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Condorcet domains on at most seven alternatives

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ABSTRACT

A Condorcet domain is a collection of linear orders which avoid Condorcet's paradox for majority voting. We have developed a new algorithm for complete enumeration of all maximal Condorcet domains and, using a supercomputer, obtained the first enumeration of all maximal Condorcet domains on $n \leq 7$ alternatives.

We investigate properties of these domains and use this study to resolve several open questions regarding Condorcet domains, and propose several new conjectures. Following this we connect our results to other domain types used in voting theory, such a non-dictatorial and strategy-proof domains. All our data are made freely available on the web.

1. Introduction

Since the seminal treatise on voting (de Condorcet, 1785) it has been known that majority voting can lead to collective preferences that are cyclic, and hence does not identify a winner in the election. Specifically, Condorcet studied systems where each voter ranks a list of candidates $A_1, A_2, \ldots, A_n$ and a candidate A_j is declared the winner if, for any other candidate A_i, a majority of the voters prefers A_j over A_i (here we assume that the number of voters is odd). The candidate A_j is what is now called a *Condorcet winner*. However, Condorcet showed, de Condorcet (1785) pages 56 to 61, that there are collections of rankings for three candidates without a Condorcet winner. There the pairwise majorities lead to a cyclic ranking of the form $A_1 < A_2 < A_3 < A_1$. In fact, each candidate loses to one other candidate by a two thirds majority. This is now often referred to as Condorcet's paradox, and the three candidates are said to form a Condorcet cycle. Ever since Condorcet's result one has worked to better understand both majority voting and more general voting systems.

Going in one direction, which results a vote can actually lead to, has been investigated in combinatorics. In order to describe an election result more fully one forms a directed graph T, with the set of candidates as its vertices, placing a directed edge from A_i to A_k if a majority of the voters rank A_k higher than A_i, and no edge if the two alternatives are tied. Condorcet's paradox demonstrates that T may contain directed cycles. In McGarvey (1953) it was proved that given any specified directed graph T, and a sufficient number of voters, there is a set of preferences for those voters that realises T by majority voting. Results by Erdős and Moser (1964) and Stearns (1959) bounded the number of voters required for tournaments of a given size. Later, Alon (2002) also determined how strong the pairwise majorities in such a realisation can be.

Going in the other direction, Black (1948), and Arrow (1951) found that if the set of rankings is restricted in a non-trivial way, either directly or indirectly, e.g. by voters basing their ranking of candidates on their positions on a common left–right political scale, there will always be a Condorcet winner, no matter how the votes are distributed over the set of allowed rankings. This motivated the general question: Which sets of rankings always lead to a Condorcet winner? A set of rankings is now called a *Condorcet domain* if, in a majority vote with an odd number of voters, it always leads to a linear order on the alternatives, or equivalently, T is a transitive tournament. In the 1960's several equivalent characterisations of Condorcet domains were given by Inada (1964, 1969), Sen (1966) and Ward (1965), and others. In particular (Ward, 1965) proved that they can be characterised as exactly those sets which do not contain a copy of Condorcet's original example on three candidates.

Following these early works the focus shifted to understanding the possible structure and sizes of Condorcet domains. First (Blin, 1972) gave some early examples with structure different from those by Black and Arrow. Later (Raynaud, 1981) showed that if the number of alternatives is at least four then there are maximal Condorcet

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domains of size just 4. In Johnson (1978) it was conjectured that the maximum possible size is 2^{n-1}. Abello and Johnson (1984) investigated the maximum possible size and proved that this is at least $3(2^{n-2})-4$, for $n \geq 5$ candidates, thereby disproving Johnson's conjecture for $n \geq 6$. They also noted that it was hard to give non-trivial upper bounds for the possible size of a Condorcet domain and conjectured that the maximum is at most 2^n. That conjecture was disproved in Abello (1991). Later (Fishburn, 1992) showed that the maximum size grows at least as c^n for some $c > 2$, and Raz (2000) showed that there is an upper bound of the same form. By now the maximum possible size has been determined for $n \leq 8$ (Leedham-Green et al., 2023).

In addition to their size many different structural properties of Condorcet domains have been studied. Monjardet (2009) surveys many mathematical results on how Condorcet domains relate to the Weak Bruhat order on the set of permutations. More recent works have studied Condorcet domains (Slinko, 2019) with a specific local structure in terms of Sen's value restriction (Sen, 1966), symmetry properties (Karpov and Slinko, 2023), structure of median graphs (Danilov and Koshevoy, 2013) and extensions (Puppe and Slinko, 2019) of the original single-peaked property of Black and Arrow. The thesis (Dittrich, 2018) produced the first full enumeration of all Condorcet domains on $n \leq 5$ alternatives. A recent survey on maximal Condorcet domains can be found in Puppe and Slinko (2024). Still, much remains unknown both regarding possible sizes and structures, with open questions motivated both by political science and new applications in computer science.

In this paper we extend the previous results significantly with the first explicit enumeration of all non-isomorphic maximal Condorcet domains on $n \leq 7$ alternatives. This has been made possible by the combination of a new search algorithm developed by us, described in Section 3, and access to a supercomputer. After presenting basic statistics such as the number of isomorphism classes of maximal Condorcet domains of a given size, we go on to an in-depth investigation of the properties of all Condorcet domains on $n \leq 7$ alternatives. Here we give data on the number of domains with various well-studied properties, and we present answers to several open questions from the research literature. Motivated by patterns in our data we present several conjectures on the behaviour of Condorcet domains for large numbers of alternatives. All our data have been made freely available to download for other researchers via a website which we intend to expand in future works.

1.1. Outline of the paper

In Section 2 we define terminology and discuss background material. Section 3 describes our algorithm for generating Condorcet domains. In Section 4 we discuss of the results of our calculations for degrees $n \leq 7$, where we have complete enumerations. We also pose a number of questions and give conjectures motivated by the data and our theorems. In Section 5 we discuss connections to other, non-Condorcet, domain types.

2. Background material and definitions

In this paper we will typically use lower case greek letters for permutations, calligraphic letters for sets and list, and capitals for Condorcet domains.

As stated in the introduction a Condorcet domain is a set of linear orders such that if an odd number of voters choose rankings from this set then the result of taking pairwise majorities between all candidates lead to a transitive ranking of the candidates. In Ward (1965) it was shown that Condorcet's original example give us an equivalent definition. A set $\mathcal{S} = \{s_1, s_2, \ldots, s_q\}$ of linear orders on $\mathcal{N} = \{1, 2, \ldots, n\}$ is a Condorcet domain if and only if, given any three of the linear orders s_i, s_j, s_k, and any three of the elements a, b, c of $\mathcal{N}$, when we create a table in which each row r is the three elements ordered according the rth permutation, at least one column is not a permutation.

It is often convenient to equate a linear order $i_1 > i_2 > \cdots > i_n$ on $\mathcal{N}$ with the permutation $\sigma(j) = i_j$, so a Condorcet domain may be regarded as a subset of the symmetric group S_n. Then the natural ordering $1 > 2 > \cdots > n$, which we denote by α, is equated with the identity map, and the reverse ordering $n > n-1 > \cdots > 1$, which we denote by ω, is equated with the permutation $\omega(i) = n+1-i$. We refer to an element of a Condorcet domain as a permutation or as an ordering, as best fits the context.

