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Landau-Fokker-Planck equation for hot electrons in semiconductors

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In this paper we derive the Landau-Fokker-Planck (LFP) equation for the energy distribution function of a nondegenerate hot electrons in bulk semiconductors with strong electron-electron (*e-e*) interaction by the screened Coulomb potential described by a static dielectric function (the effect of dynamic screening is also analyzed qualitatively). The *e-e* scattering rates are calculated quantum-mechanically using the first Born approximation. The LFP equation was derived employing two different approaches: (i) a direct general transformation of the integral Boltzmann scattering operator; (ii) using the Chapman-Kolmogorov integral equation (CKE) for conditional transition probability densities. The screened potential allows to avoid any ad hoc cutoff procedures for tackling the problem of the Coulomb divergence of the matrix element in the scattering probability rates. However, the screening at the same time brings to the theory another problem, namely, that increase of the screening strength changes the character of the *e-e* scattering when it eventually becomes strongly inelastic. This corresponds to large energy transfer between the scattering electrons, which in turn, prohibits in principle considering the electron scattering process as a diffusion in the energy space (the necessary condition for the LFP approximation). We show that the screening effect must be explicitly included into the kinetic theory to correctly address the above two problems, and the cases of a weak and a strong screening need to be analyzed in detail. The LFP approximation is physically justified only for a weak screening. The obtained screening function coincides with the Brooks-Herring screening function. We have established general relationships between the kinetic coefficients in the LFP operator obtained by the above two approaches. The explicit expressions for the dynamic friction and the diffusion coefficients in the energy space are also obtained. Although, both approaches lead to the same solutions, we demonstrate that the CKE method is more universal approach, and the obtained here general relationships can be applied for any other scattering mechanism, provided that the required conditions are satisfied. We discuss the necessary validity conditions for the LFP equation, and show that for typical physical situations, e.g., for high-energy photoexcitation of the carriers, the LFP equation can be used up to the densities $\sim 10^{18}$ cm^{-3}. However, for higher densities the screening energy becomes too large, which results in an increase of the minimum energy transfer in the scattering events, and the required for the LFP conditions are violated. In this case, one must use the integral *e-e* scattering operator, which we obtained in the form suitable for numerical solution. Dynamic screening extends the LFP validity range. As an example of the application of the LFP equation, we discuss the perovskite solar cells, which are characterized by relatively low and comparable energies of the optical and acoustic phonons, and the generated carrier densities of the order of 10^{18} cm^{-3}. In this case, the LFP equation can be applied for each of these interactions (electron-phonon and the *e-e* scattering). This allows to find correct distribution function of the hot electrons, and calculate the electron power loss, to which the *e-e* scattering makes the direct contribution through the corresponding energy relaxation channel in the kinetic equation.

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I. INTRODUCTION

The problem of the electron-electron (*e-e*) scattering is one of the most complex and challenging problems in condensed matter statistical physics. Its complexity stems from the two facts. The first issue is due to nonlinearity of the problem in terms of the distribution functions, as two interacting particles (electrons and/or holes) are described by their own distribution functions. The second issue is due to absence of a universal small parameter, which describes strength of the *e-e* interaction. In addition, the Boltzmann's scattering operator in such systems has an integral form, which implies that the corresponding kinetic equation for a nonequilibrium system in presence of the external fields or optical excitations is a nonlinear integrodifferential equation. Solution of such equations presents a formidable mathematical challenge for both analytical and numerical handling.

Most efforts in this area were invested into consideration of a nonequilibrium kinetics of plasmas and ionized gases, in which the interparticle interaction (typically, by the Coulomb inverse-square force) is the main scattering mechanism, and the particles' motion is described by classical mechanical

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equations. The particle scattering probability rate is expressed through the effective scattering cross-section, which in turn is given by the well-known Rutherford formula. The first mathematical breakthrough in kinetics of systems with Coulomb interactions was made by Landau in his seminal work [1]. Using the idea that due to the long-rage action of the Coulomb forces, the velocity (or momentum) distribution of the scattering particles is mainly affected (in an integral sense) by many small-angle deflections produced by the long-distance interactions between the particles, Landau has derived the differential form of the Boltzmann integral scattering operator valid at the small-angle scattering conditions. The obtained differential operator has a universal Fokker-Planck (FP) form [2,3], which manifests the diffusion-like (Brownian) motion of the particles in the momentum space. For this reason, it is usually called the Landau-Fokker-Planck (LFP) operator. It is necessary to note that the derived LFP operator is in general a nonlinear operator, since the friction and diffusion coefficients, which describe the particle flux in the momentum space, are the functionals (integrals) of the unknown distribution functions, thus, without further approximations the corresponding kinetic equation still remains of the integrodifferential type. Later, Spitzer and coworkers [4,5] applied the LFP equation to investigation of transport phenomena in ionized gases. (It is interesting to note that the authors of Ref. [4] pointed out that the key conjecture of the Landau theory about dominance of the small-angle scattering events over the rare large-angle scattering for the inverse-square interaction force was first suggested by Jeans in 1929 [6].) The derived in Ref. [1] LFP differential operator was presented in general form, but without explicit calculation of the drift and diffusion coefficients. Due to the use of classical description of the *e-e* scattering dynamics, the latter problem required quite involved further calculations. This problem was solved by Rosenbluth and coworkers in the highly cited paper [7]. They essentially rediscovered the LFP differential form of the scattering operator (it looks like the Authors of Ref. [7] were not aware about the Landau's earlier work [1]). The obtained in Ref. [7] expressions for the friction and diffusion coefficients in the velocity/momentum space were of a very elegant and general form, using the so-called Rosenbluth potentials. These potentials allow relatively straightforward calculation of the friction and diffusion coefficients for various symmetries of the problem, as well as for using them for numerical solution of the electron transport problem in ionized gases and plasmas. Particularly interesting and important case of an angularly isotropic distribution was considered in Ref. [8] using the results from Ref. [7], where the distribution functions in the LFP operator depends only on the electron energy. Similar results were obtained by Ginzburg and Gurevich [9] using expansion of the distribution function in a series of Legendre polynomials and exploiting the small-angle (and small-energy) change in the scattering events. The results of Refs. [8] and [9] for isotropic distribution functions coincide with each other.

The works discussed above, have essentially laid the ground for applied investigations of nonequilibrium kinetics in plasmas and ionized gases in which the Coulomb interparticle scattering plays the key role. Later, there were also works devoted to further generalization of the LFP theory to include the terms which allow to treat the large-angle scattering events [10], and on development of numerical schemes for solution of the LFP equation [11], including constructing the efficient Monte Carlo algorithms [12].

Although, the main progress in developing the LFP theory was achieved in considering its utilization for plasmas and ionized gases, it was also realized that similar approach can be applied to electrons in semiconductor materials. However, in semiconductors the Coulomb interaction between carriers (electron-electron, hole-hole, and electron-hole) and with the charged impurities is only one type of the interaction. The other important scattering mechanisms include interaction with various types of phonons. The first applications of the FP equation to investigation of a nonequilibrium energy transport in semiconductors were actually used for electron systems interacting with phonons [13,14], without taking into account the interparticle Coulomb interactions. In this case, there exists a small parameter, in the form of the ratio of the energy exchange in the process of the scattering, the phonon energy, and the free carrier energy (the so-called quasielastic scattering approximation). As a result, the carrier motion in the energy space becomes analogous to the Brownian motion with small energy changes at each scattering event, and this justified the application of the FP formalism. Nonetheless, for the case of a quasielastic electron-phonon scattering in semiconductors, the FP equation brings little new, and actually gives no advantage in comparison with a direct solution of the Boltzmann equation under the same conditions, as was demonstrated in Ref. [15].

