



Diversity-seeking jump games in networks

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Abstract

Recently, strategic games inspired by Schelling's influential model of residential segregation have been studied in the TCS and AI literature. In these games, agents of k different types occupy the nodes of a network topology aiming to maximize their utility, which is a function of the fraction of same-type agents they are adjacent to in the network. As such, the agents exhibit similarity-seeking strategic behavior. In this paper, we introduce a class of strategic jump games in which the agents are *diversity-seeking*: The utility of an agent is defined as the fraction of its neighbors that are of *different* type than itself. We show that in general it is computationally hard to determine the existence of an equilibrium in such games. However, when the network is a tree, diversity-seeking jump games always admit an equilibrium assignment. For regular graphs and spider graphs with a single empty node, we prove a stronger result: The game is potential, that is, the improving response dynamics always converge to an equilibrium from any initial placement of the agents. We also show (nearly tight) bounds on the price of anarchy and price of stability in terms of the social welfare (the total utility of the agents).

Keywords Schelling games · Diversity · Equilibria · Price of anarchy

A preliminary version of this paper was published as [32]. This version includes the complete proofs of all results, as well as a new set of results on the price of anarchy and price of stability (Section 6).

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1 Introduction

In his seminal work [37, 38], the economist Thomas Schelling proposed an elegant model to explain the phenomenon of residential segregation in American cities. In his model, agents of two types are placed uniformly at random on the nodes of a network. An agent is happy if at least some fraction τ of its neighbors are of the same type as itself, and unhappy otherwise. An unhappy agent will *jump* to an unoccupied node in the network, or exchange positions with another unhappy agent of a different type. Schelling showed experimentally that this random process would lead to *segregated neighborhoods*, even for $\tau \approx 1/3$. His work showed that even small and local individual preferences for one's own type can lead to large-scale and global phenomena such as residential segregation. As such, it inspired a significant number of follow-up studies in sociology and economics [5, 33, 36], with many empirical studies being conducted to study the influence of different parameters on segregation models [4, 10, 13, 22, 41].

Inspired by Schelling's model, there have been a few lines of inquiry pursued by computer scientists as well. Some researchers attempted to prove analytically the conditions under which segregation takes place [9, 26], while a more recent line of research used *game-theoretic* tools to study the problem [11, 12, 15, 18, 20, 27]. In the latter setting, strategic agents are placed at arbitrary positions in the network and move to new positions to improve their *utility*, defined as a function of the fraction of their neighbors which are of the same type as themselves. Two main classes of games have been considered. In *jump* games, the agents can move to previously unoccupied locations aiming to improve their utility, while in *swap* games, unhappy agents of different types can swap their locations if this increases the utility of both of them. Researchers have mainly studied questions related to the *existence*, *quality*, and *computational complexity* of *equilibria*, which are stable assignments of agents to the nodes of the network such that no agent wants or can deviate.

1.1 Our contribution

In this paper, we introduce a new class of strategic games, called *diversity-seeking jump games*. In such games, the utility of an agent is the fraction of its neighbors of a *different* type than itself. To the best of our knowledge, this is the first time such a utility function has been studied in the context of jump games on networks that are inspired by Schelling's model. Data from the General Social Survey [39] show that people prefer diverse neighborhoods rather than segregated ones. This survey, which is regularly conducted in the US since 1950, shows that the percentage of people preferring diverse neighborhoods has been steadily increasing. There are also other settings that could be modeled using this new utility function; e.g., teams composed of people with diverse backgrounds and skill sets to bring a broader range of perspectives to their business, or research groups composed of people bringing expertise from different disciplines. Indeed, many studies show that ethnically and gender-wise diverse teams lead to better outcomes for business [3, 17, 19, 21, 23, 30, 34, 35, 40].

It may seem at first glance that Schelling jump games are intimately related to our diversity-seeking jump games, but the relationship is not obvious. Observe that the utility of an agent in a Schelling *similarity-seeking* jump game is not always equal to the complement of its utility in the diversity-seeking setting. To see this, consider an agent placed at a node

of degree 1 that is adjacent to an empty node. Such an isolated agent has utility 0 in both similarity-seeking and diversity-seeking settings, thus showing that the models are not simply complements of each other.

We present a series of results related to the computational complexity, the existence, and quality of equilibrium assignments in diversity-seeking jump games. We first show that given a network topology and a number of agents drawn from a set of types, it is NP-hard to determine whether there exists an equilibrium if some of the agents are *stubborn* (and do not ever move from the nodes they are initially placed).¹ This hardness reduction can be found in Section 3.

In terms of positive existence results, in Sections 4 and 5, we show that there are always equilibrium assignments in games where the network topology is a tree graph with any number of empty nodes, if all agents are strategic. For some classes of graphs, we show a stronger result: The diversity-seeking jump game is a *potential game* when the topology is a regular graph or a spider graph with a single empty node, or a line with any number of empty nodes. This means that from any initial placement of agents to the nodes of the topology, improving response dynamics (IRD) by the agents always converges to an equilibrium. Our positive results show a sharp difference between diversity-seeking jump games and similarity-seeking jump games, for which there are instances with tree topologies that do not admit equilibria [20], and instances with spider graph topologies which are not potential games.

Finally, we show that there is not much loss of efficiency at equilibrium assignments in diversity-seeking jump games in terms of social welfare, defined as the total utility of all agents. In particular, we show that the price of anarchy (that is, the worst-case ratio of the optimal social welfare over the minimum social welfare achieved at equilibrium) is at most $n/(k-1)$ when there are n strategic agents partitioned into $k \geq 2$ types, and at most $k/(k-1)$ when the agents are partitioned into k *symmetric* types each consisting of the same number of agents. These bounds are (nearly) tight even for simple topologies, such as a line graph in some cases or a star graph in other cases. We remark that, in contrast to similarity-seeking games where the price of anarchy has been shown to be increasing in k for symmetric types [20, 28], it is decreasing in k for diversity-seeking games and tends to 1. Intuitively, this is because in the games we study here, the social welfare is a measure of integration which is easier to achieve when there are a lot of different types of agents, whereas the social welfare is a measure of segregation in the model of Elkind et al. [20] which is harder to achieve when there are many types. We also prove a lower bound of $65/62 \approx 1.048$ on the price of stability (which is the worst-case ratio of the optimal social welfare over the maximum social welfare achieved at equilibrium), thus showing that optimal assignments are not necessarily equilibria. These results can be found in Section 6.

1.2 Related work

In the last decade, there have been attempts in the computer science literature [9, 26] to theoretically analyze the random process described by Schelling. It was shown that the expected size of the resulting segregated regions is polynomial in the size of the neighbor-

¹We remark that stubborn agents naturally exist in scenarios like the ones captured by Schelling's model where agents are people that form neighborhoods or groups. In particular, there are people that due to the underlying cost of moving or other factors (such as age or beliefs) are not willing or cannot move.

hood of agents on a line topology [9] and exponential in its size on a grid topology [26]; however, in both cases it is independent of the overall number of agents.

The study of the above process as a strategic game played by two types of agents rather than as random process appears to have been initiated by Zhang [42], who introduced a game-theoretic model where agents have a noisy single-peaked utility function that depends on the ratio of the numbers of agents of two agent types in any local neighborhood. Chauhan et al. [15] introduced a game-theoretic model that incorporates Schelling's notion of a tolerance threshold. In their model, each agent has a threshold parameter $\tau \in (0, 1)$ and may have a preferred location. The primary goal of an agent is to find a location where the happiness ratio exceeds τ ; if such a location does not exist, the agent aims to maximize its happiness ratio. Its secondary goal is to minimize the distance to its preferred location. Chauhan et al. [15] studied the convergence of best-response dynamics to an equilibrium assignment in both jump and swap games with two types of agents and for various values of the threshold parameter. Echzell et al. [18] generalized the model of [15] by considering games with agents of $k \geq 2$ different types, in which the cost of an agent is related to the ratio of agents of its own type over the whole neighborhood, or the ratio of agents of its own type over the part of the neighborhood that includes the agents of the majority type different than its own. They showed results related to the convergence of the dynamics and the computational hardness of finding placements that maximize the number of happy agents.

More related to our model, Elkind et al. [20] studied similarity-seeking jump games (that is, $\tau = 1$) for $k \geq 2$ types of agents that can be either strategic (aiming to maximize their utility) or stubborn (who stay at their initial location regardless of the composition of the neighborhood). Elkind et al. [20] showed that, while equilibria always exist when the topology is a star or a graph of degree 2, they may not exist even for trees of degree 4. They also showed that it is computationally hard to find equilibrium assignments or assignments with optimal social welfare, and bounded the price of anarchy and stability for both general and restricted games. Agarwal et al. [1] considered the utility function of [20], but focused on swap games and, besides the social welfare, also considered a new social objective called the *degree of integration*. They showed similar results about the existence and the computational complexity of equilibrium assignments. An extended version of these two works that includes more results was later published as [2]. The hardness results of [2] were improved by Kreisel et al. [29], who showed that the problem of determining the existence of equilibria is NP-hard in both jump and swap games, even if all agents are strategic. Bilò et al. [7] investigated the influence of the underlying topology and locality in swap games when agents can only swap with other agents at most a number of hops away, and also improved some price of anarchy results of [2].

Other variants of the model have been considered by Kanellopoulos et al. [27] who focuses on a slightly different utility function where an agent is also counted as part of its neighborhood, by Chan et al. [14] who considered social Schelling games where the agents are not partitioned into type but rather are connected via a social network, and by Bilò et al. [8] who considered a model in which the types are continuous rather than discrete (as in most papers). The computational complexity of finding assignments with high objective value according to different functions, including the social welfare, the Nash welfare, and other optimality notions has been considered by Bullinger et al. [11], and also by Deligkas et al. [16] from the parameterized complexity perspective.

Some other recent papers that are related to ours are that of Bilò et al. [6], Friedrich et al. [24] and Kanellopoulos et al. [28], which are also motivated by the observation that real-world agents might favor diverse neighborhoods to some extent. Bilò et al. [6] and Friedrich et al. [24] considered swap and jump games, respectively, in which agents of two types have single-peaked utility functions that increase monotonically with the fraction of same-type neighbors in the interval $[0, \Lambda]$ for some $\Lambda \in (0, 1)$, and then decrease monotonically afterward; this is in contrast to our model here, where the utility is a monotone function. Kanellopoulos et al. [28] considered the case where there are $k \geq 2$ types of agents and there is an implicit ordering of the types such that the distance between the types according to this ordering determines how much different types affect the utility of an agent whenever there are agents of those types in its neighborhood. All these papers showed instances where equilibria do not always exist, but also identified restricted classes of games for which existence of equilibria is guaranteed either by explicit constructive algorithms, or because IRD converges. They also present tight bounds on the price of anarchy and the price of stability in terms of the social welfare or the degree of integration.

2 The model and notation

In an instance $I = (R, S, \mathcal{T}, G, \lambda)$ of the *diversity-seeking jump game*, there is a set R of *strategic agents* and a set S of *stubborn agents* with $|R \cup S| = n \geq 2$. The agents are further partitioned into $k \geq 2$ different *types* $\mathcal{T} = (T_1, \dots, T_k)$. Each agent occupies a node of an undirected graph $G = (V, E)$ that satisfies $|V| > n$; G is referred to as the *topology*. The locations (nodes of G) of the stubborn agents are fixed and given by an injective mapping λ from S to V . In contrast, the strategic agents can choose their locations freely, subject to the constraint that no two agents can share the same node. An *assignment* C determines the node $v_A(C)$ that each agent A (strategic or stubborn) occupies on the topology, that is, it is an injective mapping of agents to the nodes that respects λ ; let $\mathcal{C}(I)$ be the set of all possible assignments for game I .

We define $n_T(v, C)$ to be the number of neighbors of node v that are occupied by agents of type $T \in \mathcal{T}$ according to assignment C . Let $n(v, C) = \sum_{T \in \mathcal{T}} n_T(v, C)$ be the total number of agents in v 's neighborhood according to assignment C . The satisfaction of the agents for an assignment C is measured by a *utility function*. In particular, the utility of an agent A of type T that occupies node $v_A(C)$ in assignment C is defined as the fraction of agents of type different than T in A 's neighborhood. That is,

$$u_A(C) = \frac{\sum_{t \neq T} n_t(v_A(C), C)}{n(v_A(C), C)}$$

Observe that $u_A(C) \in [0, 1]$. By convention, the utility of an agent without neighbors is 0.

2.1 Equilibria and improving response dynamics

We are interested in stable assignments of diversity-seeking jump games, in which strategic agents do not have incentive to deviate by jumping to empty nodes of the topology to increase their utility.

Definition 1 An assignment C is said to be a *pure Nash equilibrium* (or, simply, *equilibrium*; NE) if $u_A(C) \geq u_A(C')$ for every strategic agent A and any assignment C' that is obtained from C by moving agent A to an empty node while all other agents occupy the same nodes as in C . We will denote by $NE(I)$ the set of all equilibrium assignments of game I .

More specifically, we study the convergence of *improving response dynamics (IRD)* to equilibrium assignments in diversity-seeking jump games: In each step, a single strategic agent tries to move to an unoccupied node where it gets higher utility. We say such a move of an agent A from assignment C to assignment C' is an *improving move* if $u_A(C) < u_A(C')$. Note that in assignment C , a single agent may have many different improving moves, and multiple agents may have improving moves. We do not make any assumptions about which improving move will take place if multiple ones are available.

A game is a *potential game* if and only if there exists a *generalized ordinal potential function*, that is, a non-negative real-valued function Φ on the set of assignments such that for any two assignments C and C' where there is an improving move from C to C' , we have $\Phi(C') < \Phi(C)$. As observed by Monderer and Shapley [31], such games always admit an equilibrium, and furthermore, regardless of the starting assignment, any sequence of improving moves is finite and will terminate in an equilibrium.