We will also make use of a third, equivalent, definition of a Condorcet domain, first given in Sen (1966). A *never condition* for a triple of elements $\{i, j, k\}$ can be of three different forms: xNt, xNm, or xNb. Here x is an element of the triple and the three conditions state that when a linear order is restricted to this triple then x cannot be ranked, first, middle, or last respectively. A Condorcet domain is a set of linear orders such that every triple of alternatives satisfies at least one never condition. Fishburn (1992) pointed out that if a domain is assumed to contain the natural order α then the never conditions can be restated in the form iNj, $i, j \in \{1, 2, 3\}$. Now iNj means that the ith alternative from the triple, according to α, does not get ranked jth within this triple in any order from the domain. This gives nine distinct never conditions, and we shall mostly be using never conditions in this form.

By a *Maximal Condorcet Domain* of degree n we mean a Condorcet domain of degree n that is maximal under inclusion among the set of all Condorcet Domains of degree n. By a *Maximum Condorcet domain* of degree n we mean a Condorcet domain of the largest possible cardinality among those of degree n. By a *Unitary Condorcet Domain* we mean one that contains the natural order α, or the identity permutation, depending on how the domain is represented. As we shall see in the next subsection every Condorcet domain is isomorphic to some unitary Condorcet domain, so one can usually assume that a domain is unitary without loss of generality. However, making this assumption also leads to various algebraic and algorithmic simplifications. Henceforth we shall use the acronyms CD, MCD, UCD, and MUCD for the terms Condorcet Domain, Maximal Condorcet Domain, Unitary Condorcet Domain, and Maximal Unitary Condorcet Domain.

For degree 3 there are nine MCDs, corresponding to the nine different never conditions $x\mathrm{N}i$. Each of these domains contain exactly four elements, of which, when regarded as permutations, two are odd, and hence are transpositions, and two are even, and hence are the identity or a 3-cycle. Exactly six of these are unitary, since the never conditions 1N1, and 2N2, and 3N3 each rule out a UCD of degree 3.

2.1. Transformations and isomorphism of Condorcet domains

Given a permutation σ and a set $\mathcal{A}$ of integers, $\mathcal{A}\sigma$ is the set obtained by applying σ to each element of $\mathcal{A}$. For a list $\mathcal{B}$ of integers and a permutation σ we let $\sigma\mathcal{B}$ denote the list in which the element at position i in $\mathcal{B}$ is placed at position $\sigma(i)$.

If A is a CD, and $\sigma \in S_n$ is any permutation, then $A\sigma$ is also a CD; if A satisfies the never condition $x\mathrm{N}i$ on a triple $\{a, b, c\}$ for some $x \in \{a, b, c\}$ then $A\sigma$ satisfies the never condition $\sigma(x)\mathrm{N}i$ on the triple $\{\sigma(a), \sigma(b), \sigma(c)\}$. We say that the CDs A and $A\sigma$ are *isomorphic*, and two isomorphic CDs are identical apart from a relabelling of the elements of $\mathcal{N}$. Every CD A is isomorphic to a UCD, since we can apply σ^{-1} to A for any $\sigma \in A$ and obtain an isomorphic UCD. Similarly we get the following lemma, since some element of the first UCD must be mapped to the identity order in the second UCD.

Lemma 2.1. *If two* UCD*s A and B are isomorphic then $A\sigma^{-1} = B$ for some σ in A.*

The lemma leads to the following observation.

Proposition 2.2. *Isomorphism between two* CD*s of equal size can be tested in time which is polynomial in the size of the domain and n.*

Proof. Let A and B be two CDs. Form $A_1 = A\sigma^{-1}$ for some $\sigma \in A$ and $B_1 = B\tau^{-1}$ for some $\tau \in B$. Clearly A_1 and B_1 are unitary and isomorphic to A and B respectively, and A is isomorphic to B if and only if A_1 and B_1 are isomorphic.

In order to test for isomorphism of A_1 and B_1 we simply need to check if $A_1\sigma^{-1} = B_1$ for any $\sigma \in B_1$. This requires at most $|B_1|$ tests, and each test can be done in time $O(|A_1|n)$ using the Radix sort-algorithm, assuming that the permutations are stored as strings of length n. □

The run time given by the simple algorithm described here is not optimised for small domain sizes. For small domains the radix-sort step could be replaced by e.g. insertion sort.

Definition 2.1. The *core* of a UCD A is the set of permutations $\sigma \in A$ such that $A\sigma = A$.

Since A is unitary the core of A is a group. We shall study the properties of the core and other symmetries of a UCD, both for small n and in general, in more detail in a later paper.

Definition 2.2. The *dual* of a CD A is the CD obtained by reversing each linear order in A.

Equivalently the dual is given by ωA, when A is viewed as a set of permutations. Note that if A satisfies the never condition $x\mathrm{N}i$ on some triple then ωA satisfies the never condition $x\mathrm{N}(4-i)$ on the same triple. Thus $A^{\omega} := \omega A \omega$ is also a CD, and if A is a UCD then so is A^{ω}.

Lemma 2.3. *For every $n > 1$ the map $A \mapsto A^{\omega}$ permutes the set of isomorphism classes* UCDs *of degree n,*

Proof. Let A and $B = A\sigma^{-1}$ be UCDs of degree n, where $\sigma \in A$. Then $B^{\omega} = (A\sigma^{-1})^{\omega} = A^{\omega}(\sigma^{-1})^{\omega}$. But $(\sigma^{-1})^{\omega} = (\sigma^{\omega})^{-1}$, and $\sigma^{\omega} \in A^{\omega}$; so B^{ω} is isomorphic to A^{ω}, as required. □

Definition 2.4. If E is an isomorphism class of UCDs such that $E^{\omega} = E$ we say that E is *reflexive*. If this is not the case we say that E and E^{ω} are *twinned*. If A and B are UCDs that are isomorphic, or in twinned isomorphism classes, we say that A and B are flip-isomorphic.

2.2. The weak Bruhat order and Condorcet domains as posets

The weak Bruhat order[2] is a partial order on the set of permutations S_n, and hence also a partial order on the set of linear orders. A number of results on CDs have been proved using the structure of this partial order and we shall classify CDs according to some such properties.

Given a linear order σ, here seen as a permutation, an *inversion* is a pair $i < j$ such that $\sigma(i) > \sigma(j)$ and we let $Inv(\sigma)$ denote the set of all inversions for σ. The weak Bruhat order is defined by saying that $\sigma_1 \le \sigma_2$ if $Inv(\sigma_1) \subseteq Inv(\sigma_2)$. We say that σ_2 covers σ_1 if $\sigma_1 \le \sigma_3 \le \sigma_2$ implies that σ_3 is equal to one of σ_1 and σ_2. The *Hasse diagram* is the directed graph with vertex set S_n and a directed edge from σ_1 to σ_2 if σ_2 covers σ_1.

The weak Bruhat order turns the set of linear orders, or equivalently the symmetric group S_n, into a partially ordered set known as the *permutohedron*. Since a CD A can be viewed as a subset of the permutohedron we also get an induced partial order on the elements of A. Note that the dual CD for A induces the dual, in the poset sense, partial order of A. It was noted in Blin (1972) that a maximal chain in the permutohedron is a Condorcet domain.

