

PAPER • OPEN ACCESS

## Reductions and degenerate limits of Yang–Baxter maps with $3 \times 3$ Lax matrices

To cite this article: P Adamopoulou *et al* 2025 *J. Phys. A: Math. Theor.* **58** 205202

View the [article online](#) for updates and enhancements.

You may also like

- [Darboux transformations, finite reduction groups and related Yang–Baxter maps](#)  
S Konstantinou-Rizos and A V Mikhailov
- [3D compatible ternary systems and Yang–Baxter maps](#)  
Theodoros E Kouloukas and Vasilios G Papageorgiou
- [Non-Abelian hierarchies of compatible maps, associated integrable difference systems and Yang-Baxter maps](#)  
Pavlos Kassotakis

# Reductions and degenerate limits of Yang–Baxter maps with $3 \times 3$ Lax matrices

P Adamopoulou<sup>1,\*</sup> , T E Kouloukas<sup>2</sup>  and G Papamikos<sup>3</sup> 

<sup>1</sup> Maxwell Institute for Mathematical Sciences and Department of Mathematics, Heriot-Watt University, Edinburgh, United Kingdom

<sup>2</sup> School of Computing and Digital Media, London Metropolitan University, London, United Kingdom

<sup>3</sup> School of Mathematics, Statistics and Actuarial Science, University of Essex, Colchester, United Kingdom

E-mail: [p.adamopoulou@hw.ac.uk](mailto:p.adamopoulou@hw.ac.uk), [t.kouloukas@londonmet.ac.uk](mailto:t.kouloukas@londonmet.ac.uk) and [g.papamikos@essex.ac.uk](mailto:g.papamikos@essex.ac.uk)

Received 30 November 2024; revised 2 April 2025

Accepted for publication 30 April 2025

Published 16 May 2025



CrossMark

## Abstract

We generalise a family of quadrirational parametric Yang–Baxter (YB) maps with  $3 \times 3$  Lax matrices by introducing additional essential parameters. These maps preserve a prescribed Poisson structure which originates from the Sklyanin bracket. We investigate various low-dimensional reductions of this family, as well as degenerate limits with respect to the parameters that were introduced. As a result, we derive several birational YB maps, and we discuss some of their integrability properties. This work is part of a more general classification of YB maps admitting a strong  $3 \times 3$  Lax matrix with a linear dependence on the spectral parameter.

Keywords: Yang–Baxter equation, birational maps, Lax matrices, discrete dynamical systems, symplectic maps, Liouville integrability

\* Author to whom any correspondence should be addressed.



Original Content from this work may be used under the terms of the [Creative Commons Attribution 4.0 licence](https://creativecommons.org/licenses/by/4.0/). Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.

## 1. Introduction

### 1.1. Yang–Baxter (YB) maps and Lax matrices

In [8], the study of set-theoretical solutions to the YB equation [4, 28] was proposed, with such solutions now known as YB maps. A map  $\mathbf{R} : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ , with  $\mathcal{X}$  any set, is called a YB map [6, 26] if it satisfies the set-theoretical YB equation

$$\mathbf{R}^{12} \circ \mathbf{R}^{13} \circ \mathbf{R}^{23} = \mathbf{R}^{23} \circ \mathbf{R}^{13} \circ \mathbf{R}^{12}. \tag{1}$$

$\mathbf{R}^{ij}$ , for  $i \neq j \in \{1, 2, 3\}$ , denotes the action of the map  $\mathbf{R}$  on the  $i$  and  $j$  factors of  $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$  and of the identity map on  $\mathcal{X}$  on the remaining factor, i.e.  $\mathbf{R}^{12} = \mathbf{R} \times id_{\mathcal{X}}$  where  $id_{\mathcal{X}}$  is the identity map over  $\mathcal{X}$ . While in general the set  $\mathcal{X}$  can be any set, here we assume that it is an algebraic variety over a field  $\mathbb{F}$  of characteristic zero. A parametric YB map [26, 27] is a YB map which depends on parameters  $a, b \in \mathcal{P} \subset \mathbb{F}^d$ , acting as  $\mathbf{R} : (\mathcal{X} \times \mathcal{P}) \times (\mathcal{X} \times \mathcal{P}) \rightarrow (\mathcal{X} \times \mathcal{P}) \times (\mathcal{X} \times \mathcal{P})$ , with

$$\mathbf{R}((x, a), (y, b)) = ((u(x, a, y, b), a), (v(x, a, y, b), b)) := (u, v). \tag{2}$$

We will refer to the parameters  $a, b \in \mathcal{P}$  as YB parameters and denote such a YB map simply by  $\mathbf{R}_{a,b} : (x, y) \mapsto (u, v)$ , i.e. as a map from  $\mathcal{X} \times \mathcal{X}$  to itself. The YB maps that we consider are all birational maps. We call a birational map  $(x, y) \rightarrow (u, v)$  quadrirational [2, 19] or non-degenerate if both maps  $u(\cdot, y)$  and  $v(x, \cdot)$  are birational isomorphisms of  $\mathcal{X}$  to itself.

Of particular relevance are YB maps which arise from refactorisation problems of Lax matrices. A matrix  $\mathbf{L}$  depending on  $x \in \mathcal{X}$ ,  $a \in \mathcal{P}$  and another parameter  $\lambda \in \mathbb{F}$ , is called a Lax matrix [25, 26] for a parametric YB map  $\mathbf{R}_{a,b} : (x, y) \mapsto (u, v)$ , if

$$\mathbf{L}(u, a, \lambda) \mathbf{L}(v, b, \lambda) = \mathbf{L}(y, b, \lambda) \mathbf{L}(x, a, \lambda). \tag{3}$$

If the refactorisation problem (3) is equivalent to  $(u, v) = \mathbf{R}_{a,b}(x, y)$ , then  $\mathbf{L}$  is called strong Lax matrix. The maps  $\mathbf{R}_{a,b}$  obtained in this way are all birational but not necessarily quadrirational. In addition, if the equation

$$\mathbf{L}(\hat{x}, a, \lambda) \mathbf{L}(\hat{y}, b, \lambda) \mathbf{L}(\hat{z}, c, \lambda) = \mathbf{L}(x, a, \lambda) \mathbf{L}(y, b, \lambda) \mathbf{L}(z, c, \lambda) \tag{4}$$

implies the unique solution  $\hat{x} = x$ ,  $\hat{y} = y$ ,  $\hat{z} = z$ , then it follows that the map  $\mathbf{R}_{a,b}$  is a YB map [16].

Various YB maps and their generalisations in higher dimensions or over associative algebras (but not necessarily commutative) have been constructed recently, see [1, 5, 7, 10–13, 15, 18, 20] and references therein. In this work, we focus on refactorisation problems of  $3 \times 3$  Lax matrices. Starting from an 18-dimensional YB map, which we call principal parametric YB map, we derive lower-dimensional multi-parametric YB maps via several types of reductions. The obtained maps admit invariant quantities (first-integrals) that Poisson commute with respect to an  $r$ -matrix Poisson structure (Sklyanin bracket). After a degenerate limit is considered, the resulting maps lose their quadrirationality and become birational. These reduced birational maps can be thought of as vectorial and multi-parametric generalisations of the Adler–Yamilov (AY) map which is related to the nonlinear Schrödinger equation [3].

