcal{S}_{a,b}$ and its moduli provides the $\mathcal{J}$-set of invariants of the map. Indeed, expanding the determinant and using the explicit forms of $A, B, C$, we obtain

$$\mu^3 f_{P_S}(\mu,\lambda) = \mu^4 + \mu^3 f_3(\lambda) + \mu^2 f_2(\lambda) + \mu f_1(\lambda) + f_0(\lambda)\,, \tag{24}$$

where the $f_i(\lambda)$, with $i = 0,\dots,3$, are generating functions of the invariants of map $\mathcal{S}_{a,b}$ and have the following form

$$\begin{aligned}
&f_3(\lambda) = \lambda^2 + \lambda(\mathcal{J}_1 + a) + \mathcal{J}_4\,,\\
&f_2(\lambda) = -\lambda(\mathcal{J}_1 + \mathcal{J}_2 + a + b) - (2\mathcal{J}_5 + \mathcal{J}_4 + \mathcal{J}_3 + ab)\,,\\
&f_1(\lambda) = \mathcal{J}_3\,,\\
&f_0(\lambda) = -2\mathcal{J}_5\,.
\end{aligned} \tag{25}$$

Similarly, we define the monodromy matrix of $\mathcal{R}_{a,b}$ to be $P_{\mathcal{R}}(\boldsymbol{x},\chi,\boldsymbol{y},\boldsymbol{\psi}) = \mathcal{L}_b(\boldsymbol{y},\boldsymbol{\psi})\mathcal{L}_a(\boldsymbol{x},\chi)$ and then it follows that the characteristic function $g_{P_{\mathcal{R}}}(\mu,\lambda)$ associated with the map $\mathcal{R}_{a,b}$ can be written in the form

$$\mu^3 g_{P_{\mathcal{R}}}(\mu,\lambda) = \mu^4 + \mu^3 g_3(\lambda) + \mu^2 g_2(\lambda) + \mu g_1(\lambda)\,,$$

with

$$\begin{aligned}
&g_3(\lambda) = \lambda^2 + \lambda(\mathcal{I}_1 + \mathcal{I}_2 + a + b) + \mathcal{I}_5 + \mathcal{I}_4 + ab - 1\,,\\
&g_2(\lambda) = -\lambda(\mathcal{I}_1 + \mathcal{I}_2) - (3\mathcal{I}_5 + \mathcal{I}_4 + \mathcal{I}_3)\,,\\
&g_1(\lambda) = 2\mathcal{I}_5 + \mathcal{I}_3\,,
\end{aligned} \tag{26}$$

thus obtaining the $\mathcal{I}$–set of invariants of $\mathcal{R}_{a,b}$. □

**Remark 3.4.** The invariants $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_4, \mathcal{I}_5$ of map $\mathcal{R}_{a,b}$ were derived in [48] using the supertrace of the monodromy. We notice that $\mathcal{I}_2$ and $\mathcal{I}_5$ are related by $2\mathcal{I}_5 = \mathcal{I}_2^2$. Here we obtain the new invariant $\mathcal{I}_3$ of the map using the characteristic function.

**Remark 3.5.** One can verify that the quantities $\chi_1\psi_1$ and $\chi_2\psi_2$ are anti-invariants of all maps $\mathcal{R}_{a,b}, \mathcal{S}_{a,b}, \mathcal{T}_{a,b}$, and that the invariant $\mathcal{I}_5$ (and $\mathcal{J}_5$) can be obtained from the product of those anti-invariants. Moreover, for map $\mathcal{R}_{a,b}$ the quantities $x_i\psi_i - y_i\chi_i$ for $i = 1,2$ are anti-invariants, and $\mathcal{I}_3$ is the product of these two anti-invariants.

**Remark 3.6.** Using Remark 3.2 we deduce that the invariants of map $\mathcal{T}_{a,b}$ can be obtained from the $\mathcal{J}$-set using the reflection $a \leftrightarrow b$, $\boldsymbol{x} \leftrightarrow \boldsymbol{y}$, $\chi \leftrightarrow \boldsymbol{\psi}$.

In the following two sections we derive entwining YB maps in dimensions 8 and 16 with commutative variables. To achieve this, we consider the derived maps $\mathcal{R}_{a,b}, \mathcal{S}_{a,b}, \mathcal{T}_{a,b}$ in $\Gamma(1)$ and $\Gamma(2)$. This way, we demonstrate how the first two members of a hierarchy of birational entwining YB maps can be obtained, and we show how the integrability properties of the hierarchy can be obtained from those of maps (15)–(17). The members of the hierarchy are maps of increasing dimension $8, 16, \dots, 2^{n+2}, \dots$.

## 4. The Grassmann algebra $\Gamma(1)$

The algebra $\Gamma(1)$ has unit 1 and one generator $\theta$ with $\theta^2 = 0$. This is the case of dual numbers (over $\mathbb{F}$). We consider the maps $\mathcal{R}_{a,b}, \mathcal{S}_{a,b}, \mathcal{T}_{a,b}$ in (15)–(17) over the Grassmann algebra $\Gamma(1)$, and in this way we derive birational maps with commutative variables, denoted by $S_{a,b}$, $R_{a,b}$, $T_{a,b}$, which satisfy the entwining YB equation (9). Moreover, we study integrability properties of the obtained maps, such as Lax representation, invariants, and measure preservation.