At the same time, it is obvious that when the free carrier density in a semiconductor increases, e.g., by doping or under an external optical excitation, the role of the interparticle Coulomb interaction cannot be ignored. In this case, most of the published literature uses the *e-e* scattering operator in the LFP form which is "borrowed" from plasma physics (see, e.g., Ref. [15] and references therein) using the friction and diffusion coefficients derived in Refs. [7–9]. There is, however, a well-known issue with this method, which stems from a long-range character of the Coulomb interaction potential, which in turn leads to a divergence of the effective integral cross-section for small values of the scattering angle. To circumvent this mathematical problem, it was necessary to introduce some ad hoc cutoff procedure, which physically limits the minimal allowed scattering angle. The actual estimate of the minimal scattering angle depends on whether the scattering is described classically or quantum-mechanically [16], although the Rutherford formula is valid in both cases. Since in plasmas and ionized gases the charged particle motion is described classically, the cutoff parameter must be introduced as an external parameter using the relevant physical reasoning, for example, specifying the maximum impact parameter to be equal to the average interparticle separation, or to the Debye screening radius [16]. Although, it is necessary to note, that there were various attempts [17,18] to introduce some self-consistent schemes for calculation of the friction and diffusion coefficients in which the cutoff parameter was unnecessary, but this led to considerable complication of the original LFP theory. For plasmas, the problem of removing the Coulomb divergence was resolved [16] by Lenard [19] and Balescu [20]. In Ref. [19] the problem was solved by an exact transformation of the integral Bogoliubov equation to another equivalent form from which the LFP equation was obtained. In Ref. [20], the diagram technique was employed for description of many-particle collective interactions in plasma. This essentially led to incorporation of the dielectric permittivity of the plasma into the derived kinetic equation, but it came at a price. Although, the obtained equation had the FP form, its actual structure, particularly the nonlinearity, was far more complicated, since the dielectric permittivity itself depended on the unknown distribution function.

In semiconductors the carrier scattering events are usually treated quantum-mechanically, considering the scattering as a transition of the scattered carrier and the scatterers (phonons, ionized centers, other free carriers) between some given initial state and one of the available final states in the momentum space, taking into account the energy and the momentum conservation. This has a particular advantage in the case of the Coulomb *e-e* scattering, since the effect of the screening can be directly included into the interaction potential for calculation of the transition probability using, e.g., the Born approximation [21]. This in turn allows to avoid an ad hoc cutoff procedure in the derivation of the LFP equation. It is necessary to note that the latter approach was used before in many publications, but mainly for analysis of the *e-e* scattering times, see, e.g., Refs. [22–24], and not for analysis of the nonequilibrium *e-e* scattering operator and nonequilibrium kinetic equation.

In this paper, we derive the LFP equation in the energy space using two different approaches. In the first approach, the LFP *e-e* operator is derived directly from the Boltzmann kinetic equation for the *e-e* scattering with the statically screened Coulomb potential. The obtained friction and diffusion coefficients differ from the standard expressions for these coefficients. A different result is obtained for the Coulomb logarithm. To establish the relationship between the friction and the diffusion coefficients obtained directly from the Boltzmann equation and the standard expressions for these coefficients in the LFP operator, we use completely different second method, and derive the LFP equation for the *e-e* scattering in semiconductors using the Chapman-Kolmogorov equation [25]. The latter procedure is probably the most general statistical approach to the solution of the problem, as it can be applied in a universal manner to any scattering mechanism under certain required conditions. From the obtained two forms of the LFP operator, we derive the general relationships between the kinetic coefficients, which enter each of these expressions. We also examine in detail the necessary conditions, which are required for validity of the LFP operator in the energy space, and discuss physical systems where the obtained expressions should be used. The obtained results are of particular interest to highly excited nonequilibrium systems, e.g., the hot-electron solar cells, in which the *e-e* scattering can strongly affect the power loss of the carriers, and thus, the efficiency of these devices.

II. SCATTERING RATE AND ELECTRON-ELECTRON COLLISION OPERATOR

We consider interaction (scattering) of two electrons with the momenta $\vec{k}$ and $\vec{p}$, which after the scattering event change their momenta to $\vec{k}'$ and $\vec{p}'$, respectively. The law of the *e-e* interaction is the screened Coulomb potential $V(\vec{r}) = (e^2/\kappa_0 r)\exp(-r/R_D)$, where e is electronic charge, κ_0 is static dielectric constant of the material, $r = |\vec{r}| = |\vec{r}_1 - \vec{r}_2|$ is the spacing between the two colliding electrons, and $R_D = (\kappa_0 k_B T/4\pi n e^2)^{1/2}$ is the Debye screening radius (T is the carrier temperature, n is the carrier density, k_B is Boltzmann's constant). The transition matrix element $M_{\vec{k},\vec{p}\rightarrow\vec{k}',\vec{p}'}$ can be calculated in the first Born approximation [21] using either the Bloch wave functions or plane waves. This gives for the transition probability rate:

$$W(\vec{k},\vec{p}\rightarrow\vec{k}',\vec{p}') = \frac{2\pi}{\hbar}|M_{\vec{k},\vec{p}\rightarrow\vec{k}',\vec{p}'}|^2\delta(\varepsilon_{\vec{k}'}+\varepsilon_{\vec{p}'}-\varepsilon_{\vec{k}}-\varepsilon_{\vec{p}}), \tag{1}$$

where the square of the matrix element is

$$|M_{\vec{k},\vec{p}\rightarrow\vec{k}',\vec{p}'}|^2 = \left(\frac{4\pi e^2}{\kappa_0 L^3}\right)^2\left[\frac{\hbar^2}{q^2+q_0^2}\right]^2\delta_{\vec{k}'+\vec{p}'-\vec{k}-\vec{p}}. \tag{2}$$

Here $\vec{q} = \vec{k}' - \vec{k} = \vec{p} - \vec{p}'$ is the electron momentum change in the scattering event, $q_0 = \hbar/R_D$ is a characteristic Debye screening momentum, $\hbar$ is Planck's constant, and L^3 is the normalization volume. This is equivalent to assuming the static dielectric function $\kappa_e(\vec{q}) = 1 + q_0^2/q^2$ for description of the free-carrier screening. (The effect of dynamic screening, when the dielectric function $\kappa_e(\vec{q},\omega)$ depends on the momentum $\vec{q}$ and the relevant frequency ω of the perturbation field will be discussed later.)