[2] In group theory the weak and the strong Bruhat orders are two partial orders on the elements on certain groups. In our case it is the symmetric group. See Björner and Brenti (2005) for an in detail discussion of these partial orders.

Definition 2.5. A CD A is self-dual if it is isomorphic to the dual of A.

Note that this means that A, as a poset, is isomorphic to the dual poset of A. Also note that if we request A to instead be identical to its dual, we get a symmetric CD.

Definition 2.6. A CD A is connected if for any two $a, b \in A$ there exists a sequence $a = \sigma_1, \sigma_2, \ldots, \sigma_k = b$, with each $\sigma_i \in A$, such that either σ_i covers σ_{i+1}, or $\sigma_i + 1$ covers σ_i in the permutohedron.

This definition states that A induces a weakly connected subgraph in the Hasse diagram of the permutohedron.

2.3. Bounds for the size of a Condorcet domain

One of the most studied properties of Condorcet domains has been the maximum size of an Condorcet domain of degree n, denoted $f(n)$. Fishburn developed two methods to construct CDs, and bounds for $f(n)$. The *alternating scheme*, Fishburn (1996) and Fishburn (1997), gives rise to maximum CDs for degrees up to 7, and the replacement scheme can do better in degrees greater than 15. There are two isomorphic alternating schemes A_n and B_n of degree n; A_n is defined by the following never conditions. For every triple $a < b < c$, $b\mathrm{N}1$ is imposed if b is even, and the $b\mathrm{N}3$ is imposed if b is odd. Similarly $B_n = \omega A_n$. Both A_n and B_n are UCDs. Galambos and Reiner proved in Galambos and Reiner (2008) that $|A_n| = 2^{n-3}(n+3) - \binom{n-2}{n/2-1}(n-3/2)$ if $n > 3$ is even, and $|A_n| = 2^{n-3}(n+3) - \binom{n-1}{(n-1)/2}(n-1)/2$ if $n > 2$ is odd, and also prove that these UCDs are maximal. Fishburn's second method is the *replacement scheme*. Let A and B be CDs on the sets $\mathcal{Y} = 1, 2, \ldots, k+1$ and $\mathcal{Z} = k+1, k+2, \ldots, k+l$. Then a CD C on $\mathcal{X} = \mathcal{Y} \cup \mathcal{Z}$ is obtained by taking all the elements of $\mathcal{Y}$, as orderings, and replacing all occurrences of $k+1$ by elements of $\mathcal{Z}$. So C is a CD on $\mathcal{X}$, and $|C| = |A||B|$. Here k and l may be equal to 2, and one can sees the CD of degree 3 defined by the never condition 1N2, is a replacement scheme with $k = l = 2$. If A and B are unitary then so is C and if A and B are maximal then so is C.

Finally, Raz (2000) proved that there is an upper bound for $f(n)$ of the form c^n for some universal constant c. His proof covers a wider class of sets of linear orders than Condorcet domains, and his argument will not yield a tight value for c in the case of CDs. Fishburn's schemes imply (Fishburn, 1997) that $c > 2.17$ and Conjecture 3 of that paper would imply that $c \le 3$.

2.4. Closed permutation sets and sets of never conditions

Finally, there is a correspondence between subsets of S_n, or *permutation sets* and sets of never conditions. Here a permutation set corresponds to the set of never conditions that are obeyed by every permutation in the set, and a set of never conditions corresponds to the set of permutation sets that satisfy them. This gives rise to the concept of a *closed* set of never conditions, which is a set L of never conditions that contains all never conditions that are consequences of L. We similarly define a *closed* permutation set,[3] which is a permutation set A that contains all permutations that satisfy all the never conditions satisfied by all the elements of A. When viewed as a permutation set a MCD will be a closed permutation set.

Call the set of elements of S_n that satisfy a given never condition a *principal* closed permutation set. These all have cardinality $\frac{4}{6}n!$, and the closed permutation sets are precisely the intersections of sets of principal permutation sets. In our algorithm, to construct all MUCDs of a given degree, we only consider closed permutation sets and we are concerned with the closure of sets of never conditions. However, we do

[3] Not to be confused with the closed Condorcet domains of Puppe and Slinko (2019).

not have a good theoretical grip on these concepts. The only algorithm that we use for determining the closure of a set of never conditions is to go back to the definition, construct the set of permutations that obey these conditions, and see what further conditions these permutations all obey and similarly for the closure of a permutation set.

3. The generation algorithm

Next we describe our algorithm for generating all MUCDs of a given degree n. We have implemented this algorithm in C, both in a serial version which is sufficient for degree $n \leq 6$, and in a parallelised version which was used for $n = 7$. Throughout this section we will always refer to a never condition as a law, in order to shorten the text. In particular we will use the set of *unitary laws*, $1N2, 1N3, 2N1, 2N3, 3N1, 3N2$, which correspond to the six unitary Condorcet domains on three alternatives.

Our first step is to arrange the $\binom{n}{3}$ triples of integers in $\mathcal{N}$ in some fixed order, and we also order the set of six unitary laws. We then construct and store all the principal closed sets $P_{t,k}$ obtained by applying the kth unitary law to the tth triple.

To a first approximation the algorithm operates in the *full Condorcet tree*, which is a homogeneous rooted tree of depth $\binom{n}{3}$, where every non-leaf has six children, and every edge is labelled by a unitary law. Each vertex v of the tree will be assigned a closed permutation set C_v. For the root vertex this is the set of all $n!$ permutations of $\mathcal{N}$. For lower vertices v, of depth t say, we set $C_v = C_u \cap P_{t,k}$, where u is the parent of v, and the edge joining u and v is labelled by the kth unitary law, so if v is a child of the root then C_v is a principal permutation set. The numbering is arranged so that the root has depth 0.

Clearly every MUCD will appear at least once as C_v for some leaf v of this tree. However, for degree six we have a tree with 6^{20} leaves, making the computation infeasible. Many MUCDs appear more than once, and, more seriously, for many leaves v the UCD C_v is not maximal.

In constructing our algorithm we restrict our search to a sub-tree of the full Condorcet tree such that the sets C_v, for v a leaf of this sub-tree of depth $\binom{n}{3}$, constitute the set of all MUCDs, with no repetitions. We first describe the restrictions made to the search, and then prove that the resulting *reduced Condorcet tree* has this property.

3.1. Implied laws and redundancy

A depth-first search of the tree is used, taking the edges from a given vertex to its children in the chosen order of the six unitary laws that label these edges, but abandoning certain vertices and their descendants as described below. Every vertex, of depth t say, is joined to the root by a unique path of length t, and representing each edge in this path, from the root downwards, by the integer that represents the corresponding law, according to the assumed order of the unitary laws, gives rise to a word of length t on the alphabet $1, 2, \ldots, 6$. The search visits the vertices in the lexicographic order of the words associated with them in this way.