We expand all variables of the maps $\mathcal{R}_{a,b}, \mathcal{S}_{a,b}, \mathcal{T}_{a,b}$ in (15)–(17) and their images in terms of 1 and $\theta$. The Latin variables are in $\Gamma(1)_0$ and are therefore proportional to 1, while the Greek variables are in $\Gamma(1)_1$ and thus proportional to $\theta$. Comparing coefficients of 1 and $\theta$ on both sides of Eqs. (15)–(17) we obtain maps $R_{a,b}$, $S_{a,b}$, $T_{a,b} : \mathbb{F}^4\times\mathbb{F}^4 \to \mathbb{F}^4\times\mathbb{F}^4$ with

$$((x_1, x_2, \chi_1, \chi_2), (y_1, y_2, \psi_1, \psi_2)) \mapsto ((u_1, u_2, \xi_1, \xi_2), (v_1, v_2, \eta_1, \eta_2)).$$

These maps are given by the following expressions

$$R_{a,b} : \begin{cases} u_1 = y_1 - \dfrac{a-b}{1+x_1y_2}x_1 , & v_1 = x_1 , \\ u_2 = y_2 , & v_2 = x_2 + \dfrac{a-b}{1+x_1y_2}y_2 , \\ \xi_1 = \psi_1 - \dfrac{a-b}{1+x_1y_2}\chi_1 , & \eta_1 = \chi_1 , \\ \xi_2 = \psi_2 , & \eta_2 = \chi_2 + \dfrac{a-b}{1+x_1y_2}\psi_2 , \end{cases} \tag{27}$$

$$S_{a,b} : \begin{cases} u_1 = \dfrac{y_1^2}{bx_1} + \dfrac{y_1(y_2-a)}{b} , & v_1 = x_1 , \\ u_2 = \dfrac{b}{y_1} , & v_2 = a + x_1x_2 - \dfrac{y_1}{x_1} , \\ \xi_1 = \psi_1 - \dfrac{y_1}{x_1}\chi_1 , & \eta_1 = \chi_1 , \\ \xi_2 = \psi_2 , & \eta_2 = \chi_2 + \dfrac{y_1}{x_1}\psi_2 , \end{cases} \tag{28}$$

and $T_{a,b}$ is related to $S_{a,b}$ by (17). For simplicity, we have used the same letters for the variables in (27)–(28) as in the case of $\Gamma(n)$. From Theorem 3.1 it follows that the above maps satisfy the entwining YB equation (9).

The eight-dimensional birational maps $R_{a,b}, S_{a,b}, T_{a,b} = \pi \circ S_{b,a}^{-1} \circ \pi$ defined by (27)–(28) admit a strong Lax triple $L_a, M_a, L_a$, with $L_a, M_a$ given by

$$L_a(x_1,x_2,\chi_1,\chi_2) = \begin{pmatrix} x_1x_2+a+\lambda & 0 & x_1 & 0 & 0 & \chi_1 \\ 0 & x_1x_2+a+\lambda & 0 & x_1 & 0 & 0 \\ x_2 & 0 & 1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 1 & 0 & 0 \\ 0 & \chi_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{29}$$

and

$$M_a(x_1,x_2,\chi_1,\chi_2) = \begin{pmatrix} x_2+\lambda & 0 & x_1 & 0 & 0 & \chi_1 \\ 0 & x_2+\lambda & 0 & x_1 & 0 & 0 \\ \frac{a}{x_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{a}{x_1} & 0 & 0 & 0 & 0 \\ 0 & \chi_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{30}$$

where $x_i, \chi_i \in \mathbb{F}$ for $i = 1, 2$. Namely, each of the matrix refactorisations

$$L_a(u_1,u_2,\xi_1,\xi_2)L_b(v_1,v_2,\eta_1,\eta_2) = L_b(y_1,y_2,\psi_1,\psi_2)L_a(x_1,x_2,\chi_1,\chi_2), \tag{31a}$$

$$L_a(u_1,u_2,\xi_1,\xi_2)M_b(v_1,v_2,\eta_1,\eta_2) = M_b(y_1,y_2,\psi_1,\psi_2)L_a(x_1,x_2,\chi_1,\chi_2), \tag{31b}$$

$$M_a(u_1,u_2,\xi_1,\xi_2)L_b(v_1,v_2,\eta_1,\eta_2) = L_b(y_1,y_2,\psi_1,\psi_2)M_a(x_1,x_2,\chi_1,\chi_2), \tag{31c}$$

leads to a system of polynomial equations which can be solved uniquely for $(u_1, u_2, \xi_1, \xi_2, v_1, v_2, \eta_1, \eta_2)$ leading to maps $R_{a,b}, S_{a,b}, T_{a,b}$, respectively.

The Lax matrices $L_a$, $M_a$ in (29), (30) are obtained by using the tensor product of $\mathrm{Mat}_3(\mathbb{F})$ with a representation of $\Gamma(1)$. In particular, first we express the Lax supermatrices $\mathcal{L}_a$ and $\mathcal{M}_a$ in (14) as elements in $\mathrm{Mat}_3(\mathbb{F})\otimes\Gamma(1)$ by expanding their entries in terms of 1 and $\theta$, therefore writing them as

$$\mathcal{L}_a(\mathbf{x},\chi) = L_1 \otimes 1 + L_2 \otimes \theta , \quad \mathcal{M}_a(\mathbf{x},\chi) = M_1 \otimes 1 + M_2 \otimes \theta , \tag{32}$$

with

$$L_1 = \begin{pmatrix} x_1x_2+a+\lambda & x_1 & 0 \\ x_2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} x_2+\lambda & x_1 & 0 \\ \frac{a}{x_1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$L_2 = M_2 = \begin{pmatrix} 0 & 0 & \chi_1 \\ 0 & 0 & 0 \\ \chi_2 & 0 & 0 \end{pmatrix}.$$

Then, we represent 1 and $\theta$ by $2\times 2$ matrices using the algebra homomorphism $\rho : \Gamma(1) \to \mathrm{Mat}_2(\mathbb{F})$ defined by its action on the basis of the algebra

$$1 \overset{\rho}{\mapsto} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \theta \overset{\rho}{\mapsto} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{33}$$

and thus we obtain $L_a$, $M_a$ in (29) and (30) from

$$L_a = L_1 \otimes \rho(1) + L_2 \otimes \rho(\theta) \text{ and } M_a = M_1 \otimes \rho(1) + M_2 \otimes \rho(\theta). \tag{34}$$

**Remark 4.1.** The matrix refactorisation problems in (31) with

$$\tilde{L}_a = \rho(1) \otimes L_1 + \rho(\theta) \otimes L_2 , \quad \tilde{M}_a = \rho(1) \otimes M_1 + \rho(\theta) \otimes M_2 \tag{35}$$

also imply the maps $R_{a,b}, S_{a,b}, T_{a,b}$, since the Lax matrices $L_a$, $M_a$ in (34) and $\tilde{L}_a$, $\tilde{M}_a$ in (35) are similar under a permutation matrix.