With these definitions, we can write for the *e-e* scattering operator, considering the electron in a given state $\vec{k}$:

$$\left(\frac{\partial F(\vec{k})}{\partial t}\right)_{ee} \equiv \hat{I}_{ee}(\vec{k}) = \sum_{\vec{k}'\vec{p}\vec{p}'}[W(\vec{k}',\vec{p}'\rightarrow\vec{k},\vec{p})F(\vec{p}')F(\vec{k}') - W(\vec{k},\vec{p}\rightarrow\vec{k}',\vec{p}')F(\vec{p})F(\vec{k})]. \tag{3}$$

This is the Boltzmann *e-e* scattering integral in the momentum space, which manifests the balance between transitions into a given state $\vec{k}$ from all other states, and the competing transition out of a state $\vec{k}$ into all other available states. In this form it enters the Boltzmann kinetic equation. Within the first Born approximation [26], the transition probabilities satisfy the detailed balance (DB) condition, which, for example, can be obtained from the equation $\hat{I}_{ee}(\vec{k}) = 0$ for the equilibrium distribution functions:

$$W(\vec{k}',\vec{p}'\rightarrow\vec{k},\vec{p}) = W(\vec{k},\vec{p}\rightarrow\vec{k}',\vec{p}'). \tag{4}$$

We will be interested in nonequilibrium systems, which are isotropic in the momentum space, when the distribution functions depends only on the corresponding carrier energies. The collision operators can always be decomposed into a symmetric part and an asymmetric part, so the calculations for each part can be carried out separately. There is, however, a large class of the problems, for example, the optically excited nonequilibrium systems, where only the symmetric part of the collision operator is needed to describe the nonequilibrium kinetics in the energy space. In such cases, the asymmetric part is exactly zero. For systems with anisotropy, e.g., in presence of external fields, the relaxation of the asymmetric part of the distribution function is usually governed by the interaction with the ionized impurities. The contribution of the *e-e* scattering to asymmetric distribution can thus be ignored (the contribution of the *e-e* scattering to the symmetric part of the distribution remains the same as in completely isotropic systems). An exception is the case of a very high carrier density when the *e-e* scattering controls both the electron energy and momentum relaxation. However, again, in such systems the degenerate electron gas is described by the so-called displaced Fermi distribution [22], which follows directly from Eq. (3), where the Pauli factors should be included in the right-hand side of the expression.

In our case it is convenient to introduce the averaging of the scattering operator $\hat{I}_{ee}(\vec{k})$ over the surface of the constant energy $\varepsilon(\vec{k}) = \varepsilon$:

$$\begin{aligned}\hat{I}_{ee}(\varepsilon) &= \frac{1}{g(\varepsilon)} \sum_{\vec{k}\vec{k}'\vec{p}\,\vec{p}'} \hat{I}_{ee}(\vec{k})\,\delta\,(\varepsilon_{\vec{k}} - \varepsilon)\\ &= \frac{1}{g(\varepsilon)} \sum_{\vec{k}\,\vec{k}'\vec{p}\vec{p}'} [W(\vec{k}', \vec{p}' \rightarrow \vec{k}, \vec{p})F(\varepsilon_{\vec{p}'})F(\varepsilon_{\vec{k}'})\\ &\quad - W(\vec{k}, \vec{p} \rightarrow \vec{k}', \vec{p}')F(\varepsilon_{\vec{p}})F(\varepsilon_{\vec{k}})]\delta\,(\varepsilon_{\vec{k}} - \varepsilon). \qquad (5)\end{aligned}$$

Here $g(\varepsilon) = \sum_{\vec{k}} \delta\,(\varepsilon_{\vec{k}} - \varepsilon) = (\sqrt{2}m^{3/2}L^3/\pi^2\hbar^3)\varepsilon^{1/2}$ is the electron density of states in parabolic bands taking into account two orientations of spin, m is the electron effective mass. Using Eq. (1) and the DB condition from Eq. (3), and interchanging the summation indices $\vec{k} \rightleftarrows \vec{k}'$ and $\vec{p} \rightleftarrows \vec{p}'$ in the first sum of Eq. (5), we can present $\hat{I}_{ee}(\varepsilon)$ in the form, which is more convenient for further transformation:

$$\begin{aligned}\hat{I}_{ee}(\varepsilon) &= \frac{1}{g(\varepsilon)} \sum_{\vec{k}\,\vec{k}'\vec{p}\vec{p}'} w(\vec{q})\delta_{\vec{p}'-\vec{p},\vec{q}}F(\varepsilon_{\vec{p}})F(\varepsilon_{\vec{k}})\\ &\quad \times \delta(\varepsilon_{\vec{k}'} + \varepsilon_{\vec{p}'} - \varepsilon_{\vec{k}} - \varepsilon_{\vec{p}}) \times [\delta(\varepsilon_{\vec{k}'} - \varepsilon) - \delta(\varepsilon_{\vec{k}} - \varepsilon)], \qquad (6)\end{aligned}$$

where we introduced new notations:

$$w(\vec{q}) = \frac{w_0}{(q^2 + q_0^2)^2}, \quad w_0 = \frac{2\pi}{\hbar}\left(\frac{4\pi e^2\hbar^2}{\kappa_0 L^3}\right)^2. \qquad (7)$$

III. DERIVATION OF THE LFP FORM FOR $\hat{I}_{ee}(\varepsilon)$ OPERATOR FROM THE BOLTZMANN INTEGRAL

Using δ functions in Eq. (6), we can rewrite it as

$$\begin{aligned}\hat{I}_{ee}(\varepsilon) = \frac{1}{g(\varepsilon)} \sum_{\vec{k}\,\vec{k}'\vec{p}\vec{p}'} w(\vec{q})\delta_{\vec{p}'-\vec{p},\vec{q}}F(\varepsilon_{\vec{p}})\{&F[\varepsilon - (\varepsilon_{\vec{p}} - \varepsilon_{\vec{p}'})]\delta[\varepsilon_{\vec{k}} - \varepsilon + (\varepsilon_{\vec{p}} - \varepsilon_{\vec{p}'})]\,\delta(\varepsilon_{\vec{k}'} - \varepsilon)\\ &- F(\varepsilon)\delta\,[\varepsilon_{\vec{k}'} - \varepsilon - (\varepsilon_{\vec{p}} - \varepsilon_{\vec{p}'})]\,\delta(\varepsilon_{\vec{k}} - \varepsilon)\}. \qquad (8)\end{aligned}$$

We now will make an important assumption that the energy change in each collision event is small in comparison with the energy of the colliding electrons:

$$\Delta\varepsilon \equiv \varepsilon_{\vec{k}'} - \varepsilon_{\vec{k}} = \varepsilon_{\vec{p}} - \varepsilon_{\vec{p}'} \ll \varepsilon. \qquad (9)$$

It is necessary to note that for the *e-e* scattering there is no small parameter, which would guarantee that the above condition is satisfied in general case. This will be discussed in detail later, when we will analyze the limits of the validity of the obtained equations. Using Eq. (9), we can expand the functions in Eq. (8) in series:

$$F\,[\varepsilon - (\varepsilon_{\vec{p}} - \varepsilon_{\vec{p}'})] = F(\varepsilon) - (\varepsilon_{\vec{p}} - \varepsilon_{\vec{p}'})\frac{dF(\varepsilon)}{d\varepsilon} + \frac{1}{2}(\varepsilon_{\vec{p}} - \varepsilon_{\vec{p}'})^2\frac{d^2F(\varepsilon)}{d\varepsilon^2}, \qquad (10)$$

$$\delta\,[(\varepsilon_{\vec{k}} - \varepsilon) \pm (\varepsilon_{\vec{p}} - \varepsilon_{\vec{p}'})] = \delta(\varepsilon_{\vec{k}} - \varepsilon) \mp (\varepsilon_{\vec{p}} - \varepsilon_{\vec{p}'})\frac{d}{d\varepsilon}\delta(\varepsilon_{\vec{k}} - \varepsilon) + \frac{1}{2}(\varepsilon_{\vec{p}} - \varepsilon_{\vec{p}'})^2\frac{d^2}{d\varepsilon^2}\delta(\varepsilon_{\vec{k}} - \varepsilon). \qquad (11)$$

