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Static Risk Measures in a Frequency-Severity Framework with Systematic Risk: Application in Reinsurance

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This article presents the concept of *static risk measures* as an approach to assessing risk by focusing on loss severity within a frequency-severity framework that encompasses systematic and common shocks. This article will discuss a set of important properties of *static risk measures* and compare them with the ruin-based risk measures. It will also give a robust representation of *static risk measures* and discuss the implications of such representation to designing optimal reinsurance contracts. By introducing a flexible framework, the model accommodates additional elements such as systematic and common shocks, enhancing its applicability in the field of reinsurance.

1. INTRODUCTION

This article introduces and studies *static risk measures*, based on an underlying risk measure, in a frequency-severity framework where the systematic and common shock are part of the setting. A *static risk measure* is a function of the loss severity, and the frequency and common shock are assumed to be the same. More precisely, for a fixed frequency and common shock model and an underlying risk measure, a *static risk measure* associates with a loss severity the value of the underlying risk measure on the aggregate losses. It will be seen that though a *static risk measure* can inherit many of the properties of the underlying risk measure, such as positive homogeneity, monotonicity, subadditivity, boundedness, and preserving second-order stochastic dominance (SSD), it cannot inherit some others such as co-monotonic additivity or cash invariance. *Static risk measures* can be appealing to actuaries because they fill an important gap in the literature dealing with optimal contracts for the individual rather than the aggregate losses including systematic and common shocks. For instance, in the case of car insurance, the setup can be used for optimal reinsurance design when the reinsurance contract is agreed up front for losses due to each single accident within a year rather than the aggregate losses until the end of the year. This setup also allows for systematic and common shock events such as years with bad weather conditions, which would increase the number and severity of car accidents. The benefit of this approach is to convert a dynamic problem to a static one. The robust representation of static risk measures is also obtained. Finally, the framework is used for reinsurance application. The differences of the optimal reinsurance contracts on aggregate versus individual loss severity are discussed, as well as examples with systematic and common shocks.

The most similar approach to *static risk measures* in the literature is the ruin-based risk measures because they are also considered in the risk processes; that is, a dynamic frequency-severity framework. First, Trufin, Albrecht, and Denuit (2011) introduced and studied a Value at Risk (VaR)-based risk measure. They studied the major properties of a risk measure, including the stochastic dominance, in their paper. Then Cosette and Marceau (2013) used the idea of risk measures in a risk theory context to examine the capital assessment for an insurance portfolio by VaR and tail VaR (TVaR). Wüthrich (2015) discussed ruin-based risk measures by modifying the classical Cramér–Lundberg ruin theory with regards to solvency issues. Cossette et al. (2020) studied properties of ruin-based risk measures defined within discrete-time risk models and used them for

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insurance applications. Though the reason behind the introduction of ruin-based risk measures is to present a risk measure theory perspective of ruin theory, this article follows a different view, directly based on risk measure theory; that is, the *acceptability* of the risk. More details will be discussed later.

In addition to ruin-based risk measures, note that the type of risk measures discussed in this article can be related to the dynamic setups. These types of risk measures have also been widely studied; for example, see the early work of Cheridito, Delbaen, and Kupper (2004). Cheridito, Delbaen, and Kupper (2005) introduced risk measures on general random processes, and Assa (2011) extended those risk measures to introduce and identify the Lebesgue property on the space of general random processes.

Systematic and common shocks are other important subjects that are covered by *static risk measures* that recently have become important topics in the actuarial research particularly after COVID-19 and other global natural disasters. Milevsky, Promislow, and Young (2006) studied systematic risk in a mortality problem while the law of large number breaks. They considered model uncertainty in a problem when the parameters of the loss variables are random. Dahl and Moller (2006) and Dahl, Melchior, and Moller (2008) considered evaluating and hedging life insurance contracts that are subject to systematic mortality risk. Deelstra et al. (2020) and Linders (2021) considered valuation methods that value not only the financial and actuarial risks but also the systematic risk that is neither hedgeable nor diversifiable. Bassamboo, Juneja, and Zeevi (2008) and Tang, Tang, and Yang (2019) studied a particular model that includes systematic risk and integrates it with the concept of common shock. In finance, systematic risk is discussed in the financial markets. For instance, in the capital asset pricing model, systematic risk was formulated and studied more profoundly; see, for example, Choo and deJong (2009, 2016). In Assa and Boonen (2022), it is shown that in a common shock framework pooling is beneficial, with the assumption that the losses have the same distribution.

As mentioned earlier, a particularly interesting application of *static risk measures is in the* design of optimal reinsurance. The literature on optimal reinsurance design is rather large and starts with seminal papers Borch (1960a), Borch (1960b), and Borch (1962), that argue why layer policies, such as stop-loss and excess-of-loss policies, are optimal in an economic setup, using utility functions. However, this study focused on reinsurance design using risk measures. Kaluszka and Okolewski (2008) extended Arrow's result on optimal reinsurances, Balbás, Balbás, and Heras (2009) studied an optimal reinsurance with general risk measures, Bernard and Tian (2009) studied optimal reinsurance under tail risk measures, Boonen (2016) considered reinsurance with heterogeneous reference probabilities, and Han, Liang, and Zhang (2019) considered reinsurance in a setup where there is a common shock. In this article, a standard formulation of reinsurance problems known as marginal indemnity functions (MIFs) is employed, as introduced by Assa (2015). Assa (2015) provided an economic interpretation of admissible reinsurance contracts in terms of MIF and characterized the optimal reinsurance problem from the perspectives of the ceding entity, reinsurer, and social planner. The use of MIF has become a standard technique, as demonstrated by studies such as Zhuang et al. (2016) and Boonen (2022). For a more in-depth exploration of reinsurance applications, the reader is referred to Albrecher, Beirlant, and Teugels (2017).

The rest of the article is organized as follows: Section 2 introduces the necessary notions and notations. In Section 3, a frequency-severity risk model for the business and *static risk measure* is introduced. In Section 4, the optimal reinsurance for individual and aggregate risk in the general case is characterized. Section 5 presents the conclusion. Proofs are included in the Appendix.

2. PRELIMINARIES AND NOTATION

Let $(\Omega, \mathcal{F}, P)$ be a standard and non-atomic probability space, where Ω represents the "states of nature," $\mathcal{F}$ is a σ-field of measurable sets, and P is the physical probability measure. Let $p, q \in [1, \infty]$ be two numbers such that $1/p + 1/q = 1$. For $p \in [1, \infty)$, L^p denotes the space of real-valued random variables X, on Ω, such that $\|X\|_p = E(|X|^p)^{\frac{1}{p}} < \infty$, where E represents the mathematical expectation. The space L^∞ consists of all P almost surely bounded random variables; that is, $L^\infty = \{X \in L^1 | \exists M > 0, P(|X| < M) = 1\}$. The space L^p_+ consists of those members of L^p that are P almost surely nonnegative.

In addition to a general non-atomic probability space $(\Omega, \mathcal{F}, P)$, considered in this article, we will be working with Lebesgue measurable functions on $\mathbb{R}_+$. In particular, denote the space of all nonnegative Lebesgue integrable functions on $\mathbb{R}_+$ by $L^1(\mathbb{R}_+)_+$. Usually, the notation f, g and h is used to denote measurable functions on $\mathbb{R}_+$.

Consider that L^p_+ represents the space of all individual loss variables.[1] The cumulative distribution function of a random variable $X \in L^p$, is denoted by F_X. There are two random variables $X, X' \in L^p$, with the same distribution with $X \sim X'$.

[1]Unlike the finance literature, which considers a profit variable, the loss variable was found to be more convenient to deal with.

For any random variable $X \in L^p$, the left inverse of the cumulative distribution function F_X, denoted by $F_X^{-1}(\alpha)$, is defined as follows:

$$F_X^{-1}(\alpha) = \inf\{x \in \mathbb{R} | P(X > x) \leq 1 - \alpha\}.$$

Here the VaR on L^p as $\mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha)$ is introduced. To be consistent with the literature on risk theory, most of the time $\mathrm{VaR}_\alpha(X)$ is used instead of $F_X^{-1}(\alpha)$ (but not always).

Finally, X is SSD over Y if and only if $\int_{-\infty}^x F_Y(t)dt \geq \int_{-\infty}^x F_X(t)dt$ for all x, with strict inequality at some x. This is equivalent to $E(\phi(Y)) \leq E(\phi(X))$, for all concave and nondecreasing functions ϕ, see Dana (2005).

2.1. Risk Measures

Consider a mapping $\varrho : L^p \to \mathbb{R}$. ϱ can have one or a few conditions from the following list.

1. **Positive homogeneity of degree one**, $\forall X \in L^p, \forall a > 0,\ \varrho(aX) = a\varrho(X)$.
2. **Cash invariance**, $\forall X \in L^p, \forall c \in \mathbb{R},\ \varrho(X + c) = \varrho(X) + c$.
3. **Monotonicity**, $\forall X, Y \in L^p$, if $X \leq Y, a.s.,\ \varrho(X) \leq \varrho(Y)$.
4. **Subadditivity**, $\forall X, Y \in L^p\ \varrho(X + Y) \leq \varrho(X) + \varrho(Y)$.
5. **Law invariance**, $\forall X, Y \in L^p$, if $F_X = F_Y\ \varrho(X) = \varrho(Y)$.
6. **Co-monotone additivity**, $\forall f, g$ nondecreasing functions, $\forall\ X \in L^p$,

$$\varrho(f(X) + g(X)) = \varrho(f(X)) + \varrho(g(X)).$$

7. **Boundedness**, $\exists B_\varrho > 0,\ \forall X \in L^p,\ |\varrho(X)| \leq B_\varrho \|X\|_p$.
8. **Preserving SSD (or PSSD)**, $\forall X, Y \in L^p$, if X SSD dominates Y, then $\varrho(X) \leq \varrho(Y)$.
9. **Lipschitz continuity**, $\exists L_\varrho > 0,\ \forall X, Y \in L^p,\ |\varrho(X) - \varrho(Y)| \leq L_\varrho \|X - Y\|_p$.

Definition 1. A lower semicontinuous mapping $\varrho : L^p \to \mathbb{R}$ with properties 1, 4, and 5 is called a PSL risk measure. PSL stands for positive homogeneity of degree one, subadditivity, and law invariance. If, in addition, it is monotone, it is called PMSL.

Definition 2. A coherent risk measure is a lower semicontinuous mapping $\varrho : L^p \to \mathbb{R}$ with properties 1, 2, 3, and 4.

A popular example of a risk measure that is not generally subadditive is VaR. This risk measure plays a key role in the characterization of the law-invariant coherent risk measures. A popular example of a coherent risk measure that is law invariant is the conditional VaR, defined as on L^p :

$$\mathrm{CVaR}_\alpha(X) = \frac{1}{1 - \alpha} \int_\alpha^1 \mathrm{VaR}_t(X)dt. \tag{2.1}$$

Here $\alpha \in (0, 1)$ is a risk aversion parameter. Another popular example is the mean–variance principle on L^2 given by

$$MV(X) = \beta\sigma(X) + E(X),$$

where σ is the standard deviation and $\beta > 0$ is a number that represents the risk aversion.

Remark 1. Note that PSL risk measures give us the flexibility to consider more general problems including nonmonotone risk measuresl fir example, mean–variance.

The Fenchel-Moreau dual representation of PSL risk measures can be given as follows.

Proposition 1. *Let $\varrho : L^p \to \mathbb{R}$ be a PSL risk measure. Then,*

$$\varrho(X) = \sup\nolimits_{Z \in \Delta_\varrho} E(ZX),$$

where Δ_ϱ, is a law-invariant closed convex subset of L^q. Law invariance of Δ_ϱ here means $\forall Z' \in L^q$, such that $F_Z = F_{Z'}$, and we have that $Z' \in \Delta_\varrho$.