The expansion in the basis of $\Gamma(1)$ can also be performed for the $\mathcal{I}, \mathcal{J}$ families of invariants in (19), (20) of maps $\mathcal{R}_{a,b}$ and $\mathcal{S}_{a,b}$ leading to invariants of $R_{a,b}, S_{a,b}$. This way, invariants $\mathcal{I}_i$ and $\mathcal{J}_i$, $i = 1, \ldots, 5$, are expressed in terms of 1 and powers of $\theta$. It is interesting to note that, while $\theta^k = 0$ for $k \geq 2$, the coefficients of $\theta^k$ in the expansions can still be invariant quantities. For example, the invariant $\mathcal{I}_2 = \chi_1\chi_2 + \psi_1\psi_2$ of $\mathcal{R}_{a,b}$ is expressed as $\mathcal{I}_2 = I_2\theta^2$ in $\Gamma(1)$. While $\theta^2 = 0$, it turns out that $I_2$, which involves the commutative coefficients of $\chi_i, \psi_i \in \Gamma(1)$, is an invariant of map $R_{a,b}$ (27). Following this idea, we obtain the following sets of functionally independent invariants for $R_{a,b}$ and $S_{a,b}$, respectively:

$$I_1 = x_1x_2 + y_1y_2 , \quad I_2 = \chi_1\chi_2 + \psi_1\psi_2 , \quad I_3 = (x_1\psi_1 - y_1\chi_1)(x_2\psi_2 - y_2\chi_2),$$
$$I_4 = bx_1x_2 + ay_1y_2 + x_1x_2y_1y_2 + x_1y_2 + x_2y_1 , \tag{36}$$

and

$$J_1 = y_2 + x_1x_2 , \quad J_2 = \chi_1\chi_2 + \psi_1\psi_2 ,$$
$$J_3 = (a + x_1x_2)\psi_1\psi_2 + (b + y_2)\chi_1\chi_2 + (1 - x_2y_1)\chi_1\psi_2 + \left(1 - \frac{bx_1}{y_1}\right)\psi_1\chi_2 ,$$
$$J_4 = b\frac{x_1}{y_1} + y_2(a + x_1x_2) + x_2y_1 . \tag{37}$$

We note that the invariants $\mathcal{I}_5$ and $\mathcal{J}_5$ of $\mathcal{R}_{a,b}, \mathcal{S}_{a,b}$ do not produce any invariants for $R_{a,b}, S_{a,b}$.

**Remark 4.2.** The invariants $I_1, I_4$ of $R_{a,b}$ can also be obtained from the characteristic function of the monodromy matrix $P_R = L_b(y_1,y_2,\psi_1,\psi_2) L_a(x_1,x_2,\chi_1,\chi_2)$. Similarly, $J_1, J_4$ can be obtained from the characteristic function $P_S = M_b(y_1,y_2,\psi_1,\psi_2)L_a(x_1,x_2,\chi_1,\chi_2)$ associated to map $S_{a,b}$.

Maps $R_{a,b}, S_{a,b}, T_{a,b} = \pi \circ S_{b,a}^{-1} \circ \pi$ in (27)–(28) can be written in 'triangular form', meaning that each of them can be expressed as the composition of a linear map which acts on $(\chi_1, \chi_2, \psi_1, \psi_2)$ with coefficients being rational functions of $(x_1, x_2, y_1, y_2)$, and a map that acts non-trivially only on the variables $(x_1, x_2, y_1, y_2)$. For example, for $R_{a,b}$ we have the decomposition $R_{a,b} = \bar{R}_{a,b} \circ \hat{R}_{a,b}$, where

$$\hat{R}_{a,b} : (x_1, x_2, \chi_1, \chi_2, y_1, y_2, \psi_1, \psi_2) \mapsto (x_1, x_2, \xi_1, \xi_2, y_1, y_2, \eta_1, \eta_2)$$

with

$$(\xi_1, \xi_2, \eta_1, \eta_2) = \left(\psi_1 + \frac{b-a}{1+x_1y_2}\chi_1, \chi_1, \chi_2 + \frac{a-b}{1+x_1y_2}\psi_2, \psi_2\right),$$

and $\bar{R}_{a,b}$ an extension of the Adler–Yamilov map [64]

$$\bar{R}_{a,b} : (x_1, x_2, \chi_1, \chi_2, y_1, y_2, \psi_1, \psi_2) \mapsto (u_1, u_2, \chi_1, \chi_2, v_1, v_2, \psi_1, \psi_2)$$

with

$$(u_1, u_2, v_1, v_2) = \left(y_1 + \frac{b-a}{1+x_1y_2}x_1, y_2, x_1, x_2 + \frac{a-b}{1+x_1y_2}y_2\right).$$

In these decompositions of $R_{a,b}, S_{a,b}, T_{a,b}$ the rational maps which act only on the variables $(x_1, x_2, y_1, y_2)$ are extensions of the maps $\mathrm{R}_{a,b}, \mathrm{S}_{a,b}, \mathrm{T}_{a,b}$ in (13), which were shown to be Liouville integrable in [61].

Finally, regarding the dynamical properties of maps $R_{a,b}, S_{a,b}, T_{a,b}$, we show that these maps are measure preserving. This means that for each of them there exists a function $m$ of the dynamical variables such that the Jacobian determinant $J$ of the map can be written as [26]

$$J = \frac{m(x_1, x_2, \chi_1, \chi_2, y_1, y_2, \psi_1, \psi_2)}{m(u_1, u_2, \xi_1, \xi_2, v_1, v_2, \eta_1, \eta_2)}.$$

It can be verified that maps $R_{a,b}, S_{a,b}, T_{a,b}$ are measure preserving with $m(x_1, x_2, \chi_1, \chi_2, y_1, y_2, \psi_1, \psi_2)$ equal to 1, $\frac{1}{y_1}$ and $\frac{1}{x_1}$, respectively. In particular, the YB map $R_{a,b}$ is volume preserving.