Substitution of Eq. (10) and Eq. (11) into Eq. (8) eliminates the terms with $F(\varepsilon)\delta(\varepsilon_{\vec{k}'} - \varepsilon)\delta(\varepsilon_{\vec{k}} - \varepsilon)$, the other terms can be grouped as follows:

$$\begin{aligned}\hat{I}_{ee}(\varepsilon) = \frac{1}{g(\varepsilon)} \sum_{\vec{k}\,\vec{k}'\vec{p}\vec{p}'} w(\vec{q})\delta_{\vec{p}'-\vec{p},\vec{q}}F(\varepsilon_{\vec{p}}) \times \Bigg\{&-(\varepsilon_{\vec{p}} - \varepsilon_{\vec{p}'})\frac{d}{d\varepsilon}[F(\varepsilon)\delta(\varepsilon_{\vec{k}'} - \varepsilon)\delta(\varepsilon_{\vec{k}} - \varepsilon)]\\ &+ \frac{1}{2}(\varepsilon_{\vec{p}} - \varepsilon_{\vec{p}'})^2\Bigg[F(\varepsilon)\delta(\varepsilon_{\vec{k}'} - \varepsilon)\frac{d^2}{d\varepsilon^2}\delta(\varepsilon_{\vec{k}} - \varepsilon) + 2\frac{dF(\varepsilon)}{d\varepsilon}\delta(\varepsilon_{\vec{k}'} - \varepsilon)\frac{d}{d\varepsilon}\delta(\varepsilon_{\vec{k}} - \varepsilon)\\ &- \frac{d^2F(\varepsilon)}{d\varepsilon^2}\delta(\varepsilon_{\vec{k}'} - \varepsilon)\delta(\varepsilon_{\vec{k}} - \varepsilon) - F(\varepsilon)\delta(\varepsilon_{\vec{k}} - \varepsilon)\frac{d^2}{d\varepsilon^2}\delta(\varepsilon_{\vec{k}'} - \varepsilon)\Bigg]\Bigg\}. \qquad (12)\end{aligned}$$

The term $\sim (\varepsilon_{\vec{p}} - \varepsilon_{\vec{p}'})^2$ needs further transformation. For this, we interchange the summation indices $\vec{k} \rightleftarrows \vec{k}'$ in the last term in the square brackets. This will also change the sign of $\vec{q} \rightarrow -\vec{q}$, but this will not affect $w(\vec{q})$, as it is an even function of $\vec{q}$. The sum over $\vec{p}$, $\vec{p}'$ $\sum_{\vec{p}\vec{p}'} (\varepsilon_{\vec{p}} - \varepsilon_{\vec{p}'})^2F(\varepsilon_{\vec{p}})\delta_{\vec{p}'-\vec{p},\pm\vec{q}}$ is not affected by the sign of $\vec{q}$. As a result, the first and the last terms in the square brackets cancel each other. Similarly, using the interchange $\vec{k} \rightleftarrows \vec{k}'$ in the summations $\sum_{\vec{k},\vec{k}'}$, the intermediate term is transformed as

$$2\delta\,(\varepsilon_{\vec{k}'}-\varepsilon)\frac{d}{d\varepsilon}\delta\,(\varepsilon_{\vec{k}}-\varepsilon)\to\delta\,(\varepsilon_{\vec{k}'}-\varepsilon)\frac{d}{d\varepsilon}\delta\,(\varepsilon_{\vec{k}}-\varepsilon)+\delta\,(\varepsilon_{\vec{k}}-\varepsilon)\frac{d}{d\varepsilon}\delta\,(\varepsilon_{\vec{k}'}-\varepsilon)=\frac{d}{d\varepsilon}[\delta\,(\varepsilon_{\vec{k}'}-\varepsilon)\delta\,(\varepsilon_{\vec{k}}-\varepsilon)].$$

(Here the change ($\vec{k}\rightleftarrows\vec{k}'$ is used only in the second term in the above expression.)
With this, the $\hat{I}_{ee}(\varepsilon)$ operator in Eq. (12) becomes

$$\hat{I}_{ee}(\varepsilon)=\frac{1}{g(\varepsilon)}\sum_{\vec{k}\vec{k}'\vec{p}\vec{p}'}w(\vec{q})\delta_{\vec{p}'-\vec{p},\vec{q}}F(\varepsilon_{\vec{p}})\times\left\{-(\varepsilon_{\vec{p}}-\varepsilon_{\vec{p}'})\frac{d}{d\varepsilon}[F(\varepsilon)\delta(\varepsilon_{\vec{k}'}-\varepsilon)\delta(\varepsilon_{\vec{k}}-\varepsilon)]\right.$$
$$\left.+\frac{1}{2}(\varepsilon_{\vec{p}}-\varepsilon_{\vec{p}'})^2\frac{d}{d\varepsilon}\left[\frac{dF(\varepsilon)}{d\varepsilon}\delta(\varepsilon_{\vec{k}'}-\varepsilon)\delta(\varepsilon_{\vec{k}}-\varepsilon)\right]\right\}. \tag{13}$$

It may appear that the linear term $\sim(\varepsilon_{\vec{p}}-\varepsilon_{\vec{p}'})$ can change its sign, as we sum it up over $\vec{p}'$ for a given value of $\varepsilon_{\vec{p}}$. To show explicitly that this does not happen, we can present it in a more symmetric form, using the invariance property of $\hat{I}_{ee}(\varepsilon)$ with respect to change $\vec{k}\rightleftarrows\vec{k}'$, $\vec{p}\rightleftarrows\vec{p}'$, and that the distribution function $F(\varepsilon_{\vec{p}})$ is in general a decreasing function of $\varepsilon_{\vec{p}}$ (this is the necessary condition for normalization of $F(\varepsilon_{\vec{p}})$):

$$\hat{I}_{ee}(\varepsilon)=\frac{1}{g(\varepsilon)}\sum_{\vec{k}\vec{k}'\vec{p}\vec{p}'}w(\vec{q})\delta_{\vec{p}'-\vec{p},\vec{q}}\left\{\frac{1}{2}(\varepsilon_{\vec{p}}-\varepsilon_{\vec{p}'})[F(\varepsilon_{\vec{p}'})-F(\varepsilon_{\vec{p}})]\frac{d}{d\varepsilon}[F(\varepsilon)\delta(\varepsilon_{\vec{k}'}-\varepsilon)\delta(\varepsilon_{\vec{k}}-\varepsilon)]\right.$$
$$\left.+\frac{1}{2}(\varepsilon_{\vec{p}}-\varepsilon_{\vec{p}'})^2F(\varepsilon_{\vec{p}})\frac{d}{d\varepsilon}\left[\frac{dF(\varepsilon)}{d\varepsilon}\delta(\varepsilon_{\vec{k}'}-\varepsilon)\delta(\varepsilon_{\vec{k}}-\varepsilon)\right]\right\}. \tag{14}$$