### 1.2. Poisson YB maps with binomial Lax matrices

To construct multidimensional YB maps it is very natural to study the solutions of the refactorisation problem,

$$\mathbf{L}(U, a, \lambda) \mathbf{L}(V, b, \lambda) = \mathbf{L}(Y, b, \lambda) \mathbf{L}(X, a, \lambda) \tag{5}$$

with respect to  $U$  and  $V$  and with Lax matrices  $\mathbf{L}(X, a, \lambda)$  which are first degree polynomials in the spectral parameter  $\lambda$ ,

$$\mathbf{L}(X, a, \lambda) = X - \lambda K_a. \tag{6}$$

Here,  $X, Y, U, V$  are generic elements in  $\mathfrak{gl}_n(\mathbb{F})$ , while  $K : \mathbb{F}^d \rightarrow \text{GL}_n(\mathbb{F})$  is a  $d$ -parametric family of commuting matrices and  $K_a, K_b$  denote the values  $K(a), K(b)$  respectively. In [16, 17], solutions of this refactorisation problem were presented which satisfy the YB equation. We can express these solutions recursively as

$$\begin{aligned} U &= \left( -f_0(X; a)I - \sum_{i=1}^n (-1)^i f_i(X; a)M_{i-1} \right) \left( \sum_{i=1}^n (-1)^i f_i(X; a)M_{i-1} \right)^{-1} K_a, \\ V &= K_a^{-1} (YK_a + K_b X - UK_b), \end{aligned} \tag{7}$$

where  $M_0 = I, N_0 = 0, M_1 = (YK_a + K_b X)K_b^{-1}, N_1 = -YK_b^{-1}K_a$  and

$$M_i = M_1 M_{i-1} + N_1 N_{i-1}, \quad N_i = N_1 M_{i-1}, \quad \text{for } i = 2, \dots, n.$$

Here, the functions  $f_i$ , for  $i = 0, \dots, n$ , are defined by the coefficients of the polynomial  $p_\lambda^a(X) = \det(X - \lambda K_a)$ , by the expression:

$$\begin{aligned} p_\lambda^a(X) &= (-1)^n f_n(X, a) \lambda^n + (-1)^{n-1} f_{n-1}(X, a) \lambda^{n-1} + \dots \\ &\quad + (-1) f_1(X, a) \lambda + f_0(X, a), \end{aligned}$$

with  $f_n(X, a) = \det K_a$  and  $f_0(X, a) = \det X$ .

The solution (7) satisfies the additional conditions

$$f_i(U, a) = f_i(X, a), \quad f_i(V, b) = f_i(Y, b), \quad i = 0, \dots, n, \tag{8}$$

or equivalently the condition  $\det(UK_b - YK_a) \neq 0$  (or  $\det(K_a V - K_b Y) \neq 0$ ). The corresponding map  $\mathcal{R}_{a,b} : (X, Y) \mapsto (U, V)$  defined by (7), is a quadrirational YB map. Furthermore,  $\mathcal{R}_{a,b}$  is a Poisson map with respect to the Sklyanin bracket [23]

$$\{\mathbf{L}(X, a, \lambda_1) \otimes \mathbf{L}(X, a, \lambda_2)\} = \left[ \frac{P}{\lambda_1 - \lambda_2}, \mathbf{L}(X, a, \lambda_1) \otimes \mathbf{L}(X, a, \lambda_2) \right], \tag{9}$$

and  $\{\mathbf{L}(X, a, \lambda_1) \otimes \mathbf{L}(Y, b, \lambda_2)\} = 0$ , where  $P(x \otimes y) = y \otimes x$ . The functions  $f_i$ , along with all elements of  $K_a$  and  $K_b$ , are Casimirs for this Poisson bracket. Hence, the invariant conditions (8) allow us to further reduce the  $2n^2$ -dimensional map  $\mathcal{R}_{a,b}$  to a  $2n(n-1)$ -dimensional symplectic YB map on the level sets

$$\mathcal{C} = \{(X, Y) : f_i(X) = \alpha_i, f_i(Y) = \beta_i, i = 0, \dots, n-1\} \subset \mathfrak{gl}_n(\mathbb{F}) \times \mathfrak{gl}_n(\mathbb{F}), \tag{10}$$

where  $\alpha_i$  and  $\beta_i$  represent additional YB parameters.

In this paper, we study the case where  $K_a$  and  $K_b$  are  $3 \times 3$  diagonal matrices. Our analysis covers all cases of binomial Lax matrices with diagonalisable higher-degree term, as equation (5) remains invariant under conjugation with a constant matrix. We will investigate lower dimensional reductions and specific limits leading to non-quadrirational YB maps.

## 2. YB maps with $3 \times 3$ Lax matrices

### 2.1. The principal parametric YB map

We consider the refactorisation problem (5) for generic  $3 \times 3$  matrices  $X, Y, U, V$  and nonzero diagonal 3-parametric matrices  $K_a, K_b$ . Using the scaling symmetry  $\mathbf{L} \rightarrow r\mathbf{L}$ , with  $r \in \mathbb{F} \setminus \{0\}$ , of equation (5), one can rescale any of the parameters of  $K_a$  and  $K_b$  to 1, without loss of generality. Nevertheless, in this section we keep all parameters in  $K_a, K_b$  arbitrary, as elements of a projective space, and we will use the rescaling when we consider certain reductions in later sections.

We start with the general  $3 \times 3$  Lax matrix of the form  $\mathbf{L}(X, a, \lambda) = X - \lambda K_a$  with  $\lambda \in \mathbb{F}$  and

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \quad K_a = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad (11)$$

where  $X \in \mathfrak{gl}_3(\mathbb{F})$  and  $a = (a_1, a_2, a_3)$  is an element of the  $\mathbb{F}$ -projective plane  $\mathbb{P}^2(\mathbb{F})$ . The Sklyanin bracket (9) implies the following linear Poisson bracket between the variables  $x_{ij}$

$$\{x_{ij}, x_{kl}\} = a_i x_{kj} \delta_{li} - a_j x_{il} \delta_{kj}, \quad (12)$$

while  $\{x_{ij}, a_k\} = 0$ , where  $\delta_{ij}$  denotes the Kronecker delta. This Poisson bracket admits six linearly independent Casimir functions  $a_1, a_2, a_3, f_0, f_1, f_2$ , where  $f_i$  are defined by the coefficients of the polynomial  $p_\lambda^a(X) = \det(X - \lambda K_a)$ , i.e.

$$\begin{aligned} f_0(X, a) &= \det X, \\ f_1(X, a) &= a_3(x_{11}x_{22} - x_{12}x_{21}) + a_2(x_{11}x_{33} - x_{13}x_{31}) + a_1(x_{22}x_{33} - x_{23}x_{32}), \\ f_2(X, a) &= a_2a_3x_{11} + a_1a_3x_{22} + a_1a_2x_{33}, \\ f_3(X, a) &= \det K_a. \end{aligned} \quad (13)$$

In this case, the refactorisation problem (5), implies uniquely an 18-dimensional Poisson YB map

$$\mathbf{R}_{a,b} : (X, Y) \mapsto (U, V)$$

defined by (7), which can be reduced to a 12-dimensional symplectic YB map  $\mathbf{R}_{\bar{a}, \bar{b}}$  on  $\mathcal{C}$ , the intersection  $\cap_{i=0}^2 f_i^{-1}(\alpha_i) \times f_i^{-1}(\beta_i) \subset \mathfrak{gl}_3(\mathbb{F}) \times \mathfrak{gl}_3(\mathbb{F})$ , where  $\alpha_i$  and  $\beta_i$  are additional YB parameters taking values in  $\mathbb{F}$ . Here we denote by  $\bar{a} = ((a_1, a_2, a_3), (\alpha_0, \alpha_1, \alpha_2)) \in \mathbb{P}^2(\mathbb{F}) \times \mathbb{F}^3$  and similarly for  $\bar{b}$ . This parametric family of YB maps  $\mathbf{R}_{\bar{a}, \bar{b}}$ , containing ten effective parameters, constitutes a generalisation of the family derived in proposition 4.4 of [17] where the case  $K_a = K_b = I$  was considered. In the following sections, we show that further lower dimensional reductions of this map are possible.

