

Some colouring problems for Paley graphs

E. Maistrelli and D. B. Penman

October 3, 2012

Abstract The Paley graph P_q , where $q \equiv 1 \pmod{4}$ is a prime power, is the graph with vertices the elements of the finite field \mathbf{F}_q and an edge between x and y if and only if $x - y$ is a non-zero square in \mathbf{F}_q . This paper gives new results on some colouring problems for Paley graphs and related discussion.

Keywords Paley graph, pseudo-random graph, graph colourings

1 Introduction

For a prime power $q \equiv 1 \pmod{4}$, a **Paley graph** P_q is the graph with vertex set \mathbf{F}_q and an edge between x and y (we write $x \sim y$) if and only if $x - y = a^2$ for some non-zero $a \in \mathbf{F}_q$. Paley graphs are self-complementary, vertex and edge transitive, and $(q, (q - 1)/2, (q - 5)/4, (q - 1)/4)$ -strongly regular (see [8] for these and other basic properties of Paley graphs).

A key feature of Paley graphs is that they mimic the typical behaviour of a random graph $G(n, 1/2)$, that is the graph on n labelled vertices where each of the $n(n - 1)/2$ possible edges are present with probability $1/2$ and absent with probability $1/2$ independently of all other edges. Note that ‘typical behaviour’ here concerns graph properties \mathcal{P} such that $\lim_{n \rightarrow \infty} \text{Prob}(G(n, 1/2) \text{ has } \mathcal{P}) = 1$, in which case we write $G(n, 1/2)$ has \mathcal{P} **whp**.

This paper aims to prove some new results about colouring problems for Paley graphs. The main new results are a very slight improvement in the long-standing upper bound on the clique number (Theorem 2.3), determination of the total chromatic number for q square (Theorem 4.7), a determination of the achromatic number (Theorem 5.1), an improved upper bound

on the list edge colouring number (Theorem 4.6) and a proof that P_9 is the only perfect Paley graph (Proposition 5.2).

We are grateful to the referees for their comments on the paper.

2 The chromatic number

For a graph G on n vertices with chromatic number $\chi(G)$ and independence number $\alpha(G)$, $\chi(G) \geq \lceil n/\alpha(G) \rceil$. We also know that if G is a vertex-transitive graph on n vertices, then $\chi(G) \leq \lceil \ln n(n/\alpha(G)) \rceil$ (this follows from the Symmetric Hypergraph Theorem of [15]). This suggests that $\chi(P_q)$ may be close to $\lceil q/\alpha(P_q) \rceil$. For $G(n, 1/2)$, a result of Bollobás ([8]) says that $\chi(G(n, 1/2)) = (1 + o(1))n/\alpha(G(n, 1/2))$ **whp**. The following fact from [9] which also supports this idea will be used often in the sequel: $\omega(G)$ as usual denotes the clique number.

Lemma 2.1 *For q square, $\alpha(P_q) = \omega(P_q) = \chi(P_q) = \sqrt{q}$.*

The difficulty of determining the chromatic number of some graphs arises from the fact that it is closely related to their independence number which is known to be a very hard problem. For P_q , we know that $\alpha(P_q) \leq \sqrt{q}$ (see Chapter XIII, Theorem 14 in [8], Proposition 4.7 in [11], Proposition 4.5 in [21] (which is very similar to Theorem 31.3 in [22]), Problem 13.13 (solution on page 446) in [18], Theorem 3.9 in [12] for various independent proofs of this fact). We present now a very small improvement in this bound.

A graph G on $n = \alpha(G)\omega(G) + 1$ vertices with the property that for every vertex v , there is a partition of $V(G \setminus v)$ into cliques of size $\omega(G)$ and a partition of $V(G \setminus v)$ into independent sets of size $\alpha(G)$, is called an $(\alpha(G), \omega(G))$ -graph (some authors use this term for a slightly different notion). A fact about our $(\alpha(G), \omega(G))$ -graphs ([6]) is that the number of complete subgraphs of size $\omega(G)$ equals the number of independent subgraphs of size $\alpha(G)$ and are both equal to the number of vertices $\alpha(G)\omega(G) + 1$.

Lemma 2.2 *There is no P_q with $|V(P_q)| = \alpha(P_q)^2 + 1$ apart from P_5 .*

Proof. Suppose there is such a P_q . Then this P_q is an $(\alpha(P_q), \omega(P_q))$ -graph (with $\alpha(P_q) = \omega(P_q)$). This is because $|V(P_q)| = \alpha(P_q)^2 + 1$ and for all vertices x of P_q , the subgraph $P_q \setminus x$ has an $\omega(P_q)$ -colouring with $\omega(P_q)$ colour classes of size $\alpha(P_q)$ and a covering with $\alpha(P_q)$ vertex-disjoint $\omega(P_q)$ -cliques.

To see the last part of the previous claim, let A be an independent set of order $\alpha(P_q)$ and C a clique of order $\omega(P_q) = \alpha(P_q)$ and note that the sets $c - A$ are independent sets as c varies over C . Also note that they are disjoint so there is only one element not in $\bigcup_{c \in C}(c - A)$. (For, if $c_1 - a_1 = c_2 - a_2 \Rightarrow c_1 - c_2 = a_1 - a_2$ and we would have that a square or 0 equals a non-square or 0 and so $c_1 = c_2$ and $a_1 = a_2$). Similarly the sets $C - a$ are disjoint cliques as a varies over A , with the same element not in $\bigcup_{a \in A}(C - a)$. By vertex-transitivity the vertex x can be any vertex of P_q .