This expression allows to present the $\hat{I}_{ee}(\varepsilon)$ operator in the final differential LFP form:

$$\hat{I}_{ee}(\varepsilon)=\frac{1}{g(\varepsilon)}\frac{d}{d\varepsilon}\left[g(\varepsilon)\left(A_{ee}(\varepsilon)F(\varepsilon)+B_{ee}(\varepsilon)\frac{dF(\varepsilon)}{d\varepsilon}\right)\right]=-\frac{1}{g(\varepsilon)}\frac{d}{d\varepsilon}[g(\varepsilon)J_{ee}(\varepsilon)], \tag{15}$$

$$J_{ee}(\varepsilon)=-\left[A_{ee}(\varepsilon)F(\varepsilon)+B_{ee}(\varepsilon)\frac{dF(\varepsilon)}{d\varepsilon}\right], \tag{16}$$

where $J_{ee}(\varepsilon)$ is the carrier flux along the energy axis. The coefficients $A_{ee}(\varepsilon)$ and $B_{ee}(\varepsilon)$ are defined according to Eq. (14) as follows:

$$A_{ee}(\varepsilon)=\frac{1}{2g(\varepsilon)}\sum_{\vec{k}\vec{k}'\vec{p}\vec{p}'}w(\vec{q})\delta_{\vec{p}'-\vec{p},\vec{q}}(\varepsilon_{\vec{p}}-\varepsilon_{\vec{p}'})[F(\varepsilon_{\vec{p}'})-F(\varepsilon_{\vec{p}})]\,\delta(\varepsilon_{\vec{k}'}-\varepsilon)\delta(\varepsilon_{\vec{k}}-\varepsilon), \tag{17}$$

$$B_{ee}(\varepsilon)=\frac{1}{2g(\varepsilon)}\sum_{\vec{k}\vec{k}'\vec{p}\vec{p}'}w(\vec{q})\delta_{\vec{p}'-\vec{p},\vec{q}}(\varepsilon_{\vec{p}}-\varepsilon_{\vec{p}'})^2F(\varepsilon_{\vec{p}})\,\delta(\varepsilon_{\vec{k}'}-\varepsilon)\delta(\varepsilon_{\vec{k}}-\varepsilon). \tag{18}$$

The derived expressions for the coefficients $A_{ee}(\varepsilon)$ and $B_{ee}(\varepsilon)$ differ from those reported in Refs. [8,9] which were based on fully classical picture of the *e*-*e* scattering. Also, here the screening effect is self-consistently included in the function $w(\vec{q})$, which is given by Eq. (7).

IV. CALCULATION OF THE KINETIC COEFFICIENTS $A_{ee}(\varepsilon)$ AND $B_{ee}(\varepsilon)$

As is seen from Eqs. (15)–(18), the *e*-*e* scattering operator $\hat{I}_{ee}(\varepsilon)$ in the LFP form is still a nonlinear expression with respect of the distribution function $F(\varepsilon)$. In calculating the coefficients $A_{ee}(\varepsilon)$ and $B_{ee}(\varepsilon)$, we will first try to proceed as far as possible without any restrictions on the distribution function $F(\varepsilon_{\vec{p}})$, which enters Eqs. (17) and (18). For the coefficient $A_{ee}(\varepsilon)$ we obtain after summations over $\vec{p}'$:

$$A_{ee}(\varepsilon)=\frac{1}{g(\varepsilon)}\sum_{\vec{k}\vec{k}'}w(\vec{q})\,\varepsilon_{\vec{q}}\,\delta(\varepsilon_{\vec{k}'}-\varepsilon)\delta(\varepsilon_{\vec{k}}-\varepsilon)\sum_{\vec{p}}F(\varepsilon_{\vec{p}}). \tag{19}$$

The last sum allows us to use the normalization condition to obtain

$$\sum_{\vec{p}}F(\varepsilon_{\vec{p}})=\int_0^\infty g(\varepsilon_{\vec{p}})F(\varepsilon_{\vec{p}})d\varepsilon_{\vec{p}}=nL^3, \tag{20}$$

without any assumptions about $F(\varepsilon_{\vec{p}})$, n is the electron density.

Although, the summations in Eq. (19) over $\vec{k}$, $\vec{k}'$ is straightforward, using δ functions and the expression $\varepsilon_{\vec{q}} = q^2/2m = [\varepsilon_{\vec{k}} + \varepsilon_{\vec{k}'} - 2(\varepsilon_{\vec{k}}\varepsilon_{\vec{k}'})^{1/2}\cos\theta]$, where θ is the scattering angle between $\vec{k}$ and $\vec{k}'$, it is instructive to show an intermediate expression for $A_{ee}(\varepsilon)$, which will help to understand the obtained final result:

$$A_{ee}(\varepsilon) = \frac{1}{2}\frac{w_0 nL^3}{(2m)^2} g(\varepsilon)\int_{-1}^{+1}\frac{\tilde{\varepsilon}(\theta)\,d(\cos\theta)}{[\tilde{\varepsilon}(\theta)+\varepsilon_0]^2} = \frac{w_0 nL^3}{(4m)^2}\frac{g(\varepsilon)}{\varepsilon}\int_0^{4\varepsilon}\frac{\tilde{\varepsilon}(\theta)\,d\,[\tilde{\varepsilon}(\theta)]}{[\tilde{\varepsilon}(\theta)+\varepsilon_0]^2}. \tag{21}$$

Here $\tilde{\varepsilon}(\theta) = \varepsilon_{\vec{q}}[\varepsilon_{\vec{k}} = \varepsilon,\ \varepsilon_{\vec{k}'} = \varepsilon] = 2\varepsilon\,(1-\cos\theta)$, $\varepsilon_0 = q_0^2/2m$. After integration in Eq. (21) over the scattering angles, we finally obtain

$$A_{ee}(\varepsilon) = \frac{w_0 nL^3}{(4m)^2}\frac{g(\varepsilon)}{\varepsilon}\Lambda(\varepsilon), \tag{22}$$

$$\Lambda(\varepsilon) = \ln\left(1 + \frac{4\varepsilon}{\varepsilon_0}\right) - \frac{4\varepsilon/\varepsilon_0}{1 + 4\varepsilon/\varepsilon_0}. \tag{23}$$

The function $\Lambda(\varepsilon)$ is responsible for removing the Coulomb divergence from $\hat{I}_{ee}(\varepsilon)$. (In the absence of screening, $\varepsilon_0 = 0$, and $\Lambda(\varepsilon)$ diverges logarithmically to infinity, then one will need to introduce some ad hoc cutoff parameter to make the logarithm finite.) It is interesting to note that $\Lambda(\varepsilon)$ here has exactly the same form as was obtained in the Brooks-Herring theory [22,23] for the electron momentum relaxation time due to the electron elastic scattering by the screened charged impurity. It seems this coincidence is mathematically accidental. In the case of the momentum relaxation time, the additional weighting factor $(1-\cos\theta)$ (which stems from the asymmetric part of the distribution function) is introduced in the corresponding integral for the momentum relaxation time. In our case the same factor originates internally from the energy $\tilde{\varepsilon}(\theta)$. The coefficient $A_{ee}(\varepsilon)$ contains large logarithm, as typically, particularly for the hot electrons, we have that $(\varepsilon/\varepsilon_0) \gg 1$.