### 2.2. Reduction to 8-dimensional YB map

The 9-dimensional Poisson manifold  $\mathcal{L} := \{\mathbf{L}(X, a, \lambda) : a \text{ constant}\}$ , equipped with the Sklyanin bracket (12), has rank six. However, the rank of this Poisson structure can be reduced to four by imposing constraints on  $x_{ij}$  that result in the vanishing of matrix minors. We denote

by  $M_{ijk,lmn}$  the minor formed by deleting rows  $i, j, k$  and columns  $l, m, n$  from the Poisson structure matrix  $\{\mathbf{L}(X, a, \lambda_1) \otimes \mathbf{L}(X, a, \lambda_2)\}$  defined in (9). We consider the minors

$$\begin{aligned} M_{789,125} &= -a_1 a_2 a_3^2 \left( a_2 (x_{13}^2 x_{21} - x_{13} x_{11} x_{23}) + a_1 (x_{13} x_{22} x_{23} - x_{12} x_{23}^2) \right)^2, \\ M_{589,127} &= -a_1^2 a_2 a_3 \left( a_3 (x_{12}^2 x_{23} - x_{12} x_{13} x_{22}) + a_2 (x_{13} x_{12} x_{33} - x_{13}^2 x_{32}) \right)^2, \\ M_{478,478} &= -a_1^2 a_2^2 a_3^2 (x_{12} x_{23} x_{31} - x_{13} x_{21} x_{32})^2. \end{aligned} \tag{14}$$

We notice that the system of equations  $M_{789,125} = 0$ ,  $M_{589,127} = 0$ ,  $M_{478,478} = 0$  is linear in  $x_{11}$ ,  $x_{31}$ ,  $x_{32}$ , and for  $x_{13}, x_{23} \neq 0$  implies the solution

$$\begin{aligned} x_{11} &= \frac{x_{13} x_{21}}{x_{23}} + \frac{a_1 (x_{13} x_{22} - x_{12} x_{23})}{a_2 x_{13}}, \\ x_{31} &= \frac{x_{21} x_{33}}{x_{23}} - \frac{a_3 x_{21} x_{22}}{a_2 x_{23}} + \frac{a_3 x_{21} x_{12}}{a_2 x_{13}}, \\ x_{32} &= \frac{x_{12} x_{33}}{x_{13}} - \frac{a_3 x_{12} x_{22}}{a_2 x_{13}} + \frac{a_3 x_{12}^2 x_{23}}{a_2 x_{13}^2}. \end{aligned} \tag{15}$$

Substituting relations (15) in the Casimir functions (13) we obtain the following set of reduced rational Casimirs  $\tilde{f}_i := \iota^* f_i = f_i \circ \iota : \mathbb{F}^6 \rightarrow \mathbb{F}$

$$\begin{aligned} \tilde{f}_0 &= \frac{(x_{13} x_{22} - x_{12} x_{23})^2 (a_2 x_{13} (a_1 x_{23} x_{33} + a_3 x_{13} x_{21}) + a_1 a_3 x_{12} x_{23}^2)}{a_2^2 x_{13}^3 x_{23}}, \\ \tilde{f}_1 &= \frac{(x_{13} x_{22} - x_{12} x_{23}) (2a_2 x_{13} (a_1 x_{23} x_{33} + a_3 x_{13} x_{21}) + a_1 a_3 x_{23} (x_{12} x_{23} + x_{13} x_{22}))}{a_2 x_{13}^2 x_{23}}, \\ \tilde{f}_2 &= a_1 a_2 x_{33} + 2a_1 a_3 x_{22} - \frac{a_1 a_3 x_{12} x_{23}}{x_{13}} + \frac{a_2 a_3 x_{13} x_{21}}{x_{23}}. \end{aligned} \tag{16}$$

Here we define  $\iota$  to be the inclusion map  $\mathbb{F}^6 \hookrightarrow \mathfrak{gl}_3(\mathbb{F})$

$$\iota : (x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{33}) \mapsto X = (x_{ij}),$$

with  $x_{11}, x_{31}, x_{32}$  given by (15). In what follows, we denote  $\mathcal{M} = \text{Img}(\iota)$ , i.e. the set of reduced matrices in  $\mathcal{L}$ , with  $x_{11}, x_{31}, x_{32}$  defined by (15). The following theorem holds:

**Theorem 2.1.**  $\mathcal{M}$  is a Poisson submanifold of  $\mathcal{L}$  of rank four. Furthermore, the discriminant of the cubic polynomial in  $\lambda$  of  $\iota^* p_\lambda^a$  vanishes, i.e.

$$4\tilde{f}_0 \tilde{f}_2^2 - \tilde{f}_1^2 \tilde{f}_2 + 4\tilde{f}_3 \tilde{f}_1^2 - 18\tilde{f}_0 \tilde{f}_1 \tilde{f}_2 \tilde{f}_3 + 27\tilde{f}_3^2 \tilde{f}_0 = 0. \tag{17}$$

**Proof.** By direct computation we can show that for  $x_{13}, x_{23} \neq 0$ , the inclusion map  $\iota$  defined by (15) is Poisson with respect to the Poisson bracket (12) (and the induced bracket on  $\mathcal{M}$ ). Hence,  $\mathcal{M}$  is a Poisson submanifold of  $\mathcal{L}$  and by substituting (15) in (9) reduces the rank of the Poisson structure matrix to four. The coefficients of the pullback  $\iota^* p_\lambda^a$  are the reduced Casimirs  $\tilde{f}_i$  given in (16). The rank of the Jacobian matrix of  $\tilde{f}_0, \tilde{f}_1, \tilde{f}_2$  is equal to two, and therefore there is one functional relation between them. Relation (17) can be obtained using elimination algorithms.  $\square$

As we mentioned in the proof of theorem (2.1), on the submanifold  $\mathcal{M}$  there are two functionally independent Casimirs. Solving the system  $\tilde{f}_1 = \alpha_1, \tilde{f}_2 = \alpha_2$  for  $x_{22}$  and  $x_{33}$  we obtain the following expressions

$$\begin{aligned} x_{22} &= \frac{c_1}{a_3} + \frac{c_2}{a_3} + \frac{x_{12}x_{23}}{x_{13}}, \\ x_{33} &= \frac{c_1}{a_2} - \frac{2c_2}{a_2} - \frac{a_3x_{13}x_{21}}{a_1x_{23}} - \frac{a_3x_{12}x_{23}}{a_2x_{13}}, \end{aligned} \tag{18}$$

where  $c_1$  and  $c_2$  depend on the level sets of the Casimirs and the matrix  $K_a$  as follows:

$$c_1 = \frac{\alpha_2}{3a_1}, \quad c_2 = \pm \frac{\sqrt{\alpha_2^2 - 3\alpha_1 a_1 a_2 a_3}}{3a_1}. \tag{19}$$