When considering the children of a vertex u of depth $t-1$ the first test is to see if C_u is contained in $P_{t,k}$ for some k. If this is the case, as a first restriction on the search tree, the only child v of u that is considered arises from the edge labelled by the least such k, and C_v is defined to be C_u, the other children of u being abandoned. At late stages in the search we often find that all remaining triples have such implied laws, and hence there is no further branching of the search tree. If no such k exists, and the ℓth child v of u is to be processed, then $C_v = C_u \cap P_{t,\ell}$ is constructed.

The second restriction that is applied is to abandon v if $C_v \subseteq C_w$ for some vertex w of depth t that precedes v according to this lexicographic order.

The test for this condition is carried out as follows. If $1 \leq s < t$, define u_s and v_s to be the least integers such that $C_u \subset P_{s,u_s}$ and $C_v \subset P_{s,v_s}$ respectively. That is, such that C_u and C_v obey law number u_s, respectively v_s, on triple number s. Since $C_v \subseteq C_u$ it follows that $v_s \leq u_s$. If $v_s < u_s$ for some s the vertex w defined by the same word as v, but with the sth letter replaced by v_s, satisfies the condition; and whenever a vertex that satisfies the condition exists then a vertex that satisfies the condition may be constructed in the described manner.

For any triple t, define a t-UCD to be a permutation set that obeys some law for every triple $s \leq t$, and define a t-MUCD to be a t-UCD that is maximal with respect to inclusion. So every t-MUCD is a closed permutation set. If $t = \binom{n}{3}$ then a t-UCD is a UCD, and a t-MUCD is a MUCD.

The third and final restriction is to abandon the vertex v of depth t if C_v is not a t-MUCD. The test for this condition is carried out using the following lemma.

Lemma 3.1. *Let v be a vertex of depth t. For every $s \leq t$ let L_s be the set of laws that C_v obeys on the sth triple, and let $M_s = \cup_{\ell \in L_s} P_{s,\ell}$. Then C_v is a t-MUCD if and only if $C_v = \cap_{1 \leq s \leq t} M_s$.*

Proof. Let $A = \cap_{1\leq s\leq t} M_s$. Since C_v is contained in A, the condition $C_v = A$ is equivalent to the condition that A is contained in C_v. Suppose it is not and let $\sigma \in A \setminus C_v$. Then clearly $C_v \cup \{\sigma\}$ is a t-UCD that properly contains C_v. Conversely, if C_v is not a t-MUCD then there is some $\sigma \in A \setminus C_v$ such that $C_v \cup \{\sigma\}$ is a t-UCD. □

3.2. Adequacy of the reduced Condorcet tree

Define the *reduced Condorcet tree* of degree n to be the sub-tree of the full Condorcet tree of degree n obtained by deleting the vertices that were abandoned by the above restrictions, and their descendants. Define an *ancestor* of a vertex v to be any vertex other than v in the path from the root to v.

Lemma 3.2. *Let M be a t-MUCD, where $t > 0$. There is a vertex v in the full Condorcet tree of depth t such that $C_v = M$ and C_u is a $(t-1)$-MUCD, where u is the parent of v.*

Proof. Let v be a vertex of depth t such $C_v = M$, let u be the parent of v, and let ℓ be the law that gives rise to v. If C_u is not a $(t-1)$-MUCD there is a vertex u' of depth $t-1$ such that $C_{u'}$ is a $(t-1)$-MUCD and $C_{u'} \supset C_u$. Let v' be the child of u' defined by ℓ. Then $C'_v \supseteq M$, and since M is a t-MUCD it follows that $C_{v'} = M$. □

Lemma 3.3. *If M is a t-MUCD there is a vertex v of depth t in the full Condorcet tree such that $C_v = M$ and, if w is any ancestor of v, of depth s say, then C_w is a s-MUCD.*

Proof. By the previous lemma there is a vertex v of depth t whose parent u is a $(t-1)$-MUCD, and if $t > 2$ there is a vertex u' with $C_{u'} = C_u$, and whose parent w is a $(t-2)$-MUCD. But since $C_{u'} = C_u$, and C_u has a child v with $C_v = M$, it follows that u' has a child v' with $C_{v'} = M$. So now we have a vertex v' of depth t whose parent and grandparent are a $(t-1)$-MUCD and a $(t-2)$-MUCD respectively, and with $C_{v'} = M$. Iterating this argument proves the lemma. □

Define a *proper* vertex to be a vertex v, of depth t say, such that C_v is a t-MUCD and, for every ancestor u of v, C_u is a s-MUCD for the corresponding value of s. Define a *canonical* vertex v of depth t to be a proper vertex, with $C_v = M$ say, that is lexicographically first among all proper vertices u of depth t with $C_u = M$.

By the previous lemma, for every t-MUCD M there is a unique canonical vertex v of depth t with $C_v = M$.

Proposition 3.4. *The vertices of the reduced Condorcet tree are the canonical vertices.*

Proof. Observe first that the parent of a canonical vertex of positive degree is a canonical vertex. Clearly none of the three restrictions applied to the tree can abandon a canonical vertex. Conversely, if v is a non-canonical vertex of depth t in the reduced tree then C_v is a t-MUCDand there is a canonical vertex u of depth t with $C_u = C_v$. Then u and v have a deepest common ancestor w and w has one child that is an ancestor of u and a different child that is an ancestor of v. But then the second restriction will abandon the latter, and v does not lie in the reduced tree, giving a contradiction. □

Proposition 3.5. *No proper sub-tree of the reduced tree contains a copy of every* MUCD *as* C_v *for some leaf* v.

Proof. If N is a t-MUCD then N is contained in a MUCD M, and if v is a proper leaf with $C_v = M$ then the ancestor u of v of depth t satisfies $C_u = N$. So a sub-tree of the full tree that contains a copy of every MUCD must contain a copy of every t-MUCD. □

3.3. Final reduction, parallelisation, and implementation

The final reduction is to reduce the list of MUCDs, which are stored in a hash table, by deleting all but one representative of each isomorphism class. The parallel version of the main algorithm first finds all vertices at a user-specified distance from the root of the search tree and outputs them into a file. Next, independent copies of the programme complete the search of the sub-trees rooted at each of the vertices in the file. Finally the outputs from these searches are merged in the same way as for the serial version.

The correctness of our programme was tested against full enumerations of MUCDs for small n generated by other programmes that used brute force enumeration.

4. The maximal Condorcet domains of degree at most 7 and their properties.

Using our algorithm we have made a complete enumeration of all non-isomorphic MCDs of degree $n \leq 7$. The total numbers for n from 3 to 7 are

$3, 31, 1362, 256895, 171870480.$

Reducing further to flip-isomorphism classes we get

$2, 18, 688, 128558, 85935807.$

Here the first four numbers in both cases agree with published results (Dittrich, 2018) and the final one is new. The MCDs, given as unitary domains, are available for download (Markström, 2024).

In the next subsections we discuss our computational analysis of these MCDs and their properties. We will provide counts for the number of MCDs with certain well studied properties and the distribution of properties which have a range of values. We also test several conjectures from the existing literature and report on those results. Throughout this discussion we will always consider MCDs up to isomorphism. So, if we say that there are x MCDs with some property we mean that there are up to isomorphism x such MCDs, or equivalently that there are x equivalence classes of MCDs with that property.