Not all games are potential ones in case there is an *improving response cycle (IRC)* that is a sequence of assignments obtained by unilateral improving moves of the agents such that the first and the last assignment in the sequence are the same. In other words, a game is potential if and only if there are no IRC. We also remark that, while the existence of an IRC implies that the IRD does not converge to an equilibrium, it does not imply the non-existence of an equilibrium.

For particular topologies that we will consider in later sections, we will show that the following function is a generalized ordinal potential function. Similar potential functions have been used before in different models [15, 18, 20].

Definition 2 For an assignment C and some value $m \in (0, 1)$, let $\Phi(C) = \sum_e w_C(e)$, where the weight $w_C(e)$ of any edge $e = \{u, v\} \in E$ is defined as:

$$w_C(e) = \begin{cases} 1 & \text{if } u \text{ and } v \text{ are occupied by agents of the same type} \\ m & \text{if either } u \text{ or } v \text{ is unoccupied} \\ 0 & \text{otherwise.} \end{cases}$$

The following lemma shows that, if a diversity-seeking jump game without stubborn agents is a potential game for a particular class of topologies, then it is also a potential game if some of the agents are stubborn for the same class of topologies. This will allow us to consider only the case when all agents are strategic when showing that a game is a potential one for a class of networks.

Lemma 2.1 For any given $k \geq 2$ and topology G , if the game $I = (R, \emptyset, T, G, \lambda)$ is a potential game, then every game $I' = (R', S', T, G, \lambda)$ in which S' and R' form a partition of R , is also a potential game.

Proof Suppose $I = (R, \emptyset, T, G, \lambda)$ is a potential game for a given topology G , and let \mathcal{C} be the set of valid assignments for I . Now consider the game $I' = (R', S', T, G, \lambda)$ where $S' \neq \emptyset$, $R' \subset R$ and $S' \subset R$; let \mathcal{C}' be the set of valid assignments for I' . Since $R' \subset R$, $\mathcal{C} \subset \mathcal{C}'$, and any move between two assignments in I' is also possible in I , though the converse does not hold. Consequently, if there is an IRC in I' , then this IRC also exists in I . But this is impossible since I is a potential game, which implies that I' is also a potential game. \square

2.2 Price of anarchy and stability

We are also interested in the welfare guarantees of equilibrium assignments in terms of the *social welfare* that is defined as the total utility of all strategic agents. In particular, for an assignment C , the social welfare is

$$SW(C) = \sum_{A \in R} u_A(C).$$

For a given game I , let $\text{OPT}(I) = \max_{C \in \mathcal{C}(I)} SW(C)$ be the optimal social welfare that can be achieved among all possible assignments of I . The *price of anarchy* is a pessimistic measure of the loss of welfare at equilibrium and is defined as the worst-case ratio, over all possible games I with $\text{NE}(I) \neq \emptyset$, of the optimal social welfare $\text{OPT}(I)$ over the *minimum* social welfare achieved among all equilibrium assignments, that is,

$$\text{PoA} = \sup_{I: \text{NE}(I) \neq \emptyset} \frac{\text{OPT}(I)}{\min_{C \in \text{NE}(I)} SW(C)}.$$

On the other hand, the *price of stability* is an optimistic measure of the loss of welfare at equilibrium and is defined as the worst-case ratio of the optimal social welfare over the *maximum* social welfare at equilibrium, that is,

$$\text{PoS} = \sup_{I: \text{NE}(I) \neq \emptyset} \frac{\text{OPT}(I)}{\max_{C \in \text{NE}(I)} SW(C)}.$$

Clearly, the price of anarchy and the price of stability are both always at least 1; the closer they are to 1, the smaller the loss of welfare at equilibrium. Besides showing bounds on these two measures for all possible games, we will also focus on specific classes of games with particular topologies.

3 Complexity of computing equilibria

In this section, we study the computational complexity of determining if a given diversity-seeking jump game admits an equilibrium assignment.

Theorem 3.1 *For any $k \geq 2$, it is NP-complete to decide whether a given diversity-seeking jump game $I = (R, S, T, G, \lambda)$ admits an equilibrium assignment.*

Proof We give a proof for $k = 2$; it is straightforward to extend it to $k \geq 2$. We can clearly verify if a given assignment is an equilibrium, so the decision problem belongs to NP. To show hardness, we will use a reduction from the Independent Set (IS) decision problem [25]. Recall that an independent set in a graph is a subset of vertices of the graph such that no two vertices in the subset are connected by an edge. An instance of the IS problem is an undirected graph $H = (X, Y)$, where X and Y are the set of vertices and edges respectively, and an integer s ; it is a yes-instance if and only if H has an independent set of size $\geq s$. We construct a diversity-seeking jump game I as follows (see also Fig. 1):

1. There are two agent types: red and blue.
2. There are a total of $8s + 16$ agents; $s + 1$ of them are strategic red agents, $2s + 4$ are stubborn red agents, and $5s + 11$ are stubborn blue agents.
3. The topology $G = (V, E)$ consists of two components G_1 and G_2 , as follows:
 - $G_1 = (V_1, E_1)$, where $V_1 = X \cup W$, $|W| = 7s + 1$, and $E_1 = Y \cup \{v, w\} : v \in X, w \in W\}$; $5s + 1$ stubborn blue agents and $2s$ stubborn red agents are placed at the nodes of W .
 - G_2 has exactly three nodes (denoted x, y , and z) available to strategic agents, while all the other nodes are occupied by stubborn agents. Node x is connected to y and one blue agent. Node y is connected to six nodes, containing one red agent and five blue agents. Finally, z is connected to 7 nodes containing 4 blue agents and 3 red agents. Observe that when the other two nodes in G_2 are unoccupied, x, y , and z offer utilities 1, $5/6$ and $4/7$, respectively.
 - There is an edge between two (arbitrarily chosen) nodes containing stubborn agents in G_1 and G_2 , thereby connecting the graph.

The main idea of the reduction is as follows: We will show that, if an IS of size $\geq s$ exists in H , then placing s strategic agents at the nodes of the IS and the remaining strategic agent at node x is an equilibrium assignment in I . Conversely, if no IS of size at least s exists in H , then we will show with a case analysis that there cannot be an equilibrium assignment in I .

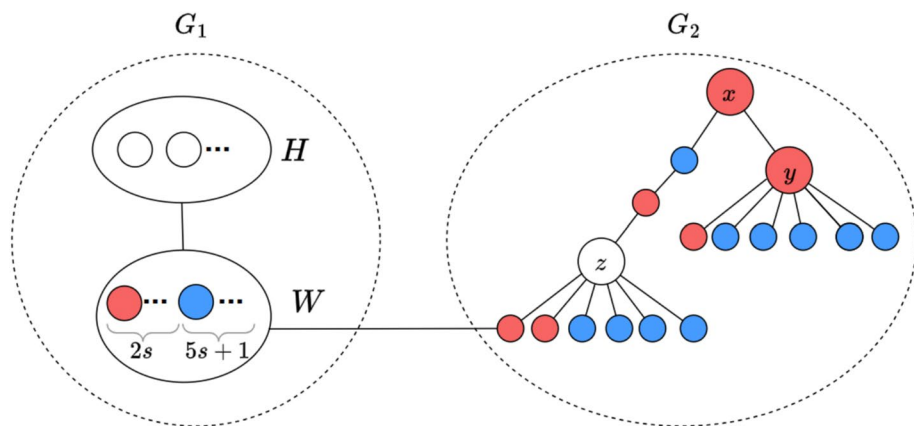


Fig. 1 G_1 and G_2 are the two components of G , connected by a single edge. Every node in H is connected to every node in W

We proceed with the proof that I admits an equilibrium if and only if H has an IS of size at least s . First, suppose that H contains an IS of size at least s . Consider the following assignment in I : s red agents occupy the nodes of the IS in X and one red agent occupies node x in G_2 . We argue this is an equilibrium. The utility of each agent at a node in IS is $\frac{5s+1}{7s+1}$. Note that any remaining empty node in G_1 offers a utility that is at most $\frac{5s+1}{7s+1} < 1$, while the utility of the agent at x is 1. Therefore, the agent at x does not benefit by moving to an empty node in G_1 . Also, note that when node x is occupied, the highest utility a red agent can achieve in G_2 is $5/7$ by moving to y . We have $\frac{5s+1}{7s+1} > 5/7$, therefore no agent in G_1 has an improving move to an empty node in G_2 , and the assignment is an equilibrium.

On the other hand, suppose that H does not contain an IS of size at least s . Suppose towards a contradiction that I admits an equilibrium assignment. There are four possible cases:

1. All available nodes in G_2 are empty. Then since there is no IS of size s in H , and therefore in G_1 , at least one strategic red agent is adjacent to another strategic red agent and has utility at most $\frac{5s+1}{7s+2}$. Since $\frac{5s+1}{7s+2} < 1$, that agent can increase its utility by moving to node x in G_2 . Therefore, this cannot be an equilibrium assignment.
2. Exactly one available node in G_2 is occupied by a strategic agent. Then this agent must occupy x as it provides the highest utility. As in the previous case, since there is no IS of size s , at least one strategic red agent is adjacent to another strategic red agent in G_1 and has utility at most $\frac{5s+1}{7s+2}$. Since $\frac{5s+1}{7s+2} < 5/7$, that agent can improve its utility by moving to node y in G_2 . Therefore, this also cannot be an equilibrium assignment.
3. Exactly two available nodes in G_2 are occupied. If x and y are occupied, the agent at x has utility $1/2$ and is motivated to move to z to get utility $4/7$. If x and z are occupied, the agent at z has utility $4/7$ and is motivated to move to y to get utility $5/7$. If y and z are occupied, the agent at y has utility $5/6$ and is motivated to move to x to get utility 1. In all cases, there is an agent that wants to move to increase its utility, so this cannot be an equilibrium assignment.
4. All available nodes in G_2 are occupied. Every empty node in G_1 offers a utility that is at least $\frac{5s+1}{(7s+1)+(s-2)} = \frac{5s+1}{8s-1}$. Note that the agent at x has utility $1/2 < \frac{5s+1}{8s-1}$. Therefore, the agent at x is motivated to move to an empty node in G_1 . Therefore, this cannot be an equilibrium assignment.

We have shown by contradiction that, if H does not contain an IS of size s , there is no equilibrium assignment in I . This completes the proof of the NP-hardness. \square

The proof of the hardness result above relies heavily on the existence of stubborn agents, as in the proofs of similar results for similarity-seeking jump games in [20]. In a recent paper, Kreisel et al. [29] showed that deciding the existence of an equilibrium in a similarity-seeking game is NP-complete, even when all agents are strategic. Their techniques are not immediately applicable to our setting, and in fact we have not been able to construct a diversity-seeking jump game with only strategic agents that does not have an equilibrium.

4 Seeking diversity in tree topologies

In this section, we consider diversity-seeking jump games with tree topologies. We first show that the IRD does not always converge to an equilibrium for such games, even when there is a single empty node. However, there is always an equilibrium assignment; we give a polynomial-time algorithm to find one. In contrast, recall that similarity-seeking games do not always admit an equilibrium when the topology is a tree [20]. In Section 4.1, we will show that the game is a potential game when the topology is a spider graph (i.e., a tree in which there is a single node of degree at least 3).

Theorem 4.1 *For every $k \geq 2$, there exists a diversity-seeking jump game $I = (R, S, T, G, \lambda)$ such that G is a tree and I is not a potential game, even when there is just one empty node.*

Proof Figure 2 shows an IRC in a diversity-seeking jump game on a tree with a single empty node and two types of agents. In the figure on the left, the red agent at node x has utility $1/2$ and jumps to node z where it gets utility $4/7$. Next, as shown in the center figure, the red agent at node y has utility $5/6$ and jumps to node x to get utility 1. Finally, as shown in the figure on the right, the red agent at z has utility $4/7$ and jumps to node y where it gets utility $5/7$, thereby returning to the configuration of the figure on the left. Thus, these three moves lead to an IRC. \square

In fact, if only the agents that occupy x, y or z in Fig. 2 are strategic, and the remaining agents are all stubborn, then this diversity-seeking jump game does not admit an equilibrium. Thus, equilibria are not guaranteed to exist when the topology is a tree and there exist stubborn agents. Nevertheless, we show that there is always an equilibrium assignment when the topology is a tree if *all* agents are strategic; we give an algorithm to find such an assignment. As a warm-up, we first consider the case where the tree has a single empty node.

Algorithm to find equilibrium assignment C on a tree when $|V| = |R| + 1$ and $S = \emptyset$ Pick a node r of degree 1 to be the root of the tree and call v its unique neighbor. Order the agents by type. The algorithm proceeds in two phases. In the first phase, starting with the deepest *odd* level in the tree, place agents in the current level ℓ , moving from left to right. When the current level is completely filled, skip a level, go to level $\ell - 2$ and repeat, until the unique node v in level 1 is filled, say by an agent of a type T_x : For convenience, we call this type the *red type*, and agents of this type *red agents*. This ends the first phase. In the second phase, move downwards from level 2, filling even levels from left to right. At the end of this procedure, only the root node is empty. See Fig. 3 for an example.

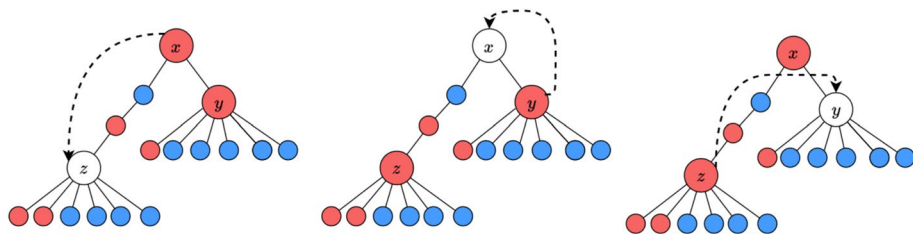


Fig. 2 An IRC in a game with a tree topology. The dotted lines indicate improving moves

Lemma 4.2 *The assignment C computed by the algorithm has the following properties:*

(P1) Every non-red agent has utility 1.