Using relations (18) in (15) we obtain the reduced expressions for  $x_{11}, x_{31}, x_{32}$

$$\begin{aligned} x_{11} &= \frac{a_1(c_1 + c_2)}{a_2a_3} + \frac{x_{13}x_{21}}{x_{23}}, \\ x_{31} &= -\frac{3c_2x_{21}}{a_2x_{23}} - \frac{a_3x_{13}x_{21}^2}{a_1x_{23}^2} - \frac{a_3x_{12}x_{21}}{a_2x_{13}}, \\ x_{32} &= -\frac{3c_2x_{12}}{a_2x_{13}} - \frac{a_3x_{12}x_{21}}{a_1x_{23}} - \frac{a_3x_{12}^2x_{23}}{a_2x_{13}^2}. \end{aligned} \tag{20}$$

After the reduction, the Poisson brackets (12) between the remaining variables  $x_{12}, x_{13}, x_{21}, x_{23}$  take the form

$$\begin{aligned} \{x_{13}, x_{21}\} &= a_1x_{23}, \quad \{x_{12}, x_{21}\} = a_1 \frac{x_{12}x_{23}}{x_{13}} - a_2 \frac{x_{13}x_{21}}{x_{23}}, \\ \{x_{12}, x_{23}\} &= -a_2x_{13}, \end{aligned} \tag{21}$$

with all other brackets vanishing. Similarly, after the reduction (18)–(20) the Lax matrix (6) takes the form  $\tilde{X} - \lambda K_a$  where  $\tilde{X}$  is given by

$$\begin{pmatrix} \frac{a_1(c_1+c_2)}{a_2a_3} + \frac{x_{13}x_{21}}{x_{23}} & x_{12} & x_{13} \\ x_{21} & \frac{c_1+c_2}{a_3} + \frac{x_{12}x_{23}}{x_{13}} & x_{23} \\ -\frac{3c_2x_{21}}{a_2x_{23}} - \frac{a_3x_{13}x_{21}^2}{a_1x_{23}^2} - \frac{a_3x_{12}x_{21}}{a_2x_{13}} & -\frac{3c_2x_{12}}{a_2x_{13}} - \frac{a_3x_{12}x_{21}}{a_1x_{23}} - \frac{a_3x_{12}^2x_{23}}{a_2x_{13}^2} & \frac{c_1-2c_2}{a_2} - \frac{a_3x_{13}x_{21}}{a_1x_{23}} - \frac{a_3x_{12}x_{23}}{a_2x_{13}} \end{pmatrix}.$$

The change of variables

$$x_{12} = -a_2x_2X_1, \quad x_{13} = X_1, \quad x_{21} = -a_1x_1X_2, \quad x_{23} = X_2, \tag{22}$$

brings the brackets (21) to the canonical form, i.e.

$$\{x_1, X_1\} = 1, \quad \{x_2, X_2\} = 1, \quad \{x_1, x_2\} = 0, \quad \{X_1, X_2\} = 0, \tag{23}$$

while the reduced Lax matrix  $\tilde{X} - \lambda K_a$  takes the form

$$\begin{aligned} L(\mathbf{x}, \mathbf{X}, p, \lambda) &= \begin{pmatrix} \frac{a_1(c_1+c_2)}{a_2a_3} - a_1x_1X_1 - \lambda a_1 & -a_2x_2X_1 & X_1 \\ -a_1x_1X_2 & \frac{c_1+c_2}{a_3} - a_2x_2X_2 - \lambda a_2 & X_2 \\ \frac{3a_1c_2x_1}{a_2} - a_1a_3x_1(x_1X_1 + x_2X_2) & 3c_2x_2 - a_2a_3x_2(x_1X_1 + x_2X_2) & \frac{c_1-2c_2}{a_2} + a_3(x_1X_1 + x_2X_2) - \lambda a_3 \end{pmatrix}, \end{aligned} \tag{24}$$

where  $\mathbf{x} = (x_1, x_2)$ ,  $\mathbf{X} = (X_1, X_2)$  and  $p = (c_1, c_2, a_1, a_2, a_3)$ .

The matrix refactorisation problem

$$L(\mathbf{u}, \mathbf{U}, p, \lambda) L(\mathbf{v}, \mathbf{V}, q, \lambda) = L(\mathbf{y}, \mathbf{Y}, q, \lambda) L(\mathbf{x}, \mathbf{X}, p, \lambda), \quad (25)$$

with  $q = (d_1, d_2, b_1, b_2, b_3)$  has a unique solution for  $\mathbf{u}, \mathbf{U}, \mathbf{v}, \mathbf{V}$  in terms of  $\mathbf{x}, \mathbf{X}, \mathbf{y}, \mathbf{Y}$  given by

$$\begin{aligned} (u_1, u_2) &= \frac{b_3}{a_3} (y_1, y_2) + \frac{b_3 C_1}{a_3 D_1} \left( \frac{a_3}{b_1} x_1 - y_1, \frac{a_3}{b_2} x_2 - y_2 \right), \\ (v_1, v_2) &= \left( \frac{a_1}{b_1} x_1, \frac{a_2}{b_2} x_2 \right) - b_1 \frac{C_2}{D_2} \left( a_1 \left( \frac{a_3}{b_1} x_1 - y_1 \right), a_2 \left( \frac{a_3}{b_2} x_2 - y_2 \right) \right), \end{aligned} \quad (26)$$

where

$$\begin{aligned} C_1 &= a_2 a_3 b_1 (d_1 - 2d_2) - b_1 b_2 b_3 (c_1 - 2c_2), \\ C_2 &= \frac{b_2 b_3}{a_3} (c_1 + c_2) - a_2 (d_1 + d_2), \\ D_1 &= a_2 a_3 b_2 b_3 (a_3 x_1 - b_1 y_1) Y_1 + a_2 a_3 b_1 b_3 (a_3 x_2 - b_2 y_2) Y_2 + C_1 + 3a_2 a_3 b_1 d_2, \\ D_2 &= a_2 b_1 b_2 b_3 (a_3 x_1 - b_1 y_1) X_1 + a_2 b_1 b_2 b_3 (a_3 x_2 - b_2 y_2) X_2 - C_1 - 3a_2 a_3 b_1 d_2, \end{aligned} \quad (27)$$

and

$$\begin{aligned} U_1 &= \frac{(a_1 x_1 - b_1 v_1) X_1 + (a_1 y_1 - a_3 v_1) Y_1}{a_1 u_1 - b_3 v_1}, \quad U_2 = \frac{(a_2 x_2 - b_2 v_2) X_2 + (a_2 y_2 - a_3 v_2) Y_2}{a_2 u_2 - b_3 v_2}, \\ V_1 &= \frac{(b_1 u_1 - b_3 x_1) X_1 + (a_3 u_1 - b_3 y_1) Y_1}{a_1 u_1 - b_3 v_1}, \quad V_2 = \frac{(b_2 u_2 - b_3 x_2) X_2 + (a_3 u_2 - b_3 y_2) Y_2}{a_2 u_2 - b_3 v_2}. \end{aligned} \quad (28)$$

Finally, by direct computation, we can prove the following theorem:

**Theorem 2.2** *The map*

$$R_{p,q} : ((x_1, x_2, X_1, X_2), (y_1, y_2, Y_1, Y_2)) \mapsto ((u_1, u_2, U_1, U_2), (v_1, v_2, V_1, V_2)), \quad (29)$$

with  $u_i, U_i, v_i, V_i$  defined by (26)–(28) is a parametric quadrirational YB map with strong Lax matrix (24). Furthermore,  $R_{p,q}$  is symplectic with respect to<sup>4</sup>

$$\omega = dx_1 \wedge dX_1 + dx_2 \wedge dX_2 + dy_1 \wedge dY_1 + dy_2 \wedge dY_2.$$

The map (29) admits four functionally independent invariants which can be obtained from the characteristic polynomial of the monodromy matrix  $L(\mathbf{y}, \mathbf{Y}, q, \lambda) L(\mathbf{x}, \mathbf{X}, p, \lambda)$ . The two simplest invariants obtained in this way are

$$I_1 = x_1 X_1 + y_1 Y_1, \quad I_2 = x_2 X_2 + y_2 Y_2. \quad (30)$$

**Remark 2.3.** For  $a_i = b_i = 1$  and particular choices of the parameters  $c_i, d_i$ , the YB map (29) reduces to a non-degenerate Boussinesq and Goncharenko–Veselov map [9, 17].

<sup>4</sup> The operation  $\wedge$  denotes the wedge product between 1-forms, i.e  $dx \wedge dy(\mathbf{u}, \mathbf{v}) = dx(\mathbf{u})dy(\mathbf{v}) - dy(\mathbf{u})dx(\mathbf{v})$ , where  $\mathbf{u}, \mathbf{v}$  are vector fields.

2.3. Reduction to a 4-dimensional YB map

In this section, we consider a further folding reduction of the YB map (29) on an invariant manifold. We impose the relations

$$\begin{aligned} x_1 = x_2 := x, \quad X_1 = X_2 := X, \quad a_1 = a_2 := a, \\ y_1 = y_2 := y, \quad Y_1 = Y_2 := Y, \quad b_1 = b_2 := b, \end{aligned} \tag{31}$$

which are consistent with the map since relations (26)–(28) imply

$$u_1 = u_2, \quad U_1 = U_2, \quad v_1 = v_2, \quad V_1 = V_2.$$

Thus, for  $p = (c_1, c_2, a, a_3)$  and  $q = (d_1, d_2, b, b_3)$ , the map (29) is reduced to a 4-dimensional quadrirational YB map

$$\mathbb{R}_{p,q} : ((x, X), (y, Y)) \mapsto ((u, U), (v, V))$$

on the invariant manifold

$$\mathcal{N} = \{((x, x, X, X), (y, y, Y, Y)) : x, X, y, Y \in \mathbb{F}\},$$

where

$$\begin{aligned} u = \frac{b_3}{a_3}y + \frac{b_3}{a_3} \frac{\tilde{C}_1}{\tilde{D}_1} \left(\frac{a_3}{b}x - y\right), \quad v = \frac{a}{b}x - ab \frac{\tilde{C}_2}{\tilde{D}_2} \left(\frac{a_3}{b}x - y\right), \\ U = \frac{(ax - bv)X + (ay - a_3v)Y}{au - b_3v}, \quad V = \frac{(bu - b_3x)X + (a_3u - b_3y)Y}{au - b_3v}, \end{aligned} \tag{32}$$

and  $\tilde{C}_i, \tilde{D}_i$  are obtained from (27) under the reduction (31).

The reduced symplectic structure on  $\mathcal{N}$  is

$$\omega = dx \wedge dX + dy \wedge dY,$$

and  $\mathbb{R}_{p,q}$  is symplectic with respect to  $\omega$ . Furthermore, map  $\mathbb{R}_{p,q}$  admits the reduced Lax matrix

$$\mathbb{L}(x, X, p, \lambda) = \begin{pmatrix} \frac{c_1+c_2}{a_3} - axX - \lambda a & -axX & X \\ -axX & \frac{c_1+c_2}{a_3} - axX - \lambda a & X \\ 3c_2x - 2aa_3x^2X & 3c_2x - 2aa_3x^2X & \frac{c_1-2c_2}{a} + 2a_3xX - \lambda a_3 \end{pmatrix}, \tag{33}$$

which can be obtained from (24) by imposing the reduction (31).

Using the trace of the monodromy matrix associated to Lax matrix (33) we obtain two functionally independent invariants of the map (32). One of the invariants is  $\mathbb{I}_1 = xX + yY$ , which is  $I_1$  (or  $I_2$ ) under the reduction (31), while the other has the form

$$\mathbb{I}_2 = a_{11}^{00} xX + a_{00}^{11} yY + a_{10}^{01} xY + a_{01}^{10} XY + XY (a_{21}^{01} x^2 + a_{01}^{21} y^2 + a_{11}^{11} xy), \tag{34}$$

where the coefficients  $a_{ij}^{kl}$  of the monomials  $x^i X^j y^k Y^l$  depend on the parameters of the map (32). For simplicity, we have omitted the exact dependence of the coefficients  $a_{ij}^{kl}$  on the parameters of the map. Moreover, the invariants  $\mathbb{I}_1$  and  $\mathbb{I}_2$  Poisson commute with respect to the canonical Poisson structure  $\{x, X\} = 1$  and  $\{y, Y\} = 1$  and therefore the map (32) is a 4-dimensional symplectic quadrirational map that is also integrable in the Liouville sense.

### 3. Degenerate limits

#### 3.1. A single degenerate limit

We are interested in studying zero limits for certain parameters involved in the maps derived in section 2, which effectively result in degenerations for the maps introduced in [17]. We call such limits degenerate. In particular, in this section we focus on the degenerate limit  $a_3 \rightarrow 0$ .

We consider the limit  $a_3 \rightarrow 0$  of expressions (18). For both values of  $c_2$  given in (19), the level set  $\alpha_2$  of the Casimir  $f_2$  can be chosen so that the branch of the square root is such that the limit  $a_3 \rightarrow 0$  of  $x_{22}$  results in a unique well-defined expression. The same limit of  $x_{33}$  is regular. The obtained expressions are the following

$$\lim_{a_3 \rightarrow 0} x_{22} = \frac{a_2 \alpha_1}{2\alpha_2} + \frac{x_{12} x_{23}}{x_{13}}, \quad \lim_{a_3 \rightarrow 0} x_{33} = \frac{\alpha_2}{a_1 a_2}. \quad (35)$$

Substituting formulas (35) in the expressions (15) for  $x_{11}, x_{31}, x_{32}$  and taking the now regular limit  $a_3 \rightarrow 0$  we obtain

$$x_{11} = \frac{a_1 \alpha_1}{2\alpha_2} + \frac{x_{13} x_{21}}{x_{23}}, \quad x_{31} = \frac{\alpha_2 x_{21}}{a_1 a_2 x_{23}}, \quad x_{32} = \frac{\alpha_2 x_{12}}{a_1 a_2 x_{13}}. \quad (36)$$