## 5. The Grassmann algebra $\Gamma(2)$

In this section we derive 16-dimensional birational, parametric, entwining YB maps $\mathsf{R}_{a,b}, \mathsf{S}_{a,b}, \mathsf{T}_{a,b} : \mathbb{F}^8 \times \mathbb{F}^8 \to \mathbb{F}^8 \times \mathbb{F}^8$ which act as

$$\begin{aligned}&((x_{11}, x_{12}, x_{21}, x_{22}, \chi_{11}, \chi_{12}, \chi_{21}, \chi_{22}), (y_{11}, y_{12}, y_{21}, y_{22}, \psi_{11}, \psi_{12}, \psi_{21}, \psi_{22})) \mapsto \\ &\quad ((u_{11}, u_{12}, u_{21}, u_{22}, \xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}), (v_{11}, v_{12}, v_{21}, v_{22}, \eta_{11}, \eta_{12}, \eta_{21}, \eta_{22})).\end{aligned}$$

These maps are obtained from maps $\mathcal{R}_{a,b}, \mathcal{S}_{a,b}, \mathcal{T}_{a,b}$ in (15)–(17) for the case of the Grassmann algebra $\Gamma(2)$, following the ideas presented in Section 4. Unlike the case of maps $R_{a,b}, S_{a,b}, T_{a,b}$ obtained in the previous section, the maps presented here are not in 'triangular form'. More precisely, although the maps act linearly on $(\chi_{11}, \chi_{12}, \chi_{21}, \chi_{22}, \psi_{11}, \psi_{12}, \psi_{21}, \psi_{22})$ with coefficients which are rational functions of only $(x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22})$, their action on $(x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22})$ has coefficients which are functions of all the dynamical variables $x_{ij}, \chi_{ij}, y_{ij}, \psi_{ij}$ for $i, j \in \{1, 2\}$. Similar to the case of the 8-dimensional birational maps which were derived in Section 4, the 16-dimensional maps $\mathsf{R}_{a,b}, \mathsf{S}_{a,b}, \mathsf{T}_{a,b}$ obtained in this section admit a strong Lax triple, are measure preserving, and each of them has a family of invariants.

The elements of the $\Gamma(2)$ Grassmann algebra can be written as linear combinations of $1, \theta_1, \theta_2, \theta_1\theta_2$ with $\theta_i\theta_j + \theta_j\theta_i = 0$ for $i, j = 1, 2$. Expressing each of the components of a point $(\boldsymbol{x}, \boldsymbol{\chi}) = (x_1, x_2, \chi_1, \chi_2) \in \mathbb{F}_2^{2,2}$ in the basis of $\Gamma(2)$ we have

$$x_i = x_{i1} + x_{i2}\theta_1\theta_2, \quad \chi_i = \chi_{i1}\theta_1 + \chi_{i2}\theta_2, \tag{38}$$

and for even elements we have

$$x_i^{-1} = \frac{1}{x_{i1}} - \frac{x_{i2}}{x_{i1}^2}\theta_1\theta_2, \quad \text{with} \quad x_{ij}, \chi_{ij} \in \mathbb{F} \quad \text{for} \quad i, j \in \{1, 2\}. \tag{39}$$

Starting from maps $\mathcal{R}_{a,b}, \mathcal{S}_{a,b}, \mathcal{T}_{a,b}$ that were derived in Section 3, we express each of their variables as described above. By comparing coefficients of $1, \theta_1\theta_2$ and $\theta_1, \theta_2$ on both sides of each of the Eqs. (15)–(17), we obtain sixteen-dimensional maps which we denote by $\mathsf{R}_{a,b}, \mathsf{S}_{a,b}, \mathsf{T}_{a,b}$, respectively. The components of these maps are given in Box I,

$$\mathsf{S}_{a,b} : \begin{cases} u_{11} = \frac{y_{11}^2}{bx_{11}} + \frac{y_{11}(y_{21}-a)}{b}, & v_{11} = x_{11}, \\ u_{12} = -\frac{x_{12}y_{11}^2}{bx_{11}^2} + \frac{y_{11}(2y_{12}+y_{11}(\chi_{12}\psi_{21}+\chi_{11}\psi_{22}))}{bx_{11}} & \\ \qquad + \frac{y_{12}(y_{21}-a)}{b} + \frac{y_{11}(y_{22}+\psi_{12}\psi_{21}-\psi_{11}\psi_{22})}{b}, & v_{12} = x_{12}, \\ u_{21} = \frac{b}{y_{11}}, & v_{21} = a + x_{11}x_{21} - \frac{y_{11}}{x_{11}}, \\ u_{22} = -\frac{by_{12}}{y_{11}^2}, & v_{22} = x_{11}x_{22} + x_{12}x_{21} \\ & \qquad + \frac{x_{12}y_{11}}{x_{11}^2} - \frac{y_{12}}{x_{11}} \\ & \qquad - \chi_{12}\chi_{21} + \chi_{11}\chi_{22}, \\ \xi_{11} = \psi_{11} - \frac{y_{11}}{x_{11}}\chi_{11}, & \eta_{11} = \chi_{11}, \\ \xi_{12} = \psi_{12} - \frac{y_{11}}{x_{11}}\chi_{12}, & \eta_{12} = \chi_{12}, \\ \xi_{21} = \psi_{21}, & \eta_{21} = \chi_{21} + \frac{y_{11}}{x_{11}}\psi_{21}, \\ \xi_{22} = \psi_{22}, & \eta_{22} = \chi_{22} + \frac{y_{11}}{x_{11}}\psi_{22}, \end{cases} \tag{41}$$

and again $\mathsf{T}_{a,b} = \pi \circ \mathsf{S}_{b,a}^{-1} \circ \pi$.