Since P_q ($q = \alpha(P_q)^2 + 1$) is an $(\alpha(P_q), \omega(P_q))$ -graph, it has the property that the total number of maximum cliques is q , so that if r denotes the number of maximum cliques that each edge is in, we will have (by double counting (C, e) where C is a maximum clique and $e \in E(C)$)

$$q \frac{\omega(P_q)(\omega(P_q) - 1)}{2} = \frac{q(q - 1)}{4} r \Rightarrow r = \frac{2(\alpha(P_q) - 1)}{\alpha(P_q)} \Rightarrow r < 2.$$

But $r \geq 1$ (every edge of P_q is in a $\omega(P_q)$ -clique by edge-transitivity). Thus $r = 1$, so that the $\omega(P_q)$ -cliques are edge-disjoint and we have that

$$r = \frac{2(\alpha(P_q) - 1)}{\alpha(P_q)} \Rightarrow 2(\alpha(P_q) - 1) = \alpha(P_q) \Rightarrow \alpha(P_q) = 2$$

so that $q = 5$ (as $q = \alpha(P_q)^2 + 1$), completing the proof. •

Proposition 2.3 *If q is a non-square and $q \neq 5$, then $\alpha(P_q) \leq \sqrt{q - 4}$.*

Proof. We already know that $q \geq \alpha(P_q)^2$. Now $q \neq \alpha(P_q)^2$ by hypothesis, $q \neq \alpha(P_q)^2 + 1$ by Lemma 2.2 as $q \neq 5$, and $q \neq \alpha(P_q)^2 + 2, \alpha(P_q)^2 + 3$ as $q \equiv 1 \pmod{4}$ and $x^2 \equiv 0$ or $1 \pmod{4}$. •

We know $\chi(P_q) = \lceil q/\alpha(P_q) \rceil$ not only for q square as mentioned above, but also in the following cases:

(i) $q \leq 109$. See [23] for the detailed calculations for $q \leq 89$ and [10] for a statement for all $q \leq 109$.

(ii) $p \equiv 1 \pmod{4}$ a prime and $\omega(P_p) = n(p)$, where $n(p) \in \mathbf{N}$ is the least quadratic non-residue modulo p . Since $\{0, 1, 2, \dots, n(p) - 1\}$ is a clique of order $\omega(P_p)$, we can express $V(P_p)$ as the union of $\lceil p/\alpha(P_p) \rceil$ complete subgraphs (just by translating this clique along) and so, by self-complementarity, we can express $V(P_p)$ as the union of $\lceil p/\alpha(P_p) \rceil$ independent sets. However note that the equality $\omega(P_p) = n(p)$ seems unlikely to happen very often. Indeed, for $p \leq 7000$, it happens only 6 times (see [26]), and it may well happen for only finitely many p .

3 The choice number

A k -list-assignment L to the vertices of a graph G is the assignment of a list, $L(v)$, of at least k colours to every vertex v of G . The graph G is k -choosable if for every k -list-assignment, we can choose a colour for each vertex from its list such that no two adjacent vertices have the same colour. Then the choice number $\text{ch}(G)$ of G is the smallest number k such that G is k -choosable. Since $\text{ch}(G) \geq \chi(G)$, for any graph G we have the inequalities $\lceil n/\alpha(G) \rceil \leq \chi(G) \leq \text{ch}(G)$. J. Kahn (see [2]) showed that $\text{ch}(G(n, 1/2)) = (1 + o(1))\chi(G(n, 1/2))$ **whp**. This suggests that $\text{ch}(P_q)$ may well be close to $\chi(P_q)$. For P_q we have the following observations.

Proposition 3.1 For $q = 5, 9, 13$, $\chi(P_q) = \text{ch}(P_q)$.

Proof. (i) Using the choosability form of Brooks' theorem ([2]), we have

$$3 = \chi(P_5) \leq \text{ch}(P_5) \leq \Delta(P_5) + 1 = 3$$

so $\chi(G) = \text{ch}(G) = 3$.

(ii) It is easy to check that P_9 is the line graph of $K_{3,3}$. By Galvin's theorem ([14]) for any bipartite graph G , $\text{ch}(L(G)) = \Delta(G)$, where $\Delta(G)$ denotes the maximum degree of G . Thus $\text{ch}(P_9) = 3 = \chi(P_9)$.

(iii) We know $\chi(P_{13}) \geq \lceil 13/3 \rceil = 5$. By the choosability form of Brooks' theorem $\text{ch}(P_{13}) \leq 6$ so is either 5 or 6. For the rest of the proof we will assume (using the main theorem in [25]) that the total number of colours is less than or equal to 12.