3. RISK MODEL AND STATIC RISK MEASURE

In the following, modeling the loss frequency/severity with common shocks is considered. This setting combines an extended version of the frequency-severity model with a general counting process and the common shock model similar to Tang, Tong, and Yang (2021). In Tang, Tong, and Yang (2021), the authors considered both common shocks and systematic risk factors without any random counting process, unlike the setting in this study.

Let us denote the amounts of the rth individual idiosyncratic risk variable by the nonnegative random variable X_r, where the sequence $\{X_r\}_{r=1,2,\dots} \sim X$ is independent and identically distributed (i.i.d.). The common shock variable is denoted by ς, and a general counting variable N is considered. An individual risk variable X_r occurs if $r \leq N$. The three components of the model that include $\{X_r\}_{r=1,2,\dots}$, ς, and N are mutually independent.

The aggregate losses is modeled by a random variable L as follows:

$$L := \varsigma \sum_{r=1}^{N} X_r,$$

with the convention that $L = 0$ for the event $N = 0$.

In a standard Cramér-Lundberg model, the counting process is a Poisson process with a constant arrival rate λ. However, because the general counting variable is used, the approach in this study can be regarded more relevant to the Sparre Andersen model, with an extra major extension by including common shocks.

Remark 2. Our general assumptions give us new possibilities; for example, incorporating systematic risk along with the common shocks in the model. As is evident, the component ς can incorporate common shocks. This makes the individual losses correlated. Indeed, the co-variance of $\varsigma X_1, \varsigma X_2$ is $(E(X))^2 var(\varsigma) \neq 0$, if $var(\varsigma) \neq 0$. The common shock variable ς can be regarded as the scale multiplier after the shock is realized. For instance, consider a case for car insurance. In cold years, when there is a higher chance of frost events, accidents with greater severity are expected. This can be incorporated by ς as the multiplier of the severity. Another way of considering systematic events is when the rate of the loss frequency changes. In more detail, consider S and its complement set S^C to represent systematic and unsystematic events, respectively. Let us consider two Poisson processes N_t^S and $N_t^{S^C}$ with rates $\lambda^S > \lambda^{S^C}$, respectively. Assume that $\{X_r\}_{r=0,1,2,..}$, 1_S, ς, $\{N_t^S\}_{t\geq 0}$, and $\left\{N_t^{S^C}\right\}_{t\geq 0}$ are independent. Then, introduce $N = N_T^S 1_S + N_T^{S^C} 1_{S^C}$, for a horizon time T (1 year for car insurance). This model can incorporate the systematic shock in the frequency part. This time in the car accident example in the cold years, a systematic event means a higher accident rate during frost events. Systematic risks have been well-studied in the literature. Lindskog and McNeil (2003), Meyer (2007), Avanzi, Taylor, and Wong (2018), Yuen, Liang, and Zhou (2015), Han, Liang, and Zhang (2019), and Tang, Tong, and Yang (2021) incorporated common shock in risk management problems. More recently, common shocks have been used in the context of COVID-19; see, for example, Assa and Boonen (2022) and Ceci, Colaneri, and Cretarola (2022).

Let us now make some assumptions on the common shock, severity, and frequency parts. First, as mentioned earlier, it is assumed that $X \in L_+^p$ for some $p \in [1, \infty)$. Second, it is assumed that $\varsigma > 0$ and also $\varsigma \in L^\infty$. Actually, this is a technical assumption; however, as one can observe later, many of the results work if we relax this assumption. Third, it is assumed that the growth rate of the frequency variable N belongs to L^p. For example, this property readily holds for a counting variable given by a Poisson process. Furthermore, this condition implies the following evident but useful property:

Proposition 2. *If $X \in L^p$, then $\| \sum_{r=1}^{N} X_r \|_p \leq \|N\|_p \|\varsigma\|_\infty \|X\|_p$.*

3.1. Static Risk Measure

Any optimal allocation of the individual losses is a contract on $X_r, r = 1, 2, \dots$. This implies that any policy based on the individual risks directly depends on X_rs and indirectly on N and ς. On the other hand, it is known that all individual losses have the same distribution as ςX. With these in mind, for a given risk variable X, let us introduce a *static risk measure* as a function of loss severity X:

Definition 3. Let $\varrho : L^p \to \mathbb{R}$ be a law invariant risk measure. Consider a counting variable N, a common shock variable ς, and an i.i.d. sequence $\{X_r\}_{r=1,2,\ldots} \sim X,$ and assume that they are independent. Introduce the *static risk measure* $\varrho^{N,\varsigma} : L^p \to \mathbb{R}$ as follows:

$$\varrho^{N,\varsigma}(X) := \varrho\left(\varsigma \sum_{r=1}^{N} X_r\right). \tag{3.1}$$

If there is no common shock—that is, $\varsigma = 1$—it is simple to show the *static risk measure* by ϱ^N.

Note that the law invariance property of ϱ plays an important role in the definition of $\varrho^{N,\varsigma}$.

Remark 3. If $N \equiv 1$ and $\varsigma = 1,$ then $\varrho^{N,\varsigma} = \varrho^N = \varrho$. This means that the framework in this article can be considered as an extension of the *static* framework.

3.2. Properties of Static Risk Measures

This section is devoted to studying the properties of *static risk measures*. More precisely, which properties $\varrho^{N,\varsigma}$ will inherit will be observed.

Let us begin with the following theorem:

Theorem 1. *The following statements hold:*

1. *If ϱ is positive-homogeneous, so is $\varrho^{N,\varsigma}$.*
2. *If ϱ is subadditive, so is $\varrho^{N,\varsigma}$.*
3. *If ϱ is monotone, so is $\varrho^{N,\varsigma}$.*
4. *If ϱ is bounded, so is $\varrho^{N,\varsigma}$.*
5. *If ϱ is PSSD, so is $\varrho^{N,\varsigma}$.*

The first three statements are self-evident. Statement 4 is a result of Proposition 2. As one can see, this statement to some extent shows that $\varrho^{N,\varsigma}$ is scaled by $\|\varsigma\|_\infty \|N\|_p$.

However, the last statement needs a little bit more explanation. It is clear that for the proof it is enough to show that for i.i.d. sequences $\{X_r\}_{r=1,2,\ldots} \sim X$ and $\{Y_r\}_{r=1,2,\ldots} \sim Y,$ independent of N and $\varsigma,$ if X SSD dominates Y, then $\varsigma \sum_{r=1}^{N} X_r$ also SSD dominates $\varsigma \sum_{r=1}^{N} Y_r$. This is actually clear for constant $N = n$. Here is a brief demonstration for $n = 2$: let ϕ be a concave and nondecreasing real function. Then we have

$$\begin{aligned}
E(\phi(\varsigma X_1 + \varsigma X_2)) &= \textstyle\int_0^\infty \int_0^\infty E(\phi(sX_1 + sx_2))dF_X(x_2)d_\varsigma(s) \\
&\geq \textstyle\int_0^\infty \int_0^\infty E(\phi(sY_1 + sx_2))dF_X(x_2)d_\varsigma(s) \\
&= \textstyle\int_0^\infty \int_0^\infty E(\phi(sy_1 + sX_2))dF_Y(y_1)d_\varsigma(s) \\
&\geq \textstyle\int_0^\infty \int_0^\infty E(\phi(sy_1 + sY_2))dF_Y(y_1)d_\varsigma(s) \\
&= E(\phi(\varsigma Y_1 + \varsigma Y_2)).
\end{aligned}$$

Now by conditioning on $N = n,$ it is confirmed that

$$E\left(\phi\left(\varsigma \sum_{r=1}^{N} X_r\right)\right) \geq E\left(\phi\left(\varsigma \sum_{r=1}^{N} Y_r\right)\right),$$

for an arbitrary ϕ that is concave and nondecreasing.

Furthermore, it is not very hard to see that if ϱ is PMSL, then $\varrho^{N,\varsigma}$ is always PSSD. Actually, Dana (2005), theorem 4.1, and Carlier and Dana (2006), proposition 2.4, showed that for a concave and upper semicontinuous utility function defined on a non-atomic probability space, monotonicity and law invariance are equivalent to preserving SSD. Now as an outcome of statements 1 it 3 it is confirmed that if ϱ is PMSL, so is $\varrho^{N,\varsigma}$, and therefore it is PSSD.

In addition, this discussion can help to find out more about the Schur concavity. A functional is said to be Schur concave if it preserves the concave order. Dana (2005), proposition 2.1, showed that being SSD-preserving is equivalent to being Schur-concave and monotone. As a by-product of discussion above, similar results hold for PMSL ϱ and hence $\varrho^{N,\varsigma}$.

Finally, this also helps to obtain a dual representation for $\varrho^{N,\varsigma}$ when ϱ is PMSL. Dana (2005), theorem 3.1, provided a representation on L^∞ of the Schur-concave functionals as follows:

$$\varrho^{N,\varsigma}(X) = \sup_{Z\in L^1} \int_0^1 F_X^{-1}(t)F_Z^{-1}(t) - \left(\varrho^{N,\varsigma}\right)^*(Z), \tag{3.2}$$

where $(\varrho^{N,\varsigma})^*$ is a Schur-concave functional on L^1. There are two points to mention here: first the theorems in Dana (2005) are in L^∞, which can easily be extended to L^p. This, however, warrants a short explanation. In the proof of theorem 3.1 in Dana (2005), the main technical materials in use are the Schur concavity, Hardy-Littlewood's inequality, and the lower semicontinuity in $\sigma(L^\infty, L^1)$. All of these can be legitimately replicated in L^p and $\sigma(L^p, L^q)$.

Second, because there is a positive homogeneous function, $(\varrho^{N,\varsigma})^*$ is a characteristic function. This means that there is a law-invariant closed convex set $\Delta_{\varrho^{N,\varsigma}} \subseteq L^1$ so that $(\varrho^{N,\varsigma})^*(Z) = \begin{cases} 0, & Z \in \Delta_{\varrho^{N,\varsigma}} \\ \infty, & o.w. \end{cases}$. Note that the same notation $\Delta_{\varrho^{N,\varsigma}}$ is used in the representation of Theorem 3, because they are evidently identical.

Let us also add a technical point. The representation (3.2) is very useful in the proof of Theorem 3 if we wanted only to use PMSL. However, the proof shows that this also works for PSL risk measures.

It is also important to note that in general for a PSL risk measure ϱ, $\varrho^{N,\varsigma}$ is not SSD preserving. For that consider the PSL mean–variance risk measure $\varrho(X) = \sigma(X) + E(X)$. As we have seen, $\varrho^N(X) = \sqrt{E(N)}\sigma(X) + E(N)E(X)$, which based on Dana (2005), theorem 4.1, and Carlier and Dana (2006), proposition 2.4, cannot be SSD because it is not monotone.

Though positive homogeneity, subadditivity, monotonicity, boundedness, and PSSD are inherited by $\varrho^{N,\varsigma}$, the same is not necessarily true for other properties. For instance, if ϱ is cash-invariant or co-monotone additive, the same is not true for $\varrho^{N,\varsigma}$. To see this, let us simply look at $\varrho^N(X + c)$ (i.e., $\varsigma = 1$) for a constant number c. In the following theorem it is shown that cash-invariant and co-monotone additivity do not hold for CVaR_α^N (see the proof in the Appendix). The importance of this theorem can be seen when realizing that $\mathrm{CVaR}_\alpha, \alpha \in (0,1)$, are the building blocks of all law-invariant coherent risk measures (see Kusuoka (2001)):

Theorem 2. *If $\{N_t^\lambda\}_{t\geq 0}$ is a Poisson process with parameter λ, then for any i.i.d. sequence $\{1_{A_r}\}_{r=1,2,\dots} \sim 1_A$, with $P(A) > 0$, independent of $\{N_t^\lambda\}_{t\geq 0}$ and an $\alpha \in (0,1)$, we have*

$$\mathrm{CVaR}_\alpha^{N_T^\lambda}(1_A + 1) < \mathrm{CVaR}_\alpha^{N_T^\lambda}(1_A) + \mathrm{CVaR}_\alpha^{N_T^\lambda}(1). \tag{3.3}$$

This shows that static risk measures do not inherit co-monotone additivity and cash invariance.