Hence, in the limit  $a_3 \rightarrow 0$  the Lax matrix  $\tilde{X} - \lambda K_a$  takes the form

$$\lim_{a_3 \rightarrow 0} (\tilde{X} - \lambda K_a) = \begin{pmatrix} \frac{a_1 \alpha_1}{2\alpha_2} + \frac{x_{13} x_{21}}{x_{23}} - \lambda a_1 & x_{12} & x_{13} \\ x_{21} & \frac{a_2 \alpha_1}{2\alpha_2} + \frac{x_{12} x_{23}}{x_{13}} - \lambda a_2 & x_{23} \\ \frac{\alpha_2 x_{21}}{a_1 a_2 x_{23}} & \frac{\alpha_2 x_{12}}{a_1 a_2 x_{13}} & \frac{\alpha_2}{a_1 a_2} \end{pmatrix}. \quad (37)$$

The change of variables (22) is not affected by the limit  $a_3 \rightarrow 0$ , and in the variables  $x_1, x_2, X_1, X_2$  the Lax matrix (37) takes the form

$$\tilde{L}(x, X, p, \lambda) = \begin{pmatrix} \frac{a_1 \alpha_1}{2\alpha_2} - a_1 x_1 X_1 - \lambda a_1 & -a_2 x_2 X_1 & X_1 \\ -a_1 x_1 X_2 & \frac{a_2 \alpha_1}{2\alpha_2} - a_2 x_2 X_2 - \lambda a_2 & X_2 \\ -\frac{\alpha_2}{a_2} x_1 & -\frac{\alpha_2}{a_1} x_2 & \frac{\alpha_2}{a_1 a_2} \end{pmatrix}, \quad (38)$$

with  $p = (\alpha_1, \alpha_2, a_1, a_2)$ . The refactorisation problem associated to the Lax matrix (38) results in the following birational (but non quadrirational) YB map  $\tilde{R}_{p,q} : (x_1, x_2, X_1, X_2, y_1, y_2, Y_1, Y_2) \rightarrow (u_1, u_2, U_1, U_2, v_1, v_2, V_1, V_2)$  where

$$\begin{aligned} u_i &= \frac{a_1 a_2 \beta_2}{b_1 b_2 \alpha_2} y_i, \\ U_i &= \frac{b_1 b_2 \alpha_2}{a_1 a_2 \beta_2} Y_i - \frac{b_1 b_2 b_i k}{a_1 a_2 \beta_2 (b_1 y_1 X_1 + b_2 y_2 X_2) - \alpha_2 \beta_2} X_i, \\ v_i &= \frac{a_i}{b_i} x_i + \frac{a_1 a_2 a_i k}{a_1 a_2 \alpha_2 (b_1 y_1 X_1 + b_2 y_2 X_2) - \alpha_2^2} y_i \\ V_i &= \frac{b_i}{a_i} X_i, \end{aligned} \quad (39)$$

for  $i = 1, 2$  and with  $k = \frac{\alpha_2 \beta_1 - \alpha_1 \beta_2}{2\beta_2}$ . This 8-dimensional YB map admits four functionally independent polynomial invariants. Two of these are given in (30), which remain unaffected by the limit  $a_3 \rightarrow 0$ . Hence, we have  $\tilde{I}_1 = x_1 X_1 + y_1 Y_1$ ,  $\tilde{I}_2 = x_2 X_2 + y_2 Y_2$ . Additionally, from the spectrum of the monodromy matrix, we obtain the following two invariants:

$$\begin{aligned} \tilde{I}_3 &= -a_1^2 a_2 b_1^2 b_2 (\alpha_2 \beta_1 x_1 X_1 + \alpha_1 \beta_2 y_1 Y_1) - a_1 a_2^2 b_1 b_2^2 (\alpha_2 \beta_1 x_2 X_2 + \alpha_1 \beta_2 y_2 Y_2) \\ &\quad + 2\alpha_2 \beta_2 (b_1 b_2 (a_1 x_1 Y_1 + a_2 x_2 Y_2) - \beta_2) (a_1 a_2 (b_1 y_1 X_1 + b_2 y_2 X_2) - \alpha_2), \end{aligned} \quad (40)$$

and

$$\begin{aligned} \tilde{I}_4 = & a_1 a_2 b_1 b_2 (\alpha_2 \beta_1 (x_1 X_1 + x_2 X_2) + \alpha_1 \beta_2 (y_1 Y_1 + y_2 Y_2)) + 2\alpha_2^2 \beta_2 (b_2 x_1 Y_1 + b_1 x_2 Y_2) \\ & + 2\alpha_2 \beta_2^2 (a_2 y_1 X_1 + a_1 y_2 X_2) - 2a_1 a_2 b_1 b_2 \alpha_2 \beta_2 (x_1 X_1 + x_2 X_2) (y_1 Y_1 + y_2 Y_2). \end{aligned} \quad (41)$$

The folding reduction (31) can also be applied to the map (39). Indeed, in this case the map will simplify to the four dimensional YB map  $\tilde{\mathbb{R}}_{a,b} : (x, X, y, Y) \rightarrow (u, U, v, V)$  given by

$$\begin{aligned} u = \frac{a^2 \beta_2}{b^2 \alpha_2} y, \quad U = \frac{b^2 \alpha_2}{a^2 \beta_2} Y - \frac{b^3 k}{2a^2 b \beta_2 y X - \alpha_2 \beta_2} X, \\ V = \frac{b}{a} X, \quad v = \frac{a}{b} x + \frac{a^3 k}{2a^2 b \alpha_2 y X - \alpha_2^2} y. \end{aligned} \quad (42)$$

The invariants of map (39) will reduce to invariants of map  $\tilde{\mathbb{R}}_{a,b}$  given in (42). Under the folding reduction, both invariants  $\tilde{I}_1, \tilde{I}_2$  will become equal to  $\tilde{\mathbb{I}} = xX + yY$ , and similarly both invariants  $\tilde{I}_3$  and  $\tilde{I}_4$  will take the same form (up to a scaling factor), resulting in the following invariant

$$\begin{aligned} \tilde{\mathbb{J}} = & a^2 b^2 (\alpha_2 \beta_1 xX + \alpha_1 \beta_2 yY) + 2\alpha_2 \beta_2 (b\alpha_2 xY + a\beta_2 yX) \\ & - 4a^2 b^2 \alpha_2 \beta_2 xXyY. \end{aligned} \quad (43)$$

Both invariants  $\tilde{\mathbb{I}}, \tilde{\mathbb{J}}$  can be obtained from the trace of the monodromy associated to the Lax matrix (38) after applying the reduction (31). The invariants  $\tilde{\mathbb{I}}$  and  $\tilde{\mathbb{J}}$  Poisson commute therefore the map (42) is symplectic and Liouville integrable.

**Remark 3.1.** The 4-dimensional map (42) can also be obtained from the refactorisation of the Lax (33) by taking the limit  $a_3 \rightarrow 0$ .

**Remark 3.2.** The choice of parameters

$$a = b = 1, \quad \alpha_2 = \beta_2 = -1, \quad \alpha_1 = 2A, \quad \beta_1 = 2B,$$

results in the map

$$u = y, \quad U = Y - \frac{A - B}{1 + 2yX} X, \quad v = x + \frac{A - B}{1 + 2yX} y, \quad V = X, \quad (44)$$

which after the rescaling of the variables  $x, y, u, v$  by a factor of  $\frac{1}{2}$  and the flip  $x \leftrightarrow X, y \leftrightarrow Y, u \leftrightarrow U$  and  $v \leftrightarrow V$  becomes the well-known AY map [3].