For the coefficient $B_{ee}(\varepsilon)$ we obtain after summation over $\vec{p}'$:

$$\begin{aligned}
B_{ee}(\varepsilon) &= \frac{1}{2g(\varepsilon)}\sum_{\vec{k}\vec{k}'\vec{p}} w(\vec{q})F(\varepsilon_{\vec{p}})\big[4\varepsilon_{\vec{p}}\varepsilon_{\vec{q}}\cos^2\chi + \varepsilon_{\vec{q}}^2\big]\delta(\varepsilon_{\vec{k}'}-\varepsilon)\delta(\varepsilon_{\vec{k}}-\varepsilon)\\
&= \frac{1}{2g(\varepsilon)}\sum_{\vec{k}\vec{k}'} w(\vec{q})\varepsilon_{\vec{q}}\delta(\varepsilon_{\vec{k}'}-\varepsilon)\delta(\varepsilon_{\vec{k}}-\varepsilon)\sum_{\vec{p}} 4\varepsilon_{\vec{p}}F(\varepsilon_{\vec{p}})\cos^2\chi\\
&\quad + \frac{1}{2g(\varepsilon)}\sum_{\vec{k}\vec{k}'} w(\vec{q})\varepsilon_{\vec{q}}^2\delta(\varepsilon_{\vec{k}'}-\varepsilon)\delta(\varepsilon_{\vec{k}}-\varepsilon)\sum_{\vec{p}} F(\varepsilon_{\vec{p}}),
\end{aligned} \tag{24}$$

where χ is the angle between the $\vec{p}$ and $\vec{q}$ vectors. The summation over $\vec{k}$, $\vec{k}'$ in the second line in Eq. (24) is exactly the same as in Eq. (19), the sum over $\vec{p}$ gives $(4/3)\bar{\varepsilon}nL^3$, where $\bar{\varepsilon}$ is the average energy of the electron gas:

$$\bar{\varepsilon} = \frac{1}{nL^3}\int_0^{\infty} g(\varepsilon)\varepsilon F(\varepsilon)d\varepsilon. \tag{25}$$

Therefore, this term is equal to $(2/3)\bar{\varepsilon}A_{ee}(\varepsilon)$. The last term in Eq. (24) differs from $A_{ee}(\varepsilon)$ in Eq. (19) (except for the factor of 1/2) that here we have $\varepsilon_{\vec{q}}^2$ instead of $\varepsilon_{\vec{q}}$ in Eq. (19). As a result, we obtain the integral over the scattering angle similar to the one in Eq. (21), except that in the numerator in the integrand we now have $\tilde{\varepsilon}^2(\theta)$. This in turn makes this integral finite (not divergent) even for $\varepsilon_0 = 0$ (no screening). This essentially means that retaining this term is an exaggeration of the accuracy of our approximation, and thus, it should be neglected in comparison with the first term. We can calculate this term explicitly, and see that the corresponding factor is of order unity:

$$\begin{aligned}
\frac{1}{2g(\varepsilon)}\sum_{\vec{k}\vec{k}'} w(\vec{q})\varepsilon_{\vec{q}}^2\delta(\varepsilon_{\vec{k}'}-\varepsilon)\delta(\varepsilon_{\vec{k}}-\varepsilon)\sum_{\vec{p}} F(\varepsilon_{\vec{p}}) &= \frac{1}{2}\frac{w_0 nL^3}{(2m)^2}g(\varepsilon)\int_{-1}^{+1}\frac{\tilde{\varepsilon}^2(\theta)\,d(\cos\theta)}{[\tilde{\varepsilon}(\theta)+\varepsilon_0]^2}\\
= \frac{w_0 nL^3}{(2m)^2}g(\varepsilon)\left[1 + \frac{1}{1+(4\varepsilon/\varepsilon_0)} - \frac{\varepsilon_0}{2\varepsilon}\ln\left(1+\frac{4\varepsilon}{\varepsilon_0}\right)\right] &= A_{ee}(\varepsilon)\times 2\varepsilon\frac{\Phi(\varepsilon)}{\Lambda(\varepsilon)},
\end{aligned} \tag{26}$$

where

$$\Phi(\varepsilon) = \left[1 + \frac{1}{1+(4\varepsilon/\varepsilon_0)} - \frac{\varepsilon_0}{2\varepsilon}\ln\left(1+\frac{4\varepsilon}{\varepsilon_0}\right)\right]. \tag{27}$$

As is seen from Eq. (27), for hot electrons with $(\varepsilon/\varepsilon_0) \gg 1$, we have $\Phi(\varepsilon) \approx 1$, as was evaluated above. Therefore, the last term in Eq. (24) can be neglected in comparison with the first term, and we finally obtain for $B_{ee}(\varepsilon)$:

$$B_{ee}(\varepsilon) = \frac{2}{3}\bar{\varepsilon}A_{ee}(\varepsilon) = \frac{2}{3}\bar{\varepsilon}\frac{w_0 nL^3}{(4m)^2}\frac{g(\varepsilon)}{\varepsilon}\Lambda(\varepsilon). \tag{28}$$

Note, that since the coefficient $B_{ee}(\varepsilon)$ depends on the unknown function $F(\varepsilon)$, so we do not know the average energy $\bar{\varepsilon}$ of a nonequilibrium electron gas in Eq. (25). At the same time, the coefficient $A_{ee}(\varepsilon)$ does not depend on $F(\varepsilon)$, since the value nL^3 (total number of the electrons) in the normalization integral in Eq. (20) does not depend on the form of the distribution function $F(\varepsilon)$. Also, the relationship in Eq. (28) between $B_{ee}(\varepsilon)$ and $A_{ee}(\varepsilon)$ for the case of thermodynamic equilibrium, when $\bar{\varepsilon} = (3/2)k_B T_0$ (T_0 is the equilibrium temperature) gives $B_{ee}(\varepsilon) = A_{ee}(\varepsilon)k_B T_0$. This ensures that the equation $\hat{I}_{ee}(\varepsilon) = 0$ with the *e-e* operator from Eq. (15), gives the correct equilibrium Maxwellian solution $F_0(\varepsilon) = A\exp(-\varepsilon/k_B T_0)$, A is the normalization constant. [The equation $\hat{I}_{ee}(\varepsilon) = 0$ is actually satisfied by the Maxwellian function with any electron temperature $T \neq T_0$. In this case we should use $(2/3)\bar{\varepsilon} = k_B T$ in Eq. (28).]