Invariants of maps $\mathsf{R}_{a,b}$ and $\mathsf{S}_{a,b}$ can be obtained starting from the families of invariants $\mathcal{I}$, $\mathcal{J}$ of $\mathcal{R}_{a,b}$, $\mathcal{S}_{a,b}$, respectively. Following the ideas discussed in the previous section, we first express the variables that appear in invariants $\mathcal{I}_i$ and $\mathcal{J}_i$ in terms of $1, \theta_1\theta_2$ and $\theta_1, \theta_2$. Then the coefficients of 1, $\theta_1\theta_2$ as well as those of powers of $\theta_1$ and $\theta_2$ can lead to invariants for the 16-dimensional maps (40), (41). In particular, using only invariants $\mathcal{I}_1, \mathcal{I}_2$ and $\mathcal{I}_4$ from the list (19) we obtain the following functionally independent invariants for map $\mathsf{R}_{a,b}$

$$\begin{aligned} &\mathrm{I}_1 = x_{11}x_{21} + y_{11}y_{21}, \quad \mathrm{I}_2 = x_{11}x_{22} + x_{12}x_{21} + y_{11}y_{22} + y_{12}y_{21}, \\ &\mathrm{I}_3 = \chi_{11}\chi_{22} + \psi_{11}\psi_{22}, \quad \mathrm{I}_4 = \chi_{12}\chi_{21} + \psi_{12}\psi_{21}, \\ &\mathrm{I}_5 = \chi_{11}\chi_{21} + \psi_{11}\psi_{21}, \quad \mathrm{I}_6 = \chi_{12}\chi_{22} + \psi_{12}\psi_{22}, \\ &\mathrm{I}_7 = bx_{11}x_{21} + ay_{11}y_{21} + x_{11}y_{21} + x_{21}y_{11} + x_{11}x_{21}y_{11}y_{21}, \\ &\mathrm{I}_8 = b(x_{11}x_{22} + x_{12}x_{21} + \chi_{11}\chi_{22} - \chi_{12}\chi_{21}) \\ &\quad + a(y_{11}y_{22} + y_{12}y_{21} + \psi_{11}\psi_{22} - \psi_{12}\psi_{21}) \\ &\quad + x_{11}x_{21}(y_{11}y_{22} + y_{12}y_{21}) + y_{11}y_{21}(x_{11}x_{22} + x_{12}x_{21}) \\ &\quad + y_{11}y_{21}(\chi_{11}\chi_{22} - \chi_{12}\chi_{21}) \\ &\quad + x_{11}x_{21}(\psi_{11}\psi_{22} - \psi_{12}\psi_{21}) + x_{11}y_{22} + x_{22}y_{11} + x_{21}y_{12} + x_{12}y_{21} + \chi_{11}\psi_{22} \\ &\quad + \chi_{22}\psi_{11} - \chi_{12}\psi_{21} - \chi_{21}\psi_{12}. \end{aligned} \tag{42}$$

Moreover, expanding the anti-invariants given in Remark 3.5 in the basis of $\Gamma(2)$ we obtain the following six anti-invariants of map $\mathsf{R}_{a,b}$

$$\mathsf{A}_{ij} = x_{i1}\psi_{ij} - y_{i1}\chi_{ij} \quad \text{and} \quad \mathsf{B}_i = \chi_{i1}\psi_{i2} - \chi_{i2}\psi_{i1}, \quad \text{for } i, j = 1, 2. \tag{43}$$

The squares of $\mathsf{A}_{ij}$ and $\mathsf{B}_i$, as well as any product of two of them is an invariant of map $\mathsf{R}_{a,b}$. Obviously, not all of invariants (42) and those obtained from combinations of the anti-invariants (43) form a generating set for the ring of invariants of $\mathsf{R}_{a,b}$, since, for example, the invariants $\mathsf{A}_{ij}^2, \mathsf{B}_k^2$ and $\mathsf{A}_{ij}\mathsf{B}_k$ satisfy the syzygy $(\mathsf{A}_{ij}\mathsf{B}_k)^2 = (\mathsf{A}_{ij})^2(\mathsf{B}_k)^2$. Similarly, we use the invariants $\mathcal{J}_1$-$\mathcal{J}_4$ in (20) to find the following functionally independent invariants of map $\mathsf{S}_{a,b}$

$$\begin{aligned} &\mathrm{J}_1 = y_{21} + x_{11}x_{21}, \quad \mathrm{J}_2 = y_{22} + x_{11}x_{22} + x_{12}x_{21} + \chi_{11}\chi_{22} - \chi_{12}\chi_{21}, \\ &\mathrm{J}_3 = \chi_{11}\chi_{22} + \psi_{11}\psi_{22}, \quad \mathrm{J}_4 = \chi_{12}\chi_{21} + \psi_{12}\psi_{21}, \\ &\mathrm{J}_5 = \chi_{11}\chi_{21} + \psi_{11}\psi_{21}, \quad \mathrm{J}_6 = \chi_{12}\chi_{22} + \psi_{12}\psi_{22}, \\ &\mathrm{J}_7 = (a + x_{11}x_{21})(\psi_{11}\psi_{22} - \psi_{12}\psi_{21}) + (b + y_{21})(\chi_{11}\chi_{22} - \chi_{12}\chi_{21}) \\ &\quad + (1 - x_{21}y_{11})(\chi_{11}\psi_{22} - \chi_{12}\psi_{21}) + \left(1 - \frac{bx_{11}}{y_{11}}\right)(\chi_{22}\psi_{11} - \chi_{21}\psi_{12}), \end{aligned} \tag{44}$$

$$\mathsf{R}_{a,b}:\begin{cases} u_{11}=y_{11}-\frac{a-b}{1+x_{11}y_{21}}x_{11}, & v_{11}=x_{11},\\ u_{12}=y_{12}-\frac{(a-b)(x_{12}-x_{11}(x_{11}y_{22}+\chi_{11}\psi_{22}-\chi_{12}\psi_{21}))}{(1+x_{11}y_{21})^2}, & v_{12}=x_{12},\\ u_{21}=y_{21}, & v_{21}=x_{21}+\frac{a-b}{1+x_{11}y_{21}}y_{21},\\ u_{22}=y_{22}, & v_{22}=x_{22}+\frac{(a-b)(y_{22}-y_{21}(x_{12}y_{21}+\chi_{11}\psi_{22}-\chi_{12}\psi_{21}))}{(1+x_{11}y_{21})^2},\\ \xi_{11}=\psi_{11}-\frac{a-b}{1+x_{11}y_{21}}\chi_{11}, & \eta_{11}=\chi_{11},\\ \xi_{12}=\psi_{12}-\frac{a-b}{1+x_{11}y_{21}}\chi_{12}, & \eta_{12}=\chi_{12},\\ \xi_{21}=\psi_{21}, & \eta_{21}=\chi_{21}+\frac{a-b}{1+x_{11}y_{21}}\psi_{21},\\ \xi_{22}=\psi_{22}, & \eta_{22}=\chi_{22}+\frac{a-b}{1+x_{11}y_{21}}\psi_{22}, \end{cases} \quad (40)$$