Equation (3.3) clearly indicates why co-monotone additivity does not hold. For cash invariance, however, we need a short explanation. It is clear by cash invariance that we need to have $\mathrm{CVaR}_\alpha^{N_T^\lambda}(1_A + 1) = \mathrm{CVaR}_\alpha^{N_T^\lambda}(1_A) + 1$. On the other hand, if we consider a measurable set $\mathcal{N}$ with $P(\mathcal{N}) = 0$, as $1_\mathcal{N} + 1 = 1$ and $\sum_{i=1}^{N_T} 1_{\mathcal{N}_i} = 0$, both almost surely, by cash invariance one must have $\mathrm{CVaR}_\alpha^{N_T^\lambda}(1) = \mathrm{CVaR}_\alpha^{N_T^\lambda}(1_\mathcal{N} + 1) = \mathrm{CVaR}_\alpha^{N_T^\lambda}(1_\mathcal{N}) + 1 = \mathrm{CVaR}_\alpha(\sum_{i=1}^{N_T} 1_{\mathcal{N}_i}) + 1 = 1$. Combining these relations with (3.3), we get $\mathrm{CVaR}_\alpha^{N_T^\lambda}(1_A) + 1 = \mathrm{CVaR}_\alpha^{N_T^\lambda}(1_A + 1) < \mathrm{CVaR}_\alpha^{N_T^\lambda}(1_A) + \mathrm{CVaR}_\alpha^{N_T^\lambda}(1) = \mathrm{CVaR}_\alpha^{N_T^\lambda}(1_A) + 1$, which is a contradiction. Actually, the crux here is that by a (false) cash invariance assumption it could be shown that $\mathrm{CVaR}_\alpha^{N_T^\lambda}(1) = 1$, which is not in general true.

3.3. Robust Representation of Static Risk Measures

This section provides a robust representation of the static risk measures. The robust characterization provided in Theorem 3 in the following is similar to the robust representation approach of coherent and convex risk measures known mainly from seminal works by Artzner et al. (1999), Delbaen (2002), Föllmer and Schied (2002), Föllmer and Schied (2002), Frittelli and Rosazza Gianin (2002), and Frittelli and Rosazza Gianin (2005).

First, we need the following definition. A set $\Delta \subseteq L^q$, for $1 < q < \infty$, is bounded if $\sup_{Z\in\Delta} \|Z\|_q < \infty$. One can easily see that because $\|Z\|_q = \sup_{\|X\|_p \le 1} E(ZX)$, boundedness in L^q is equivalent to

$$\sup_{\|X\|_p \le 1} \sup_{Z\in\Delta} E(ZX) < \infty.$$

It is well understood that in L^p boundedness of Δ is equivalent to weak compactness. Additionally, one can see that for a PSL risk measure ϱ, the boundedness of Δ_ϱ is equivalent to the boundedness of ϱ, with $B_\varrho = \sup_{\|X\|_p \le 1} \sup_{Z\in\Delta_\varrho} E(ZX)$. Finally, both are equivalent to Lipschitz continuity of a PSL risk measure ϱ with a Lipschitz parameter $L_\varrho = B_\varrho$.

Now we have the following theorem.

Theorem 3. *Let $\varrho : L^p \to \mathbb{R}$ be a PSL risk measure. The following statements hold:*

1. *$\varrho^{N,\varsigma}$ is characterized as follows: $\forall X \in L^p$*

$$\varrho^{N,\varsigma}(X) = \sup_{Z\in\Delta_{\varrho^{N,\varsigma}}} E(ZX),$$

where $\Delta_{\varrho^{N,\varsigma}}$ is a law-invariant closed convex subset of L^q.

1. *If Δ_ϱ is bounded in L^q, so is $\Delta_{\varrho^{N,\varsigma}}$.*
2. *If ϱ is PMSL, $\Delta_{\varrho^{N,\varsigma}}$ is nonnegative.*
3. *The set $\Delta_{\varrho^{N,\varsigma}}$ is equal to the closed convex hull of I given as follows:*

$$I := \left\{ \hat{Z} \sim \sum_{r=1}^{N} Z_r \,\middle|\, \begin{array}{l} Z_r = F^{-1}_{\varsigma Z}(W_r), r \in \mathbb{N}, \text{ for some } Z \in \Delta_\varrho \text{ where} \\ (W_r, U) \sim (U, U_r), r \in \mathbb{N}, \text{on } (P_{\Omega_n}, \Omega_n) \\ \text{for some}, U \sim U(0,1) \text{ and} \\ \{U_r\}_{r=1,2,\dots}, i.i.d. \sim U(0,1), N, \varsigma \text{ are independnt} \end{array} \right\}, \tag{3.4}$$

where $\Omega_n = \{N = n\}$.

Finally, let us look at a few examples. Let us consider $\pi(X) = (1+\rho)E(X)$, for a positive number ρ. In this case,

$$\Delta_\pi = \{(1+\rho)1_\Omega\},$$

which is evidently bounded, implying that $\Delta_{\pi^{N,\varsigma}}$ is bounded. In addition, it is trivial that because $(1+\rho)E(L) = (1+\rho)E(\varsigma \sum_{r=1}^{N} X_r) = (1+\rho)E(N)E(\varsigma)E(X)$, then $\Delta_{\pi^{N,\varsigma}} = \{(1+\rho)E(N)E(\varsigma)1_\Omega\}$.

Another example is CVaR:

$$\Delta_{\mathrm{CVaR}_\alpha} = \left\{ Z \in L^q \,\middle|\, E(Z) = 1, 0 \le Z \le \frac{1}{\alpha} \right\},$$

which is again clearly a bounded set in L^q. As a result, $\Delta_{\mathrm{CVaR}_\alpha^{N,\varsigma}}$ is also bounded.

For the mean–variance risk measure, let us take $p = 2$. Then we have

$$\Delta_{MV} = \{\beta(Z - E(Z)) + 1; \|Z\|_2 \le 1\},$$

which is bounded in L^2, implying that $\Delta_{MV^{N,\varsigma}}$ is also bounded. It is also possible to explicitly identify $\Delta_{MV^{N,\varsigma}}$, where $N = N_T^\lambda$ is the Poisson process with rate λ. First, the mean–variance risk measure in this case is

$$MV^{N,\varsigma}(X) = \beta\sigma\left(\varsigma \sum_{r=1}^{N} X_r\right) + \lambda T \times E(\varsigma)E(X).$$

So, we have to find the two parts of the risk measure. By applying the law of total variance, we get the following:

$$\begin{aligned} Var\left(\varsigma\sum_{r=1}^{N}X_r\right) &= E\left(Var\left(\varsigma\sum_{r=1}^{N}X_r|\varsigma\right)\right) + Var\left(E\left(\varsigma\sum_{r=1}^{N}X_r|\varsigma\right)\right) \\ &= E\left(\varsigma^2 Var\left(\sum_{r=1}^{N}X_r\right)\right) + Var\left(\varsigma E\left(\sum_{r=1}^{N}X_r\right)\right) \\ &= E\left(\varsigma^2\right)Var\left(\sum_{r=1}^{N}X_r\right) + E(N)^2E(X)^2Var(\varsigma) \\ &= E\left(\varsigma^2\right)\times\lambda T\times E\left(X^2\right) + (\lambda T)^2E(X)^2Var(\varsigma) \\ &= C_1E\left(X^2\right) + C_2E(X)^2, \end{aligned}$$

where $C_1 = \lambda T \times E(\varsigma^2) > 0$ and $C_2 = (\lambda T)^2 Var(\varsigma) \geq 0$. Now let us benefit from the Hilbert space properties to give a dual representation of $\sigma(\varsigma\sum_{r=1}^{N}X_r)$. Consider an inner product on L^2 as follows:

$$\langle X,Y\rangle_{New} = C_1E(XY) + C_2E(X)E(Y). \tag{3.5}$$

It is easy to check that $\langle\cdot,\cdot\rangle_{New}$ satisfies all the properties of an inner product. So, the following new norm on L^2 is introduced:

$$\|X\|_{New} = \sqrt{\langle X,X\rangle_{New}} = \sqrt{C_1E(X^2) + C_2E(X)^2} = \sigma\left(\varsigma\sum_{r=1}^{N}X_r\right).$$

By Jensen's inequality applied to (3.5), it is clear that

$$\sqrt{C_1}\|X\|_2 \leq \|X\|_{New} \leq \sqrt{C_1+C_2}\|X\|_2.$$

This confirms that $(L^2,\langle\cdot,\cdot\rangle_{New})$ is isomorphic to $(L^2,\langle\cdot,\cdot\rangle)$, where $\langle X,Y\rangle = E(XY)$, so it is a Hilbert space. It is understood that the norm of a Hilbert space can be given as follows:

$$\|X\|_{New} = \sup_{\|Z\|_{New}\leq 1}\langle X,Z\rangle_{New}.$$

This gives

$$\sigma\left(\varsigma\sum_{r=1}^{N}X_r\right) = \sup_{C_1E(Z^2)+C_2E(Z)^2\leq 1} C_1E(XZ) + C_2E(X)E(Z).$$

Therefore, we have

$$\begin{aligned} MV^{N,\varsigma}(X) &= \beta\sigma\left(\varsigma\sum_{r=1}^{N}X_r\right) + E(N)E(\varsigma)E(X) \\ &= \sup\nolimits_{C_1E(Z^2)+C_2E(Z)^2\leq 1}\left\{\beta C_1E(XZ) + \beta C_2E(X)E(Z)\right\} + \lambda T\times E(\varsigma)E(X) \\ &= \sup\nolimits_{C_1E(Z^2)+C_2E(Z)^2\leq 1}\left\{E(X(\beta C_1Z + \beta C_2E(Z) + \lambda T\times E(\varsigma)))\right\}. \end{aligned}$$

From here we have

$$\Delta_{MV^{N,\varsigma}} = \left\{ \beta C_1 Z + \beta C_2 E(Z) + \lambda T \times E(\varsigma) \middle| C_1 E(Z^2) + C_2 E(Z)^2 \le 1 \right\}.$$

For the special case where we remove the common shock risk factor—that is,—$\varsigma = 1,$ we get

$$\Delta_{MV^N} = \left\{ \beta(\lambda T) Z + \lambda T | \lambda T \times E(Z^2) \le 1 \right\} = \lambda T \times \left\{ \beta Z + 1 \middle| \|Z\|_2 \le \frac{1}{\sqrt{\lambda T}} \right\}.$$

Remark 4. There is an interesting connection to mathematical finance here, as we have the robust representation. The members of Δ_ϱ play the role of a state price density process, if ϱ is a pricing rule or premium function; see Bernard, Rüschedorf, and Vanduffe (2014). In the literature they are also known as the stochastic discount factor. The structure of the set given in Theorem 3 shows that they more look like a frequency-severity model, even though there are some major differences; for instance, the severity is no longer independent of the frequency.

3.4. Literature and Static Risk Measures

Let us finish discussion on the properties of *static risk measures* by comparing them with their closest peers, ruin-based risk measures. These risk measures have been studied in several papers, including Trufin, Albrecht, and Denuit (2011), Cosette and Marceau (2013), Wüthrich (2015), Cossette et al. (2020), and Assa and Constantinescu (2021). The mindset behind these risk measures is to develop a setup that can embed ruin theory into risk measure theory.

In addition, such risk measures can also be viewed as a link between the measure of individual loss distribution and the measure of aggregate loss distribution. In practice, the insurer is interested in the aggregate risk (which is the actual liability of the insurer), whereas many estimations and modifications are done on the individual loss level. Given the frequency N and the common shock variable ς, $\varrho^{N,\varsigma}(X)$ measures individual loss X directly, and the value represents the aggregate loss exposure. It can help the insurer to better understand the impact of any changes in the individual risk level on the aggregate risk level.