(P2) Every red agent with a non-red parent can only have non-red children² and therefore has utility 1.

Proof We prove each property separately:

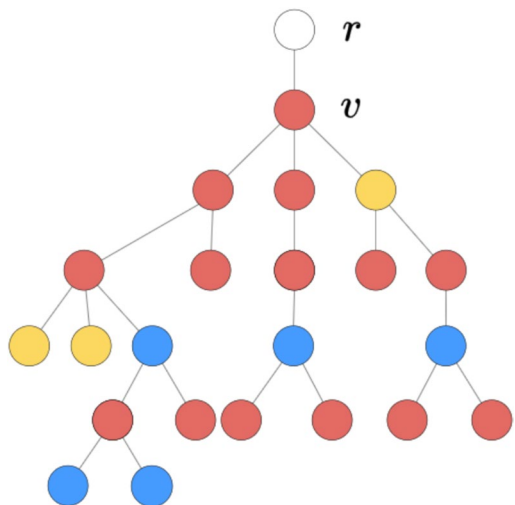
(P1) This follows because all non-red agents are either placed exclusively on odd levels, or exclusively on even levels. Thus, any non-red agent cannot be adjacent to an agent of its own type, and has utility 1.

(P2) Consider a red agent \mathcal{A} all of whose children are not red. Note that the parent and the children of this red agent are either both on odd levels or both on even levels. Suppose \mathcal{A} is at an even level in the tree. Then since odd levels were filled from bottom to top and red agents are the last type of agents to start filling the odd levels, if an odd level contains a non-red agent, no odd level below that level contains red agents. Thus if \mathcal{A} 's parent is non-red, \mathcal{A} cannot have red children. Similarly, even levels are filled from top to bottom, with red agents being the first to fill them, so if an even level contains a non-red agent, no even level below that level contains red agents. Thus if \mathcal{A} is at an odd level, and has a non-red parent, then none of \mathcal{A} 's children can be red. \square

Theorem 4.3 *Given $k \geq 2$, every diversity-seeking jump game (R, S, T, G, λ) where G is a tree, $|V| = |R| + 1$ and $S = \emptyset$ admits an equilibrium assignment.*

Proof Consider the assignment C computed by the above algorithm for a given game with a tree topology. By Lemma 4.2, all non-red agents are guaranteed to have utility one, and have no incentive to move. The only empty node is the root of the tree, which offers util-

Fig. 3 An equilibrium assignment for an instance with $k = 3$. By convention, r is on level 0. In the first phase, red agents are placed first, starting at level 5, and then at odd levels 3 and 1. In the second phase, we go downwards placing agents at even levels. The remaining red agents are placed at level 2, followed by yellow agents and then blue agents on even levels 4 and 6. Agents are placed left to right on each level



²For brevity, we say an agent is a child (neighbor/parent) of another agent to mean an agent is placed at a node that is the child (neighbor/parent) of the node where the other agent is placed.

ity 0 to a red agent. Thus, though there may be red agents that have utility less than 1, they cannot improve their utility by moving to the root. This proves that the assignment C is an equilibrium. \square

Next we consider the case where the number of empty nodes is more than 1. The key idea is as follows: we first find a connected sub-tree with one more node than the number of agents, and find an equilibrium assignment for the sub-tree using the method of Theorem 4.3. Next we adjust this assignment to get an assignment for the original tree by moving some agents to empty nodes in the original tree that were not present in the sub-tree in such a way that no agent will have an improving move. This adjustment has to be done carefully, as moving an agent may reduce its neighbors' utilities.

From Lemma 4.2, non-red agents as well as red agents with non-red parents have utility 1, and do not have improving moves. So the only possible agents that have improving moves are red agents whose parents are red. Of these, some may have children that are all red, or all non-red, or they may have some children that are red and some that are non-red. We call a red agent which has both red and non-red children a *mixed* agent. Such agents need special care in our algorithm, and we show the following technical lemma about these agents.

Lemma 4.4 *In the assignment C , there are at most two mixed agents. In addition, if there are two mixed agents A and B with A closer to the root than B :*

1. A and B are on adjacent levels.
2. If B is not a child of A , then the red children of A have only red children.

Proof Observe that there are at most two levels that can have both red and non-red agents, namely the levels where we start and finish placing red agents; call these levels ℓ_s and ℓ_f respectively. Consider one of these levels, say ℓ_s . Since agents are placed left to right in each level in order of their type, there can be at most one agent in the level $\ell_s - 1$ that has both red and non-red children. If this agent is red, then it is a mixed agent. A similar argument applies to ℓ_f . It follows that there are at most two mixed red agents, and they are on different levels.

Suppose there are two mixed agents A and B , with A being closer to the root. Recall that ℓ_f is even and ℓ_s is odd.

$\ell_f < \ell_s$: Then A is on the odd level $\ell_f - 1$ and B is on the even level $\ell_s - 1$. Since ℓ_f is the last (and therefore largest) even level on which we place red agents, it must be that $\ell_s - 1 = \ell_f$. This shows (1). Furthermore, observe that since we place agents from left to right, the level ℓ_f must start with red agents and end with non-red agents. If B is not a child of A , since it is a red agent, it must be to the left of all children of A , and all the red agents that are to the right of B on the same level (including children of A) can only have red children, if any (see Fig. 4), thus proving (2).

$\ell_s < \ell_f$: Then A is on the even level $\ell_s - 1$ and B is on the odd level $\ell_f - 1$. Since ℓ_s is the first odd level on which we place red agents, it must be that $\ell_f - 1 = \ell_s$, which establishes (1). In this case, since ℓ_s is a level that starts with non-red

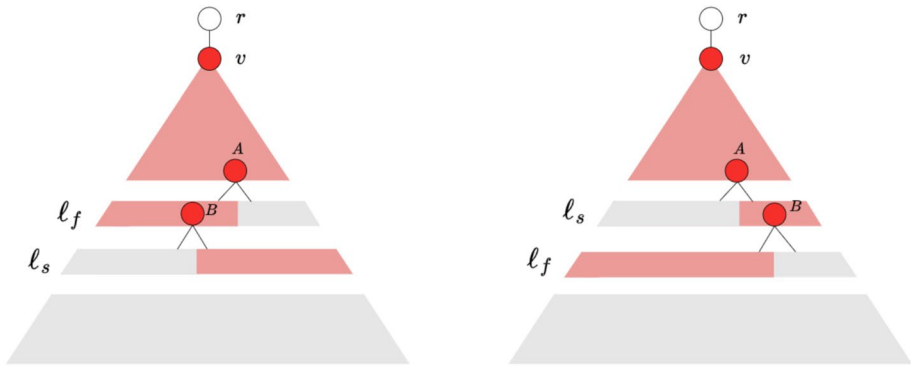


Fig. 4 The red areas are occupied by red agents and the grey areas by non-red agents. In the figure on the left, $\ell_f = \ell_s - 1$, and agent B is on the even level ℓ_f . In the figure on the right, $\ell_s = \ell_f - 1$, and agent B is on the odd level ℓ_s

agents and ends with red agents, it follows that B must be to the right of all children of A . Therefore all the red agents that are to the left of B on the same level (including children of A) can only have red children, if any (see Fig. 4), thus proving (2).

□ We are now ready to describe how to obtain an equilibrium assignment in a tree with more than one empty node.

Theorem 4.5 *Given $k \geq 2$, every diversity-seeking jump game (R, S, T, G, λ) where G is a tree, $|V| > |R| + 1$ and $S = \emptyset$ admits an equilibrium assignment.*

Proof Given the tree $G = (E, V)$, fix a root node of degree one, and repeatedly remove leaf nodes until we have a tree with exactly $|R| + 1$ nodes, call it $G' = (E', V')$. Then G' is a graph that will have only one empty node in any assignment of the agents in R . Let $C_{G'}$, denote the equilibrium assignment in G' as described in the proof of Theorem 4.3. As before, let us call the type of the agent assigned to the unique neighbor of the root a red agent.

Now consider the original tree G , and consider the same placement of agents as in $C_{G'}$, call this assignment C_G . Clearly the utility of agents in C_G is exactly the same as their utility in $C_{G'}$, as acquiring new empty neighbors does not change an agent's utility. From Property (P1) in Lemma 4.2 all non-red agents have utility 1. As in $C_{G'}$, there may be some red agents that have utility less than one in C_G , but in G , they may have improving moves available, i.e., C_G may not be an equilibrium.

Next we show how to convert C_G into an equilibrium assignment on G , by changing the locations of some agents one by one in such a way that the agent being moved increases its utility and no other agent decreases its utility. Our algorithm proceeds in three phases: pre-processing the children of mixed agents; creating an ordered list of candidate agents to move; identifying a list of available empty nodes to move to, and moving the candidate agents from the created list to these nodes. □

Pre-processing of children of mixed agents By Property (P2) of Lemma 4.2, there are at most 2 red mixed agents, and if there are two, they must be on adjacent levels. Let \mathcal{A} be such an agent; if there are two, call the agent closer to the root \mathcal{A} and the one further from the root \mathcal{B} . We make the following local adjustment to the placement of children of \mathcal{A} . Let S_1 be the set of red children of \mathcal{A} that are adjacent to at least one more agent, and let S_2 be its non-red children with *no* other agents as neighbors. Thus, an agent in S_2 is either located on a leaf in G , or at a node that has only empty nodes as children.

- If $S_2 \neq \emptyset$, we swap³ the positions of $\min(|S_1|, |S_2|)$ agents in S_2 with the agents in the other set. If the second mixed agent \mathcal{B} exists and is a child of \mathcal{A} , then it must be in the set S_1 , and we make sure that \mathcal{B} participates in the swap. See Fig. 5 for an example.
- If $S_2 = \emptyset$, and the second mixed agent \mathcal{B} exists and is a child of \mathcal{A} , we swap the position of \mathcal{B} with any non-red child of \mathcal{A} . Note that the non-red child in its new position still has utility 1, as do all non-red children of \mathcal{A} in this case.

Claim 4.6 After the pre-processing operation, all non-red agents retain utility 1. Also, if there remains another mixed agent in the tree, then this agent \mathcal{B} is not a child of \mathcal{A} .

Proof Note that the pre-processing operation moves a non-red agent to a different location in the *same level*. As mentioned in the proof of Property (P1), there are no agents of its own type in adjacent levels. It follows that they cannot become adjacent to agents of their own type. Furthermore, it remains a child of \mathcal{A} , and therefore retains utility 1. Next, suppose there was a second mixed agent \mathcal{B} and it was a child of \mathcal{A} . If $S_2 \neq \emptyset$, \mathcal{B} was swapped with an agent in S_2 that had no other agents as neighbors. Thus \mathcal{B} is no longer a mixed agent. If instead $S_2 = \emptyset$, \mathcal{B} was swapped with a non-red child of \mathcal{A} . Since \mathcal{B} had both red and non-red children in its original location, it cannot be that it also has red and non-red children in its new location (since agents are placed in order of type). Thus \mathcal{B} is no longer a mixed agent. \square

³Note that this is a jump game, so the swap referred to here is not to be confused with improving response dynamics.

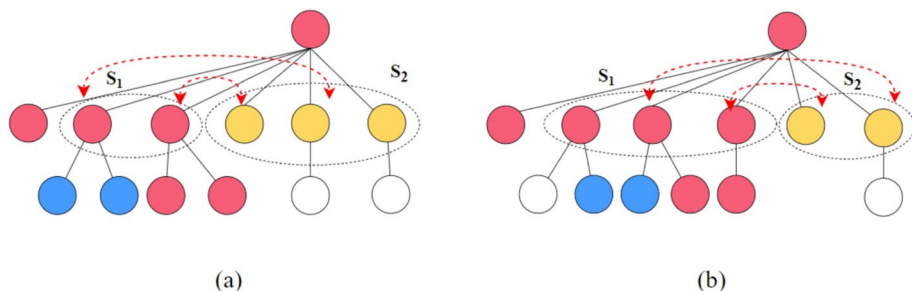


Fig. 5 Swapping the red children adjacent to at least one more agent (S_1) with non-red children with no other agents as neighbors (S_2). (a): $|S_1| \leq |S_2|$. (b): $|S_1| > |S_2|$. Red dotted arrows show the swaps

If there remains another mixed agent in the tree, then as shown in Claim 4.6, this agent \mathcal{B} is not a child of \mathcal{A} , and we perform the same pre-processing operation for \mathcal{B} 's children as well.

Claim 4.7 After the pre-processing operation, *at least one* of the following two conditions holds:

1. All red children of \mathcal{A} have utility 0.
2. All of \mathcal{A} 's non-red children would have utility 1 even if \mathcal{A} was moved.

Proof If $|S_1| \leq |S_2|$, by (P2), the agents in S_2 that swapped positions now have children of a different color than themselves, and still have utility 1. The remaining agents in S_2 have no neighbors other than \mathcal{A} and retain utility 1. Also, all the red children of \mathcal{A} , have no neighbors other than \mathcal{A} and have utility 0, so (1) holds. See Fig. 5 (a).

Otherwise, if $|S_1| > |S_2|$, all agents in S_2 are moved to locations where they have children, and so all non-red children of \mathcal{A} have children. As mentioned in the proof of Claim 4.6, any such non-red child of \mathcal{A} must have children of different types than itself. Thus if \mathcal{A} moves from its position, these non-red agents would still have utility 1, so (2) holds. See Fig. 5 (b). \square

Creating an ordered list of candidate agents to move From Claim 4.6, even after pre-processing, all non-red agents have utility 1, and thus there is no need to move them. We now create an ordered list of *candidate* red agents to move that could improve their utilities. Starting with level 1, moving down level by level, we put in red agents that have utility 0 in this list. Next we consider agents of utility strictly between 0 and 1. It follows from Property (P2) of Lemma 4.2 that these are red agents with red parents and at least one non-red child: these are either agents that are mixed, or they are children of red agents who have all non-red children. Call this second set S_3 . The agents in S_3 will not be in the list of candidate agents to move; our algorithm for moving agents will ensure that for any agent in S_3 , either its red parent will move, causing it to have utility 1, or if its red parent does not move, there is no empty node offering it a higher utility.