Suppose that for any $v \in P_{13}$, we write $V(P_{13})$ as $v \cup N_1(v) \cup N_2(v)$ where $N_1(v) = \{x \in V(P_{13}) : x \sim v\}$ and $N_2(v) = \{x \in V(P_{13}) : x \not\sim v, x \neq v\}$. For any $v \in P_{13}$, $P_{13}[N_1(v)]$ is a hexagon and $P_{13}[N_2(v)]$ is a 3-regular graph on 6 vertices. Also every vertex in $N_1(v)$ is adjacent to exactly three vertices in $N_2(v)$ and every vertex in $N_2(v)$ is adjacent to exactly three vertices in $N_1(v)$.

Consider the two independent sets $\{1, 3, 9\}$ and $\{4, 10, 12\}$ in $N_1(0)$. The lists of the vertices 1, 3, 9 have 15 colour places to fill but there are at most 12 different colours. So there is some colour, c_1 say, in two lists. Similarly, the lists of the vertices 4, 10, 12 have 15 colour places to fill and if c_1 was the only colour common to at least two lists then there would be at least 12 other colours in the other colour places. But we can only have at most 11 colours besides c_1 . Thus there exists another colour $c_2 \neq c_1$ in two of these

lists. Now we can assume without loss of generality that $c_1 \in L(1) \cap L(3)$ and so we can have three possible cases for c_2 , namely $c_2 \in L(4) \cap L(12)$ or $c_2 \in L(4) \cap L(10)$ or $c_2 \in L(10) \cap L(12)$. Note that the first two of these are symmetric. The only problem when we are in the first case (i.e. $c_2 \in L(4) \cap L(12)$) is when $L(9) = \{c_2, a\}$ and $L(10) = \{c_1, a\}$. But then we can change the c_1 -set $\{1, 3\}$ to $\{3, 10\}$ and we are in a situation isomorphic to the third case (where $c_2 \in L(10) \cap L(12)$). Since the first case can be reduced to the third one, so can the second. Therefore we only have to consider the third case, where $c_1 \in L(1) \cap L(3)$ and $c_2 \in L(10) \cap L(12)$. But then we can always choose a colour $c \neq c_1$ for vertex 4 and similarly a colour $c' \neq c_2$ for vertex 9.

Since we can colour $N_2(0)$ using 3-colour lists (by the choosability form of Brooks' theorem) omitting c_1 and c_2 from these lists, and $N_1(0)$ using the reduced lists of at least two colours, we can always have a spare colour for vertex 0 (since at most 4 colours are used for $N_1(0)$). •

The following theorem polishes the constant from the result in [3], where no particular effort was made to achieve best possible constants (see [23] for the proof).

Theorem 3.2 *Given $\delta > 0$, there exists a $q_0(\delta)$ such that for every $q \geq q_0(\delta)$ we have*

$$\chi(P_q) \leq \text{ch}(P_q) \leq \frac{(2 + \delta)q}{\log_2 q}.$$

This can sometimes be improved, e.g. for q a square, $\text{ch}(P_q) \leq \sqrt{q} \ln q$ (treat P_q as a subgraph of the complete \sqrt{q} -partite graph with \sqrt{q} vertices in each class, and use the result from [2]). In fact we know of no reason to prevent $\text{ch}(P_q)$, for q square, from being as small as \sqrt{q} . Proving this would improve the example in [2] of an n -vertex graph G with $\text{ch}(G) + \text{ch}(\bar{G}) \leq c\sqrt{n \log n}$. However we see no way to resolve the question at present.

4 Edge and total colourings of P_q

An edge colouring of a graph G is an assignment of colours to its edges so that no two incident edges have the same colour. The edge-chromatic number $\chi'(G)$ of a graph G is the least number k of colours for which G has an edge colouring with exactly k colours. It is well known that a regular graph of

common degree Δ with an odd number of vertices has $\chi'(G) = \Delta + 1$, see e.g. [28]. so $\chi'(P_q) = (q + 1)/2$.

The edge choice number $\text{ch}'(G)$ of a graph G is the smallest integer k such that whenever every edge of G is given a list of at least k colours, there exists an edge colouring of G in which every edge receives a colour from its own list and no two incident edges have the same colour. The List Edge Colouring Conjecture (LECC) states that for every graph G , $\text{ch}'(G) = \chi'(G)$. Kahn ([20]) showed that, for any graph G , $\text{ch}'(G) = \Delta(G)(1 + o(1))$ as $\Delta(G) \rightarrow \infty$, so the conjecture is asymptotically almost true. More precisely Häggkvist and Janssen ([16], Theorem 1.2) have shown that if $\Delta(G)^{2/3} > 60 \log(3\Delta(G))$ then $\text{ch}'(G) \leq \Delta(G) + 23\Delta(G)^{2/3} \sqrt{\log(3\Delta(G))}$.

For Paley graphs we know that $\text{ch}'(P_5) = \chi'(P_5)$ (easy) and that $\text{ch}'(P_9) = \chi'(P_9)$ (by [19], every graph of maximum degree 4 is 5-choosable, also $\text{ch}'(P_9) \geq \chi'(P_9) = 5$ by Lemma 4.1). For the special case when q is a square we can sharpen the error term using a result from [16]: this is Theorem 4.6 below. We first assemble some standard facts about Paley graphs of square order.