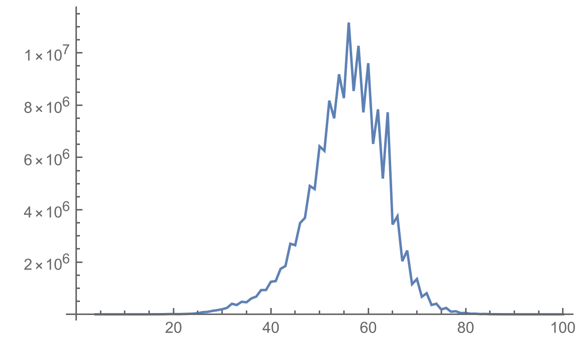

Fig. 4.1. The number of MCD classes as function of domain size for $n = 7$.

4.1. The sizes and numbers of maximal Condorcet domains

In Tables 1, 2, 3 and 4 we display the number of non-isomorphic MCDs of degree four to seven listed according to various properties. In each table the column labelled Total gives the number of non-isomorphic MCDs with the size stated in the previous column.

Using our results we can settle a conjecture whose status has been uncertain for some time. In Fishburn (1997) it was conjectured that for $n = 6, 7$ a CD is maximum if and only if it is isomorphic to those constructed by his alternating scheme. He also proved that the same statement is true for $n = 4, 5$. In Fishburn (2002) he provided a long, and according to himself partial, proof for the case $n = 6$. His caveat was not due to any uncertainty in the proof, but rather to the fact that the considerable length of the proof made him leave many details out of the published version. In Galambos and Reiner (2008), Section 3.2, the authors stated that they verified the conjecture for $n = 7$, but only stated that this was done by extending Fishburn's method, rather than giving a detailed proof. The lack of a published proof led the recent survey (Elkind et al., 2022) to list even the maximum size for $n = 7$ as unknown. Using our data we now have a computational verification of Fishburn's results for $n = 4, 5, 6$ and a proof of his conjecture for $n = 7$.

Theorem 4.1. *For* $n = 4, \ldots, 7$ *every maximum* CD *is isomorphic to a* MUCD *constructed by Fishburn's alternating scheme. In particular, the maximum size of a CD for* $n = 7$ *is 100.*

We also note, using (Leedham-Green et al., 2023), that for $n \leq 8$ the maximum CDs have size $\left\lceil 4 \times 5^{\frac{n-3}{2}} \right\rceil$. That such a simple form will continue to hold is too much to hope for but the growth rate is compatible with known data and bounds.

Problem 4.2. Let $f(n)$ denote the size of a maximum CD on n alternatives. Prove or disprove that

$$\lim_{n\to\infty} \frac{\ln(f(n))}{n} = \ln\sqrt{5}$$

Next, as we can see the total number sequence for a fixed degree is not unimodal, though roughly so. The sequence achieves its largest values at slightly more than half the size of the maximum MUCD for each degree, but it is strongly affected by parity and divisibility by larger powers of 2. In Fig. 4.1 we display the size counts for $n = 7$. We can create a natural notion of a random MCD by giving each isomorphism class equal probability and taking a random member of the chosen isomorphism class. The expected size of a MCD under this distribution is for each small degree lower than 2^{n-1} but can be very well fitted to an exponential function.

Conjecture 4.3. *Let* Z_n *be a random* MCD *then* $\log(\mathbb{E}(|Z_n|)) \sim n\log(q)$, *for some constant* $1 < q < 2$.

Table 1
MCDs of degree 4 and 5.

Degree	Size	Total	Connected peak-pit	Normal	(Symmetric)	Self-dual	Non-ample	Reducible	Cop
4	4	1		1	(1)	1			1
4	7	4	2	4					4
4	8	25	7	16	(2)	3		8	25
4	9	1	1	1		1			1
5	4	2		2	(2)	2			
5	8	12		8	(2)	2		2	12
5	11	28	2	18					26
5	12	41	16	32		1	1		36
5	13	52	2	32					44
5	14	279	26	118		1	1	20	236
5	15	212	42	58					208
5	16	573	57	141	(3)	7	1	100	572
5	17	106	20	34					106
5	18	43	6	19		1		5	43
5	19	12	8	6					12
5	20	2	2	2					2

Table 2
MCDs of degree 6.

Size	Total	Connected peak-pit	Normal	(Symmetric)	Self-dual	Non-ample	Reducible	Cop
4	8		8	(8)	8			
8	11		7	(7)	7	4	7	
9	26		18					
10	46		28					
11	8		6					
12	11		4		1	7		
13	106		38			4		90
14	80		32			2		76
15	66		34			2		54
16	1 036	2	246	(6)	8	6	62	970
17	808	12	244					642
18	808	14	280			16		600
19	1 399	76	537		3	40		1 125
20	1 734	144	664		4	45		1 333
21	2 156	124	708		2	114		1 486
22	5 072	100	1194			164	168	3 876
23	4 986	114	1378			108		3 372
24	8 617	246	1850		9	207	237	5 964
25	9 892	240	1624		2	156		7 014
26	16 629	491	2502		5	164	312	11 345
27	17 137	739	1756		3	138		12 269
28	32 708	883	3100		16	281	1604	27 013
29	25 453	1176	1760		5	168		21 909
30	31 310	1420	2289		6	188	1272	28 820
31	22 543	1099	1381		7	114		21 159
32	38 894	1022	2195	(6)	46	307	3127	37 885
33	12 168	548	821		24	84		11 722
34	11 554	490	1075		10	70	636	11 332
35	4 635	332	532		7	38		4 573
36	3 720	232	458		22	92		3 620
37	1 297	144	177		11	8		1 283
38	1 300	114	284		2	18	72	1 282
39	366	79	70		2			366
40	192	35	41		2	5	8	187
41	50	22	16					50
42	57	31	15		7			57
43	7	5	2		1			7
44	4	4	2					4
45	1	1	1		1			1

Fitting an exponential function to the four, admittedly few, values we have for $\mathbb{E}(|Z_n|)$ gives a very good fit to 0.59163×1.91324^n. Fitting the variance also give a good fit to an exponential growth of $O(4.663^n)$. The third moment is negative and gives a negative skewness which is growing in magnitude for our range of n. With all of this in mind it seems likely that the size distribution converges after a proper normalisation but it is not clear what the asymptotic form will be.

Question 4.4. *Let M_n and σ_n be the mean and standard deviation of $|Z_n|$ and define $Y_n = (|Z_n| - M_n)/\sigma_n$.*

Does Y_n converge in distribution as $n \to \infty$? If so, what is the asymptotic distribution?

Here one might ask the same question with M_n replaced by the median instead of the mean if it turns out that they differ significantly. It is also natural to ask how quickly the number of MCDs grow. If we let $t(n)$ denote the number of non-isomorphic MCDs on n alternatives then it is trivial that $t(n) \leq 6^{\binom{n}{3}}$, since there are six unitary never conditions and $\binom{n}{3}$ triples. One would also expect $t(n)$ to grow at least as $6^{\binom{n}{2}/3}$, since one must use at least $\binom{n}{2}/3$ triples in order to include every pair of alternatives in a never condition. It is possible to get a good fit to the

Table 3
MCDs of degree 7.