It is useful to modify the obtained kinetic coefficients $A_{ee}(\varepsilon)$ and $B_{ee}(\varepsilon)$ in Eqs. (22) and (28), and present the LFP operator $\hat{I}_{ee}(\varepsilon)$ in physically more transparent form. For this, we introduce the so-called elementary *e-e* scattering rate [22]:

$$\nu_{ee}(\varepsilon) = \frac{1}{\tau_{ee}(\varepsilon)} = \frac{2\sqrt{2}\pi e^4 n}{\kappa_0^2 m^{1/2}} \frac{1}{\varepsilon^{3/2}} \Lambda(\varepsilon). \tag{29}$$

Using Eqs. (7) and (29), the coefficients can be presented as

$$A_{ee}(\varepsilon) = \frac{\varepsilon}{\tau_{ee}(\varepsilon)},$$
$$B_{ee}(\varepsilon) = \frac{2}{3}\bar{\varepsilon}A_{ee}(\varepsilon) = \frac{2}{3}\bar{\varepsilon}\frac{\varepsilon}{\tau_{ee}(\varepsilon)}. \tag{30}$$

This gives for $\hat{I}_{ee}(\varepsilon)$

$$\hat{I}_{ee}(\varepsilon) = \frac{1}{g(\varepsilon)}\frac{d}{d\varepsilon}\left[\frac{g(\varepsilon)\varepsilon}{\tau_{ee}(\varepsilon)}\left(F(\varepsilon) + \frac{2}{3}\bar{\varepsilon}\frac{dF(\varepsilon)}{d\varepsilon}\right)\right]. \tag{31}$$

The obtained expressions in Eqs. (15), (16), (22), (23), and (28) for the LFP *e-e* scattering operator are valid only if the key condition given by Eq. (9) is satisfied. The only way to check that the obtained LFP equation for the *e-e* scattering gives correct solution, is to substitute it into the exact integral form of the $\hat{I}_{ee}(\varepsilon)$ operator in Eq. (6) and compare the values given by the exact and the approximate expressions.

Also, our coefficients $A_{ee}(\varepsilon)$ and $B_{ee}(\varepsilon)$ differ from the corresponding coefficients obtained in Ref. [9] for the *e-e* scattering in plasma, where the scattering is treated classically (see Eqs. (2.18) and (2.18a) in Ref. [9]). The compared coefficients becomes the same only if in Ref. [9] the additional condition $\varepsilon \gg \bar{\varepsilon}$ is imposed. However, here, in deriving Eqs. (30) and (31), we did not use this condition.

It is necessary to note that although we have calculated the LFP operator $\hat{I}_{ee}(\varepsilon)$ and the kinetic coefficients $A_{ee}(\varepsilon)$ and $B_{ee}(\varepsilon)$ in the energy space, the approach that we used did not require calculating explicitly the transition rates $W_{ee}(\varepsilon \leftrightarrow \varepsilon')$ of the scattered electrons in the energy space between the states with energies $\varepsilon(\vec{k}) = \varepsilon$ and $\varepsilon(\vec{k}') = \varepsilon'$. This would require the direct calculation of the corresponding transition probability rates $W_{ee}(\varepsilon \leftrightarrow \varepsilon')$ and the following integration over the states $d\varepsilon'$ in the corresponding expressions for $A_{ee}(\varepsilon)$ and $B_{ee}(\varepsilon)$. As will be seen, it is difficult to accomplish such calculations analytically due to complexity of the explicit form of the functions $W_{ee}(\varepsilon \leftrightarrow \varepsilon')$. In our approach, we have circumvented these calculation challenges by using the averaging of the *e-e* scattering operator over the constant energy surface in Eq. (5). At the same time, derivation of the explicit expressions for the probability rates $W_{ee}(\varepsilon \leftrightarrow \varepsilon')$ can shed a light on the validity of the LFP approximation for the *e-e* scattering, as will be shown later.

We will derive later the explicit expressions for the transition rates $W_{ee}(\varepsilon \leftrightarrow \varepsilon')$. However, this will be done as a part of a more general problem of derivation of the LFP equation from the Chapman-Kolmogorov equation (CKE). The reasons and justification for derivation of the LFP equation from the CKE are as follows. First, using the CKE is the most general mathematical approach for derivation of the FP equations. Secondly, the CKE equation is a linear equation, so in the case of the *e-e* scattering it needs some generalization to include the nonlinearity due to the two-particle *e-e* interaction, and this will be done in this paper. Thirdly, as will be shown, the derived LFP equation has differences in definitions of the particle flux along the energy axis and the corresponding kinetic coefficients in comparison with our expressions obtained in Eqs. (15)–(18), so for practical reasons it is important to establish general relationships between these quantities obtained from different approaches.

V. THE CHAPMAN-KOLMOGOROV EQUATION AND THE MASTER EQUATION FOR CASE OF THE *e-e* SCATTERING

We consider statistical behavior of the electrons which change their states in the momentum space due to interaction with each other, assuming that the random transitions between the states can be considered as a Markov process. The standard Chapman-Kolmogorov equation [3,25] is an integral relationship between the conditional transition probabilities $V(\vec{k}, t|\vec{k}', t')$ which describe transition of a particle from the state $\vec{k}'$ which particle occupies at the moment t' to the new state $\vec{k}$ at the moment t. However, for our purposes, it is more convenient to use an equivalent form of the CKE [27], which directly involves the probability density $w_1(\vec{k}, t)$ of the occupation of the state with a given momentum $\vec{k}$ at some moment t. These probability density is directly related to the corresponding distribution functions $F(\vec{k}, t)$. However, due to different normalizations, $w_1(\vec{k}, t)$ is normalized to unity, and $F(\vec{k}, t)$ is normalized on the total number of the electrons according to Eq. (20), it is more convenient to use in the CKE the one-particle probability densities $w_1(\vec{k}, t)$, and later make a transition to the distribution functions $F(\vec{k}, t)$. For generality of the derivation, it is assumed that the distribution function depends also on time; this condition will be relaxed later. The starting CKE is

$$w_1(\vec{k}, t) = \int V(\vec{k}, t|\vec{k}', \tau) w_1(\vec{k}', \tau) d\vec{k}', \tag{32}$$

where $V(\vec{k}, t|\vec{k}', \tau)$ is the transition probability density of the electron from the state with momentum $\vec{k}'$ which electron occupies at some moment τ into the state with momentum $\vec{k}$ at the time t. Strictly speaking, we should use in Eq. (32) summation over a discretized momentum $\vec{k}'$, but this can be replaced by the integration over a quasidiscontinuous momentum (for simplification of notations, we have included here the factor $2(2\pi/L)^3$ into $d\vec{k}'$). Since for the *e-e* scattering there are two interacting particles, we need to generalize Eq. (32) for this case by introducing the two-particle transition probability density $V(\vec{k},\vec{p},t|\vec{k}',\vec{p}',\tau)$, which describes random interaction (scattering) of two electrons, which have the momenta $\vec{k}'$ and $\vec{p}'$ at the moment τ, with transition into the states with the new momenta $\vec{k}$ and $\vec{p}$ at the moment t. The occupations of the states before the scattering transition event takes place, are given by the one-particle probability densities $w_1(\vec{k}',\tau)$ and $w_1(\vec{p}',\tau)$, respectively. These probability densities are directly related to the corresponding distribution functions $F(\vec{k}',\tau)$ and $F(\vec{p}',\tau)$. We can then write the CKE for the *e-e* scattering case:

$$w_1(\vec{k},t)=\iiint V(\vec{k},\vec{p},t|\vec{k}',\vec{p}',\tau)\,w_1(\vec{k}',\tau)\times w_1(\vec{p}',\tau)d\vec{k}'d\vec{p}'d\vec{p}. \tag{33}$$