Lemma 4.1 *Suppose q is a square. Then*

- (i) $\omega(P_q) = \alpha(P_q) = \sqrt{q}$
- (ii) $\mathbf{F}_{\sqrt{q}}$ is a \sqrt{q} -clique in P_q , and the \sqrt{q} distinct cosets of $\mathbf{F}_{\sqrt{q}}$ cover $V(P_q)$. We say any \sqrt{q} vertex-disjoint \sqrt{q} -cliques form a parallel class
- (iii) The \sqrt{q} -cliques in P_q are edge-disjoint. Two such cliques are vertex-disjoint if they are in the same parallel class and have exactly one vertex in common otherwise
- (iv) The \sqrt{q} -cliques in P_q form $(\sqrt{q} + 1)/2$ parallel classes
- (v) Any \sqrt{q} -clique and \sqrt{q} -independent set have exactly one vertex in common.

Proof. For (i) and (ii) see [9]. (iii) follows from the result of Blokhuis ([7]). By (iii) and edge-transitivity of Paley graphs, P_q acts transitively on \sqrt{q} -cliques: that there are $\sqrt{q}(\sqrt{q} + 1)/2$ \sqrt{q} -cliques follows from double-counting pairs (C, e) as in the proof of Lemma 2.1. For (v) see Proposition 4.7 in [11]. •

Similar results apply to independent sets as P_q is self-complementary.

Lemma 4.2 *In a bipartite graph G , $\chi'(G) = \Delta(G)$.*

Proof. See [28], Theorem 20.4 on page 94. •

Lemma 4.3 *If, for q square, $V(P_q) = I_1 \dot{\cup} I_2 \dot{\cup} \dots \dot{\cup} I_{\sqrt{q}}$ where $I_1, I_2, \dots, I_{\sqrt{q}}$ is a parallel class of independent \sqrt{q} -sets, then the set of edges $E(I_i, I_j)$ running between I_i and I_j decomposes into $(\sqrt{q} + 1)/2$ edge-disjoint \sqrt{q} -matchings.*

Proof. By Lemma 1 in [5] and the fact that $\text{Aut}(P_q)$ (q square) acts transitively on the \sqrt{q} -cliques C_i , the ‘bipartite’ graph (C_i, C_j) is $(\sqrt{q} - 1)/2$ -regular. So, taking complements, the bipartite graph (I_i, I_j) is $\sqrt{q} - (\sqrt{q} - 1)/2 = (\sqrt{q} + 1)/2$ -regular. Thus, using Lemma 4.2, (I_i, I_j) has edge-chromatic number $(\sqrt{q} + 1)/2$. As each matching in this colouring has less than or equal to \sqrt{q} edges and since there are $\sqrt{q}(\sqrt{q} + 1)/2$ edges $I_i - I_j$ ($|I_i| = \sqrt{q}$), we conclude that each of the $(\sqrt{q} + 1)/2$ colour classes contains exactly \sqrt{q} edges and so the matching is perfect, completing the proof. •

The total-chromatic number $\chi''(G)$ of a graph G is the least number of colours needed for a colouring of both the vertices and the edges of G so that adjacent vertices, incident edges and edges and their incident vertices all have different colours. We will need the following lemma both in our current argument improving the upper bound on $\text{ch}'(P_q)$ and later to deal with $\chi''(P_q)$ for q square.

Lemma 4.4 $\chi''(K_n) = n$, if n is odd.

Proof. See [29], Theorem 3.1. The classes are $\mathcal{C}_i = \{v_i\} \cup \{v_{i+j}v_{i-j} : j = 1, 2, \dots, (n-1)/2\}$, where, for $i = 1, 2, \dots, n$, $v_i \in V(K_n)$ and $i+j$ and $i-j$ are calculated modulo n . •

Definition 4.1 *An edge composition graph $G < \mathcal{H} >$ is constructed from a graph $G = (V, E)$ and a family of graphs $\mathcal{H} = \{H_e : e \in E\}$, where each H_e is a bipartite graph (W_e^1, W_e^2) with $|W_e^1| = |W_e^2| = m$ and W_e^1, W_e^2 labelled, by replacing each edge of G by the corresponding bipartite graph of \mathcal{H} .*

Lemma 4.5 *Suppose G is a d -regular graph which is the edge-disjoint union of k graphs, each of which is an edge composition graph $K_n < \mathcal{H} >$, where \mathcal{H} is a family of 1-regular graphs on $2m$ vertices. Then $\text{ch}'(G) \leq d + 2k - 1$.*

Proof. See [16], Theorem 4.2. •

Theorem 4.6 *Let q be square. Then $\text{ch}'(P_q) \leq (q + 2\sqrt{q} - 1)/2$.*

Proof. P_q (q square) is $(q - 1)/2$ -regular and is the edge-disjoint union of $(\sqrt{q} + 1)/2$ graphs, $K_{\sqrt{q}} < (I_i, I_j) >$, described as follows. If we write $V(P_q) = I_1 \dot{\cup} I_2 \dot{\cup} \dots \dot{\cup} I_{\sqrt{q}}$, where the I_i s form a parallel class of independent \sqrt{q} -sets, we can easily see that every edge of P_q is an edge from I_i to I_j for some $1 \leq i < j \leq \sqrt{q}$ and that the edges from I_i to I_j decompose into $(\sqrt{q} + 1)/2$ matchings of order \sqrt{q} (Lemma 4.4).