For example, let us look at the definition of the ruin-based risk measure in Trufin, Albrecht, and Denuit (2011). Consider a surplus model $u + ct - \sum_{k=1}^{N_t^\lambda} X_k,$ where $\{N_t^\lambda\}_t$ is a Poisson process with rate λ. The minimum capital requirement for the insurance company to keep the business solvent is given by

$$\inf\left\{ u \in \mathbb{R} \middle| P\left(\inf_{t \ge 0} \left(u + ct - \sum_{k=1}^{N_t^\lambda} X_k \right) < 0 \right) \le 1 - \alpha \right\}, \tag{3.6}$$

where α is a (small) ruin probability. Interestingly, this value can be identified as follows:

$$u = \mathrm{VaR}_\alpha \left(\sup_{t \ge 0} \left(\sum_{r=1}^{N_t^\lambda} X_k - ct \right) \right). \tag{3.7}$$

With this in mind Trufin, Albrecht, and Denuit (2011) introduced a risk measure as follows:

$$\varrho_\alpha(X) = \mathrm{VaR}_\alpha \left(\sup_{t \ge 0} \left(\sum_{r=1}^{N_t^\lambda} X_r - \eta \lambda E(X) t \right) \right), \tag{3.8}$$

where $\eta > 1$ is a constant and $X_r \sim X$.

The ruin-based risk measure fills a gap in the literature to relate two theories of ruin theory and risk measure theory that are essentially designed to find the minimum capital required to keep the insurance solvent. In essence, it looks like the definition in 3.8 is similar to a *static risk measure* because it also focuses on the severity. However, the major concern in ruin-based risk measures is the ruin probability. This implies a major difference to include the supremum over time in addition to including the premium $\eta\lambda E(X)t$. The same mindset is not relevant in the risk measure theory, based on which the *static risk measures* is introduced. In risk measure theory the concern is to make sure that the insurance risk position is within a set of *acceptable* positions. It is important to note that from a technical perspective this supremum adds an extra layer of complexity to the model that is usually difficult to deal with, particularly when it is considered for some application such as optimal reinsurance design. Another point in relation to ruin-based risk measures is that they do not perform well when it comes to continuity; see Assa and Constantinescu (2021). Actually, they are not continuous unless for L^∞ topology. From a technical perspective this is again a result of the supremum above that is taken over time; actually, it has the same supremum for all loss severity like $X = 1_A,$ regardless of the probability $P(A),$ either large or small. However, in many cases *static risk measures* can be continuous owing to their robust representation. Particularly when ϱ is PSL and bounded, as discussed, both ϱ and $\varrho^{N,\varsigma}$ are Lipschitz continuous in L^p.

It is also important to mention that the assumption of having i.i.d. $\{X_r\}_{r=1,2,\ldots},$ though necessary in our setup, is to some extent restrictive. This can be dealt with in ruin-based risk measures; see Cossette et al. (2020).

Finally, as one can see, even though $\varrho^{N,\varsigma}$ is a risk measure on the aggregate losses in a dynamic framework, it only deals with the *individual losses* in a *static* framework. By $\varrho^{N,\varsigma},$ a technique to embed a problem in a dynamic setup into a *static* one is introduced. Similar ideas have been used in the past in the literature; for instance, Assa (2011) introduced some sort of static risk measure associated with a dynamic risk measure with the same Lebesgue property.

4. REINSURANCE APPLICATION

In this section, *static risk measures* are used to design optimal reinsurance, dealing with individual loss severity. From a practical point of view, this problem sounds difficult because we need to deal with a dynamic setup where the number of the contracts to consider is not prespecified. The approach in this article, however, will reduce the problem to a nondynamic one, by introducing *static risk measures*. One must also note that this setup is rich enough to include other concepts such as systematic and common shocks. Reinsurance in the presence of common shocks has previously been studied; see, for example, Han, Liang, and Zhang (2019).

Considering the presence of a common shock variable, there are two distinct approaches for risk sharing between the insurance and reinsurance companies. The first option involves jointly sharing the risk of individual risks, which encompasses both systematic and idiosyncratic risks. The alternative approach entails a risk sharing strategy of the idiosyncratic component beforehand (ex ante) and subsequently sharing the risk associated with the systematic component (ex post). Mathematically, the former shares the risk of the rth individual as allocated by W_1^r and $W_2^r,$ where $W_1^r + W_2^r = \varsigma X_r,$ and the latter shares the risk in the form of W_1^r and $W_2^r,$ where $W_1^r + W_2^r = X_r$. In this article, the second option was chosen because it was assumed that due to the systematic nature of the risk variable $\varsigma,$ the risk of the systematic part needs to be realized before making any decision to share it.

Therefore, let us assume that an insurance company wants to share the risk of each individual risk X_r with a reinsurance company to minimize the total risk. In other words, we are looking for an optimal allocation (W_1^r, W_2^r) of any individual risk X_r; that is, $X_r = W_1^r + W_2^r,$ where $0 \le W_1^r, W_2^r \le X_r$. However, if this allocation is based on a contract that is agreed by the parties up front, it is rather natural to assume that the sequence $\{(W_1^r, W_2^r)\}_{r=1,2,\ldots}$ is i.i.d. Also assume that this sequence, N, and ς are independent. Let us denote the insurance and the reinsurance loss respectively as

$$L_1 = \varsigma \sum_{r=1}^{N} W_1^r, L_2 = \varsigma \sum_{r=1}^{N} W_2^r.$$

It is clear that $L = L_1 + L_2$.

In this article, it is assumed that the reinsurance company uses the expectation principle for pricing. Therefore, the global risk position of the insurance company is given as follows:

$$L_1 + (1+\rho)E(L_2),$$

where ρ is a risk loading factor. Now let us assume that the insurance company is using a cash-invariant PSL risk measure ϱ to assess the risk. So, the insurance total risk is given by

$$\varrho\big(L_1 + (1+\varrho)E(L_2)\big) = \varrho(L_1) + (1+\varrho)E(L_2).$$

If we assume that $(W_1^r, W_2^r) \sim (W_1, W_2)$, it is clear that the total risk is identical to

$$\varrho^{N,\varsigma}(W_1) + (1+\rho)E(N)E(\varsigma)E(W_2).$$

In addition, in the existing literature, the consideration of co-monotonicity to mitigate moral hazard risk is quite common when analyzing loss risks. To materialize this concept, the following set of indemnity functions is introduced:

$$\mathcal{C} = \left\{ f \in L^1(\mathbb{R}_+)_+ \middle| \begin{array}{c} f \text{ and } id - f \text{ are} \\ \text{nondecreasing} \end{array} \right\}$$

Here *id* is the identity function. The following proposition plays an important role in our understanding of the indemnity functions (see Assa (2015)).

Proposition 3. *If $f \in \mathcal{C}$, then f is Lipschitz of degree 1 and there are functions $0 \le h \le 1$ so that $f(x) = \int_0^x h(s)ds$.*

Now for the loss $X \ge 0$, let us introduce the set of contracts

$$\mathcal{C}(X) = \{f(X) | f \in \mathcal{C}\}.$$

By Proposition 3, it is clear that $\mathcal{C}(X)$ is a closed convex set in L^p. It is also bounded because $0 \le f(X) \le X$ for any $f \in \mathcal{C}$. So essentially the insurance company has to search for an optimal policy in the set $\mathcal{C}(X)$. Because the insurance company wants to minimize the total risk, this article focused on the solutions to the following problems:

$$\min_{W \in \mathcal{C}(X)} \varrho^{N,\varsigma}(W) + \gamma E(X - W), \tag{4.1}$$

where $\gamma = (1+\rho)E(N)E(\varsigma)$. Now, the following theorem (see the proof in the Appendix) is presented.

Theorem 4. *Let ϱ be a PSL risk measure with bounded Δ_ϱ and $X \ge 0$ be a loss variable. Then there exists $Z^* \in \Delta_{\varrho^{N,\varsigma}}$ such that a solution to (4.1) is given by $W^* = \int_0^X h^*(t)dt$, where*

$$h^*(t) = \begin{cases} 1, & \text{if } \mathrm{CVaR}_{F_X(t)}(Z^*) < \gamma \\ 0, & \text{if } \mathrm{CVaR}_{F_X(t)}(Z^*) > \gamma \end{cases}. \tag{4.2}$$

h^* can take any value between 0,1 if $\mathrm{CVaR}_{F_X(t)}(Z^*) = \gamma$. As a result, an optimal solution for individual risks can be represented by $(W_1^r, W_2^r) = (X_r \wedge a^*, 0 \vee (X_r - a^*))$, where a^* can be any number in the following range:

$$\left[\sup\left\{t; \mathrm{CVaR}_{F_X(t)}(Z^*) < \gamma\right\}, \inf\left\{t; \mathrm{CVaR}_{F_X(t)}(Z^*) > \gamma\right\}\right].$$

The proof of this theorem is presented in the Appendix; however, a sketch of the proof is discussed here. As one can see by Proposition 3, any $f \in \mathcal{C}$ can be identified by an MIF h. The major idea behind the proof of Theorem 4 is based on a standard technique known as the MIF formulation introduced and developed in Assa (2015) and later in Zhuang et al. (2016) and Boonen (2022). That is why representation such as that in (4.2) would be expected. To make use of this method, we need to find an equivalent problem to (4.1) where the risk measure is co-monotone and subadditive. To make this happen, we have to follow a path that gives a similar representation to (3.2).

The results in Theorem 4 can readily be compared to those from Cai and Tan (2007), Cai et al. (2008), Cheung (2010), Chi and Tan (2013), Cheung et al. (2014), and, in particular, Assa (2015) and Zhuang et al. (2016), where a similar methodology is adopted. As the results also show, the solutions are layered policies; that is, excess-of-loss and stop-loss policies. A major point, though, is that like most of the literature, the optimal solution is stop-loss for insurance and excess-of-loss for reinsurance. This may be perceived as a moral hazard promoter, even though the no-moral-hazard condition is met. But this is mainly a result of the fact that the problem is solved from the insurance company's perspective, though one can also solve it from the reinsurance company perspective. For more discussion on that and alternative setups, see Assa (2015).

It is worth mentioning that the results in this study can be compared with those in Ludkovski and Young (2009), in which there is layering via a lower envelope of distortions.

It should also be mentioned that the frequency N and common shock ς explicitly play important roles in the definition of the *static risk measures.* Therefore, the results of Theorem 4 depend on the choice of N and ς. Actually, these are the main factors to specify the *static risk measure* and, accordingly, Z^*. This is important to make a decision up front. In fact, it is almost impossible to know the number of losses in a given year until the end of that year. Any particular process would affect our optimal decisions, which in finding the optimal risk sharing for individual losses it should not be of great importance.

Remark 5. Following discussions in Chateauneuf, Dana, and Tallon (2000) and Filipović and Svindland (2008), for any set of monotone, law-invariant, and convex mappings, the solutions to the risk sharing problems are co-monotone. This means that if ϱ is monotone, we can remove the moral hazard condition because it is automatically achieved.

4.1. Individual versus Aggregate

Now it is possible to compare the two frameworks managing the individual and aggregate risks. As we will see, there is a huge difference between the risk sharing problem when we consider the aggregate losses versus when we consider the individual losses. This is shown in an example.

In this example, we look at an easier case where $\varsigma = 1,$ because there is a clear difference between the two frameworks, managing the individual and aggregate risks in this case, which shows that the issue would be even more complicated when ς is not a constant random variable. To find an optimal allocation on aggregate loss variable L, we have to find $L_1, L_2 \in L^p_+$ with $L_1 + L_2 = L$ so that for any other allocation $L_1', L_2' \in L^p_+$ with $L_1' + L_2' = L,$ we have $\varrho(L_1) + \pi(L_2) \leq \varrho(L_1') + \pi(L_2')$. The individual problem is to find an optimal allocation on an individual basis.