**Box I.**

$$\begin{aligned} \mathsf{J}_8 &= (a+x_{11}x_{21})(\psi_{11}\psi_{21}-\psi_{12}\psi_{22})+(b+y_{21})(\chi_{11}\chi_{21}-\chi_{12}\chi_{22})\\ &\quad+(1-x_{21}y_{11})(\chi_{11}\psi_{21}-\chi_{12}\psi_{22})+\left(1-\frac{bx_{11}}{y_{11}}\right)(\psi_{11}\chi_{21}-\psi_{12}\chi_{22}),\\ \mathsf{J}_9 &= b\frac{x_{11}}{y_{11}}+y_{21}(a+x_{11}x_{21})+x_{21}y_{11},\\ \mathsf{J}_{10} &= b\left(\frac{x_{12}}{y_{11}}-\frac{x_{11}y_{12}}{y_{11}^2}\right)+y_{21}(x_{12}x_{21}+x_{11}x_{22}+\chi_{11}\chi_{22}-\chi_{12}\chi_{21})\\ &\quad+y_{22}(a+x_{11}x_{21})\\ &\quad+x_{21}y_{12}+x_{22}y_{11}+\chi_{11}\psi_{22}-\chi_{12}\psi_{21}+\chi_{22}\psi_{11}-\chi_{21}\psi_{12}. \end{aligned}$$

Additionally, from invariant $\mathcal{J}_5$ we find that $\mathsf{B}_i$, for $i=1,2$, in (43) are anti-invariants of map $\mathsf{S}_{a,b}$.

Following the ideas in Section 4, we construct a strong Lax triple $\mathsf{L}_a, \mathsf{M}_a, \mathsf{L}_a$ for maps $\mathsf{R}_{a,b}, \mathsf{S}_{a,b}, \mathsf{T}_{a,b}$. We start by expressing the Lax matrices with Grassmann variables $\mathcal{L}_a, \mathcal{M}_a$ in (14) in the basis of $\Gamma(2)$ as

$$\begin{aligned} \mathcal{L}_a(\boldsymbol{x},\chi) &= \mathsf{L}_1\otimes 1+\mathsf{L}_2\otimes\theta_1\theta_2+\mathsf{L}_3\otimes\theta_1+\mathsf{L}_4\otimes\theta_2,\\ \mathcal{M}_a(\boldsymbol{x},\chi) &= \mathsf{M}_1\otimes 1+\mathsf{M}_2\otimes\theta_1\theta_2+\mathsf{M}_3\otimes\theta_1+\mathsf{M}_4\otimes\theta_2, \end{aligned} \quad (45)$$

with the coefficients $\mathsf{L}_i, \mathsf{M}_i$ given by

$$\mathsf{L}_1=\begin{pmatrix} x_{11}x_{21}+a+\lambda & x_{11} & 0\\ x_{21} & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\ \mathsf{L}_3=\begin{pmatrix} 0&0&\chi_{11}\\ 0&0&0\\ \chi_{21}&0&0\end{pmatrix},\ \mathsf{L}_4=\begin{pmatrix} 0&0&\chi_{12}\\ 0&0&0\\ \chi_{22}&0&0\end{pmatrix}, \quad (46)$$

$$\mathsf{L}_2=\begin{pmatrix} x_{11}x_{22}+x_{12}x_{21}+\chi_{11}\chi_{22}-\chi_{12}\chi_{21} & x_{12} & 0\\ x_{22} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad (47)$$

and

$$\mathsf{M}_1=\begin{pmatrix} x_{21}+\lambda & x_{11} & 0\\ \frac{a}{x_{11}} & 0 & 0\\ 0&0&1\end{pmatrix},\quad \mathsf{M}_2=\begin{pmatrix} x_{22} & x_{12} & 0\\ -\frac{ax_{12}}{x_{11}^2} & 0 & 0\\ 0&0&0\end{pmatrix},$$
$$\mathsf{M}_3=\mathsf{L}_3,\quad \mathsf{M}_4=\mathsf{L}_4. \quad (48)$$

Then, using the algebra homomorphism $\rho:\Gamma(2)\to \mathrm{Mat}_4(\mathbb{F})$ defined by

$$1\overset{\rho}{\mapsto}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ \theta_1\overset{\rho}{\mapsto}\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix},\ \theta_2\overset{\rho}{\mapsto}\begin{pmatrix}0&0&1&0\\0&0&0&-1\\0&0&0&0\\0&0&0&0\end{pmatrix}, \quad (49)$$

we obtain a strong Lax triple for maps $\mathsf{R}_{a,b}, \mathsf{S}_{a,b}, \mathsf{T}_{a,b}$ with matrices $\mathsf{L}_a, \mathsf{M}_a$ given by

$$\begin{aligned} \mathsf{L}_a &= \mathsf{L}_1\otimes\rho(1)+\mathsf{L}_2\otimes\rho(\theta_1)\rho(\theta_2)+\mathsf{L}_3\otimes\rho(\theta_1)+\mathsf{L}_4\otimes\rho(\theta_2)\\ \mathsf{M}_a &= \mathsf{M}_1\otimes\rho(1)+\mathsf{M}_2\otimes\rho(\theta_1)\rho(\theta_2)+\mathsf{M}_3\otimes\rho(\theta_1)+\mathsf{M}_4\otimes\rho(\theta_2), \end{aligned} \quad (50)$$

and $\mathsf{L}_i, \mathsf{M}_i$, $i=1,\dots,4$ given in (46), (47) and (48). The 16-dimensional maps $\mathsf{R}_{a,b}$ and $\mathsf{S}_{a,b}$ in (40), (41) arise from the matrix refactorisation problems of the $12\times 12$ Lax matrices (50).