### 3.2. A double limit to a vectorial AY map

In this section we aim to demonstrate that a vectorial generalisation of the AY map together with its standard Lax matrix, can be obtained from the Lax matrices that we study after taking the double limit  $a_2 \rightarrow 0$  and  $a_3 \rightarrow 0$  in  $K_a$ . Since in this case the only nonzero parameter of  $K_a$  is  $a_1$ , without loss of generality we can rescale it to one. Therefore, in this section we have that  $K_a = e_{11}$ .

The double limit  $(a_2, a_3) \rightarrow (0, 0)$  of the Casimir  $f_2$  in (13) exists and implies that  $f_2 = \alpha_2 \equiv 0$ . This means that the iterated single limits commute. Taking first the limit of  $f_2$  when  $a_2 \rightarrow 0$ , we obtain that  $a_3 x_{22} = \alpha_2$ , while if we take first the limit  $a_3 \rightarrow 0$  it follows that  $a_2 x_{33} = \alpha_2$ . Hence, when  $a_2, a_3 \rightarrow 0$ , both  $x_{22}$  and  $x_{33}$  have to be constants. The case  $x_{22} = x_{33} = 0$  leads to a

trivial refactorisation problem, therefore we consider the case  $x_{22} = x_{33} = 1$  which occurs only when  $a_2 = a_3 = \epsilon \rightarrow 0$ . If we restrict on the level set  $\alpha_1 = 1$  of the Casimir  $f_1$ , then we obtain the constraint  $x_{23}x_{32} = 0$ . We assume that  $x_{23} = x_{32} = 0$ . Finally, under the above assumptions, the Casimir  $f_0$  leads to the following constraint

$$f_0 := x_{11} - x_{12}x_{21} - x_{13}x_{31} = a, \tag{45}$$

which we use to solve for  $x_{11}$ .

Taking into account all the above, the reduced Lax matrix is of the form

$$\mathfrak{L}(\mathbf{x}, \mathbf{X}, a, \lambda) = \begin{pmatrix} a - \lambda + x_1X_1 + x_2X_2 & x_1 & x_2 \\ X_1 & 1 & 0 \\ X_2 & 0 & 1 \end{pmatrix}, \tag{46}$$

where we use the notation  $x_{ij} = x_{j-1}$  and  $x_{i1} = X_{i-1}$  for  $i, j = 2, 3$ . The refactorisation problem associated to the Lax matrix (46) implies uniquely the non-quadrirational YB map  $\mathfrak{R}_{a,b} : (\mathbf{x}, \mathbf{X}, \mathbf{y}, \mathbf{Y}) \rightarrow (\mathbf{u}, \mathbf{U}, \mathbf{v}, \mathbf{V})$

$$\begin{aligned} \mathbf{u} &= \mathbf{y} - \frac{a-b}{1 + \langle \mathbf{x}, \mathbf{Y} \rangle} \mathbf{x}, & \mathbf{U} &= \mathbf{Y}, \\ \mathbf{V} &= \mathbf{X} + \frac{a-b}{1 + \langle \mathbf{x}, \mathbf{Y} \rangle} \mathbf{Y}, & \mathbf{v} &= \mathbf{x}, \end{aligned} \tag{47}$$

where  $\langle \cdot, \cdot \rangle$  is the standard bilinear form in  $\mathbb{F}^2$ . From the spectrum of the monodromy matrix we obtain the usual invariants (30)

$$\mathfrak{I}_1 = x_1X_1 + y_1Y_1, \quad \mathfrak{I}_2 = x_2X_2 + y_2Y_2, \tag{48}$$

together with

$$\mathfrak{I}_3 = b \langle \mathbf{x}, \mathbf{X} \rangle + a \langle \mathbf{y}, \mathbf{Y} \rangle + \langle \mathbf{x}, \mathbf{Y} \rangle + \langle \mathbf{X}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{X} \rangle \langle \mathbf{y}, \mathbf{Y} \rangle. \tag{49}$$

Moreover, it can be verified by a direct calculation that the determinants  $x_1y_2 - x_2y_1, X_1Y_2 - X_2Y_1$  are anti-invariants, and therefore their product

$$\mathfrak{I}_4 = (x_1y_2 - x_2y_1)(X_1Y_2 - X_2Y_1) \tag{50}$$

is another invariant of the map (47).

**Theorem 3.3.** *The YB map  $\mathfrak{R}_{a,b}$  in (47) is Liouville integrable and symplectic with respect to the canonical symplectic structure.*

**Proof.** The Jacobian matrix of the invariants  $\mathfrak{I}_1, \dots, \mathfrak{I}_4$  has full rank and therefore the invariants are functionally independent. Moreover, the invariants commute with respect to the Poisson bracket

$$\{x_i, X_i\} = \delta_{ij}, \quad \{y_i, Y_i\} = \delta_{ij}$$

implied by the Sklyanin bracket (9). The map (47) preserves the Poisson bracket and therefore is symplectic with respect to the corresponding canonical symplectic structure

$$\omega = \sum_{i=1}^2 dx_i \wedge dX_i + dy_i \wedge dY_i.$$

□

**Remark 3.4.** The YB map  $\mathfrak{R}_{a,b}$  in (47) together with its Lax matrix (46) has an obvious generalisation to  $n$ -dimensional  $\mathbb{F}$ -vectors. The integrability properties of the map can be generalised to arbitrary dimensions in a similar way as the one described in theorem (3.3). The  $n$ -dimensional map has been derived in [13].

The reduction (31) to the invariant manifold  $\mathcal{N}$  presented in section (2.3) can also be applied in this case, and reduces the map  $\mathfrak{R}_{a,b}$  in (47) to the standard AY map [3]. Moreover, map (47) can be obtained from the YB map  $\tilde{R}_{p,q}$  in (39) after the following choice for the parameters

$$a_1 = b_1 = 1, a_2 = b_2 = 1, \alpha_2 = \beta_2 = -1, \alpha_1 = 2a, \beta_1 = 2b,$$

and the permutation in the variables

$$\mathbf{x} \leftrightarrow \mathbf{X}, \mathbf{y} \leftrightarrow \mathbf{Y}, \mathbf{u} \leftrightarrow \mathbf{U}, \mathbf{v} \leftrightarrow \mathbf{V}.$$

This implies that the vAY map (47) admits the Lax matrix

$$\tilde{L}(\mathbf{X}, \mathbf{x}, a) = \begin{pmatrix} x_1 X_1 + a + \lambda & x_1 X_2 & -x_1 \\ x_2 X_1 & x_2 X_2 + a + \lambda & -x_2 \\ -X_1 & -X_2 & 1 \end{pmatrix}. \tag{51}$$

This Lax matrix is equivalent to the Lax matrix (46) as it can be directly obtained from (46) by inversion and a similarity transformation by a permutation matrix.

#### 4. Conclusions

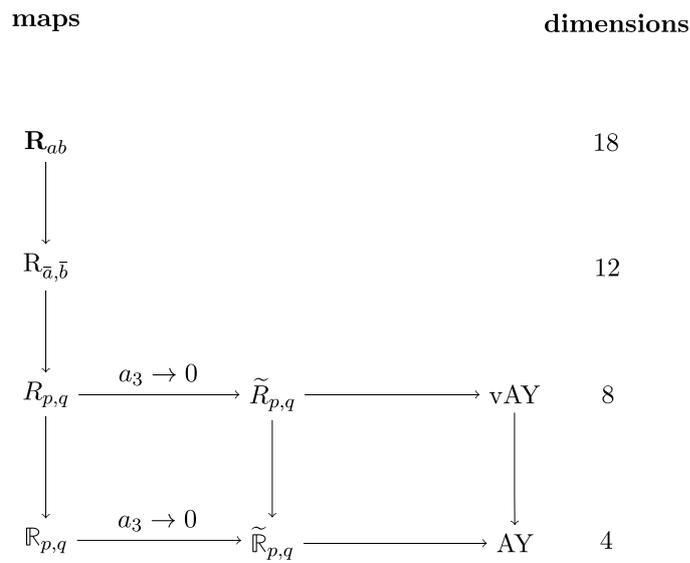
In this paper we studied reductions and degenerations of a family of YB maps with  $3 \times 3$  first-degree polynomial Lax matrices. We also introduced compatible Poisson structures associated with the Sklyanin bracket and invariant conditions, ensuring the Liouville integrability of all the presented maps.