The solutions to these two problems can be quite different. To see the difference, let us study a specific example, which is inspired by discussions from corollary 2 in Assa (2015). This example, however, can easily be generalized. Let us assume that the insurance company uses the risk measure $\varrho = \mathrm{CVaR}_\alpha$ (for some fixed $0 < \alpha < 1$), where $\alpha > \frac{\rho}{1+\rho}$. Furthermore, we make two assumptions:

Assumption 1: Assume $X > 0$ a.s. and X is unbounded above; that is, for any number $M > 0,$ $P(X > M) > 0$. For instance, X can have an exponential distribution.

Assumption 2: For all $n \in \mathbb{N},$ $P(N = n) > 0$. For instance, if $\{N^\lambda_t\}_{t\geq 0}$ is a Poisson process, $N = N^\lambda_T$ for a give horizon time T.

Now, consider the following two problems.

Problem 1. First, consider an optimal allocation of the aggregate risk variable L by finding L_1 and L_2 in L^p_+ that solve

$$\inf_{\substack{L_1, L_2 \in L^p_+ \\ L_1 + L_2 = L}} \mathrm{CVaR}_\alpha(L_1) + (1+\rho)E(L_2).$$

In corollary 2 of Assa (2015), it is shown that the solution to this problem has the following form:

$$L_1 = \begin{cases} 0, & L < a^* \\ L - a^*, & a^* \leq L < b^* \\ b^* - a^*, & L \geq b^* \end{cases} \text{ and } L_2 = L - L_1, \tag{4.3}$$

where a^* and b^* are the endpoints of an interval $\mathbb{I} = (a^*, b^*)$. It is clear that L is not bounded; however, L_1 is.

Problem 2. For the same aggregate loss risk measure and premium as in Problem 1, we consider the optimal reinsurance on and individual basis. According to Theorem 4, there is a Lebesgue measurable function $h^* : \mathbb{R}_+ \to [0,1]$ so that the optimal solution is given by $\{(W^r_1, W^r_2)\}_{r=1,2,\ldots} = \left\{(\int_0^{X_r} h^*(t)dt, \int_0^{X_r}(1-h^*(t))dt)\right\}_{r=1,2,\ldots}$. Now, we observe two cases.

Case 1: The function h^* is μ (Lebesgue measure) almost everywhere equal to 1 on $\mathbb{R}_+$. In this case, for all $r \in \mathbb{N},$ $W^r_2 = 0$ a.s. This means the insurance allocation $\sum_{r=1}^{N_T} W^r_1$ is a.s. equal to L and is unbounded.

Case 2: There is a positive number $\delta > 0$ and a Lebesgue measurable set $I \subseteq \mathbb{R}_+$ with $\mu(I) > 0$ so that $h^*|_I > \delta$. In this case, it is shown that the insurance optimal risk variable $\sum_{r=1}^{N_T} W_1^r$ is unbounded above.

First, we claim that there is $\epsilon > 0$ such that for all $r \in \mathbb{N}$ we have $P(\{W_1^r > \epsilon\}) > 0$. Consider $M > 0$ is large enough so that the set $J = I \cap [0, M]$ has a positive Lebesgue measure. Now, for any $\omega \in \{X_r > M\}$ we have $W_1^r(\omega) = \int_0^{X_r(\omega)} h^*(t)dt \geq \int_0^M h^*(t)dt > \int_{I\cap[0,M]} \delta dt = \delta\mu(J) > 0$. This implies $\{X_r > M\} \subseteq \{W_1^r > \delta\mu(J)\}$. Because X is unbounded, this implies that $P(\{W_1^r > \delta\mu(J)\}) \geq P(X_r > M) = P(X > M) > 0$. Therefore, one can take $\epsilon = \delta\mu(J)$.

Because $W_1^r, r = 1, 2, \ldots$ and $\{N_t\}_{t\geq 0}$ are all independent, then by Assumption 2, $\forall u \in \mathbb{N}$, $P((\cap_{r=1}^u \{W_1^r > \epsilon\}) \cap \{N_T = u\}) = (\Pi_{r=1}^u P(\{W_1^r > \epsilon\})) \times P(\{N_T = u\}) > 0$. On the other hand, on $(\cap_{r=1}^u \{W_1^r > \epsilon\}) \cap \{N_T = u\}$, we have $\sum_{r=1}^{N_T} W_1^r = \sum_{r=1}^u W_1^r > u\epsilon$. So, for any large number $K > 0$, if we take $u \in \mathbb{N}$ larger than K/ϵ, on the positive measure set $(\cap_{r=1}^u \{W_1^r > \epsilon\}) \cap \{N_T = u\}$ we have $\sum_{r=1}^{N_T} W_1^r > K$. This means that in Case 2, $\sum_{r=1}^{N_T} W_1^r$ is unbounded above.

Comparing the two problems above, in Problem 2, the insurance optimal allocation $\sum_{r=1}^{N_T} W_1^r$ is either L or is unbounded above, whereas the insurance optimal allocation risk variable L_1 in Problem 1 is nonzero and bounded.

Now, let us delve into the implications of this example, particularly from the reinsurance company's perspective. This viewpoint is crucial because the problem is primarily addressed from the insurance standpoint, seemingly favoring their interests. Consequently, the reinsurance company can evaluate whether the contract should be structured on an an aggregate or individual basis.

In the framework utilizing the *static risk measure*, the policy is based on each individual risk, whereas the alternative framework adopts an approach based on aggregate risk. An important implication of individual risk sharing is that the insurance company's exposure to risk can potentially become substantial, because there is no limitation on the number of incidents that may occur within the given time period, T. On the other hand, the policy based on aggregate risk is constrained by an indemnity level.

Consequently, with individual risk contracting, the reinsurance company gains a clearer understanding of their involvement in the risk. Conversely, it can be argued that the aggregate framework's risk may be advantageous for the reinsurance company by allowing it to set a high indemnity level. Thus, determining which discussion truly matters to the reinsurance company becomes challenging in the absence of clear criteria, which are currently unavailable within this context.

4.2. Systematic and Common Shock Cases

New research is now more concerned with the systematic risk after large-scale economic losses like the COVID-19 pandemic Assa and Boonen (2022). As discussed in Remark 2, we can consider a common shock model when we include a non-constant common shock variable ς and a systematic shock when we consider two independent Poisson variables N^S and N^{S^C} with rates $\lambda^S > \lambda^{S^C}$, respectively. This part considers both cases when dealing with the mean–variance risk measure and studies some numerical results.

First case: systematic shock with $\varsigma = 1$ and independent Poisson variables $N^S = N_T^{\lambda^S}$ and $N^{S^C} = N_T^{\lambda^{S^C}}$. Let us first consider a case where there is a systematic shock in the rate of the losses. Assume $\{X_r\}_{r=0,1,2,..}$, 1_S, N^S, and N^{S^C} are independent, where $P(S) = 1 - P(S^C) = \pi$. The rate λ^S represents the rate of losses in systematic events and λ^{S^C} represents the rate of losses when there is no systematic event. So, let us introduce $N = N^S 1_S + N^{S^C} 1_{S^C}$ as the number of losses in a given time period. Here we take two approaches. First, we consider the ex ante policies and then the ex post policies, which have recently received more attention because they appear to be more useful for large-scale events; see Assa and Boonen (2022). Note that the ex post approach introduced here is different than the one we discussed for the common shock variable ς in the beginning of Section 4, because we now make the realization with respect to two cases of systematic and unsystematic events S and S^C. Here the ex post means considering two different problems for two distinct cases of systematic and unsystematic shocks. This gives the premium and the reinsurance design based on the realization of the systematic event.

First, consider the ex ante policy. Letting $\kappa = \{\emptyset, S, S^C, \Omega\}$, we find the risk measure by computing the variance based on the law of total variance:

$$Var\left(\sum_{r=1}^{N} X_r\right) = Var\left(E\left(\sum_{r=1}^{N} X_r|\kappa\right)\right) + E\left(Var\left(\sum_{r=1}^{N} X_r|\kappa\right)\right)$$

$$= Var\left(\lambda^S E(X)1_S + \lambda^{S^C} E(X)1_{S^C}\right) + E\left(\lambda^S E(X^2)1_S + \lambda^{S^C} E(X^2)1_{S^C}\right)$$

$$= E(X)^2\left(\lambda^S - \lambda^{S^C}\right)^2 Var(1_S) + E(X^2)\left(\lambda^S\pi + \lambda^{S^C}(1-\pi)\right)$$

$$= E(X)^2\left(\lambda^S - \lambda^{S^C}\right)^2 \pi(1-\pi) + E(X^2)\left(\lambda^S\pi + \lambda^{S^C}(1-\pi)\right).$$

Therefore, by Theorem 4, the reinsurance problem is given by

$$\inf_{f^S \in \mathcal{C}} \beta\sqrt{D_1 E\left((X\wedge a^*)^2\right) + D_2(E(X\wedge a^*))^2} + D_3 E(X\wedge a^*), \tag{4.4}$$

where $D_1 = (\lambda^S\pi + \lambda^{S^C}(1-\pi))$, $D_2 = (\lambda^S - \lambda^{S^C})^2\pi(1-\pi)$, and $D_3 = -(\lambda^S\pi + \lambda^{S^C}(1-\pi))T\rho$. An interesting observation here is that the risk depends on the value of systematic probability π and the difference between the systematic and unsystematic rates $\lambda^S - \lambda^{S^C}$. Even though these parameters are assumed to be provided, however, the estimation of $\pi, \lambda^S, \lambda^{S^C}$ must be challenging because the estimators do not need to be independent (the rate of a systematic event and the probability of the systematic event can be correlated).

By Theorem 4 we have to find a^* so that

$$\inf_{a^*} \beta\sqrt{D_1 E\left(f(X)^2\right) + D_2 E(f(X))^2} + D_3 E(X\wedge a^*). \tag{4.5}$$

So by equalizing the derivative to zero, we get

$$\beta\frac{D_1 a^* S_X(a^*) + D_2 S_X(a^*)E(X\wedge a^*)}{\sqrt{D_1 E\left((X\wedge a^*)^2\right) + D_2 E(X\wedge a^*)^2}} + D_3 S_X(a^*) = 0. \tag{4.6}$$

Now let us consider the ex post approach. For an ex post policy, $L^S = \sum_{i=1}^{1_S N^S} X_i$ and $L^{S^C} = \sum_{i=1}^{1_{S^C} N^{S^C}} X_i$ are considered separately. Let us consider the problem when systematic event S is observed. Then the aggregate loss of the systematic event can be rewritten as follows:

$$L^S = \sum_{i=1}^{1_S N^S} X_i = 1_S \sum_{i=1}^{N^S} X_i.$$

In the following, a setup that considers that an individual risk sharing platform can be much easier when dealing with the optimal reinsurances is considered. First, let us look at the aggregate loss problem, which is given as follows:

$$\inf_{\substack{L_1, L_2 \in L_+^2 \\ L_1 + L_2 = L^S}} \beta\sigma(L_1) + E(L_1) + (1+\rho)E(L_2).$$

This can be simplified to

$$\inf_{f^S \in \mathcal{C}} \beta\sigma\left(f^S(L^S)\right) - \rho E\left(f^S(L^S)\right). \tag{4.7}$$

To solve the problem, we need to have a good understanding of the distribution of $L^S = \sum_{i=1}^{1_S N^S} X_i$, which is unknown. Second, we will focus on the individual risk sharing platform. Let us find the variance. To do so, we need

$$E\left(\left(\sum_{r=1}^{N^S 1_S} X_r\right)^2\right) = \pi\lambda^S T\Big(Var(X) + \lambda^S T(E(X))^2\Big)$$

and,

$$E\left(\sum_{r=1}^{N^S 1_S} X_r\right) = \pi\lambda^S T \times E(X).$$

These give us

$$\begin{aligned} Var\left(\sum_{r=1}^{N^S 1_S} X_r\right) &= \pi\lambda^S T\Big(Var(X) + (1-\pi)\lambda^S T(E(X))^2\Big) \\ &= \pi\lambda^S T\Big(E(X^2) + \big((1-\pi)\lambda^S T - 1\big)(E(X))^2\Big). \end{aligned}$$

Given Theorem 4, we then have to find the minimum to

$$\inf_{a^*} \beta\sqrt{\pi\lambda^S T\Big(E\Big((X\wedge a^S)^2\Big) + \big((1-\pi)\lambda^S T - 1\big)(E(X\wedge a^S))^2\Big)} - \pi\lambda^S T\rho \times E(X\wedge a^S).$$

Similar to the previous example, by equalizing the derivative to zero, we get

$$\beta\frac{a^S + \big((1-\pi)\lambda^S T - 1\big)E(X\wedge a^S)}{\sqrt{\Big(E\Big((X\wedge a^S)^2\Big) + \big((1-\pi)\lambda^S T - 1\big)(E(X\wedge a^S))^2\Big)}} - \sqrt{\pi\lambda^S T}\rho = 0.$$

A similar result is obtained for the unsystematic case.