Finally, similar to the $\Gamma(1)$ case, each of the maps $\mathsf{R}_{a,b}, \mathsf{S}_{a,b}, \mathsf{T}_{a,b}$ admits an invariant measure $m$. These measures are $m=1$ for $\mathsf{R}_{a,b}$, $m=\frac{1}{y_{11}^2}$ for $\mathsf{S}_{a,b}$, and $m=\frac{1}{x_{11}^2}$ for $\mathsf{T}_{a,b}$. In particular, we observe that the commutative consequences of map $\mathcal{R}_{a,b}$ for $n=1$ and $n=2$, that is maps $R_{a,b}$ and $\mathsf{R}_{a,b}$, are volume preserving maps. We conjecture that in $\Gamma(n)$ the map $\mathcal{R}_{a,b}$ is volume preserving for every $n$, while the maps $S_{a,b}$ and $T_{a,b}$ preserve measures of the form $y_{11}^{-n}$ and $x_{11}^{-n}$, respectively, with $x_{11}$ and $y_{11}$ defined similar to the cases $n=1$ and $n=2$.

## 6. Conclusions

We have constructed birational maps $\mathcal{R}_{a,b}$, $\mathcal{S}_{a,b}$, $\mathcal{T}_{a,b}$ with Grassmann variables given in (15)–(17), which satisfy the set-theoretical entwining YB equation (9). These maps admit a strong Lax triple, which we used to derive invariants for the maps. The invariants that we find for map $\mathcal{R}_{a,b}$ are all polynomial, while those of maps $\mathcal{S}_{a,b}$ and $\mathcal{T}_{a,b}$ are Laurent polynomials with negative powers appearing only on even variables of the Grassmann algebra. Reversing this point of view, one could make connections with non-commutative algebraic geometry by viewing the maps as birational automorphisms of non-commutative algebraic varieties.

In Sections 4 and 5 we have shown how a hierarchy of birational entwining YB maps in dimensions $2^{n+2}$, where $n$ is the order of the Grassmann algebra, can be obtained. The case $n=0$, i.e. when there are no fermionic variables, was studied in [61]. Here, we considered in detail the cases where $n=1,2$, thus obtaining birational maps over $\mathbb{F}^8$ and $\mathbb{F}^{16}$ that satisfy the entwining Yang–Baxter equation. We have derived sufficient number of independent invariants of these maps to claim their Liouville integrability, however, we have not yet been able to find a Poisson structure. Nevertheless, there are indications which point towards the integrability of the maps with commutative variables. We have found that these maps are measure preserving, and some preliminary numerical experiments that we have conducted show no existence of chaos. Moreover, we have written each of the 8-dimensional maps of Section 4 as a composition of a Liouville integrable map with a linear map. More insight regarding the integrability of the maps could be gained using other methods, such as singularity confinement or algebraic entropy. All $2^{n+2}$-dimensional maps arise from refactorization problems of Lax matrices, which we present for $n=1$ and $n=2$. It would be interesting to study the associated transfer maps à la Veselov [8] for each $n$. Finally, defining appropriately the concept of Liouville integrability in the setting of Grassmann-extended entwining YB maps is an interesting open problem.

## CRediT authorship contribution statement

**P. Adamopoulou:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Conceptualization. **G. Papamikos:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Conceptualization.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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## Appendix. Proof of Theorem 3.1

The proof of Theorem 3.1 is based on Proposition 3.1 in [13]. To prove that the maps (15)–(17) satisfy the entwining YB equation (9) we have to show that the equation

$$\mathcal{L}_a(\boldsymbol{x},\chi)\mathcal{M}_b(\boldsymbol{y},\boldsymbol{\psi})\mathcal{L}_c(\boldsymbol{z},\boldsymbol{\zeta}) = \mathcal{L}_a(\boldsymbol{x}',\chi')\mathcal{M}_b(\boldsymbol{y}',\boldsymbol{\psi}')\mathcal{L}_c(\boldsymbol{z}',\boldsymbol{\zeta}'), \tag{51}$$

with $\mathcal{L}_a(\boldsymbol{x},\chi)$ and $\mathcal{M}_a(\boldsymbol{x},\chi)$ given in (14), implies $(\boldsymbol{x},\chi) = (\boldsymbol{x}',\chi')$, $(\boldsymbol{y},\boldsymbol{\psi}) = (\boldsymbol{y}',\boldsymbol{\psi}')$ and $(\boldsymbol{z},\boldsymbol{\zeta}) = (\boldsymbol{z}',\boldsymbol{\zeta}')$. Here all the ordered pairs, e.g. $(\boldsymbol{x},\chi) = (x_1,x_2,\chi_1,\chi_2)$, are in $\mathbb{F}_n^{2,2}$.

**Proof.** We use the standard notation of $e_{ij}$ denoting the matrix with 1 in the $(i,j)$ entry and 0 elsewhere. Then the Lax matrices $\mathcal{L}_a(\boldsymbol{x},\chi)$ and $\mathcal{M}_b(\boldsymbol{y},\boldsymbol{\psi})$ are of the form

$$\mathcal{L}_a(\boldsymbol{x},\chi) = \lambda e_{11} + A_a(\boldsymbol{x},\chi), \quad \mathcal{M}_b(\boldsymbol{y},\boldsymbol{\psi}) = \lambda e_{11} + B_b(\boldsymbol{y},\boldsymbol{\psi})$$

where

$$A_a(\boldsymbol{x},\chi) = \begin{pmatrix} x_1x_2+\chi_1\chi_2+a & x_1 & \chi_1 \\ x_2 & 1 & 0 \\ \chi_2 & 0 & 1 \end{pmatrix} \quad \text{and}$$

$$B_b(\boldsymbol{y},\boldsymbol{\psi}) = \begin{pmatrix} y_2 & y_1 & \psi_1 \\ \frac{b}{y_1} & 0 & 0 \\ \psi_2 & 0 & 1 \end{pmatrix}.$$