Size	Total	Connected peak-pit	Normal	(Symmetric)	Self-dual	Non-ample	Reducible	Cop
4	46		46	(46)	46			
8	44		44	(44)	44		44	
9	24					24		
10	270		186			10		
11	188		120			2		
12	147		84		1	3		
13	176		72			4		
14	284		110		4	60		
15	548		188			112		
16	1 626		452	(24)	26	228	60	59
17	2 178		490			358		30
18	4 435		794		1	283	182	14
19	9 994		1 248			358		1 528
20	11 864		1 544		4	502	322	1 630
21	7 040		1 274			702		1 338
22	12 688	2	2 696		1	923	56	6 536
23	18 570	10	3 378			1 412		9 080
24	25 954	16	4 204		14	2 708	75	13 046
25	44 660	58	6 940			3 238		26 674
26	81 579	146	10 266		17	4 343	742	50 698
27	94 158	252	11 072			4 576		53 522
28	132 114	314	15 864		16	5 828	580	82 028
29	159 868	716	19 754			7 342		106 586
30	194 674	1 198	24 096		8	10 630	462	127 450
31	247 692	1 314	29 790			12 124		163 450
32	404 009	1 982	38 995	(15)	55	18 441	7 013	286 060
33	356 618	2 822	40 644			19 194		221 900
34	480 195	2 706	52 546		5	24 711	5 656	304 066
35	461 900	3 452	54 328			30 328		263 654
36	609 624	4 342	66 737		14	42 359	5 637	348 343
37	678 422	4 484	66 594			46 796		374 642
38	928 441	5 328	87 391		11	61 830	9 793	536 413
39	930 304	5 370	80 096			60 042		503 650
40	1 244 522	6 412	102 038		32	75 913	12 104	692 657
41	1 273 738	6 612	94 144			76 780		668 724
42	1 739 772	7 870	118 548		14	98 608	15 092	965 212
43	1 849 074	8 914	106 920			93 256		1 012 696
44	2 701 280	11 736	141 762		16	117 140	35 012	1 583 810
45	2 644 266	14 948	118 310			104 706		1 462 180
46	3 491 780	16 876	156 358		14	132 576	34 902	1 933 530
47	3 686 966	20 004	126 586			126 006		2 060 898
48	4 911 214	24 830	167 362		104	167 322	59 465	2 895 896
49	4 790 868	27 420	128 900			150 198	64	2 792 242
50	6 426 642	34 916	170 856		12	184 304	69 244	4 022 222
51	6 253 444	40 434	125 366			168 414		3 932 782
52	8 174 653	47 116	177 956		41	211 246	115 157	5 384 015
53	7 497 364	56 266	119 424			178 664		5 055 996

logarithm of the known values of $t(n)$ with a second degree polynomial, but one must again be cautious due to the short range for n.

Question 4.5. *How quickly does $t(n)$ grow?*

It would also be interesting to find an explanation for the tendency, visible as “spikes” in Fig. 4.1, to have larger numbers of MCDs of even size.

4.2. The structure of maximal Condorcet domains

The first structural property which we will look at is whether or not a MCD can be built from CDs of lower degree.

Definition 4.1. Given a MCD C on a base set $\mathcal{A}$ we say that C is *reducible* if there exists a proper subset $\mathcal{B} \subset \mathcal{A}$, of size at least 2, such that the elements of $\mathcal{B}$ are consecutive in each of the linear orders in C. If C is not reducible we say that it is irreducible.

The motivation for this definition is that a reducible MCD can be built from two CDs, C_1 on a set $\mathcal{A}'$ of size $1+|\mathcal{A} \setminus \mathcal{B}|$ and C_2 on $\mathcal{B}$, using a slight generalisation of Fishburn’s replacement scheme. There we pick some element of $\mathcal{A}'$ and then replace that element in every member of C_1 with a permutation from C_2. In the column labelled Reducible we display the number of MCDs of each size which are reducible. Obviously reducibility is strongly affected by the factorisation of the size, since the size of a reducible MCD is the product of the size of the factor CDs C_1 and C_2, each of which must be maximal. Even though the number of reducible MCDs increases with the degree we nonetheless expect them to asymptotically be outnumbered by the irreducible ones.

Conjecture 4.6. MCD*s are asymptotically almost surely irreducible.*[4]

Next we see that for each degree we find several MCDs of size *four*. The first such examples were found in Raynaud (1981) and Danilov and Koshevoy (2013) proved that these exist for all degrees. These domains can be used to construct MCDs for larger powers of 2 as well and we may ask for which fixed sizes we can find a MCD for infinitely many, or all sufficiently large, degrees.

Question 4.7. *Are there infinitely many degrees for which a* MCD *of size nine exists? For which sizes t do there exists* MCD*s for infinitely many degrees n?*

[4] That a property holds asymptotically almost surely, abbreviated a.a.s., means that as n goes to infinity the proportion of objects with the property goes to 1.

Table 4
MCDs of degree 7.

Size	Total	Connected peak-pit	Normal	(Symmetric) Self-dual	Non-ample	Reducible	Cop
54	9 180 598	67 628	162 730	26	209 307	119 959	6 487 399
55	8 270 608	72 728	108 716		173 954		6 107 600
56	11 160 909	87 290	161 446	97	224 096	223 986	8 766 421
57	8 540 924	94 064	95 220		165 036		6 742 858
58	10 269 782	97 952	134 214	10	204 109	178 171	8 360 889
59	7 723 932	94 522	83 966		149 322		6 345 838
60	9 606 176	92 548	120 399	52	202 072	214 186	8 260 766
61	6 518 148	77 586	68 314		131 766		5 655 112
62	7 839 946	69 514	98 479	22	159 649	157 801	6 963 611
63	5 191 166	55 636	55 530		108 204	32	4 642 368
64	7 728 718	54 052	85 340	(11) 254	173 910	260 912	7 162 308
65	3 436 076	38 238	42 744		93 546		3 090 834
66	3 750 621	34 346	60 993	39	112 105	85 176	3 408 640
67	2 034 070	29 028	32 490		60 780		1 836 672
68	2 440 206	26 152	49 547	42	97 782	78 262	2 221 040
69	1 152 526	20 140	24 736		47 074		1 038 388
70	1 351 871	19 750	35 886	11	65 862	32 445	1 228 087
71	671 796	14 742	17 262		26 368		616 530
72	808 375	12 776	24 520	49	53 157	25 136	732 188
73	357 970	9 872	10 936		21 338		323 602
74	405 334	7 714	15 495	12	24 711	9 079	370 471
75	186 106	6 120	7 374		9 364		171 590
76	244 369	4 848	12 120	7	16 441	8 798	223 662
77	101 268	3 966	4 818		5 286		94 074
78	116 958	3 086	6 400	2	6 592	2 562	108 916
79	48 120	2 456	2 792		2 274		45 170
80	56 464	1 816	3 719	2	3 607	1 294	52 459
81	23 402	1 720	1 490		1 396	4	21 864
82	25 154	1 208	1 864		1 506	350	23 480
83	11 344	1 146	810		456		10 806
84	14 503	938	1 271	7	686	399	13 799
85	6 108	1 020	370		254		5 834
86	4 273	552	506	1	222	49	4 049
87	2 066	506	226		96		1 970
88	2 038	308	220		46	18	1 992
89	1 248	368	106		12		1 236
90	647	154	75	1	24	7	623
91	214	66	22		4		210
92	274	98	46				274
93	106	66	6		4		102
94	76	36	10				76
95	18	10	2				18
96	36	30	8				36
97	16	14	4				16
98	4	4					4
100	2	2	2				2