Now let us look at some numerical examples to compare the first approach and the ex post approach. In the numerical results, we consider $X \sim \exp(1/\omega)$ and $X \sim \text{LogNormal}(\omega, 1)$, for the idiosyncratic risk where ω is a parameter to change the shape of the idiosyncratic distribution. Setting $\beta = 0.01, T = 1,\ \rho = 0.25,$ we have the following cases for $\lambda^{S^C} = 0.1, \lambda^S = 0.2$ and for a range of $\omega \in [0.1, 0.2]$. Samples of size 1,000,000 are used when simulating the random variables. The results of the numerical cases are presented in Figures 4.1 and 4.2.

As one can see, in both cases of exponential and Lognormal distributions, when the models are riskier (i.e., ω increases), the level of indemnity also increases. Nonetheless, it is worth noting that the level of contribution where the risk is shared without dividing it into two different policies is consistently higher compared to the ex post policy. This disparity can be attributed to the fact that the risk is more explicitly accounted for in the ex post policy scenario. Furthermore, in the ex post policy case, the insurance company assumes a greater share of the risk sharing responsibility when confronted with unsystematic events that entail lower risk. In this case, insurance pools two independent events of systematic and unsystematic events, which is less risky, thus contributing more.

Second case: common shock with ς nonconstant and Poisson variables $N^\lambda = N_T^\lambda$. Now consider a common shock where there is no systematic shock. Assume $\{X_r\}_{r=0,1,2,..}$, N^λ, and ς are independent. By the discussions on the mean–variance risk measure and Theorem 4, and after some simple reorganization of the expressions, the following problem needs to be solved to find the optimal reinsurance solution:

$$\begin{aligned} &\inf_{a^*} \beta\sqrt{\langle X\wedge a^*, X\wedge a^*\rangle_{New}} + \big(E(N)E(\varsigma) - E(N)E(\varsigma)(1+\rho)\big)E(X\wedge a^*) \\ &= \inf_{a^*} \beta\sqrt{C_1 E\Big((X\wedge a^*)^2\Big) + C_2 E(X\wedge a^*)^2} - \rho\lambda T \times E(\varsigma)E(X\wedge a^*). \end{aligned}$$

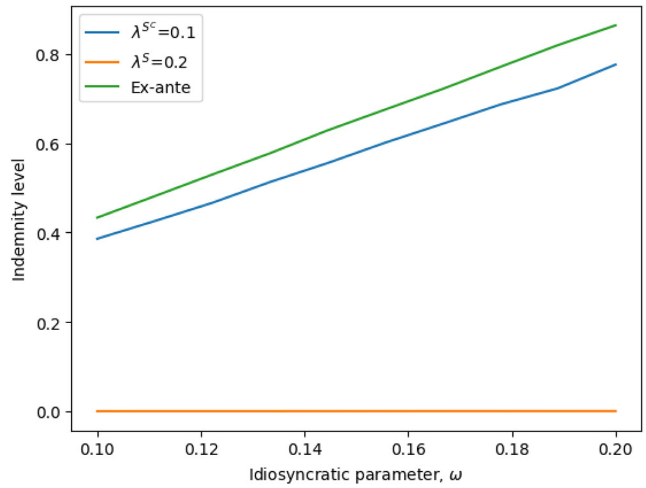

FIGURE 4.1. Ex Post and Ex Ante Indemnities for Exponential Model.

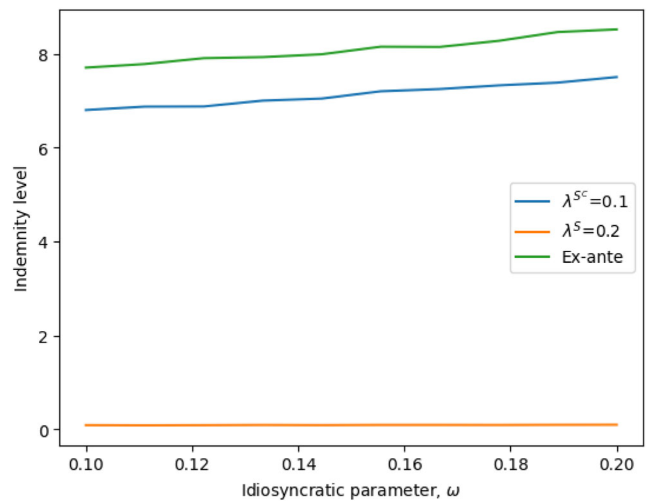

FIGURE 4.2. Ex Post and Ex Ante Indemnities for Lognormal Model.

Let us denote $C_3 = -\rho\lambda T \times E(\varsigma)$ and equalize the derivative to zero:

$$\beta \frac{C_1 a^* + C_2 E(X \wedge a^*)}{\sqrt{C_1 E\left((X \wedge a^*)^2\right) + C_2 E(X \wedge a^*)^2}} + C_3 = 0. \tag{4.8}$$

Some numerical examples were chosen for illustration. Letting $T = \beta = 1,$ based on (4.8), recall the following constants in Table 1 that we need to find:

Now let us explore the impact of the common shock; that is, ς. As one can see, we need to specify the first and second moments of ς; that is, $E(\varsigma)$ and $E(\varsigma^2)$. Because it is always possible to consider a bounded distribution given the first and second moments as long as $E(\varsigma^2) \geq E(\varsigma)^2,$ these are considered parameters. Therefore, in the simulations we consider $(E(\varsigma), E(\varsigma^2)) = (1+\eta, 4+\eta),$ where $\eta \in \{1, 1.1\}$.

In the numerical results we consider $X \sim \exp(1/\omega)$ and $X \sim \text{LogNormal}(\omega, 1),$ for the idiosyncratic risk where ω is a parameter to change the shape of the idiosyncratic distribution. Setting $\rho = 0.25,$ we have the following cases for $\lambda = 0.1, 0.2$ for a range of $\omega \in [0.5, 1.5]$. A sample of size 1,000,000 is used and the results are shown in Figures 4.3 and 4.4.

TABLE 1
Parameters for Mean–Variance

C_1	$E(\varsigma^2) \times \lambda$
C_2	$\lambda^2(E(\varsigma^2) - E(\varsigma)^2)$
C_3	$-\rho\lambda E(\varsigma)$

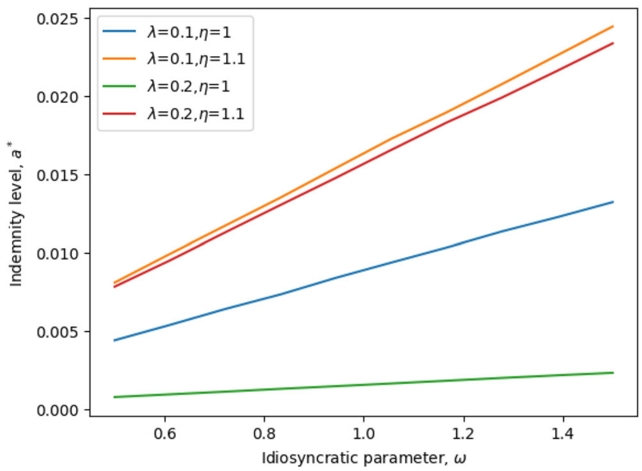

FIGURE 4.3. Indemnities for Exponential Model.

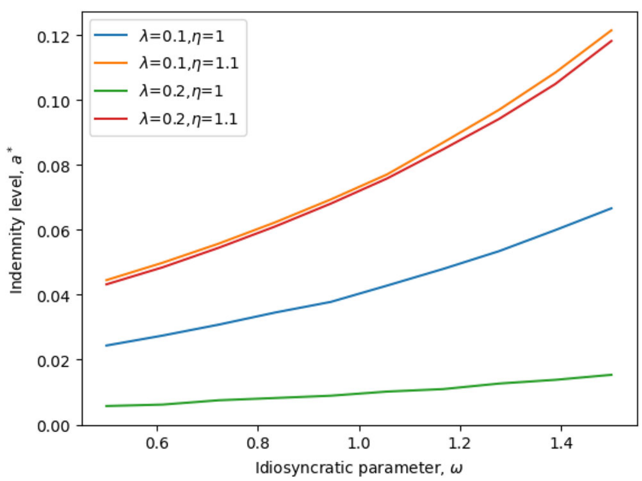

FIGURE 4.4. Indemnities for Lognormal Model.

Upon closer examination, it becomes evident that when λ is smaller, indicating lower risk, the likelihood of the insurance company's involvement increases. Similarly, a smaller value of η, indicating a larger impact of the common shock, leads to a decrease in the indemnity level. These findings align with our observations from the previous example, where higher levels of risk discourage the insurance company from actively participating in risk sharing. Finally, the differences between the common shock and the systematic shock in the models are examined. In Figure 4.1, as $\omega \in [0.1, 0.2]$, which is also the mean of X, increases, the optimal a^* falls in the range *[*0, 0.8*]*. In Figure 4.3, for $\omega \in [0.5, 1.5]$, the optimal a^* is in the range *[*0, 0.025*]*, which is extremely small compared to the value of ω. This shows the difference between the common shock model and the systematic shock model.

5. CONCLUSIONS

This article introduced and studied *static risk measures*. It also discussed how *static risk measures* can be characterized and how they can inherit properties from their underlying risk measures. Some important properties are directly inherited, such as positive homogeneity, subadditivity, law invariance, boundedness, and PSSD, whereas they cannot inherit others, such as comonotone additivity and cash insurance. The optimal reinsurance design was also studied in this setup, in addition to the differences in the reinsurance contracts in the setting in this study and the aggregate loss models. How one can include systematic and common shocks in the model was also discussed. This will yield ex post policies.

DISCLOSURE STATEMENT

No potential conflict of interest was reported by the author.