It remains to determine whether or not the mixed agents will be added to the list of candidate agents to move. The potential problem with moving a mixed agent is that if it has a non-red child with no other agents as neighbors, now that child could have utility 0. First consider agent \mathcal{A} . If Condition (1) of Claim 4.7 holds, then all of \mathcal{A} 's red children are already in the list of candidates. If they all move, then \mathcal{A} 's utility will change to 1, so we do not put \mathcal{A} into the candidate list. Otherwise, Condition (2) of Claim 4.7 holds, and so it is safe to move \mathcal{A} . In this case, we put \mathcal{A} in the list of candidates.

If there is a second mixed agent \mathcal{B} , by Claim 4.6, it is not a child of \mathcal{A} , and by Lemma 4.4, the red children of \mathcal{A} only have red children. Therefore, condition (1) of Claim 4.7 is satisfied, and \mathcal{A} was not placed in the candidate list. We now use the same procedure as used for \mathcal{A} to determine if \mathcal{B} should be placed in the candidate list. Therefore, there is at most one mixed agent in the candidate list. To recap, the ordered list of candidates consists of red agents of utility 0, placed in increasing order of their distance from the root, and possibly one mixed agent, which is the last agent in the list.

Moving the candidate agents to appropriate empty nodes Next we describe *where* to move the candidate agents. Note that all empty nodes have the property that they are adjacent to at most a single node containing an agent. This follows from the manner in which G' was extracted from G by repeatedly removing leaves. Therefore, all empty nodes either offer utility 0 or 1 to a red agent. When moving agents, we will maintain this property of the set of empty nodes.

Call a node *available* if it is empty and adjacent only to a non-red agent, and perhaps other empty nodes. We are now ready to describe our procedure to convert C_G into an equilibrium assignment. We repeatedly take the first candidate from the ordered list of candidates and move it to an available node. Observe that if a candidate red agent \mathcal{X} moves to an available node, we have the following properties:

- The utility of \mathcal{X} increases to 1.
- The utility of all other agents stays the same or increases. In particular, if \mathcal{X} is an agent of utility 0 before the move, then all its neighbors are red agents. For a neighbor of \mathcal{X} that is a mixed agent, the move of \mathcal{X} will increase the mixed agent's utility. For any other neighbors of \mathcal{X} , their utility does not change because of \mathcal{X} 's move, and remains 0. For the case when \mathcal{X} itself is a mixed agent, it is condition 2 of Claim 4.7 must have held after the preprocessing, so all non-red neighbors of \mathcal{X} will still have utility 1. Also, the utility of any red neighbors of \mathcal{X} will not decrease as the result of \mathcal{X} moving. Thus, for any move of a candidate agent, the utility of all other agents stays the same or increases.
- If the agent that is moved had utility 0, the newly vacated node is adjacent only to red agents and possibly empty nodes, and therefore offers utility 0 to any red agent.
- If the agent that is moved was a mixed agent, the newly vacated node may offer utility greater than 0 to a red agent, but since it must be the last candidate agent in the list, all other red agents already have utility 1 and therefore have no incentive to move.
- Any empty neighbor of the newly occupied node offers utility 0 to any red agent. Thus, we do not create any new available nodes, and the number of available nodes and the number of candidate agents each decreases by 1.

This process ends when there are no more available nodes or there are no more candidates. If there are no more available nodes, we have an equilibrium assignment, as all other empty nodes offer utility 0 to a red agent. If there are no more candidates, then all red agents have utility 1, and we have an equilibrium assignment. \square

4.1 Spider topologies

As we showed at the beginning of this section, if the topology is an arbitrary tree, the game might have an IRC, even when there is a single empty node (see Theorem 4.1). Nevertheless, there is a class of tree graphs for which the game turns out to be potential, namely that of *spider graphs*. A spider graph is a tree in which all nodes besides one, called the *center node* c , have degree of at most 2 (see Fig. 6 for an example); the degree of c is $\delta(c) \geq 3$. We will show that any diversity-seeking jump game in which the topology is a spider graph with a **single empty node** is a potential game.

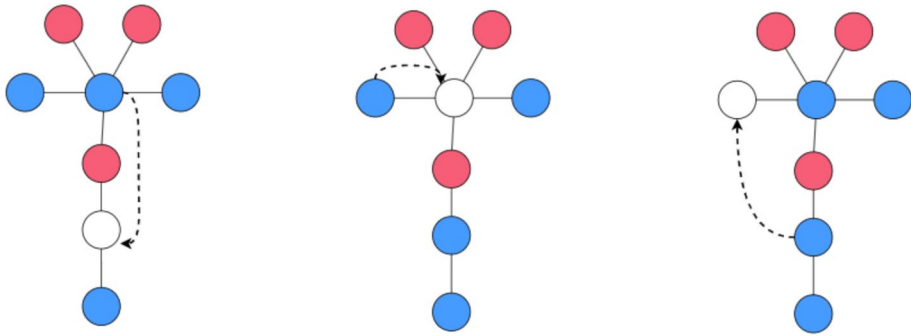


Fig. 6 An IRC in a spider graph with similarity-seeking strategic agents

Theorem 4.8 For $k \geq 2$, every diversity-seeking jump game (R, S, T, G, λ) where G is a spider graph and $|V| = |R \cup S| + 1$ is a potential game.

The proof of Theorem 4.8 (which is deferred to the appendix) is based on a careful case analysis aiming to show that, while the function Φ given in Definition 2 is not a generalized ordinal potential function for any value of $m \in (0, 1)$, any *long enough sequence of improving moves* is guaranteed to result in a lower value of the function; this suffices to show convergence of IRD. We remark that our proof assumes that there is a single empty node. It is unclear what happens when there are more empty nodes in the graph. In contrast to the above result, the similarity-seeking jump games studied in [20] are *not* potential; see the example given in Fig. 6.

5 Networks of bounded degree

In this section, we consider diversity-seeking jump games on networks of fixed degree. We first show that diversity-seeking jump games are potential games whenever the topology is a regular graph and there is a single empty node.

Theorem 5.1 For any $k, \delta \geq 2$, every diversity-seeking jump game (R, S, T, G, λ) , where G is a δ -regular graph and $|V| = |R \cup S| + 1$, is a potential game.

Proof Assume that there are no stubborn agents. We show that the function Φ from Definition 2 is a generalized ordinal potential function. Consider an improving move of an agent \mathcal{A} in assignment C_0 and call the resulting assignment C_1 . Let x_0 and x_1 be the number of \mathcal{A} 's neighbors of type different than that of \mathcal{A} in C_0 and C_1 , respectively. If \mathcal{A} does not move to a neighboring node, then we have $u_{\mathcal{A}}(C_0) = x_0/\delta$ and $u_{\mathcal{A}}(C_1) = x_1/\delta$, where $u_{\mathcal{A}}(C_0) < u_{\mathcal{A}}(C_1)$ and therefore $x_0 < x_1$. Hence,

$$\Phi(C_1) - \Phi(C_0) = m\delta + (\delta - x_1) - (\delta - x_0) - m\delta = x_0 - x_1 < 0.$$

It is easy to verify that the difference in potential is the same if \mathcal{A} moves to a neighboring empty node. This shows that Φ is a generalized ordinal potential function (for any value of

$m \in (0, 1)$) which completes the proof for the case where there are no stubborn agents. By Lemma 2.1, the game is also a potential game when $S \neq \emptyset$. \square

Theorem 5.1 does not generally extend to the case where there are more empty nodes in the topology. We show next that when there are 4 or more empty nodes, there can be IRCs in the dynamics.

Theorem 5.2 *For every $\delta \geq 4$ and $k \geq 2$, there exists a diversity-seeking jump game $I = (R, S, T, G, \lambda)$ where G is a δ -regular graph and $|V| = |R \cup S| + \delta$ such that I is not a potential game.*

Proof Note that since $|V| = |R \cup S| + \delta$, there will be δ empty nodes in any assignment. For $\delta = 4$, we show an IRC in a regular graph of degree 4 as depicted in Fig. 7. It is easy to verify that the moves shown are improving ones. Indeed, in the first move, the utility of the red agent improves from 0 to $1/2$. In the second move, the utility of the blue agent improves from $3/4$ to 1. In the third move, the utility of the red agent improves from $2/3$ to $3/4$. In the next three moves, the red and blue agents interchange roles, with the same changes in utilities.

We now give an inductive argument for the case of $\delta > 4$. Suppose inductively that there is an IRC in game I on a δ -regular graph G for some $\delta \geq 4$. We now construct a $(\delta + 1)$ -regular graph H by creating a duplicate of graph G called G' and connecting every node

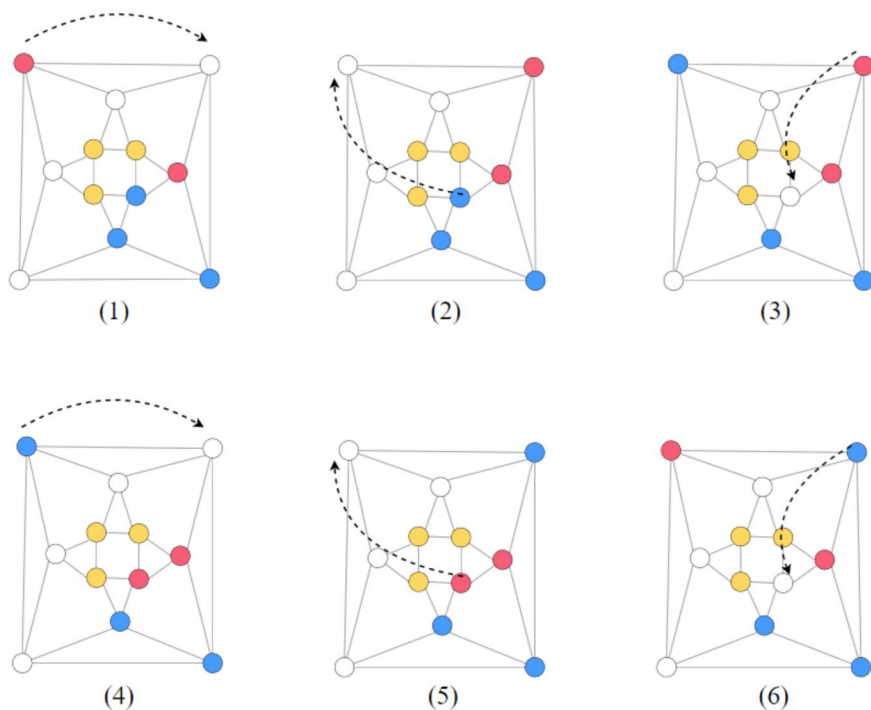


Fig. 7 An IRC in a regular graph of degree 4 with 4 empty nodes

in G' to its counterpart in G . Consider now the $(\delta + 1)$ -regular graph $G \cup G'$, and the following placement of agents. We keep the same assignment of agents in G . The counterpart in G' of the node containing the moving red agent has no agents assigned to it, while all other nodes in G' are assigned yellow agents. It can be verified that the same IRC exists in this graph for the same red agent and blue agent visiting the same nodes as in G . Indeed, in the first move, the utility of the red agent improves from 0 to $\frac{d-3}{d-2}$. In the second move, the utility of the blue agent improves from $\frac{d-1}{d}$ to 1. In the third move, the utility of the red agent improves from $\frac{d-2}{d-1}$ to $\frac{d-1}{d}$. As before, in the next three moves, the red and blue agents interchange roles, with the same changes in utilities. This completes the inductive argument. \square

Note that there is a gap between the results of Theorems 5.1 and 5.2. The question of whether diversity-seeking jump games are potential games in regular graphs of degree 3, or regular graphs of degree 4 with 2 or 3 empty nodes remains open. Finally, we remark that an IRC on a regular graph with $k = 2$ can be constructed from the tree shown in Fig. 2 but the number of empty nodes would be greater than 4.

Next, we consider topologies of maximum degree 2 (that may not be regular), and show that, regardless of the number of empty nodes, any diversity-seeking jump game is a potential game.

Theorem 5.3 *For $k \geq 2$, every diversity-seeking jump game (R, S, T, G, λ) where G is a graph of maximum degree 2 is a potential game.*

Proof Assume that there are no stubborn agents and there are at least 3 nodes in G . Consider an improving move of an agent \mathcal{A} in assignment C_0 that leads to assignment C_1 . Suppose \mathcal{A} moves from node s (in C_0) to node t (in C_1). We first consider the case where s and t are neighbors: If s has degree 1 and t has degree 2, since this is an improving move for \mathcal{A} , t 's other neighbor must have an agent of a different type from \mathcal{A} , and thus $\Phi(C_1) - \Phi(C_0) = -m$. If, instead, s has degree 2 and t has degree 2, it must be that \mathcal{A} 's other neighbor in C_0 is of the same type as \mathcal{A} or empty, while t is adjacent to an agent of a different type than that of \mathcal{A} . Thus, $\Phi(C_1) - \Phi(C_0) \leq -m$.