Consider $K_{\sqrt{q}}$ and think of its \sqrt{q} vertices as being the independent sets (of size \sqrt{q}) $I_1, I_2, \dots, I_{\sqrt{q}}$. If we replace each edge of $K_{\sqrt{q}}$ by the bipartite graph (on $2\sqrt{q}$ vertices) of the edges between the two corresponding independent sets we will obtain P_q . Now each bipartite graph (I_i, I_j) is $(\sqrt{q} + 1)/2$ -regular and so P_q is the edge-disjoint union of $(\sqrt{q} + 1)/2$ graphs, each of which consists of a perfect matching between independent \sqrt{q} -sets I_i (using Lemma 4.3). Thus (Theorem 4.5) $\text{ch}'(P_q) \leq (q - 1)/2 + 2((\sqrt{q} + 1)/2) - 1 = (q + 2\sqrt{q} - 1)/2$. •

We now return to the total chromatic number. For any graph G , clearly $\chi''(G) \geq \Delta(G) + 1$. We suspect that $\chi''(P_q) = \Delta(P_q) + 1$ for all P_q except P_5 : it is easy to show $\chi''(P_5) = 4 = \Delta(P_5) + 2$. We have checked this (see [23] for details) for $q \leq 37$. H. Hind's upper bound ([17]) shows that $\chi''(P_q) \leq \Delta(P_q) + 7$. The proof technique used by McDiarmid and Reed ([24]) to show that $\chi''(G(n, 1/2)) = \Delta(G(n, 1/2)) + 1$ **whp** breaks down here because P_q is regular. However, for q square, we can find $\chi''(P_q)$ exactly.

Notation An r -matching in a graph G is a matching with r edges in it.

Theorem 4.7 *Let q be square. Then $\chi''(P_q) = \Delta(P_q) + 1 = (q + 1)/2$.*

Proof. We aim to construct $(\sqrt{q} + 1)/2$ -total colour classes, each containing \sqrt{q} vertices and $(q - \sqrt{q})/2$ edges, and $(q - \sqrt{q})/2$ -total colour classes, each containing 1 vertex and $(q - 1)/2$ edges. Let $\mathcal{I} = \{I_1, I_2, \dots, I_{\sqrt{q}}\}$ be a fixed parallel class of independent sets of order \sqrt{q} and $\mathcal{C} = \{C_1, C_2, \dots, C_{\sqrt{q}}\}$ be a fixed parallel class of cliques of order \sqrt{q} in P_q .

Firstly, we obtain $(\sqrt{q} + 1)/2$ -total colour classes with \sqrt{q} vertices and $(q - \sqrt{q})/2$ edges;

$$\begin{aligned} m_1 = \{ & I_1, \text{ a } \sqrt{q} - \text{ matching } I_2 - I_{\sqrt{q}}, \text{ a } \sqrt{q} - \text{ matching } I_3 - I_{\sqrt{q}-1}, \dots, \\ & \text{ a } \sqrt{q} - \text{ matching } I_i - I_{\sqrt{q}-i+2}, \dots, \text{ a } \sqrt{q} - \text{ matching } I_{(\sqrt{q}+1)/2} - I_{(\sqrt{q}+3)/2} \} \\ m_3 = \{ & I_3, \text{ a } \sqrt{q} - \text{ matching } I_2 - I_4, \text{ a } \sqrt{q} - \text{ matching } I_1 - I_5, \\ & \text{ a } \sqrt{q} - \text{ matching } I_6 - I_{\sqrt{q}}, \dots, \text{ a } \sqrt{q} - \text{ matching } I_i - I_{\sqrt{q}-i+6}, \dots, \end{aligned}$$

$$\begin{aligned}
& \text{a } \sqrt{q} - \text{ matching } I_{(\sqrt{q}-5)/2} - I_{(\sqrt{q}-3)/2} \} \\
& \dots \\
& m_{\sqrt{q}} = \{ I_{\sqrt{q}}, \text{ a } \sqrt{q} - \text{ matching } I_1 - I_{\sqrt{q}-1}, \\
& \text{a } \sqrt{q} - \text{ matching } I_2 - I_{\sqrt{q}-2}, \dots, \text{ a } \sqrt{q} - \text{ matching } I_i - I_{\sqrt{q}-i}, \dots, \\
& \text{a } \sqrt{q} - \text{ matching } I_{(\sqrt{q}-1)/2} - I_{(\sqrt{q}+1)/2} \}.
\end{aligned}$$

(Note that this is essentially part of the standard total colouring of $K_{\sqrt{q}}$ (\sqrt{q} odd) with \sqrt{q} colours where $I_1, I_2, \dots, I_{\sqrt{q}}$ are the \sqrt{q} vertices (Lemma 4.3)).