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6. APPENDIX

6.1. Proof of Theorem 3

- First of all, by Proposition 1, $\mathfrak{S}^{N,\varsigma}$ is PSL. So, by Proposition 1, one can validate the dual representation and that $\Delta_{\mathfrak{S}^{N,\varsigma}}$ is law invariant.
- For the boundedness of $\Delta_{\mathfrak{S}^{N,\varsigma}}$, we need to use Proposition 2. First, because the set $\Delta_{\mathfrak{S}}$ is bounded, we know that $B_{\mathfrak{S}} = \sup_{Z\in\Delta_{\mathfrak{S}}} \|Z\|_q < \infty$. On the other hand, by using Proposition 2, we have $\left\{Y = \sum_{i=1}^N X_i; \|X\|_p \le 1\right\} \subseteq \left\{Y \in L^p; \|Y\|_p \le |N|_p\right\}$. Applying these, we have

$$\begin{aligned}
\sup_{\|X\|_p\leq 1}\left(\sup_{Z\in\Delta_{\varrho^{N,\varsigma}}}E(ZX)\right) &= \sup_{\|X\|_p\leq 1}\varrho^{N,\varsigma}(X)\\
&= \sup_{\|X\|_p\leq 1}\varrho\left(\varsigma\sum_{i=1}^{N}X_i\right)\\
\leq \sup_{\|Y\|_p\leq\|N\|_p}\sup_{Z\in\Delta_\varrho}E(\varsigma ZY) &\leq B_\varrho\times\|\varsigma\|_\infty\times\|N\|_p.
\end{aligned}$$

- Because based on Proposition 1 $6^{N,\varsigma}$ is also monotone, the nonnegativeness of $\Delta_{6^{N,\varsigma}}$ is obvious.
- Now let us focus on the dual representation by identifying $\Delta_{6^{N,\varsigma}}$. First of all, let us introduce

$$I_0=\left\{\sum_{r=1}^{N}Z_r\;\middle|\;\begin{array}{l}Z_r=F_{\varsigma Z}^{-1}(W_r),r\in\mathbb{N},\ \text{for some}\ Z\in\Delta_\varrho\ \text{where}\\(W_r,U)\sim(U,U_r),r\in\mathbb{N},\text{on}\ (P_{\Omega_n},\Omega_n)\\\text{for some}\ U\sim U(0,1)\ \text{and}\\\{U_r\}_{r=1,2,\ldots},i.i.d.\sim U(0,1),N,\varsigma\ \text{all independent}\end{array}\right\}.$$

Take the following three steps:

- **Step 1:** First we need to explain a process that will help us in the proof of steps 2 and 3. Let us consider in a general non-atomic probability space we have an i.i.d. sequence $\{W_r\}_{r=1,2,..}\sim W$ and a random variable V. Then, one can find an i.i.d. sequence $\{U_r\}_{r=1,2,..}$ and U^V of uniform random variables so that $W_r=F_W^{-1}(U_r)$ and $V=F_V^{-1}(U^V)$. Now consider $\{\tilde{U}_r\}_{r=1,2,\ldots}$ so that $(\tilde{U}_r,U^V)\sim(U^V,U_r)$. From this we can construct $\tilde{W}=F_W^{-1}(U^V)$ and $\{\tilde{V}_r=F_V^{-1}(U_r)\}_{r=1,2,\ldots}$. Note that $\{\tilde{V}_r\}_{r=1,2,\ldots}$ is i.i.d.

$$\begin{aligned}
E\left(V\sum_{r=1}^{n}W_r\right) &= \sum_{r=1}^{n}E\left(F_V^{-1}(U^V)F_W^{-1}(U_r)\right)\\
&= \sum_{r=1}^{n}E\left(F_V^{-1}(\tilde{U}_r)F_W^{-1}(U^V)\right)=E\left(\left(\sum_{r=1}^{n}\tilde{V}_r\right)\tilde{W}\right).
\end{aligned}\tag{6.1}$$

This procedure is used mainly on the probability space (Ω_n,P_{Ω_n}) in the following.

- **Step 2:** Let us take $Z\in\Delta_6$ and an i.i.d. sequence $\{X_r\}_{r=1,2,\ldots}\sim X$. Assume that this sequence, N, and ς are independent. Note that the independence of $\{X_r\}_{r=1,2,\ldots}$ and N implies that for each n, $\{1_{\Omega_n}X_r\}_{r=1,2,\ldots}$ is i.i.d. in the probability space (Ω_n,P_{Ω_n}) with the same distribution as X. Let us consider $U_r^n=F_X(X_r)$ in (Ω_n,P_{Ω_n}). By the independence assumption $X_r\sim F_X$ in (Ω_n,P_{Ω_n}), $U_r^n\sim U(0,1)$. Let us check that the sequence $\{U_r\}_{r=1,2,\ldots}$, where $U_r=\sum_{n=1}^{\infty}U_r^n1_{\Omega_n}$, is i.i.d. and uniformly distributed and that $\{U_r\}_{r=1,2,\ldots}$, N, and ς are independent. For that we just need to see the following:

$$\begin{aligned}
&P(U_1\leq u_1,\ldots,U_r\leq u_r,N=n,\varsigma\leq s)\\
&=P(U_1^n\leq u_1,\ldots,U_r^n\leq u_r,N=n,\varsigma\leq s)\\
&=P\left(F_X^{-1}(1_{\Omega_n}X_1)\leq u_1,\ldots,F_X^{-1}(1_{\Omega_n}X_r)\leq u_r,N=n,\varsigma\leq s\right)\\
&=P\left(F_X^{-1}(1_{\Omega_n}X_1)\leq u_1,\ldots,F_X^{-1}(1_{\Omega_n}X_r)\leq u_r,N=n\right)P(\varsigma\leq s)\\
&=P(U_1^n\leq u_1,\ldots,U_r^n\leq u_r,N=n)P(\varsigma\leq s)\\
&=P\left(U_1^n\leq u_1,\ldots,U_r^n\leq u_r|N=n\right)P(N=n)P(\varsigma\leq s)=u_1\times\cdots\times u_r\times P(N=n)P(\varsigma\leq s).
\end{aligned}$$

Let $U=F_{\varsigma Z}(\varsigma Z)$, and construct W_r so that $(W_r,U)\sim(U,U_r)$ on (P_{Ω_n},Ω_n) , $r=1,2,\ldots$. Let $\hat{Z}=\sum_{r=1}^{N}Z_r$, where $Z_r=F_{\varsigma Z}^{-1}(W_r)$. By step 1 one can see that

$$E\left(Z\varsigma\sum_{r=1}^{N}X_r\right)=\sum_{n=1}^{\infty}E\left(Z\varsigma\sum_{r=1}^{n}X_r|\Omega_n)P(\Omega_n)\right.$$

$$= \sum_{n=1}^{\infty} E\left(F_{\varsigma Z}^{-1}(U)\left(\sum_{r=1}^{n} F_X^{-1}(U_r)\right)|\Omega_n\right)P(\Omega_n)$$

$$= \sum_{n=1}^{\infty} E\left(\left(\sum_{r=1}^{n} F_{\varsigma Z}^{-1}(W_r)\right)F_X^{-1}(U)|\Omega_n\right)P(\Omega_n)$$

$$= E\left(\left(\sum_{r=1}^{N} Z_r\right)F_X^{-1}(U)\right).$$

Because $\hat{Z} = \sum_{r=1}^{N} Z_r \in I_0,$ this shows that

$$\varrho^{N,\varsigma}(X) \le \sup_{\tilde{X}\sim X}\left(\sup_{\sum_{r=1}^{N}\tilde{Z}_r \in I_0} E\left(\tilde{X}\sum_{r=1}^{N}\tilde{Z}_r\right)\right) = \sup_{\hat{Z}\sim\sum_{r=1}^{N} Z_r \in I_0} E(\hat{Z}X). \tag{6.2}$$

- **Step 3:** Let us take $X \in L^p$ and $\sum_{r=1}^{N} Z_r \in I_0$. Let us introduce $X_r = F_X^{-1}(U_r)$. By assumption, it is clear that $\{X_r\}_{r=1,2,\dots},$ N, and ς are independent. Let $\hat{X} = F_X^{-1}(U),$ $\bar{Z} = F_{\varsigma Z}^{-1}(U)/\varsigma$ $(W_r, U) \sim (U, U_r)$. Note that because $\bar{Z} \sim Z,$ this implies that $\bar{Z} \in \Delta_6$. On the other hand, by construction and (6.1), we have

$$\begin{aligned} E\left(\left(\sum_{r=1}^{N} Z_r\right)\hat{X}\right) &= \sum_{n=0}^{\infty} E\left(\left(\sum_{r=1}^{n} Z_r\right)\hat{X}|\Omega_n\right)P(\Omega_n) \\ &= \sum_{n=0}^{\infty} E\left(\varsigma\bar{Z}\left(\sum_{r=1}^{n} X_r\right)|\Omega_n\right)P(\Omega_n) \\ &= E\left(\varsigma\bar{Z}\left(\sum_{r=1}^{N} X_r\right)\right). \end{aligned} \tag{6.3}$$

Given that $\{X_r\}_{r=1,2,\dots}$ is i.i.d. and independent of N and $\varsigma,$ the relation above shows

$$\sup_{\hat{Z}\sim\sum_{r=1}^{N} Z_r \in I_0} E(\hat{Z}X) \le \varrho^{N,\varsigma}(X). \tag{6.4}$$

Combining (6.2), (6.4) completes the proof.

6.2. Proof of Theorem 2

The following two lemmas are presented and will be used next. The proof for the first one can be found in Assa and Constantinescu (2021) and the proof of the second one is an immediate result of the first one.

Lemma 1. *If $X_k = 1_{A_k}, k = 1, 2\dots$ where $\{A_k\}_{k=1,2,\dots}$ is a sequence of independent sets, independent of a Poisson process $\{N_t^{\lambda}\}_{t\ge 0}$ with parameter λ, such that $\forall k, P(A_k) = x$, then $Y_t = \sum_{k=1}^{N_t^{\lambda}} 1_{A_k}$ has a Poisson distribution with parameter λTx.*

Lemma 2. *Following the same assumptions and notations of the previous lemma, we have*

$$\begin{cases} \mathrm{VaR}_s(Y) = l, & \text{if } s \in \left(P(N_T^{\lambda x} \le l-1), P(N_T^{\lambda x} \le l)\right] \\ \mathrm{VaR}_s(Y) = 0 & \text{if } s \in \left[0, P(N_T^{\lambda x} = 0)\right] \end{cases} \tag{6.5}$$

Now we prove the theorem by way of contradictions. Let us assume that CVaR_α^S is cash invariant for all $\alpha \in (0,1)$. Then, for any set $A \in \mathcal{F}$, with $P(A) = x$, we must have

$$\mathrm{CVaR}_\alpha^S(1_A + 1) = \mathrm{CVaR}_\alpha^S(1_A) + \mathrm{CVaR}_\alpha^S(1).$$

Using the definition of CVaR_α^S, this means

$$\mathrm{CVaR}_\alpha\left(\sum_{r=1}^{N_T^\lambda} 1_{A_r} + N_T^\lambda\right) = \mathrm{CVaR}_\alpha\left(\sum_{r=1}^{N_T^\lambda} 1_{A_r}\right) + \mathrm{CVaR}_\alpha\left(N_T^\lambda\right), \forall\alpha \in (0,1).$$

By definition of CVaR_α, this implies

$$\int_\alpha^1 \mathrm{VaR}_s\left(\sum_{r=1}^{N_T^\lambda} 1_{A_r} + N_T^\lambda\right) ds = \int_\alpha^1 \mathrm{VaR}_s\left(\sum_{r=1}^{N_T^\lambda} 1_{A_r}\right) ds + \int_\alpha^1 \mathrm{VaR}_s\left(N_T^\lambda\right) ds, \forall\alpha \in (0,1).$$

Taking derivative with respect to α, we get

$$\mathrm{VaR}_s\left(\sum_{r=1}^{N_T^\lambda} 1_{A_r} + N_T^\lambda\right) = \mathrm{VaR}_s\left(\sum_{r=1}^{N_T^\lambda} 1_{A_r}\right) + \mathrm{VaR}_s\left(N_T^\lambda\right), \forall s \in (0,1). \tag{6.6}$$

To have a contradiction, it is shown that for some $s \in (0,1)$ the right-hand side and the left-hand side of this equality cannot hold.

First, we need to look closer at equation (6.6). On both sides of the equality we have values at risk of a random variable that takes value in $\mathbb{N} \cup \{0\}$. Therefore, to find the VaR of such random variables we have to find the intervals over which the commutative distribution function is equal to a member of $\mathbb{N} \cup \{0\}$. More precisely, let us consider X is a random variable taking values in $\mathbb{N} \cup \{0\}$; then to find the values at risk we have to look at the intervals $[0, P(X=0)]$ and $(P(X \le n-1), P(X \le n)]$, for $n = 1, 2...$ Indeed, by definition, $\mathrm{VaR}_\alpha(X) = 0$ if $\alpha \in [0, P(X=0)]$ and $\mathrm{VaR}_\alpha(X) = n$, when $\alpha \in (P(X = n-1), P(X = n)]$.