We now look at the set of never conditions a MCD satisfies. A particularly nice subfamily of CDs are those which satisfy exactly one law on each triple of alternatives. These CDs were named *copious* in Slinko (2019), the name alluding to the fact that a copious CD gives the maximum possible four orders when restricted to any triple of alternatives. In the column labelled Cop we show the number of copious MCDs of each size. For $n \geq 5$ we find examples for MCDs which are not copious. For $n = 5$ the restriction to a triple either has size three or four. For $n = 6$ all MCDs with size nine or less have restrictions of size two or four, thus being even further from being copious. We also see that for $n \leq 7$ MCDs which are close to the maximum size are always copious, and that for most of the range of sizes they make up the majority of all MCDs. However, in order to be copious the restriction of a MCD to a subset of the alternatives must be copious as well. That requirement could make copious MCD less common for larger n.

Question 4.8. *What is the minimum size of a copious* MCD *of degree n? Are* MCD*s asymptotically almost surely not copious?*

In Karpov and Slinko (2023) the term *ample* was introduced to denote those CDs which, whenever restricted to two alternatives, give both of the possible orderings for those alternatives and noted that a copious CD is ample. They asked if all MCDs are ample and we can answer this question negatively:

1	1	1	1	1	1	1	1	4	4	4	4
2	2	2	3	3	3	4	4	1	1	2	3
3	3	5	2	2	5	2	3	2	3	1	1
4	5	3	4	5	2	3	2	3	2	3	2
5	4	4	5	4	4	5	5	5	5	5	5

Fig. 4.2. The smallest non-ample MCD.

Observation 4.9. *The smallest non-ample* MCD *has degree 5 and size 12*

The number of non-ample MCDs of each size is displayed in the column labelled Non-ample. For $n = 5$ there are only 3 non-ample MCDs, but as the degree goes up they become more common. Note that for degree $n = 7$ all MCDs of size 9 are non-ample. We also find surprisingly large examples of non-ample MCDs for $n = 6$ with size 40, and $n = 7$ with size 93 (see Fig. 4.2).

Question 4.10. *Is the maximum size of a non-ample* MCD $o(f(n))$*?*

Being non-ample is not the only deviation from what one might at a first glance expect a MCD to look like. Let us say that a CD C is *fixing* if there exists a value from the base set which has the same position in every order in C. It is clear that if we take a Condorcet domain and insert a new alternative at a fixed position in every linear order we will

1	2	4	5
2	5	1	4
3	3	3	3
4	1	5	2
5	4	2	1

Fig. 4.3. A fixing MCD of order 5 and size 4.

get a new Condorcet domain, of the same size and degree one larger. One would typically not expect such a CD to be maximal, but it turns out that it is possible to construct MCDs in this way.

Observation 4.11. *The smallest fixing* MCD *has degree 5 and size 4 (see Fig. 4.3).*

For degree 5 there is a unique fixing MCD, and for degree 6 there are 2, both with size 8. For degree 7, there are 6 with size 4, 3 of size 8, 4 of size 13 and 133 of size 16.

Question 4.12. *How large can a fixing* MCD *of degree n be? Is there a characterisation of the* MCDs *which have an extension to a fixing* MCD *with one more alternative?*

4.3. Connectivity and peak-pit domains

In this section we will consider several properties of a CD which are directly connected to the view of a CD as a subset of the permutohedron.

At least since the 1960's it has been common to consider *connected* CDs, i.e. a CD which induces a connected subgraph of the permutohedron. One attractive property of such domains is that it is possible to move between any two linear orders in the domain in steps which only differ by an inversion. This can be interpreted as saying that the set of opinions is in some sense a continuum. In the column labelled Connected we display the number of connected MCDs of each size. Here two things stand out in the data. First, the majority of all MCDs are not connected. For small sizes, relative to n, this is automatic but as we can see it seems to be the case for most sizes. Secondly, up to $n = 7$ the maximum MCD is always connected. We believe that the first of these properties holds more generally:

Conjecture 4.13. *A.a.s.* MCD*s are not connected.*

Question 4.14. *Are there always exactly 2 non-isomorphic connected* MCD*s of size $\binom{n}{2}+1$?*

In Puppe and Slinko (2024) it was conjectured that a MCD is connected if and only if it is a *peak-pit domain*. Peak-pit domains stem from the early works of Black and Arrow on single-peaked domains and are defined as a CD which on every triple satisfy a condition of either the form xNt or xNb, for some x in the triple. We have tested this conjecture on our data.

Observation 4.15. *For degrees $n \le 7$ a* MCD *is connected if and only if it is a peak-pit domain.*

4.4. Condorcet domains which are of maximum width, symmetric, or self-dual

Two further classes of often-studied CDs are the *normal*, or maximum width, CDs and the *symmetric* CDs. The terminology in the literature varies a bit here but we will say that a CD is normal if it contains both the standard order α and the reverse order ω. This concept is not invariant under isomorphism and so Puppe (2018) defined a CD to be of *maximum-width* if it is isomorphic to a normal CD. Being symmetric on the other hand means that for every order β in the domain C the reversed order $\omega\beta$ also belongs to C. In the columns labelled Normal and self-dual/(Symmetric) we give the number of maximum-width, self-dual, and symmetric MCDs. As we can see, the number of maximum-width MCDs is substantially smaller than the total number, and we believe that this pattern will continue.

Conjecture 4.16. *A.a.s.* MCD*s do not have maximum width.*

We also note that for degrees $n \le 7$ the maximum MCDs always have maximum width. However, in Leedham-Green et al. (2023) the maximum MCD of degree 8 was found and it does not have maximum width. Here one may ask if maximum width implies a strong restriction on the size of a MCD.

Question 4.17. *Is the maximum size of a* MCD *of maximum width $o(f(n))$?*

The symmetric MCDs form a subfamily of the *self-dual* CDs, see Definition 2.5. The number of self-dual MCD are given in the column labelled Self-dual. Here we see that while the number of possible sizes for a self-dual MCD is much larger than for the symmetric ones the total number of self-dual MCDs is still a small proportion of the total. However, we also observe that for odd n the maximum MCD are all self-dual for $n \le 7$.

A second observation is that for odd n we have only seen self-dual MCDs with even size.

Question 4.18. *Do all self-dual* MCD*s have even size if n is odd?*

Both maximum width and being symmetric can be seen as properties of the intersection between a domain C and the dual domain ωC. A domain has maximum width if the intersection is non-empty and symmetric if the intersection is equal to the entire domain. Note that, since β is never equal to $\omega\beta$, the intersection will always have even size. In the Supplementary Tables 5 to 8 we give the number of MCDs of each degree and size with a given size for the intersection. As one might expect the two most common intersection sizes are 0 and 2. Having intersection size 4 is possible for many sizes and from degree 7 is no longer connected to having an even domain size, as sizes 49, 63 and 81 show. Also note that for domains of size 8 the proportion of symmetric domains increases with n and for $n = 7$ all MCDs of size 8 are symmetric.