Suppose we delete the set $\{m_1, m_3, \dots, m_{\sqrt{q}}\}$ of edges from P_q and so, for a given independent set I_{2k} , this removes $(\sqrt{q}+1)/2$ of the $(\sqrt{q}-1)(\sqrt{q}+1)/2$ (this is the number of I_i 's except I_{2k} times the number of vertices in each other I_i a vertex in I_{2k} is adjacent to) \sqrt{q} -matchings which had a vertex of I_{2k} as an end-vertex. Thus there are left $(\sqrt{q}-2)(\sqrt{q}+1)/2$ \sqrt{q} -matchings out of I_{2k} . Similarly, $(\sqrt{q}+1)/2-1$ (as I_{2k-1} is present in all $(\sqrt{q}+1)/2$ total colour classes apart from m_{2k-1}) of the $(\sqrt{q}-1)(\sqrt{q}+1)/2$ \sqrt{q} -matchings which had a vertex of I_{2k-1} as an end-vertex have been removed, and so there are $(\sqrt{q}-1)(\sqrt{q}+1)/2 - (\sqrt{q}-1)/2 = (q-\sqrt{q})/2$ \sqrt{q} -matchings out of I_{2k-1} .

Now consider the new graph $P'_q = P_q \setminus \{m_1, m_3, \dots, m_{\sqrt{q}}\}$. Its total number of edges is $\frac{q(q-1)}{4} - \frac{\sqrt{q}+1}{2} \frac{\sqrt{q}-1}{2} \sqrt{q} = \frac{(q-1)(q-\sqrt{q})}{4}$ (as each colour class contains $(\sqrt{q}-1)/2$ \sqrt{q} -matchings). As vertices in $\cup_k I_{2k}$ have degree $(q-1)/2 - (\sqrt{q}+1)/2 = (q-\sqrt{q}-2)/2$ (recall that we removed the $(\sqrt{q}+1)/2$ matchings, one from each colour class, that had a vertex of I_{2k} as an end-vertex) and those in $\cup_k I_{2k-1}$ have degree $(q-1)/2 - (\sqrt{q}-1)/2 = (q-\sqrt{q})/2$ (as we removed the $(\sqrt{q}+1)/2-1$ matchings that had a vertex of I_{2k-1} as an end-vertex), it suffices to prove that the edge-chromatic number of P'_q is $(q-\sqrt{q})/2$. For then, each colour class must have $(q-1)/2$ edges and for each $v \in \cup_k I_{2k}$ ($|\cup_k I_{2k}| = (q-\sqrt{q})/2$) one of these $(q-\sqrt{q})/2$ matchings is not present at v , making it possible to use the colour of that matching for v itself. Also note that every matching is present at every vertex in $\cup_k I_{2k-1}$ (as all degrees are $(q-\sqrt{q})/2$), so v is the only uncovered vertex (which will be added to the matchings to form the total colour classes).

Note that when we were taking edges out of P_q to form P'_q , we could have removed them all from cliques in \mathcal{C} . (For if I_i, I_j are independent sets, then for any clique C , $I_i \cap C = \{v_i\}$, $I_j \cap C = \{v_j\}$ so the edge $v_i v_j$ can be used in the matching $I_i - I_j$). Then each clique $C_i \in \mathcal{C}$ is left with $\frac{\sqrt{q}(\sqrt{q}-1)}{2} - \frac{\sqrt{q}+1}{2} \frac{\sqrt{q}-1}{2} = \left(\frac{\sqrt{q}-1}{2}\right)^2$ edges which partition into $(\sqrt{q}-1)/2$ $(\sqrt{q}-1)/2$ -matchings (because if $C_i = \{v_1, v_2, \dots, v_{\sqrt{q}}\}$, where $v_j = C_i \cap I_j$, the matchings are the edges of

the remaining total colour classes of Theorem 4.4). Now take any parallel class $\mathcal{K} \neq \mathcal{C}$ of cliques of order \sqrt{q} . Since $E(K_{\sqrt{q}})$ partitions into \sqrt{q} $(\sqrt{q} - 1)/2$ -matchings, a parallel class of cliques decomposes into \sqrt{q} edge-disjoint matchings of size $\sqrt{q}(\sqrt{q} - 1)/2 = (q - \sqrt{q})/2$. For each $K \in \mathcal{K}$ take the $(\sqrt{q} - 1)/2$ -matching in it which avoids (i.e. covers no vertex of) C_i (note that $|K \cap C_i| = 1$ by Lemma 4.1(iii)). Add to these $\sqrt{q}(\sqrt{q} - 1)/2$ edges a $(\sqrt{q} - 1)/2$ -matching left in C_i . Then the total number of edges in this matching is $\sqrt{q}(\sqrt{q} - 1)/2 + (\sqrt{q} - 1)/2 = (q - 1)/2$. By varying the choice of i and the choice of \mathcal{K} (there are $(\sqrt{q} - 1)/2$ parallel classes of cliques other than \mathcal{C}) we obtain $\sqrt{q}(\sqrt{q} - 1)/2 = (q - \sqrt{q})/2$ edge-disjoint matchings, completing the proof. •

5 Other colourings

A harmonious colouring of a graph G is a vertex colouring of G such that no two adjacent vertices have the same colour and for any pair of colours, there is at most one edge of G whose endpoints are coloured with this pair of colours. The harmonious chromatic number $\text{hc}(G)$ of a graph G is the least number of colours in a harmonious colouring of G . Note that in any harmonious colouring of a graph G , any two vertices which are adjacent or have a common neighbour must have distinct colours ([13]): thus, as P_q has diameter 2, we see $\text{hc}(P_q) = 2$.