Now let us go back to our problem, where we wanted to show that the equality in (6.6) cannot hold for some s. Note that

$$P\left(\sum_{r=1}^{N_T^\lambda} 1_{A_r} + N_T^\lambda = 0\right) = P\left(N_T^\lambda = 0\right)$$

and

$$\begin{aligned} &P\left(\sum_{r=1}^{N_T^\lambda} 1_{A_r} + N_T^\lambda = 1\right) = P\left(1_{A_1} = 0 \text{ and } N_T^\lambda = 1\right) \\ &= P(1_{A_1} = 0) P\left(N_T^\lambda = 1\right) = (1-x) P\left(N_T^\lambda = 1\right). \end{aligned}$$

Therefore, we have

$$\mathrm{VaR}_s\left(\sum_{r=1}^{N_T^\lambda} 1_{A_r} + N_T^\lambda\right) = 1$$
$$\text{if } s \in I_1 := \left(P\left(N_T^\lambda = 0\right), P\left(N_T^\lambda = 0\right) + (1-x)P\left(N_T^\lambda = 1\right)\right]$$

and

$$\mathrm{VaR}_s\left(\sum_{r=1}^{N_T^\lambda} 1_{A_r} + N_T^\lambda\right) \geq 2 \quad \text{if } s > P\left(N_T^\lambda = 0\right) + (1-x)P\left(N_T^\lambda = 1\right). \tag{6.7}$$

On the other hand, from Lemmas 1 and 2 we know

$$\begin{cases} \mathrm{VaR}_s\left(\sum_{r=1}^{N_T^\lambda} 1_{A_r}\right) = \mathrm{VaR}_s\left(N_T^{\lambda x}\right) = 1, & \text{if } s \in I_2 := \left(P(N_T^{\lambda x} = 0), P(N_T^{\lambda x} \leq 1)\right] \\ \mathrm{VaR}_s\left(N_T^\lambda\right) = 1, & \text{if } s \in I_3 := \left(P(N_T^\lambda = 0), P(N_T^\lambda \leq 1)\right] \end{cases}. \tag{6.8}$$

Now we want to see how intervals I_1, I_2 and I_3 overlap. First, let us show that $P(N_T^\lambda = 0) + (1-x)P(N_T^\lambda = 1) < P(N_T^{\lambda x} = 0)$. By definition of a Poisson process, we need to show that $e^{-\lambda T} + (1-x)\lambda T e^{-\lambda T} < e^{-\lambda T x}$. But note that this is equivalent to showing that $1 + (1-x)\lambda T < e^{\lambda T(1-x)}$, which, by Taylor's expansion of exponentiation function, obviously holds true. This means the right endpoint of I_1 is smaller than the left endpoint of I_2, so $I_1 \cap I_2 = \emptyset$. On the other hand, if x is very close to 1, $P(N_T^{\lambda x} = 0) < P(N_T^\lambda \leq 1)$. This means that the right endpoint of I_3 is greater than the left endpoint of I_2. Finally, note that the left endpoints of I_1 and I_3 are the same.

All of these facts imply that there exists a non-empty open interval B such that $B \subseteq I_3$, $B > I_1$ and $B < I_2$. For the reader's convenience, how B exists is shown in Figure 6.1.

For any $s \in B$, by (6.7) and (6.8), we have $\mathrm{VaR}_s(\sum_{r=1}^{N_T^\lambda} 1_{A_r} + N_T^\lambda) \geq 2$, $\mathrm{VaR}_s(\sum_{r=1}^{N_T^\lambda} 1_{A_r}) = 0$ and $\mathrm{VaR}_s(N_T^\lambda) = 1$. Therefore,

$$\mathrm{VaR}_s\left(\sum_{r=1}^{N_T^\lambda} 1_{A_r} + N_T^\lambda\right) \geq 2 > 0 + 1 = \mathrm{VaR}_s\left(\sum_{r=1}^{N_T^\lambda} 1_{A_r}\right) + \mathrm{VaR}_s\left(N_T^\lambda\right), \forall s \in B,$$

which contradicts (6.6).

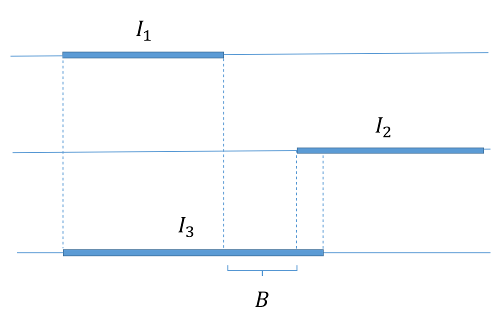

FIGURE 6.1. Illustration of the Intervals I_1, I_2, I_3, and B.

6.3. Proof of Theorem 4

Step 1. Using the representation of $\varrho^{N,\varsigma}$, the optimal problem (4.1) can be written as follows:

$$\inf_{W\in\mathcal{C}(X)}\left\{\sup_{Z\in\Delta_{\varrho^{N,\varsigma}}} E(ZW)+\gamma E(X-W)\right\}. \tag{6.9}$$

First, observe that because $\mathcal{C}(X)$ is a closed, convex, and bounded subset of L^p, by the Banach-Alaoglu theorem,[2] it is weakly compact. On the other hand, by Theorem 3, we know that $\Delta_{\varrho^{N,\varsigma}}$ is also weakly compact in L^q. Theorem 2.132 and proposition 2.105 in Barbu and Precupanu (2012) guarantee that a mini-max problem on linear objective functions and on compact sets always has a saddle point solution. Therefore, the mini-max problem (6.9) has a saddle point solution and therefore

$$\begin{aligned}&\min_{W\in\mathcal{C}(X)}\left\{\sup_{Z\in\Delta_{\varrho^{N,\varsigma}}} E(ZW)+\gamma E(X-W)\right\}\\ &=\max_{Z\in\Delta_{\varrho^{N,\varsigma}}}\left\{\min_{W\in\mathcal{C}(X)}\left\{E(ZW)+\gamma E(X-W)\right\}\right\}.\end{aligned}$$

Let us denote the saddle point (or the solution) with $W^*\in L^p,\ Z^*\in L^q$.

Step 2. In this step, using the result in step 1, a new risk measure $\tilde{\varrho}$ is introduced and it is shown that in the sense of (4.1), their optimal solution is W^*. Let

$$\Delta:=\left\{Z'\in L^q|Z'\sim Z^*\right\}$$

and introduce the risk measure $\tilde{\varrho}$ as follows:

$$\tilde{\varrho}(Y):=\sup_{Z\in\Delta}E(ZY),Y\in L^p.$$

Let us claim that $\Delta\subseteq\Delta_{\varrho^{N,\varsigma}}$. To prove this claim, let $Z\in\Delta$ be taken arbitrarily. By definition, we know $F_Z=F_{Z^*}$. But based on Proposition 3, $\Delta_{\varrho^{N,\varsigma}}$ is law invariant. Knowing this, because $Z^*\in\Delta_{\varrho^{N,\varsigma}}$ and $F_Z=F_{Z^*}$, we get $Z\in\Delta_{\varrho^{N,\varsigma}}$. Because Z was an arbitrary member of Δ, this implies that $\Delta\subseteq\Delta_{\varrho^{N,\varsigma}}$.

Now, because $\Delta\subseteq\Delta_{\varrho^{N,\varsigma}}$ and $Z^*\in\Delta$, we have that W^* is a solution to the following problem:

$$\inf_{W\in\mathcal{C}(X)}\tilde{\varrho}(W)+\gamma E(X-W). \tag{6.10}$$

Step 3. In this step, we find a particular representation for $\tilde{\varrho}$ that will help us to find the form of the optimal solution W^* in step 4.

For any $Y\in L^p$, by the extended Hardy-Littlewood theorem (see theorem A.28 in Föllmer and Schied (2004)),

$$\tilde{\varrho}(Y):=\sup_{\Delta=\{Z'\in L^q|Z'\sim Z^*\}}E(Z'Y)=\int_0^1 \mathrm{VaR}_t(Z^*)\mathrm{VaR}_t(Y)dt.$$

Note that $\tilde{\varrho}$ is positive-homogeneous of degree 1, subadditive, monotone, and law invariant. If the function $\alpha\mapsto\int_0^\alpha \mathrm{VaR}_t(Z^*)dt$ is denoted by $\Phi(\alpha)$, then $d\Phi(t)=\mathrm{VaR}_t(Z^*)dt$. This allows us to rewrite $\tilde{\varrho}$ as follows:

$$\tilde{\varrho}(Y)=\int_0^1 \mathrm{VaR}_t(Y)d\Phi(t),\forall Y\in L^p. \tag{6.11}$$

Step 4. Using the representation of the risk measures $\tilde{\varrho}$ in step 3, in this step we find the form of the optimal solutions.

[2]For the Banach-Alaoglu theorem, see Rudin (1991).

The optimal risk allocation is of the form $(f(X), X - f(X))$, when $f \in \mathcal{C}$. Thus, we can consider the following problem instead of (6.10):

$$\inf_{f \in \mathcal{C}} \tilde{\varrho}(f(X)) + \gamma E(X - f(X)).$$

For any $f \in \mathcal{C}$, using the fact that VaR_t always commutes with nondecreasing functions, we have

$$\begin{aligned} &\tilde{\varrho}(f(X)) + \gamma E(X - f(X)) \\ &\quad = \int_0^1 \mathrm{VaR}_t(f(X)) d\Phi(t) + \gamma E(X - f(X)) \\ &\quad = \int_0^1 f(\mathrm{VaR}_t(X)) d\Phi(t) + \gamma E(X - f(X)). \end{aligned} \tag{6.12}$$

Based on Proposition 3, let us assume that $f(x) = \int_0^x h(s)ds$, for a function $0 \le h \le 1$. Therefore, using these representations for f, in (6.12), we have

$$\begin{aligned} &\tilde{\varrho}(f(X)) + \gamma E(X - f(X)) \\ &\quad = \int_0^1 \int_0^{\mathrm{VaR}_t(X)} h(s) ds d\Phi(t) + \gamma \int_0^1 \int_0^{\mathrm{VaR}_t(X)} (1 - h(s)) ds dt. \end{aligned}$$

By Tonelli's theorem, we have

$$\begin{aligned} &\tilde{\varrho}(f(X)) + \gamma E(X - f(X)) \\ &\quad = \int_0^\infty \left[\int_{F_X(s)}^1 d\Phi(t) h(s) + \gamma \int_{F_X(s)}^1 dt (1 - h(s)) \right] ds \\ &\quad = \int_0^\infty [(\Phi(1) - \Phi(F_X(s))) h(s) + \gamma (1 - F_X(s))(1 - h(s))] ds. \end{aligned} \tag{6.13}$$

Now for every $s \ge 0$ let us look a bit closer at $(\Phi(1) - \Phi(F_X(s)))h(s) + \gamma S_X(s)(1 - h(s))$. For a fixed $s \ge 0$, the minimum of this last expression for $0 \le h \le 1$ is

$$\min\{\Phi(1) - \Phi(F_X(s)), \gamma S_X(s)\}.$$

This makes it clear that the minimum to be attained is

$$h^*(s) = \begin{cases} 1, & \text{if } \Phi(1) - \Phi(F_X(s)) < \gamma S_X(s) \\ 0, & \text{if } \Phi(1) - \Phi(F_X(s)) > \gamma S_X(s). \end{cases}$$

Finally, because $\Phi(1) = \int_0^1 \mathrm{VaR}_t(Z^*) dt = E(Z^*)$ and $\Phi(F_X(t)) = \int_0^{F_X(t)} \mathrm{VaR}_s(Z^*) ds$, we get the following result:

$$h^*(t) = \begin{cases} 1, & \text{if } \int_{F_X(t)}^1 \mathrm{VaR}_s(Z^*) ds < \gamma S_X(t) \\ 0, & \text{if } \int_{F_X(t)}^1 \mathrm{VaR}_s(Z^*) ds > \gamma S_X(t) \end{cases}.$$

This clearly gives

$$h^*(t) = \begin{cases} 1, & \text{if } \mathrm{CVaR}_{F_X(t)}(Z^*) < \gamma \\ 0, & \text{if } \mathrm{CVaR}_{F_X(t)}(Z^*) > \gamma \end{cases}.$$