Question 4.19. *Are all* MCD*s of size 8 symmetric for $n \ge 7$?*

Note that up to $n = 7$ the intersections always have a power of 2 as its size. This is true in general as we will now show. Additionally, in Karpov and Slinko (2023) the problem of determining the possible sizes for symmetric MCDs was raised, after noting that all known constructions give, all, powers of 2 as size. This question was in fact implicitly solved already in Danilov and Koshevoy (2013) and it also follows from our theorem.

Theorem 4.20. *Let I denote the intersection of a* MCD *C and its dual. If $I \ne \emptyset$ then the size of I is 2^k, for an integer k. If C contains the reversed order ω then I induces a Boolean sublattice of the weak Bruhat order.*

Proof. First we note that I is by definition the largest symmetric, meaning equal to its dual, subset of C. If $I \ne \emptyset$ we may assume that it contains α and u. Now, as shown in Danilov and Koshevoy (2013) C induces a distributive sublattice of the weak Bruhat order. Taking two elements $\sigma, \tau \in I$ it follows that $(\sigma \wedge \tau)^\circ = \sigma^\circ \vee \tau^\circ$, where the $\circ$ denotes the reversed order $\tau^\circ = u\tau$. That is, the reverse of the meet of any pair of orders in I is the join of their reverses. So if we add the meet of any two orders from I and the join of their reverses we get a symmetric set. But since I is the maximum symmetric subset it must be closed under taking meets and joins.

Next let us note that in this lattice the meet and join of an order β and its reverse β° are α and u respectively. This follows since the set of inversions of β° is the complement of the set of inversions of β. This means that the reverse β° satisfies the conditions for being a *complement* of β in the lattice-theoretic sense. Since the lattice is distributive it also follows that β° is the unique complement for β.

So our domain C induces a finite, distributive, complemented lattice and by e.g. Theorem 16, Chapter 10, in Birkhoff (1948) all such lattices are isomorphic to a Boolean lattice, and hence have size 2^k for some integer $k \geq 0$. □

By our proof the intersection sets I are Condorcet domains which are closed under meets and joins in the weak Bruhat order, however they are typically not maximal Condorcet domains.

5. Relation to other domain types

The main motivation for studying Condorcet domains has been to better understand majority voting, as in Condorcet's original work. However, today domains of linear orders are studied much more broadly, both in connection with other classical voting systems and regarding where well-behaved voting systems or choice rules can be constructed. The work of Dasgupta and Maskin (2008) shows that Condorcet domains are the largest domains where any voting system satisfies a specific list of axioms for a well-behaved voting system. So, in this broader context Condorcet domains stand out in this sense, but many authors focus on weaker axioms and we will here briefly comment on how the Condorcet domains for small n relate to two such lines of investigation.

Recall that a voting systems is *strategy-proof*, or non-manipulable, if the best option for each voter is to present a ranking which agrees with their actual preferences. The classical Gibbard–Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975) states that if the domain consists of all, unrestricted, linear orders then the only strategy-proof deterministic voting system is dictatorial, i.e. the outcome depends only on one voter. On the other hand, majority voting on Condorcet domains is not only strategy-proof but even proof against strategic voting by coalitions of voters, see Lemma 10.3 of Moulin (1988). A number of papers have investigated either how much a domain can be restricted while retaining the conclusion from the Gibbard–Satterthwaite theorem or how large a domain can be while allowing non-dictatorial choice functions. In Aswal et al. (2003) the authors introduced the *unique seconds property*, abbreviated USP, and showed that any domain with the USP has a non-trivial strategy-proof choice function. A domain has the USP if there exists a pair of alternatives A and B such that whenever A is ranked first in a linear order B is ranked second. The property has turned out to be quite fruitful and the authors of Chatterji and Zeng (2023) showed that in a certain well-connected class of domains the USP is in fact equivalent to the existence of non-trivial strategy-proof choice functions.

Given that we already know that CDs are strongly strategy-proof we may ask how they fit in the wider landscape of strategy-proof domains, and in particular if they have the USP. It turns out that among the CDs for small n many do in fact have the USP, but far from all do. In Supplementary Tables 9 to 12 we show the number of CDs with the USP. Since the USP is not invariant under reversal of orders it can happen that a CD does not have the USP but its dual does, and this is quite common. Therefore we also show the number of domains such that neither the domain nor its dual has the USP. These provide examples of strategy-proof domains which are not covered by the USP condition for strategy-proofness, and we find such examples close to the maximum size for CDs of these degrees. If we simply demand that the domain does not have the USP then one of the two maximum CDs for $n = 5$ is also an example.[5]

[5] Data for $n \leq 7$ can be found in the supplementary file.

Another line of work, which intertwines with strategy-proofness, concerns generalisations of Black's single-peaked domain. For each n there is up to isomorphism one Black's single-peaked domain, of size 2^{n-1}. This is a particularly well-behaved MCD arising from preferences based on positions on a linear axis, which can be characterised in various ways (Ballester and Haeringer, 2011; Puppe, 2018). This MCD was first generalised by Arrow into what is now known as Arrow's single-peaked domains. These domains are also MCDs but unlike Black's version there are several non-isomorphic examples for each n. Put briefly a MCD is Arrow's single-peaked if every triple (i, j, k) satisfies a never condition of the form xN3, where x is a member of the triple. In Slinko's study of these domains (Slinko, 2019) he enumerated them for $n = 4, 5$ and from our data we can extend this:

Observation 5.1. *The number of non-isomorphic Arrow's single-peaked* MCD*s for* $n = 4, \ldots, 7$ *is* $2, 6, 40, 560$.

Stepping outside the class of Condorcet domains (Demange, 1982) defined the class of domains which are *single-peaked on a tree*. Here a domain D on $\mathcal{N}$ is said to be single-peaked on a tree T with n vertices if we can label the vertices in T with the alternatives from $\mathcal{N}$ so that the restriction of D to the labels of any maximal path in T is a Black's single peaked domain. These domains are often not CDs but they have the weaker property of guaranteeing that pairwise majorities select a single winner, while there may be cycles among lower-ranked alternatives. For Black's single-peaked domain (Moulin, 1980) has identified all strategy-proof choice functions and Danilov (1994) extended this to domains which are single-peaked on a tree. In particular these domains always have a strategy-proof choice function and so tie in with the already mentioned works on strategy-proofness. Recently these domains have also been the focus for development of efficient algorithms, see Peters et al. (2022) and references therein.

Here it becomes natural to ask how common it is for Condorcet domains to be single-peaked on a tree and it turns out to be a rare property for MCDs. In Supplementary Tables 9 to 12 we give both the total number of MCDs which are single-peaked on a tree and those which are single-peaked on a star.

CRediT authorship contribution statement

Dolica Akello-Egwel: Software. **Charles Leedham-Green:** Writing – original draft, Software, Project administration, Methodology. **Alastair Litterick:** Writing – review & editing, Software, Methodology. **Klas Markström:** Writing – review & editing, Writing – original draft, Project administration, Methodology, Formal analysis, Conceptualization. **Søren Riis:** Writing – review & editing, Software, Methodology, Formal analysis.

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Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.mathsocsci.2024.12.002.

Data availability

Data has been published online.

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