The achromatic number $\psi(G)$ of a graph G is the greatest number of colours in a vertex colouring of G such that no two adjacent vertices have the same colour and for any pair of colours, there is at least one edge of G whose endpoints are coloured with this pair of colours.

Theorem 5.1 $\psi(P_q) = \frac{q+1}{2}$.

Proof. Since the generator g of \mathbf{F}_q^\times is not a square, it follows that (g^i, g^j) is an edge of P_q if and only if (g^{i+1}, g^{j+1}) is not an edge. Thus either $1 - g$ or $g(1 - g)$ is a square.

If $1 - g$ is a square, partition $V(P_q)$ into the $(q + 1)/2$ classes $\{0\}, \{g, g^2\}, \{g^3, g^4\}, \dots, \{g^{q-2}, g^{q-1}\}$.

Note that $g^{2i} \not\sim g^{2i-1}$ since $1 - g$ is a square and g^{2i-1} isn't, so the classes are independent sets. There is an edge between each pair of classes as $0 \sim g^{2i}$ for all i , and if g^{2j-1} is not adjacent to g^{2i-1} then g^{2j} certainly is adjacent to g^{2i} .

If $g(1-g)$ is a square, we instead take the classes to be $\{0\}, \{g^2, g^3\}, \{g^4, g^5\}, \dots, \{g^{q-1}, g\}$.

Again the classes are independent sets, and (exactly) one of $g^{2i} \sim g^{2j}$ and $g^{2i+1} \sim g^{2j+1}$ is true, so there is an edge between any two classes.

Thus $\psi(P_q) \geq (q+1)/2$. But in any partition of $V(P_q)$ into more than $q/2$ classes at least one class is a single vertex v . But then there are at most $(q-1)/2$ other images, as v has degree $(q-1)/2$. So $\psi(P_q) \leq (q+1)/2$. •

The proof of Theorem 5.1 is inspired by A. Thomason's proof that the Hadwiger number of P_q is $(q+1)/2$ ([27]). Note that $\psi(P_q)$ exceeds substantially the typical value of $\psi(G(n, 1/2))$: a similar observation about Hadwiger number is made in [27]. In [23] it is shown that some of the 'geometric graphs' shown by Thomason to have large Hadwiger number also have large achromatic number.

A graph G is perfect if and only if for every induced subgraph H of G , $\chi(H) = \omega(H)$ (for example, an odd cycle of length at least 5 is not perfect).

Proposition 5.2 P_q is not perfect except for P_9 .

Proof. If q is not a square then $\omega(P_q) < \sqrt{q} \Rightarrow \alpha(P_q) < \sqrt{q}$ and so $\chi(P_q) \geq q/\alpha(P_q) > \sqrt{q}$. This shows that $\chi(P_q) \neq \omega(P_q)$, so P_q is not perfect in this case.

In [4] it is shown that given any two disjoint sets S_1 and S_2 in P_q , where $q > 61$, such that $|S_1| = |S_2| = 2$ there is a vertex which is adjacent to the two vertices in S_1 but is not adjacent to the ones in S_2 . So if we can find an induced path of four vertices in P_q , the proof is complete (provided $q > 61$) as we can then make an induced 5-cycle in P_q . To obtain such an induced path, let $x \not\sim y$ be two non-adjacent vertices in P_q . Then there exists a vertex z adjacent to both x and y (there are $(q-1)/4$ such z 's). Now consider the set A of $(q-1)/4$ vertices adjacent to x but not to y . Note that $z \notin A$. So if $w \in A$ (i.e. $w \sim x$ and $w \not\sim y$) and $w \sim z$ for all choices of z , then the edge (w, x) would be in $(q-1)/4$ triangles, but, in fact, it is only in $(q-5)/4$ triangles. Thus for a suitable choice of z and w , we have that $w - x - z - y$ is an induced path of four vertices as required. This shows that P_q is not perfect for $q > 61$.

P_9 is perfect because $P_9 = L(K_{3,3})$ and line graphs of bipartite graphs are perfect. P_{25} is not perfect because writing $\mathbf{F}_{25} = \frac{\mathbf{F}_5}{(x^2+2)}$, the squares are $1, 2, 3, 4, x+2, x+3, 2x+1, 2x+4, 3x+1, 3x+4, 4x+2, 4x+3$ and it is then easy to check that $0 - 2 - (x+4) - (2x+2) - (2x+4) - 0$ is an induced 5-cycle

in P_{25} . P_{49} is not perfect because, since $\mathbf{F}_{49} = \frac{\mathbf{F}_7[x]}{(x^2-3)}$, the non-zero squares are $1, 2, 3, 4, 5, 6, x, x+2, x+5, 2x, 2x+3, 2x+4, 3x, 3x+1, 3x+6, 4x, 4x+1, 4x+6, 5x, 5x+3, 5x+4, 6x, 6x+2, 6x+5$ and so P_{49} contains the induced 5-cycle $0 - 3 - (x+3) - (2x+5) - (2x+4) - 0$. •

We close by mentioning a final colouring result from [1]. For a graph G , let $I(G) = \{(v, e) \in V(G) \times E(G) : v \text{ is incident with } e\}$ be the set of incidences of G , where two incidences (v, e) and (u, f) are adjacent if $v = u$ or $e = f$ or the edge vu is the same as either e or f . In an incidence colouring of G , we colour all the incidences of G so that adjacent incidences receive different colours. The incidence chromatic number $\chi_i(G)$ of G is the smallest number of colours we need in order to have an incidence colouring of G . For $q \equiv 1 \pmod{4}$ a prime, $\chi_i(P_q) \geq \Delta(P_q) + (1/4 - o(1)) \log \Delta(P_q)$ ([1]).