For simplicity we also introduce the notation

$$X_a := x_1x_2+\chi_1\chi_2+a, \quad Z_c := z_1z_2+\zeta_1\zeta_2+c.$$

Moreover, we introduce the operators $L_{e_{11}}$ and $R_{e_{11}}$ acting on $\mathrm{M}_{2,1}$ by left and right multiplication by $e_{11}$, respectively. Since, $e_{11}^2 = e_{11}$, the operators $L_{e_{11}}$ and $R_{e_{11}}$ are projections. More precisely, we have that

$$L_{e_{11}}(P) = e_{11}P = \sum_{j=1}^{3} p_{1j}e_{1j}, \quad R_{e_{11}}(P) = Pe_{11} = \sum_{i=1}^{3} p_{i1}e_{i1},$$

for any matrix $P = (p_{ij}) \in \mathrm{M}_{2,1}$. It also follows that $L_{e_{11}} \circ R_{e_{11}}(P) = R_{e_{11}} \circ L_{e_{11}}(P) = p_{11}e_{11}$.

We denote the left hand side of (51) by $\mathcal{Q}(\lambda)$ and expand it in powers of $\lambda$. We obtain that

$$\mathcal{Q}(\lambda) = \lambda^3 e_{11} + \lambda^2\mathcal{Q}_2 + \lambda\mathcal{Q}_1 + \mathcal{Q}_0,$$

with $\{\mathcal{Q}_i\}_{i=0}^2$ given by the following expressions

$$\begin{aligned}
\mathcal{Q}_2 &= L_{e_{11}} \circ R_{e_{11}}\left(B_b(\boldsymbol{y},\boldsymbol{\psi})\right) + R_{e_{11}}\left(A_a(\boldsymbol{x},\chi)\right) + L_{e_{11}}\left(A_c(\boldsymbol{z},\boldsymbol{\zeta})\right),\\
\mathcal{Q}_1 &= A_a(\boldsymbol{x},\chi)R_{e_{11}}\left(B_b(\boldsymbol{y},\boldsymbol{\psi})\right) + A_a(\boldsymbol{x},\chi)L_{e_{11}}\left(A_c(\boldsymbol{z},\boldsymbol{\zeta})\right)\\
&\quad + L_{e_{11}}\left(B_b(\boldsymbol{y},\boldsymbol{\psi})\right)A_c(\boldsymbol{z},\boldsymbol{\zeta})\\
\mathcal{Q}_0 &= A_a(\boldsymbol{x},\chi)B_b(\boldsymbol{y},\boldsymbol{\psi})A_c(\boldsymbol{z},\boldsymbol{\zeta}).
\end{aligned}$$

Expanding the right hand side of Eq. (51) in $\lambda$, we obtain similar expressions which we denote by $\{\mathcal{Q}_i'\}_{i=0}^2$. It follows that (51) implies the matrix equations $\mathcal{Q}_i = \mathcal{Q}_i'$, for $i = 0,1,2$.

Matrix equation $\mathcal{Q}_2 = \mathcal{Q}_2'$ results in nontrivial equations only for the entries in the first column and the first row. Comparing the coefficients of the matrices $e_{21}$ and $e_{31}$ in $\mathcal{Q}_2$ and $\mathcal{Q}_2'$ gives

$$x_2 = x_2', \quad \chi_2 = \chi_2'.$$

Similarly, from the coefficients of $e_{12}$ and $e_{13}$ we obtain

$$z_1 = z_1', \quad \zeta_1 = \zeta_1'.$$

In the matrix equation $\mathcal{Q}_0 = \mathcal{Q}_0'$ we focus on the equations that we obtain from the coefficients of $e_{22}$, $e_{23}$ and $e_{32}$. The equation that corresponds to $e_{22}$ reads

$$\left(x_2y_2 + \frac{b}{y_1}\right)z_1 + x_2y_1 = \left(x_2y_2' + \frac{b}{y_1'}\right)z_1 + x_2y_1',$$

where we have used the fact that $x_2 = x_2'$ and $z_1 = z_1'$. The above equation is polynomial in $z_1$ and therefore it implies that

$$y_1 = y_1', \quad y_2 = y_2'.$$

Similarly, using the equations obtained from the coefficients of $e_{23}$ and $e_{32}$ we have that

$$\psi_1 = \psi_1', \quad \psi_2 = \psi_2'.$$

From the coefficient of $e_{12}$ in $\mathcal{Q}_1 = \mathcal{Q}_1'$ we obtain the equation

$$X_az_1 + y_2z_1 + y_1 = X_a'z_1 + y_2z_1 + y_1,$$

where we have used the previously obtained equalities between primed and non-primed variables. The latter equation implies that $X_a = X_a'$. Similarly, from the coefficients of $e_{21}$ of the same matrix equation we obtain $Z_c = Z_c'$.

The coefficients of $e_{21}$ and $e_{31}$ in matrix equation $\mathcal{Q}_0 = \mathcal{Q}_0'$ give two equations involving $z_2, z_2'$ and $\zeta_2, \zeta_2'$. Using the fact that $Z_c = Z_c'$ these two equations can be written as the following homogeneous system:

$$\begin{pmatrix} y_1 & \psi_1 \\ \chi_2y_1 & 1+\chi_2\psi_1 \end{pmatrix}\begin{pmatrix} z_2'-z_2 \\ \zeta_2'-\zeta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the supermatrix of coefficients of the above system is invertible, it follows that

$$z_2 = z_2', \quad \zeta_2 = \zeta_2'.$$

Finally, from the equations that correspond to the elements $e_{12}$ and $e_{13}$, and using the fact that $X_a = X_a'$, we obtain a similar linear system that results to the remaining equalities

$$x_1 = x_1', \quad \chi_1 = \chi_1'. \quad \square$$

## Data availability

No data was used for the research described in the article.

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