Next, we look at the cases where s and t are not neighbors:

- s and t have degree 1. Then, let s 's neighbor be an agent of the same type as \mathcal{A} or it is an empty node, while t 's neighbor must be an agent of type different than that of \mathcal{A} . Hence, $\Phi(C_1) - \Phi(C_0) \leq -m$.
- s has degree 1 and t has degree 2. Then, either \mathcal{A} improves its utility from 0 to $1/2$ and thus $\Phi(C_1) - \Phi(C_0) \leq 1 - 2m$, or improves its utility from 0 to 1 and thus $\Phi(C_1) - \Phi(C_0) \leq -m$.
- s has degree 2 and t has degree 1. Then, either \mathcal{A} improves its utility from 0 to 1 and thus $\Phi(C_1) - \Phi(C_0) \leq -m$, or improves its utility from $1/2$ to 1 and thus $\Phi(C_1) - \Phi(C_0) = 2m - 1 - m = -m - 1$.
- s and t have degree 2. Then, either \mathcal{A} improves its utility from 0 to $1/2$ and thus $\Phi(C_1) - \Phi(C_0) \leq 1 - 2m$, or from 0 to 1 and thus $\Phi(C_1) - \Phi(C_0) \leq -m$, or from $\frac{1}{2}$ to 1 and thus $\Phi(C_1) - \Phi(C_0) \leq m - 1$.

In all possible cases, $\Phi(C_1) - \Phi(C_0) < 0$ for all values of $m \in (0, 1)$. Therefore, Φ is a generalized ordinal potential function, and the diversity-seeking jump game is a potential game. From Lemma 2.1, the result is also true when stubborn agents exist. \square

6 Efficiency at equilibrium

In this section, we turn our attention to showing welfare guarantees at equilibrium by bounding the price of anarchy and the price of stability. We show bounds on the price of anarchy for games consisting of $n \geq 2$ *strategic* agents that are partitioned into $k \geq 2$ types; note that we do not consider games with stubborn agents in this section, we focus exclusively on the case of strategic agents as they are the ones that care about their utility and the quality of the assignment. The types might be *asymmetric* or *symmetric*. In the asymmetric case, each type T consists of some number $n_T \geq 1$ of agents without any restriction other than that the types form a partition of the set of all agents. In contrast, in the symmetric case, each type T consists of the same number $n_T = n/k$ of agents; here we assume that n/k is an integer. We show that the price of anarchy is at most $n/(k-1)$ for asymmetric types and at most $k/(k-1)$ for symmetric types. We complement these upper bounds with tight lower bounds, which in some cases are even achieved in games with a topology as simple as a line. For the price of stability, we provide a lower bound of $65/62 \approx 1.048$ for $k = 2$ types showing that there are games in which the optimal assignments (in terms of social welfare) are not always equilibria.

We start by showing the upper bounds on the price of anarchy.

Theorem 6.1 *For $k \geq 2$ types, the price of anarchy of any diversity-seeking jump game with n strategic agents is at most $\frac{n}{k-1}$ if the types are asymmetric, and at most $\frac{k}{k-1}$ if the types are symmetric.*

Proof Consider an arbitrary game I and denote by C some equilibrium assignment, in which there is an empty node v that is adjacent to x_T agents of type T . Let $x = \sum_T x_T \geq 1$ be the total number of agents adjacent to v . First, suppose that $x = x_T = 1$ for some type T . Then the agents of all types different than T must have utility 1 to not have incentive to deviate to v , which means that $\text{SW}(C) \geq \sum_{t \neq T} n_t$. Since the maximum utility of each agent is 1, $\text{OPT}(I) \leq n$, leading to a price of anarchy of at most $\frac{n}{n-n_T}$. If the types are asymmetric, then $n_T \leq n - k + 1$, and the price of anarchy is at most $\frac{n}{k-1}$. If the types are symmetric, then $n_T = n/k$, and the price of anarchy is at most $\frac{k}{k-1}$.

So, now we assume that $x \geq 2$. To not have incentive to deviate to v , any agent of type T that is adjacent to v must have utility at least $\frac{\sum_{t \neq T} x_t}{x-1} = \frac{x-x_T}{x-1}$, while any agent of type T that is not adjacent to v must have utility at least $\frac{\sum_{t \neq T} x_t}{x} = \frac{x-x_T}{x}$. So, the social welfare at equilibrium can be lower-bounded as follows:

$$\begin{aligned}
\text{SW}(C) &\geq \sum_T \left(x_T \cdot \frac{x - x_T}{x - 1} + (n_T - x_T) \cdot \frac{x - x_T}{x} \right) \\
&\geq \frac{1}{x} \sum_T n_T \cdot (x - x_T) \\
&= n - \frac{1}{x} \sum_T n_T x_T.
\end{aligned}$$

If the types are asymmetric, then $n_T \leq n - k + 1$ for every type T , and the social welfare at equilibrium is

$$\text{SW}(C) \geq n - \frac{1}{x} \sum_T (n - k + 1) x_T = n - (n - k + 1) = k - 1.$$

If the types are symmetric, then $n_T = n/k$ for every type T , and the social welfare at equilibrium is

$$\text{SW}(C) \geq n - \frac{1}{x} \sum_T \frac{n}{k} x_T = n - \frac{n}{k} = n \frac{k-1}{k}.$$

Since $\text{OPT}(I) \leq n$, we conclude that the price of anarchy is at most $\frac{n}{k-1}$ if the types are asymmetric, and at most $\frac{k}{k-1}$ if the types are symmetric. \square

Next, we investigate under which conditions the upper bounds of Theorem 6.1 are tight. We start with the case of asymmetric types. We show that for games in which the topology is a line, a better price of anarchy bound can be shown; in particular, the price of anarchy can be shown to be at most a function of n that tends to 2, for any number of types and any distribution of agents into types.

Theorem 6.2 *The price of anarchy of any diversity-seeking jump game with a line topology and n strategic agents of $k \geq 2$ types is at most $\frac{2n}{n-1}$.*

Proof Consider an arbitrary game I with a line topology, and let C be some equilibrium assignment. First, suppose that there is an empty node v in C that is adjacent to only agents of one type, say red. Then, to not have incentive to deviate to v , the non-red agents must all have utility 1. Observe that there must exist some red agent with utility at least $1/2$; otherwise there would exist some empty node that is adjacent to non-red agents (since G is a connected line), where the red agents could jump to get positive utility. So, $\text{SW}(C) \geq n - n_r + 1/2$, where n_r is the number of red agents. We consider the following two cases:

- If $n_r \leq \lfloor n/2 \rfloor + 1$, then since $\text{OPT}(I) \leq n$ and $\text{SW}(C) \geq n - n_r + 1/2 \geq \lceil n/2 \rceil - 1/2$, the price of anarchy is at most $\frac{n}{\lceil n/2 \rceil - 1/2}$. This is at most $\frac{2n}{n-1}$ if n is even, and at most 2 if n is odd.
- If $n_r > \lfloor n/2 \rfloor + 1$, then it is not possible to achieve social welfare n since some red agents cannot have only neighbors of different type. In particular, the maximum possible social welfare is $\text{OPT}(I) \leq 2(n - n_r) + 1 = 2(n - n_r + 1/2)$; this can be achieved

by creating $n - n_r$ pairs of agents of different type (which, in the optimal assignment, are placed one after the other on the line) and possibly one more red agent placed at the beginning or the end of the line getting utility 1. Hence, the price of anarchy is at most 2.

Finally, suppose that the empty node v in C is adjacent to agents of two different types, say red and blue. Then, to not have incentive to deviate to v , the non-red and non-blue agents must all have utility 1, the two agents adjacent to v must also have utility 1, and the remaining red and blue agents must have utility at least $1/2$. Therefore, since every agent gets utility at least $1/2$ and the maximum utility is 1, the price of anarchy is at most 2. \square

We next show that the price of anarchy approaches the upper bound for asymmetric types when the topology is slightly more general than a line. In particular, we show the following lower bound.

Theorem 6.3 *There is a diversity-seeking jump game with a star topology and n strategic agents of $k \geq 2$ asymmetric types such that the price of anarchy is at least $\frac{n-1}{k-1}$.*

Proof Consider a game with star topology and $k \geq 2$ types such that $k - 1$ types have size 1 while one type, say red, has size $n - k + 1$. Any assignment such that the center node of the topology is occupied by an agent of the first $k - 1$ types is optimal with social welfare n , whereas the assignment according to which the center node is occupied by a red agent is an equilibrium with social welfare $k - 1 + \frac{k-1}{n-1} = (k - 1) \frac{n}{n-1}$. Hence, the price of anarchy is at least $\frac{n-1}{k-1}$. \square

We now turn our attention to the case of games with symmetric types. We first show that for $k = 2$ symmetric types, a slightly better bound than that of Theorem 6.2 can be shown when the topology is a line; this bound is a different function of n that again tends to the upper bound of 2 when n becomes large.

Theorem 6.4 *For diversity-seeking jump games with line topology and n strategic agents of $k = 2$ symmetric types, the price of anarchy is exactly $\frac{2n}{n+4}$.*

Proof For the lower bound, consider a game with a line topology consisting of $n + 1$ nodes and n agents of two types, red (r) and blue (b). In the optimal assignment, the agents occupy the nodes of the line so that each red agent is followed by a blue agent. This guarantees that every agent has only neighbors of different type and thus achieves a maximum utility of 1, leading to an optimal social welfare of n . Now consider the assignment in which the agents are ordered according to their types as follows: $(r, b, b, r, r, \dots, r, r, b, b, r, v, b, r)$, where v is the empty node. This is an equilibrium since the empty node is adjacent to a red and a blue agent, and every agent has either only neighbors of different type (and thus utility 1), or one red and one blue neighbor (and thus utility $1/2$); hence, no agent would have incentive to jump to the empty node since it only offers utility $1/2$. There are exactly 4 agents that achieve utility 1 and $n - 4$ agent that achieve utility $1/2$, for a social welfare of $(n - 4)/2 + 4 = (n + 4)/2$. Consequently, the price of anarchy is at least $2n/(n + 4)$.

For the upper bound, let C be an equilibrium assignment. First observe that, if there is an empty node v in C that is adjacent to a single agent or two agents of the same type, say red,

then it has to be the case that all blue agents have utility 1, and consequently the same holds for the red agents due to symmetry, leading to price of anarchy of 1. So, we can assume that v is adjacent to one red agent \mathcal{A} and one blue agent \mathcal{B} . Since C is an equilibrium, both \mathcal{A} and \mathcal{B} must have a neighbor of different type and utility 1 since otherwise they would jump to v to connect to each other. In addition, since the topology is a line, there must be at least two more agents with a single neighbor and utility 1; otherwise they would have utility 0 and incentive to jump to v for a utility of $1/2$. Similarly, all remaining agents must already achieve utility at least $1/2$ to not have incentive to jump to v . Hence, the social welfare at any equilibrium assignment must be at least $(n - 4)/2 + 4 = (n + 4)/2$. Since the maximum possible social welfare is n , the price of anarchy is at most $2n/(n + 4)$. \square

For games with $k \geq 3$ symmetric types and line topology, we show a bound of $\frac{n}{(k-1)\frac{n}{k}+1/2}$ which almost matches the upper bound of $k/(k - 1)$.

Theorem 6.5 *For diversity-seeking jump games with a line topology and n strategic agents of $k \geq 3$ symmetric types such that the price of anarchy is exactly $\frac{n}{(k-1)\frac{n}{k}+\frac{1}{2}}$.*

Proof For the lower bound, consider a game with a line topology consisting of $n + 1$ nodes. Suppose that the k types are identified by the integers $\{1, \dots, k\}$. An optimal assignment is to arrange the agents so that an agent of type T is followed by an agent of type $T + 1$, for every $T \leq [k - 1]$. Hence, every agent gets utility 1 for an optimal social welfare of n . Now consider the following assignment: There is a repeated sequence of agents so that an agent of type T is followed by an agent of type $T + 1$, for every $T \in \{1, \dots, k - 2\}$. This sequence is followed by all agents of type k and the last node is left empty. Clearly, all agents of the first $k - 1$ types have utility 1 and no incentive to deviate to the empty node, while there is exactly one agent of type k with utility $1/2$ and all others have utility 0. So, this is an equilibrium with social welfare $(k - 1)\frac{n}{k} + 1/2$, leading to the desired lower bound on the price of anarchy.

For the upper bound, let C be an equilibrium assignment. We consider the following two cases:

- There is an empty node v in C that is adjacent to a single agent or two agents of the same type, say red. Observe that all non-red agents must have utility 1 so that they do not have incentive to deviate to v . In addition, at least one of the red agents must be adjacent to an agent of different type and thus have utility at least $1/2$; otherwise, if all red agents are adjacent only to other red agents or empty nodes, then there must exist a non-red agent that is adjacent to an empty node where the red agents would like to jump. Hence, the social welfare of this equilibrium is at least $(k - 1)\frac{n}{k} + 1/2$. Since the optimal social welfare is at most n , we get the desired upper bound on the price of anarchy.
- There is an empty node v in C that is adjacent to two agents of different type, say red and blue. Then all non-red and non-blue agents must have utility 1 so that they do not have incentive to deviate to v . In addition, both agents that are adjacent to v must have utility 1 to not have incentive to deviate to v and connect to each other, while all other red and blue agents must have utility at least $1/2$. Hence, the social welfare of this equilibrium is at least $(k - 2)\frac{n}{k} + 2 + 2\left(\frac{n}{k} - 1\right) \cdot \frac{1}{2} = (k - 1)\frac{n}{k} + 1 \geq (k - 1)\frac{n}{k} + 1/2$, leading to the desired upper bound again.

This completes the proof. \square

We conclude this section with a lower bound of $65/62$ on the price of stability, showing that the assignment with optimal social welfare is not always an equilibrium.

Theorem 6.6 *There is a diversity-seeking jump game with $k = 2$ types of strategic agents such that the price of stability is at least $65/62 \approx 1.048$.*

Proof Consider a diversity-seeking jump game with 2 red agents and 4 blue agents. The topology is as shown in Fig. 8(a). The assignment of the agents in Fig. 8(b) is the optimal one with social welfare $65/12$, while that in Fig. 8(c) is the best equilibrium with social welfare $62/12 \approx 5.16$. We will now argue that any equilibrium assignment C has social welfare at most $62/12$, yielding the desired lower bound on the price of stability. For simplicity in our notation, for any node v , we will denote by $\mathcal{A}(v)$ the agent that occupies it (if any).