References

- [1] Algor, I., Alon, N. The star arboricity of graphs. *Discrete Mathematics*, Vol. 75 No. 1-3, pages 11-22, 1989.
- [2] Alon, N. Restricted colorings of graphs. *London Math. Soc. Lecture Note Ser.*, Vol. 187, pages 1-33, 1993.
- [3] Alon, N., Krivelevich, M., Sudakov, B. List coloring of random and pseudo-random graphs. *Combinatorica* Vol. 19 No. 4, pages 453-472, 1999.
- [4] Ananchuen, W., Caccetta, L. On the Adjacency properties of Paley graphs. *Networks*, Vol. 23 No. 4, pages 227-236, 1993.
- [5] Baker, R. D., Ebert, G. L., Hemmeter, J., Woldar, A. Maximal cliques in the Paley graph of square order. *Journal of Statistical Planning and Inference*, Vol. 56 No. 1, pages 33-38, 1996.
- [6] Bland, R. G., Huang, H. C., Trotter, L. E. Graphical properties related to minimal imperfection. *Discrete Math.*, Vol. 27 No. 1, pages 11-22, 1979.
- [7] Blokhuis, A. On subsets of $\text{GF}(q^2)$ with square differences. *Indagationes Mathematicae*, Vol. 46 No. 4, pages 369-372, 1984.

- [8] Bollobás, B. *Random Graphs* (2nd edition), Cambridge University Press, 2001.
- [9] Broere, I., Döman, D., Ridley, J.N. The clique numbers and chromatic numbers of certain Paley graphs. *Quaestiones Mathematicae*, Vol. 11 No. 1, pages 91-93, 1988.
- [10] Brouwer, A. E. www.win.tue.nl/~aeb/drg/graphs/Paley.html
- [11] Cameron, P. J. Automorphism groups in graphs. *Selected Topics in Graph Theory 2* (ed. Wilson, R. J., Beineke, L. W.), Academic Press, pages 89-127, 1983.
- [12] Delsarte, P. An algebraic approach to the association schemes of coding theory. *Philips Res. Rep. Suppl.*, Vol. 10, 1973.
- [13] Edwards, K. The harmonious chromatic number and the achromatic number. *London Math. Soc. Lecture Note Ser.*, Vol. 241, pages 13-47, 1997.
- [14] Galvin, F. The list chromatic index of a bipartite multigraph. *Journal of Combinatorial Theory Ser. (B)* **63** (1995) 153-158.
- [15] Graham, R. L., Rothschild, B. L., Spencer, J. H. *Ramsey theory (2nd edition)*, Wiley, 1990.
- [16] Häggkvist, R., Janssen, J. New bounds on the list-chromatic index of the complete graph and other simple graphs. *Comb. Probab. Comput.*, Vol. 6 No. 3, pages 295-313, 1997.
- [17] Hind, H. R. An upper bound for the total chromatic number of dense graphs. *J. Graph Theory*, Vol. 16 No. 3, pages 197-203, 1992.
- [18] Lovász, L. *Combinatorial Problems and Exercises*, North-Holland Publishing Co., 1979 (2nd edition in 1993).
- [19] Juvan, M., Mohar, B., Škrekovski, R. Graphs of degree 4 are 5-edge-choosable. *J. Graph Theory*, Vol. 32 No. 3, pages 250-264, 1999.
- [20] Kahn, J. Recent results on some not-so-recent hypergraph matching and covering problems. *Extremal Problems for Finite Sets, Bolyai Society Mathematical Studies 3, Budapest*, pages 305-353, 1994.

- [21] Krivelevich, M., Sudakov, B. Pseudo-random graphs. *to appear*.
- [22] van Lint, J. H., Wilson, R. M. *A Course in Combinatorics*, Cambridge Univ. Press, 1992 (2nd edition in 2001).
- [23] Maistrelli, E. Paley graphs. *thesis*, in preparation.
- [24] McDiarmid, C., Reed, B. On total colourings of graphs. *J. Combin. Theory Series B*, Vol. 57, pages 122-130, 1993.
- [25] Reed, B., Sudakov, B. List colouring of graphs with at most $(2 - o(1))\chi$ vertices. Proc. of the International Congress of Mathematicians, Vol. III (Beijing 2002), Higher Education Press, China, pages 587-603, 2002.
- [26] Shearer, J. B. <http://www.research.ibm.com/people/s/shearer/indpal.html>
- [27] Thomason, A. Complete minors in pseudorandom graphs. *Random Structures Algorithms*, Vol. 17 No. 1, pages 26-28, 2000.
- [28] Wilson, R. J. *Introduction to Graph Theory* (4th edition), Prentice Hall, 1996.
- [29] Yap, H. P. Total colourings of graphs. *Springer-Verlag Lecture Notes in Mathematics*, Vol. 1623, 1996.