YB maps serve as fundamental building blocks for constructing higher-dimensional discrete integrable systems. As it was shown in [26, 27], each YB map generates an hierarchy of commuting  $n$ -dimensional *transfer maps* which share the same integrals. A different variant of transfer maps occurs by considering periodic initial value problems on lattices, in correspondence with staircase initial value problems of integrable lattice equations [21, 22]. While the dynamics of an individual YB map sometimes may be trivial (e.g. involution), the dynamics of the associated transfer maps display highly non-trivial behaviour.

All variations of transfer maps preserve the spectrum of their corresponding monodromy matrices, constituting of products of Lax matrices. In this setting, YB maps with Lax matrices compatible with the Sklyanin bracket, as the maps presented in this paper, hold a significant advantage as they generate Poisson transfer maps with commutative integrals derived from the spectrum of the corresponding monodromy matrices.

The family of the YB maps presented in this work includes all cases involving  $3 \times 3$  Lax matrices linear in the spectral parameter and with the degree one coefficient being a constant and diagonalisable matrix. The connection between the maps that we obtained can be summarised in figure 1. The arrows in the graph indicate a connection between the maps which can be a reduction, a limit, a choice of parameters, or a simple transformation in the dynamical variables that is a symmetry of the YB equation.

In future research, we aim to study families of YB maps and corresponding degenerations, with  $3 \times 3$  Lax matrices associated with different Jordan forms of the highest-degree terms. This will complete the classification of  $3 \times 3$  binomial Lax matrices under conjugation.



**Figure 1.** Connections between the various maps presented in previous sections.

Furthermore, we intend to include the 3-dimensional consistent lattice equations associated with all these maps and investigate their integrability features.

**Data availability statement**

No new data were created or analysed in this study.

**ORCID iDs**

P Adamopoulou <https://orcid.org/0000-0002-0151-069X>

T E Kouloukas <https://orcid.org/0000-0002-9903-6788>

G Papamikos <https://orcid.org/0000-0001-5523-3627>

**References**

- [1] Adamopoulou P, Konstantinou-Rizos S and Papamikos G 2021 Integrable extensions of the Adler map via Grassmann algebras *Theor. Math. Phys.* **207** 553–9
- [2] Adler V E, Bobenko A I and Suris Y 2004 Geometry of the Yang-Baxter maps: pencils of conics and quadrirational mappings *Commun. Anal. Geom.* **12** 967–1008
- [3] Adler V E and Yamilov R I 1994 Explicit auto-transformations of integrable chains *J. Phys. A: Math. Gen.* **27** 447–92
- [4] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (Academic)
- [5] Bobenko A I and Suris Y 2002 Integrable noncommutative equations on quad-graphs. The consistency approach *Lett. Math. Phys.* **61** 241–54
- [6] Bukhshtaber V M 1998 The Yang-Baxter transformation *Russ. Math. Surv.* **53** 1343
- [7] Doliwa A 2014 Non-commutative rational Yang–Baxter maps *Lett. Math. Phys.* **104** 299–309
- [8] Drinfeld V G 1992 On some unsolved problems in quantum group theory *Quantum Groups (Lecture Notes in Mathematics)* vol 1510, ed P P Kulish (Springer) (<https://doi.org/10.1007/BFb0101175>)
- [9] Goncharenko V M and Veselov A P 2004 Yang-Baxter maps and matrix solitons, New trends in integrability and partial solvability *NATO Sci. Ser.* **132** 191–7

- [10] Kassotakis P and Kouloukas T 2022 On non-abelian quadrirational Yang-Baxter maps *J. Phys. A: Math. Theor.* **55** 175203
- [11] Kassotakis P, Nieszporski M, Papageorgiou V and Tongas A 2020 Integrable two component systems of difference equations *Proc. R. Soc. A* **476** 20190668
- [12] Kels A P 2023 Two-component Yang-Baxter maps and star-triangle relations *Physica D* **448** 133723
- [13] Konstantinou-Rizos S and Mikhailov A V 2013 Darboux transformations, finite reduction groups and related Yang-Baxter maps *J. Phys. A: Math. Theor.* **46** 425201
- [14] Konstantinou-Rizos S and Kouloukas T E 2018 A noncommutative discrete potential KdV lift *J. Math. Phys.* **59** 063506
- [15] Konstantinou-Rizos S and Nikitina A A 2024 Yang-Baxter maps of KdV, NLS and DNLS type on division rings *Physica D* **465** 134213
- [16] Kouloukas T E and Papageorgiou V G 2009 Yang-Baxter maps with first-degree-polynomial  $2 \times 2$  Lax matrices *J. Phys. A: Math. Theor.* **42** 404012
- [17] Kouloukas T E and Papageorgiou V G 2011 Poisson Yang-Baxter maps with binomial Lax matrices *J. Math. Phys.* **52** 073502
- [18] Mikhailov A V, Papamikos G and Wang J P 2016 Darboux transformation for the vector sine-Gordon equation and integrable equations on a sphere *Lett. Math. Phys.* **106** 973–96
- [19] Papageorgiou V G, Suris Y B, Tongas A G and Veselov A P 2010 On quadrirational Yang-Baxter maps *SIGMA* **6** 033
- [20] Papageorgiou V G and Tongas A G 2007 Yang-Baxter maps and multi-field integrable lattice equations *J. Phys. A: Math. Theor.* **40** 12677
- [21] Papageorgiou V G, Nijhoff F W and Capel H W 1990 Integrable mappings and nonlinear integrable lattice equations *Phys. Lett. A* **147** 106–14
- [22] Quispel G R W, Capel H W, Papageorgiou V G and Nijhoff F W 1991 Integrable mappings derived from soliton equations *Physica A* **173** 243–66
- [23] Sklyanin E K 1982 Some algebraic structures connected with the Yang-Baxter equation *Funct. Anal. Appl.* **16** 263–70
- [24] Sklyanin E K 1988 Classical limits of the SU(2)-invariant solutions of the Yang-Baxter equation *J. Math. Sci.* **40** 93–107
- [25] Suris Y and Veselov A P 2003 Lax matrices for Yang-Baxter maps *J. Nonlinear Math. Phys.* **10** 223–30
- [26] Veselov A P 2003 Yang-Baxter maps and integrable dynamics *Phys. Lett. A* **314** 214–21
- [27] Veselov A P 2007 Yang-Baxter maps: dynamical point of view *Combinatorial Aspects of Integrable Systems, MSJ Mem.* **17** 145–67
- [28] Yang C N 1967 Some exact results for the many-body problem in one dimension with repulsive delta-function interaction *Phys. Rev. Lett.* **19** 1312