First note that if there is an agent with utility 0, or two agents with utility at most $1/2$, then the social welfare is bounded above by 5, and we are done. So, we can assume that C has the following two properties: (1) No agent has utility 0, and (2) at most one agent has utility $\leq 1/2$. Property (1) implies immediately a third property: (3) Either node a is empty, or a and b are occupied by agents of different types. This follows since agent $\mathcal{A}(a)$ would have utility 0 if node b is empty, or if both a and b are occupied by agents of the same type.

Next, observe that if the degree-2 nodes d_1 and d_2 are occupied by agents of different types, then either they both have utility $1/2$ (violating Property (2)) or one of them has utility 0 (violating Property (1)). Therefore it must be the case that either one of d_1 and d_2 is left empty, or both are occupied by agents of the same type. We consider these two cases separately:

Case 1: d_2 is empty; the case where d_1 is empty is symmetric If $\mathcal{A}(d_1)$ is red, then since it must obtain utility strictly larger than 0 and one of the agents at a or b must be red by Property (3), it must be the case that c , e_1 and e_2 are all occupied by blue agents and have utility at least $1/2$, a contradiction to Property (2). If on the other hand $\mathcal{A}(d_1)$ is blue, then since

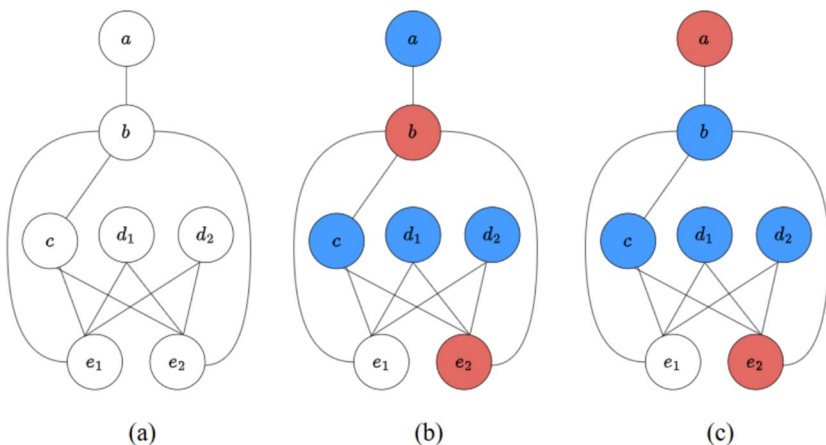


Fig. 8 (a) The topology of the game used to show the lower bound on the price of stability; (b) The optimal assignment with social welfare $65/12$; (c) The best equilibrium assignment with social welfare $62/12$

either a or b must be occupied by a red agent by Property (3), e_1 and e_2 must be occupied by agents of different type, say e_1 is occupied by a red agent and e_2 is occupied by a blue agent. However, this means that $\mathcal{A}(d_1)$ and $\mathcal{A}(e_2)$ have utility at most $1/2$, a contradiction to Property (2).

Case 2: d_1 and d_2 are occupied by agents of the same type If the agents $\mathcal{A}(d_1)$ and $\mathcal{A}(d_2)$ are red, then all the blue agents must be at the remaining nodes, which means that at least one of them must get utility 0, thus contradicting Property (1). So, the agents $\mathcal{A}(d_1)$ and $\mathcal{A}(d_2)$ must both be blue. To satisfy Properties (1) and (2), either both e_1 and e_2 are both occupied by red agents, or one of them is empty and the other is occupied by a red agent.

- **Both $\mathcal{A}(e_1)$ and $\mathcal{A}(e_2)$ are red.** To satisfy Property (3), since it is not possible to assign different-type agents at a and b , it has to be the case that a is empty. However, such an assignment cannot be an equilibrium as the red agents at e_1 and e_2 have utility less than 1 (since they are connected to each other) and would jump to a to get utility 1.
- **One of e_1 and e_2 is occupied by a red agent, while the other is empty.** Suppose without loss of generality that e_1 is empty and e_2 is occupied by a red agent. Since a and b must have agents of different types due to Property (3), only two assignments are possible. The first one is when $\mathcal{A}(a)$ is blue and $\mathcal{A}(b)$ is red. This is the optimal assignment with social welfare $65/12$, which however is not an equilibrium since $\mathcal{A}(b)$ has utility $2/3$ and would prefer to deviate to the empty node e_1 to get utility $3/4$; see Fig. 8(b). The second one is when $\mathcal{A}(a)$ is red and $\mathcal{A}(b)$ is blue. This is the best possible equilibrium with social welfare $62/12$; see Fig. 8(c).

The proof is now complete. □

7 Open problems

In this paper we proposed a new class of diversity-seeking jump games in networks. We showed that determining if there exists an equilibrium assignment in a given instance is NP-hard, using a reduction that relies on the existence of stubborn agents. Do games with only strategic agents always admit an equilibrium? If not, what is the complexity of deciding if an equilibrium exists in a given diversity-seeking jump game with only strategic agents?

We also showed that the game is not potential for tree topologies; however, if all agents are strategic, there is always an equilibrium, regardless of the number of empty nodes in the tree. It would be interesting to know if the game is weakly acyclic in trees. For spider graphs with a single empty node, we showed that the game is potential. We conjecture the same holds for any number of empty nodes. While games on δ -regular graphs are always potential games when there is a single empty node, there are games that are not potential when there are 4 or more empty nodes. It would be very interesting to know whether the game is potential for up to 3 empty nodes. Similar questions would be interesting for other classes of graphs, such as bipartite graphs and grids.

Finally, while our price of anarchy bounds are essentially tight, it would be interesting to further investigate the price of stability and either improve the lower bound of $65/62$ that we

managed to show for two types, or show an upper bound for it that is better than the upper bounds shown for the price of anarchy.

A: Proof of Theorem 4.8

We start with a lemma providing upper bounds on the change in potential caused by different types of moves. Recall the definition of the function Φ (Definition 2).

Lemma A.1 *For $k \geq 2$, in every diversity-seeking jump game (R, S, T, G, λ) where G is a spider graph, for an improving move of an agent \mathcal{A} of type T in assignment C_0 that results in assignment C_1 , we have:*

1. $\Phi(C_1) - \Phi(C_0) \leq -m$ if the improving move does not involve the center node;
2. $\Phi(C_1) - \Phi(C_0) \leq 2m - 1 + n_T(c, C_1) - m\delta(c)$ if the improving move is from a degree-2 node to the center node;
3. $\Phi(C_1) - \Phi(C_0) \leq m - m\delta(c) + n_T(c, C_0)$ if the improving move is from a leaf to the center node;
4. $\Phi(C_1) - \Phi(C_0) \leq 1 - 2m + m\delta(c) - n_T(c, C_0)$ if the improving move is from the center node to any other node.

Proof We first consider all possible types only involve non-central nodes (Case 1), and then all possible moves that involve the center node, switching subcases based on the degree of the other node involved (Cases 2, 3 and 4). For each case, we consider all the possible utility scenarios that the deviating agent can derive at source and target. Tables 1 and 2 provide a detailed counting of the change in potential based on the source and target node of a move.

1. The improving move does not involve the center node c . As an illustrative example, Fig. 9 depicts the configurations corresponding to case a).

(a) Agent \mathcal{A} jumps from a degree-2 node to another degree-2 node:

- i. $u_{\mathcal{A}}(C_0) = 0$ and $u_{\mathcal{A}}(C_1) = \frac{1}{2}$. Figure 9 (i) shows the configuration C_0 and the jump from node s to d that would lead to C_1 . The jump only affects the potential on edges incident on s and d . In C_0 , the potential on the edges incident on s is 2, and the potential on edges incident on d is $2m$, for a total of $2 + 2m$. In configuration C_1 , the potential on edges incident on s is $2m$, and that on edges incident on d is 1, for a total of $1 + 2m$. Then $\Phi(C_1) - \Phi(C_0) = (1 + 2m) - (2 + 2m) = -1$.
- ii. $u_{\mathcal{A}}(C_0) = \frac{1}{2}$ and $u_{\mathcal{A}}(C_1) = 1$. Then $\Phi(C_1) - \Phi(C_0) = (2m) - (1 + 2m) = -1$.
- iii. $u_{\mathcal{A}}(C_0) = 0$ and $u_{\mathcal{A}}(C_1) = 1$ and source and target nodes are not adjacent. Then $\Phi(C_1) - \Phi(C_0) = (2m) - (2 + 2m) = -2$.
- iv. $u_{\mathcal{A}}(C_0) = 0$ and $u_{\mathcal{A}}(C_1) = 1$ and source and target nodes are adjacent. Then $\Phi(C_1) - \Phi(C_0) = (2m) - (2m + 1) = -1$.

(b) Agent \mathcal{A} jumps from degree-2 node to leaf node:

- i. $u_{\mathcal{A}}(C_0) = 0$ and $u_{\mathcal{A}}(C_1) = 1$. Then
 $\Phi(C_1) - \Phi(C_0) = (2m) - (2 + m) = m - 2$.
- ii. $u_{\mathcal{A}}(C_0) = \frac{1}{2}$ and $u_{\mathcal{A}}(C_1) = 1$. Then
 $\Phi(C_1) - \Phi(C_0) = (2m) - (1 + m) = m - 1$.

(c) Agent \mathcal{A} jumps from leaf node to degree-2 node:

- i. $u_{\mathcal{A}}(C_0) = 0$ and $u_{\mathcal{A}}(C_1) = \frac{1}{2}$. Then
 $\Phi(C_1) - \Phi(C_0) = (m + 1) - (1 + 2m) = -m$.
- ii. $u_{\mathcal{A}}(C_0) = 0$ and $u_{\mathcal{A}}(C_1) = 1$ and source and target nodes are not adjacent.
 Then $\Phi(C_1) - \Phi(C_0) = (m) - (1 + 2m) = -1 - m$.
- iii. $u_{\mathcal{A}}(C_0) = 0$ and $u_{\mathcal{A}}(C_1) = 1$ and source and target nodes are adjacent. Then
 $\Phi(C_1) - \Phi(C_0) = (m) - (2m) = -m$.

(d) Agent \mathcal{A} jumps from leaf node to leaf node: $\Phi(C_1) - \Phi(C_0) = (m) - (1 + m) = -1$.

2. The improving move is from a degree-2 node to center node:

- (a) $u_{\mathcal{A}}(C_0) = \frac{1}{2}$ and $u_{\mathcal{A}}(C_1) > \frac{1}{2}$:
 $\Phi(C_1) - \Phi(C_0) = (2m + n_T(c, C_1)) - (1 + m\delta(c))$.
- (b) $u_{\mathcal{A}}(C_0) = 0$ and $u_{\mathcal{A}}(C_1) > 0$ and source and target nodes are not adjacent:
 $\Phi(C_1) - \Phi(C_0) = (2m + n_T(c, C_1)) - (2 + m\delta(c))$.
- (c) $u_{\mathcal{A}}(C_0) = 0$ and $u_{\mathcal{A}}(C_1) > 0$ and source and target nodes are adjacent:
 $\Phi(C_1) - \Phi(C_0) = (2m + n_T(c, C_1)) - (1 + m\delta(c))$.

3. The improving move is from a leaf to center node:

- (a) Source and target nodes are not adjacent:
 $\Phi(C_1) - \Phi(C_0) = (m + n_T(c, C_1)) - (1 + m\delta(c))$.
- (b) Source and target nodes are adjacent:
 $\Phi(C_1) - \Phi(C_0) = (m + n_T(c, C_1)) - (m\delta(c))$.

4. The improving move is from center node to any other node:

(a) Agent \mathcal{A} jumps from center node to degree-2 node:

- i. $u_{\mathcal{A}}(C_0) < \frac{1}{2}$ and $u_{\mathcal{A}}(C_1) = \frac{1}{2}$. Then
 $\Phi(C_1) - \Phi(C_0) = (1 + m\delta(c)) - (n_T(c, C_0) + 2m)$.
- ii. $u_{\mathcal{A}}(C_0) < 1$ and $u_{\mathcal{A}}(C_1) = 1$ and source and target nodes are not adjacent.
 Then $\Phi(C_1) - \Phi(C_0) = (m\delta(c)) - (n_T(c, C_0) + 2m)$.
- iii. $u_{\mathcal{A}}(C_0) < 1$ and $u_{\mathcal{A}}(C_1) = 1$ and source and target nodes are adjacent. Then
 $\Phi(C_1) - \Phi(C_0) = (m\delta(c)) - (n_T(c, C_0) + 2m)$.

(b) Agent \mathcal{A} jumps from center node to leaf node:
 $\Phi(C_1) - \Phi(C_0) = (m\delta(c)) - (n_T(c, C_0) + m)$. The proof is now complete. \square

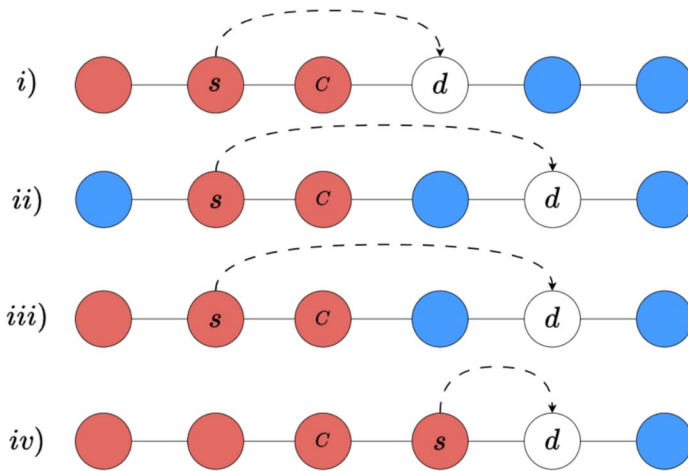


Fig. 9 Example configurations corresponding to case 1(a) illustrating improving moves between two nodes of degree 2 that does not involve the center node. Node c is the center of the spider graph

Table 1 Summary of the changes of the potential when the topology is a spider graph, resulting from improving moves that do not involve the center node

Type of move	Source utility	Target utility	$\Delta\Phi \leq$
degree 2 to degree 2	0	1/2	-1
	0	1	-1
	1/2	1	-1
degree 2 to leaf	0	1	$m - 2$
	1/2	1	$m - 1$
Leaf to degree 2	0	1/2	$-m$
	0	1	$-m$
Leaf to leaf	0	1	-1

Note that $m < 1/2$

For the remainder of the section we will assume that $m < \frac{1}{2}$. Lemma A.1 then implies that any improving move of an agent that does not involve the center node always *decreases* the value of the potential function Φ . However, an improving move of an agent that moves to or out of the center can increase the potential by a non-constant amount. The change in potential is related to the degree of the center node and the number of neighbors of the same type as the agent that moved to or from the center node. We claim, however, that such a move that increases the potential must be followed by moves that collectively decrease the potential *below* its value before the move in question, and thus that the game is a potential game.

Suppose towards a contradiction that there is an IRC. Consider the smallest IRC of length p , and denote the assignments that it involves by C_0, C_1, \dots, C_{p-1} . Since any improving move of an agent that does not involve the center node always decreases the potential, our IRC must involve moves to or out of the center node. Since every move of an agent \mathcal{A} to the center node must be followed by a move of agent \mathcal{A} out of the center node, we calculate the change in potential resulting from the set of moves that includes the move to the center, move out of the center, and any moves in between. Let $C_{j_0}, C_{j_1}, \dots, C_{j_w}$, be the assignments in the cycle in which the center node c is unoccupied, listed in increasing order. We denote a

Table 2 Summary of the potential change when the topology is a spider graph, resulting from improving moves that involve the center node

Type of move	Source utility	Target utility	$\Delta\Phi \leq$
degree 2 to center (adjacent)	0	> 0	$2m - 1 + n_T(c, C_1) - m\delta(c)$
degree 2 to center (non-adjacent)	0	> 0	$2m - 2 + n_T(c, C_1) - m\delta(c)$
	1/2	$> 1/2$	$2m - 1 + n_T(c, C_1) - m\delta(c)$
Center to degree 2	$< 1/2$	1/2	$1 - 2m + m\delta(c) - n_T(c, C_0)$
	< 1	1	$-2m + m\delta(c) - n_T(c, C_0)$
Leaf to center (adjacent)	0	> 0	$m + n_T(c, C_1) - m\delta(c)$
Leaf to center (non-adjacent)	0	> 0	$m - 1 + n_T(c, C_1) - m\delta(c)$
Center to leaf	< 1	1	$-m + m\delta(c) - n_T(c, C_0)$

Note that $m < 1/2$

segment i by a pair of consecutive assignments C_{j_i} and $C_{j_{i+1}}$; For readability, we use $i + 1$ to mean $(i + 1) \bmod (w + 1)$ when referring to the indices of the assignments. It is easy to see that the total change in the potential in the IRC is the sum of changes in the potential of segments. Furthermore, we define $C_{j_i}^+$, $C_{j_i}^-$ to be the immediate successor and predecessor of C_{j_i} , respectively. Notice that since we have a single empty node, for every segment i , the center node is occupied in $C_{j_i}^+$, $C_{j_{i+1}}^-$, and any intermediate assignment between them. Also, notice that $C_{j_i}^+$ and $C_{j_{i+1}}^-$ are two distinct assignments as it is impossible for an agent to move to the center node (going from assignment C_{j_i} to $C_{j_i}^+$) and move out of it (going from assignment $C_{j_{i+1}}^-$ to $C_{j_{i+1}}$) immediately. We show that, for every segment i , the difference in potential is always at most 0. Furthermore, we show that it is impossible for the difference to be 0 for every segment, which yields a contradiction to the existence of an IRC.

The following lemma gives an upper bound on the change in potential between assignments $C_{j_i}^+$ and $C_{j_{i+1}}^-$, for any $0 \leq i \leq w$.

Lemma A.2 Suppose the center node c is occupied by an agent \mathcal{A} of type T in segment i . Then, for any $0 \leq i \leq w$, the change in potential between assignments $C_{j_i}^+$ and $C_{j_{i+1}}^-$ is

$$\Phi(C_{j_{i+1}}^-) - \Phi(C_{j_i}^+) \leq \begin{cases} -|\Delta_T| & \text{if } s, t \text{ are not adjacent to } c & (1) \\ -|\Delta_T| + m & \text{if only } t \text{ is adjacent to } c & (2) \\ -|\Delta_T| + m - 1 & \text{if only } s \text{ is adjacent to } c & (3) \\ -|\Delta_T| + 2m - 1 & \text{if } s, t \text{ are adjacent to } c, & (4) \end{cases}$$

where $\Delta_T = n_T(c, C_{j_{i+1}}^-) - n_T(c, C_{j_i}^+)$.

Proof We will start with the easiest case when neither s nor t is adjacent to the center. It is clear that every change in the number of neighbors of the center that are occupied by agents of type T requires a pair of improving moves, an agent of type T to move out of the neighborhood and an agent of some other type to move to its place (thereby decreasing the number of neighbors of type T), or an agent of a different type to move out of the neighborhood and an agent of type T to move to its place (increasing the number of neighbors of type T). From Table 1, we get an upper bound on the change in potential of each such pair of moves. First we note that it is impossible for both such moves to be (leaf to degree 2), or

both such moves to be (degree 2 to leaf). It is straightforward to verify that all other pairs of moves have a net change in potential of at least -1 . Inequality (1) follows.

Now suppose that s is not adjacent to the center, but t is. Then, the agent \mathcal{B} at node t in the assignment before $C_{j_{i+1}}^-$ moved out to leave t vacant in $C_{j_{i+1}}^-$. If \mathcal{B} is not of type T , it did not contribute to a change in the number of neighbors of type T of the center, and the same reasoning as in the previous case applies to give us Inequality (1), which implies Inequality (2). If \mathcal{B} instead is of type T , then its move out of t does change the number of neighbors of type T , but does not have a matching move back to the center. Noting that t cannot be a leaf since \mathcal{A} would not jump to it in the next step, the move of \mathcal{B} out of t causes a potential change of at most $m - 1$ (using Table 1). Together with the potential change due to the remaining $\Delta_T - 1$ pairs of improving moves, we get Inequality (2).

If s is adjacent to the center and t is not, observe that the agent that moves to s immediately after \mathcal{A} moves to the center is not of type T and does not contribute to the change in neighbors of type T . The change in the potential resulting from this improving move is at most $m - 1$ as shown in Table 1, and together with the change in potential caused by Δ_T pairs of improving moves as in the first case, we get Inequality (3).

Last we consider the case when both s and t are adjacent to the center. The first move does not contribute to the change in neighbors of type T and changes the potential by at most $m - 1$, while the last move may contribute to this change, and in that case, would not have a matching move of an agent back to a neighbor of the center. The total change in potential is at most $m - 1$ for the first move, at most $m - 1$ for the last move, and $|\Delta_T| - 1$ for pairs of improving moves contributing to change in neighbors of type T . Thus, we get Inequality (4). \square

Now we consider the difference in potential between $C_{j_{i+1}}$ and C_{j_i} . We analyze separately the cases when s is a degree-2 node or a leaf node.

Lemma A.3 *If s is a degree-2 node, then $\Phi(C_{j_{i+1}}) - \Phi(C_{j_i}) < 0$.*

Proof The source utility of \mathcal{A} at node s could be either 0 or $1/2$ and the target utility at node t could be $1/2$ or 1. We consider all possible combinations below:

1. $u_{\mathcal{A}}(C_{j_i}) = 0$, $u_{\mathcal{A}}(C_{j_{i+1}}) = 1/2$ and the source node s is *not* adjacent to the center. In assignment C_{j_i} , agent \mathcal{A} is located at s and has two neighbors of the same type. Since $u_{\mathcal{A}}(C_{j_i}) = 0$, $u_{\mathcal{A}}(C_{j_i}^+) > 0$. Eventually agent \mathcal{A} moves out of the center node to the target node t to reach assignment $C_{j_{i+1}}$ where it has a neighbor of the same type and a neighbor of different type. Note that t must be a degree-2 node and it cannot be adjacent to the center node. Since $u_{\mathcal{A}}(C_{j_{i+1}}) = 1/2$, $u_{\mathcal{A}}(C_{j_{i+1}}^-) < 1/2$. Therefore, in assignment $C_{j_{i+1}}^-$, more than half of the neighbors of agent \mathcal{A} are of type T . From Table 2, we obtain:

$$\Phi(C_{j_i}^+) - \Phi(C_{j_i}) \leq -m\delta(c) + n_T(c, C_{j_i}^+) + 2m - 2 \quad (5)$$

and

$$\Phi(C_{j_{i+1}}) - \Phi(C_{j_{i+1}}^-) \leq m\delta(c) - n_T(c, C_{j_{i+1}}^-) + 1 - 2m. \quad (6)$$

Then, since any improving move between $C_{j_i}^+$ and $C_{j_{i+1}}^-$ decreases the potential as shown in Lemma A.1, we obtain an upper bound on the net potential change from C_{j_i} to $C_{j_{i+1}}$ by adding Inequalities (1), (5), and (6):

$$\Phi(C_{j_{i+1}}) - \Phi(C_{j_i}) \leq n_T(c, C_{j_i}^+) - n_T(c, C_{j_{i+1}}^-) - 1 - |(n_T(c, C_{j_i}^+) - n_T(c, C_{j_{i+1}}^-))| < 0.$$

2. $u_{\mathcal{A}}(C_{j_i}) = 0$, $u_{\mathcal{A}}(C_{j_{i+1}}) = 1/2$, and the source node s is adjacent to the center. In assignment C_{j_i} , agent \mathcal{A} is located at s and is adjacent to an agent of the same type and the empty center node. Since $u_{\mathcal{A}}(C_{j_i}) = 0$, $u_{\mathcal{A}}(C_{j_i}^+) > 0$. Eventually agent \mathcal{A} moves out of the center node to node t to reach assignment $C_{j_{i+1}}$ where it has a neighbor of the same type and a neighbor of different type. Note that t must be a degree-2 node and it cannot be adjacent to the center node. Since $u_{\mathcal{A}}(C_{j_{i+1}}) = 1/2$, $u_{\mathcal{A}}(C_{j_{i+1}}^-) < 1/2$. Therefore, in assignment $C_{j_{i+1}}^-$, more than half of the neighbors of agent \mathcal{A} are of type T . From Table 2, we obtain:

$$\Phi(C_{j_i}^+) - \Phi(C_{j_i}) \leq -m\delta(c) + n_{T_x}(c, C_{j_i}^+) + 2m - 1 \quad (7)$$

and

$$\Phi(C_{j_{i+1}}) - \Phi(C_{j_{i+1}}^-) \leq m\delta(c) - n_{T_x}(c, C_{j_{i+1}}^-) + 1 - 2m. \quad (8)$$

Then, since any improving move between $C_{j_i}^+$ and $C_{j_{i+1}}^-$ decreases the potential as shown in Lemma A.1, we obtain an upper bound on the net potential change from C_{j_i} to $C_{j_{i+1}}$ by adding Inequalities (3), (7), and (8):

$$\Phi(C_{j_{i+1}}) - \Phi(C_{j_i}) \leq n_T(c, C_{j_i}^+) - n_T(c, C_{j_{i+1}}^-) + m - 1 - |(n_T(c, C_{j_i}^+) - n_{T_x}(c, C_{j_{i+1}}^-))| < 0.$$

Note that, if $n_T(c, C_{j_i}^+) > n_T(c, C_{j_{i+1}}^-)$, then the number of agents of type T that are adjacent to the center node should have decreased, i.e., replaced by agents of other types. Each replacement requires a pair of moves: a move out of the neighborhood by an agent of type T and a move into the neighborhood by an agent of other type. As such, the agent that moves to s after \mathcal{A} moves to the center is not part of the replacement pair and causes an additional decrease in the potential. Therefore, the net potential is negative.

3. $u_{\mathcal{A}}(C_{j_i}) = 0$ and $u_{\mathcal{A}}(C_{j_{i+1}}) = 1$. In assignment C_{j_i} , agent \mathcal{A} is located at s and either both its neighbors are of same type, or it is adjacent to an agent of the same type and the empty center node. Since $u_{\mathcal{A}}(C_{j_i}) = 0$, $u_{\mathcal{A}}(C_{j_i}^+) > 0$. Eventually agent \mathcal{A} moves out of the center node to node t to reach assignment $C_{j_{i+1}}$ where it has a neighbor of different type. Since $u_{\mathcal{A}}(C_{j_{i+1}}) = 1$, $u_{\mathcal{A}}(C_{j_{i+1}}^-) < 1$. From Table 2, we obtain:

$$\Phi(C_{j_i}^+) - \Phi(C_{j_i}) \leq -m\delta(c) + n_T(c, C_{j_i}^+) + 2m - 1 \quad (9)$$

and

$$\Phi(C_{j_{i+1}}) - \Phi(C_{j_{i+1}}^-) \leq m\delta(c) - n_T(c, C_{j_{i+1}}^-) - m. \quad (10)$$

Since any improving move between $C_{j_i}^+$ and $C_{j_{i+1}}^-$ decreases the potential as shown in Lemma A.1, we obtain an upper bound on the net potential change from C_{j_i} to $C_{j_{i+1}}$ by adding Inequalities (2), (9), and (10):

$$\Phi(C_{j_{i+1}}) - \Phi(C_{j_i}) \leq n_T(c, C_{j_i}^+) - n_T(c, C_{j_{i+1}}^-) + 2m - 1 - |(n_T(c, C_{j_i}^+) - n_T(c, C_{j_{i+1}}^-))| < 0.$$

4. $u_A(C_{j_i}) = 1/2$ and $u_A(C_{j_{i+1}}) = 1/2$. In assignment C_{j_i} , agent \mathcal{A} while located at s has a neighbor of the same type and a neighbor of different type. Since $u_A(C_{j_i}) = 1/2$, $u_A(C_{j_i}^+) > 1/2$. Therefore, in assignment $C_{j_i}^+$, less than half of the neighbors of agent \mathcal{A} are of type T . Eventually agent \mathcal{A} moves out of the center node to the target node t to reach assignment $C_{j_{i+1}}$ where it has a neighbor of the same type and a neighbor of different type. Note that t must be a degree-2 node. Since $u_A(C_{j_{i+1}}) = 1/2$, $u_A(C_{j_{i+1}}^-) < \frac{1}{2}$. Hence, in assignment $C_{j_{i+1}}^-$, more than half of the neighbors of agent \mathcal{A} are of type T , and we obtain

$$n_T(c, C_{j_i}^+) < n_T(c, C_{j_{i+1}}^-). \quad (11)$$

We claim that the net potential change is negative for improving moves that leads from C_{j_i} to $C_{j_i}^+$, from $C_{j_{i+1}}^-$ to $C_{j_{i+1}}$, and from $C_{j_i}^+$ to $C_{j_{i+1}}^-$. Since agent \mathcal{A} must improve its utility from $1/2$ in C_{j_i} to strictly more than $1/2$ in $C_{j_i}^+$, from Table 2, we obtain:

$$\Phi(C_{j_i}^+) - \Phi(C_{j_i}) \leq -m\delta(c) + n_T(c, C_{j_i}^+) + 2m - 1. \quad (12)$$

Similarly, since agent \mathcal{A} must improve its utility from strictly less than $1/2$ in $C_{j_{i+1}}^-$ to $1/2$ in $C_{j_{i+1}}$, from Table 2, we obtain:

$$\Phi(C_{j_{i+1}}) - \Phi(C_{j_{i+1}}^-) \leq m\delta(c) - n_T(c, C_{j_{i+1}}^-) + 1 - 2m. \quad (13)$$

So, since all any improving move between $C_{j_i}^+$ and $C_{j_{i+1}}^-$, decreases the potential as shown in Lemma A.1, we obtain an upper bound on the net potential change from C_{j_i} to $C_{j_{i+1}}$ by adding Inequalities (1), (12), and (13):

$$\Phi(C_{j_{i+1}}) - \Phi(C_{j_i}) \leq n_T(c, C_{j_i}^+) - n_T(c, C_{j_{i+1}}^-) - |(n_T(c, C_{j_i}^+) - n_T(c, C_{j_{i+1}}^-))| < 0$$

where the last inequality follows from Inequality (11).

5. $u_A(C_{j_i}) = 1/2$ and $u_A(C_{j_{i+1}}) = 1$. In assignment C_{j_i} , agent \mathcal{A} while located at s has a neighbor of the same type and a neighbor of different type. Since $u_A(C_{j_i}) = \frac{1}{2}$, $u_A(C_{j_i}^+) > \frac{1}{2}$. Eventually agent \mathcal{A} moves out of the center node to node t to reach assignment $C_{j_{i+1}}$ where it has a neighbor of different type. Since $u_A(C_{j_{i+1}}) = 1$, $u_A(C_{j_{i+1}}^-) < 1$. From Table 2, we obtain:

$$\Phi(C_{j_i}^+) - \Phi(C_{j_i}) \leq -m\delta(c) + n_T(c, C_{j_i}^+) + 2m - 1 \quad (14)$$

and

$$\Phi(C_{j_{i+1}}) - \Phi(C_{j_{i+1}}^-) \leq m\delta(c) - n_T(c, C_{j_{i+1}}^-) - m. \quad (15)$$

Since any improving move between $C_{j_i}^+$ and $C_{j_{i+1}}^-$ decreases the potential as shown in Lemma A.1, we obtain an upper bound on the net potential change from C_{j_i} to $C_{j_{i+1}}$ by adding Inequalities (2), (14), and (15):

$$\Phi(C_{j_{i+1}}) - \Phi(C_{j_i}) \leq n_T(c, C_{j_i}^+) - n_T(c, C_{j_{i+1}}^-) + 2m - 1 - |(n_T(c, C_{j_i}^+) - n_T(c, C_{j_{i+1}}^-))| < 0.$$

This completes the proof for the case when there exists an assignment C_{j_i} where \mathcal{A} occupies a degree-2 node. \square

Lemma A.4 *If s is a leaf node, then $\Phi(C_{j_{i+1}}) - \Phi(C_{j_i}) \leq 0$. Furthermore, $\Phi(C_{j_{i+1}}) - \Phi(C_{j_i}) = 0$ if s is adjacent to the center and $u_{\mathcal{A}}(C_{j_{i+1}}) = 1/2$.*

Proof The source utility of \mathcal{A} at node s should be 0 and the target utility at node t could be $1/2$ or 1. We consider the four possible cases below:

1. $u_{\mathcal{A}}(C_{j_{i+1}}) = 1$ and s is not adjacent to the center. In this case, agent \mathcal{A} moves out of the center node to node t to reach assignment $C_{j_{i+1}}$ where it has a neighbor of different type. Since $u_{\mathcal{A}}(C_{j_{i+1}}) = 1$, $u_{\mathcal{A}}(C_{j_{i+1}}^-) < 1$. From Table 2, we obtain:

$$\Phi(C_{j_i}^+) - \Phi(C_{j_i}) \leq -m\delta(c) + n_T(c, C_{j_i}^+) + m - 1 \quad (16)$$

and

$$\Phi(C_{j_{i+1}}) - \Phi(C_{j_{i+1}}^-) \leq m\delta(c) - n_T(c, C_{j_{i+1}}^-) - m. \quad (17)$$

Since any improving move between $C_{j_i}^+$ and $C_{j_{i+1}}^-$ decreases the potential as shown in Lemma A.1, we obtain an upper bound on the net potential change from C_{j_i} to $C_{j_{i+1}}$ by adding Inequalities (2), (16), and (17):

$$\Phi(C_{j_{i+1}}) - \Phi(C_{j_i}) \leq n_T(c, C_{j_i}^+) - n_T(c, C_{j_{i+1}}^-) + m - 1 - |(n_T(c, C_{j_i}^+) - n_T(c, C_{j_{i+1}}^-))| < 0.$$

2. $u_{\mathcal{A}}(C_{j_{i+1}}) = 1$ and s is adjacent to center. In this case, agent \mathcal{A} moves out of the center node to node t to reach assignment $C_{j_{i+1}}$ where it has a neighbor of different type. Since $u_{\mathcal{A}}(C_{j_{i+1}}) = 1$, $u_{\mathcal{A}}(C_{j_{i+1}}^-) < 1$. From Table 2, we obtain:

$$\Phi(C_{j_i}^+) - \Phi(C_{j_i}) \leq -m\delta(c) + n_T(c, C_{j_i}^+) + m \quad (18)$$

and

$$\Phi(C_{j_{i+1}}) - \Phi(C_{j_{i+1}}^-) \leq m\delta(c) - n_T(c, C_{j_{i+1}}^-) - m. \quad (19)$$

Since any improving move between $C_{j_i}^+$ and $C_{j_{i+1}}^-$ decreases the potential as shown in Lemma A.1, we obtain an upper bound on the net potential change from C_{j_i} to $C_{j_{i+1}}$ by adding Inequalities (4), (18), and (19):

$$\Phi(C_{j_{i+1}}) - \Phi(C_{j_i}) \leq n_T(c, C_{j_i}^+) - n_T(c, C_{j_{i+1}}^-) + 2m - 1 - |(n_T(c, C_{j_i}^+) - n_T(c, C_{j_{i+1}}^-))| < 0.$$

3. $u_{\mathcal{A}}(C_{j_{i+1}}) = 1/2$ and s is not adjacent to center. In this case, agent \mathcal{A} moves out of the center node to node t to reach assignment $C_{j_{i+1}}$ where it has one neighbor of the same type and one neighbor of different type. Since $u_{\mathcal{A}}(C_{j_{i+1}}) = 1/2$, $u_{\mathcal{A}}(C_{j_{i+1}}^-) < \frac{1}{2}$. From Table 2, we obtain:

$$\Phi(C_{j_i}^+) - \Phi(C_{j_i}) \leq -m\delta(c) + n_T(c, C_{j_i}^+) + m - 1 \quad (20)$$

and

$$\Phi(C_{j_{i+1}}) - \Phi(C_{j_{i+1}}^-) \leq m\delta(c) - n_T(c, C_{j_{i+1}}^-) + 1 - 2m. \quad (21)$$

Since any improving move between $C_{j_i}^+$ and $C_{j_{i+1}}^-$ decreases the potential as shown in Lemma A.1, we obtain an upper bound on the net potential change from C_{j_i} to $C_{j_{i+1}}$ by adding Inequalities (1), (20), and (21):

$$\Phi(C_{j_{i+1}}) - \Phi(C_{j_i}) \leq n_T(c, C_{j_i}^+) - n_T(c, C_{j_{i+1}}^-) - m - |(n_T(c, C_{j_i}^+) - n_T(c, C_{j_{i+1}}^-))| < 0. \quad (22)$$

4. $u_{\mathcal{A}}(C_{j_{i+1}}) = 1/2$ and s is adjacent to center. In this case, agent \mathcal{A} moves out of the center node to node t to reach assignment $C_{j_{i+1}}$ where it has one neighbor of the same type and one neighbor of different type. Since $u_{\mathcal{A}}(C_{j_{i+1}}) = 1/2$, $u_{\mathcal{A}}(C_{j_{i+1}}^-) < 1/2$. From Table 2, we obtain:

$$\Phi(C_{j_i}^+) - \Phi(C_{j_i}) \leq -m\delta(c) + n_T(c, C_{j_i}^+) + m \quad (23)$$

and

$$\Phi(C_{j_{i+1}}) - \Phi(C_{j_{i+1}}^-) \leq m\delta(c) - n_T(c, C_{j_{i+1}}^-) + 1 - 2m. \quad (24)$$

Since any improving move between $C_{j_i}^+$ and $C_{j_{i+1}}^-$ decreases the potential as shown in Lemma A.1, we obtain an upper bound on the net potential change from C_{j_i} to $C_{j_{i+1}}$ by adding Inequalities (3), (23), and (24):

$$\Phi(C_{j_{i+1}}) - \Phi(C_{j_i}) \leq n_T(c, C_{j_i}^+) - n_T(c, C_{j_{i+1}}^-) - |(n_T(c, C_{j_i}^+) - n_T(c, C_{j_{i+1}}^-))| \leq 0.$$

The proof is complete. \square

We are now finally ready to prove Theorem 4.8.

Proof of Theorem 4.8 Suppose instead that there is a game that is not a potential game. Then, there exists an IRC, and the net potential change of the moves in this cycle must be 0. By Lemma A.1, every improving move that does not involve the center node decreases the potential. It follows that the cycle must involve moves to and out of the center node. Let $C_{j_0}, C_{j_1}, \dots, C_{j_w}$, be the assignments in the cycle in which the center node c is empty, listed in increasing order. Lemmas A.3 and A.4 show that for every i , the difference in potential between C_{j_i} and $C_{j_{i+1}}$ is at most 0. Since the potential never increases, and the net potential

change must be 0, it must in fact be that, *for every* i , $\Phi(C_{j_{i+1}}) - \Phi(C_{j_i}) = 0$. It follows from Lemma A.4 that, for every i , an agent moves to the center node from a leaf node adjacent to the center, and subsequently moves out to a degree-2 node to get a utility $1/2$.

Assume without loss of generality that an agent \mathcal{A} of type T makes a move from such a leaf node s neighbor of the center to occupy the center in $C_{j_0}^+$. Before \mathcal{A} moves out of the center node, clearly no agent of type T would move to s , as it would get a utility 0. It follows that, if *every* agent that move to the center in the cycle is of the same type T , then the number of neighbors of the center of type T must decrease during the moves comprising the cycle, which is a contradiction.

Therefore, it must be the case that, for some j_i , an agent \mathcal{A} of type T occupies the center node in $C_{j_i}^+$, and an agent \mathcal{B} of type $T' \neq T$ occupies the center in $C_{j_{i+1}}^+$. By Lemma A.4, we have $u_{\mathcal{A}}(C_{j_{i+1}}) = 1/2$, therefore $u_{\mathcal{A}}(C_{j_{i+1}}^-) < 1/2$. Since \mathcal{B} is of a different type to \mathcal{A} , it must be that $u_{\mathcal{B}}(C_{j_{i+1}}^+) > 1/2$.

Since the change in potential between $C_{j_{i+1}}$ and $C_{j_{i+2}}$ is also 0, by Lemma A.4, it must be that \mathcal{B} moves to a node t to get a utility $1/2$, and thus $u_{\mathcal{B}}(C_{j_{i+2}}^-) < 1/2$. This means that the number of agents of type T' that occupy the nodes adjacent to the center must have increased, that is, $n_{T'}(c, C_{j_{i+1}}^+) - n_{T'}(c, C_{j_{i+2}}^-) < 0$. Now, as in the proof of case 3 in Lemma A.4, we have:

$$\Phi(C_{j_{i+2}}) - \Phi(C_{j_{i+1}}) \leq n_{T'}(c, C_{j_{i+1}}^+) - n_{T'}(c, C_{j_{i+2}}^-) - |(n_{T'}(c, C_{j_{i+1}}^+) - n_{T'}(c, C_{j_{i+2}}^-))| < 0,$$

which is a contradiction. As such, there can be no cycle in the assignment graph, and we conclude that the game must be potential. \square

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Declarations

Competing interests The authors declare no competing interests.

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