Next, we need to derive from this CKE the so-called master equation. For this we calculate the time derivative of $w_1(\vec{k},t)$ as $\partial w_1(\vec{k},t)/\partial t=\lim_{\Delta t\to 0}[w_1(\vec{k},t+\Delta t)-w_1(\vec{k},t)]$. For the first term $w_1(\vec{k},t+\Delta t)$ we use Eq. (33) in which we change $t\to t+\Delta t$ and $\tau\to t$. For the second term, we use the identity [28]:

$$\begin{aligned}w_1(\vec{k},t)&=\iiint w_2(\vec{k}',\vec{p}',t+\Delta t;\vec{k},\vec{p},t)d\vec{k}'d\vec{p}'d\vec{p}\\&=\iiint V(\vec{k}',\vec{p}',t+\Delta t|\vec{k},\vec{p},t)w_1(\vec{k},t)\\&\quad\times w_1(\vec{p},t)d\vec{k}'d\vec{p}'d\vec{p}.\end{aligned} \tag{34}$$

Here $w_2(\vec{k}',\vec{p}',t+\Delta t;\vec{k},\vec{p},t)$ is a two-particle two-dimensional probability density, which is expressed in the second integral via the product of the two-particle conditional probability density $V(\vec{k}',\vec{p}',t+\Delta t|\vec{k},\vec{p},t)$ and the two one-particle probability densities $w_1(\vec{k},t)$ and $w_1(\vec{p},t)$. Substituting these expressions into the derivative $\partial w_1(\vec{k},t)/\partial t$, we obtain the master equation for the *e-e* scattering:

$$\begin{aligned}\frac{\partial w_1(\vec{k},t)}{\partial t}&=\iiint[W(\vec{k}',\vec{p}'\to\vec{k},\vec{p},t)w_1(\vec{k}',t)w_1(\vec{p}',t)\\&\quad-W(\vec{k},\vec{p}\to\vec{k}',\vec{p}',t)w_1(\vec{k},t)w_1(\vec{p},t)]d\vec{k}'d\vec{p}'d\vec{p},\end{aligned} \tag{35}$$

where we have introduced the transition probability rates:

$$W(\vec{k}',\vec{p}'\to\vec{k},\vec{p},t)=\lim_{\Delta t\to 0}\frac{V(\vec{k},\vec{p},t+\Delta t|\vec{k}',\vec{p}',t)}{\Delta t}, \tag{36}$$

$$W(\vec{k},\vec{p}\to\vec{k}',\vec{p}',t)=\lim_{\Delta t\to 0}\frac{V(\vec{k}',\vec{p}',t+\Delta t|\vec{k},\vec{p},t)}{\Delta t}. \tag{37}$$

For stationary Markov random processes, we can use that the one-particle probabilities do not depend on time, and the conditional probabilities are time-homogeneous, i.e., they depend only on the time difference Δt. Thus, the probability densities w_1 in Eq. (35) and the transition probability rates W in Eqs. (36) and (37) do not depend on time. (In this case the time derivative in the left-hand side of Eq. (35) has a symbolic meaning that it represents the rate of change of the probability density of the electron in a given state $\vec{k}$ due to the *e-e* scattering, so we can keep this symbolic notation even if w_1 does not depend on time.) After specifying these conditions, we can notice that the master equation, which was derived from the CKE equation, is actually the Boltzmann equation for the *e-e* scattering. However, for deriving the LFP form of the above master/Boltzmann equation, here we will proceed using different approach [25,27–29] than we used previously in Secs. III and IV.

We now consider the master equation for a symmetric part of a one-particle probability density $w_1(\vec{k})$. For this, we present $w_1(\vec{k})$ as a sum of a symmetric and an asymmetric parts, $w_1(\vec{k})=w_1(\varepsilon_{\vec{k}})+w_1^{\mathrm{asym}}(\vec{k})$ [and similarly, in the integrand of Eq. (35)], and, assuming that $|w_1^{\mathrm{asym}}(\vec{k})|\ll w_1(\varepsilon_{\vec{k}})$, we will retain only the main symmetric part of the equation:

$$\begin{aligned}\frac{\partial w_1(\varepsilon_{\vec{k}})}{\partial t}&=\sum_{\vec{k}'\vec{p}\,\vec{p}'}[W(\vec{k}',\vec{p}'\to\vec{k},\vec{p})\,w_1(\varepsilon_{\vec{p}'})w_1(\varepsilon_{\vec{k}'})\\&\quad-W(\vec{k},\vec{p}\to\vec{k}',\vec{p}')\,w_1(\varepsilon_{\vec{p}})\,w_1(\varepsilon_{\vec{k}})].\end{aligned} \tag{38}$$

Here we have also switched from the integration to the summation over $\vec{k}'$, $\vec{p}$, $\vec{p}'$ (this is just a change of the notations). To completely exclude the angle dependence from the above equation, we will average it over the surface of constant energy $\varepsilon_{\vec{k}}=\varepsilon$ by applying to both sides the sum $\sum_{\vec{k}}\delta(\varepsilon_{\vec{k}}-\varepsilon)/g(\varepsilon)$. In addition, in the right-hand side of Eq. (38), we use the equality $\int_0^\infty\delta(\varepsilon_{\vec{k}'}-\varepsilon')\,d\varepsilon'=1$. This gives the following equation:

$$\begin{aligned}\frac{\partial F(\varepsilon)}{\partial t}&\equiv\hat{I}_{ee}F(\varepsilon)\\&=\int_0^\infty g(\varepsilon')[W(\varepsilon',\varepsilon)F(\varepsilon')-W(\varepsilon,\varepsilon')F(\varepsilon)]d\varepsilon'.\end{aligned} \tag{39}$$

Here, we have replaced $w_1(\varepsilon)\to F(\varepsilon)$, and introduced the transition rates in the energy space:

$$\begin{aligned}W(\varepsilon,\varepsilon')&=\frac{1}{g(\varepsilon')g(\varepsilon)}\sum_{\vec{k}\vec{k}'\vec{p}\vec{p}'}W(\vec{k},\vec{p}\to\vec{k}',\vec{p}')\\&\quad\times F(\varepsilon_{\vec{p}})\delta(\varepsilon_{\vec{k}'}-\varepsilon')\delta(\varepsilon_{\vec{k}}-\varepsilon),\end{aligned} \tag{40}$$

$$\begin{aligned}W(\varepsilon',\varepsilon)&=\frac{1}{g(\varepsilon')g(\varepsilon)}\sum_{\vec{k}\vec{k}'\vec{p}\vec{p}'}W(\vec{k}',\vec{p}'\to\vec{k},\vec{p})\\&\quad\times F(\varepsilon_{\vec{p}'})\delta(\varepsilon_{\vec{k}'}-\varepsilon')\delta(\varepsilon_{\vec{k}}-\varepsilon).\end{aligned} \tag{41}$$

Note, that the expression for $\hat{I}_{ee}F(\varepsilon)$ in Eq. (39) differs from $\hat{I}_{ee}(\varepsilon)$ in Eq. (5). Also, since here we included into the scattering rates in Eqs. (40) and (41) the distribution functions of the second scattering electron, $F(\varepsilon_{\vec{p}})$ and $F(\varepsilon_{\vec{p}'})$, respectively, the DB conditions for $W(\varepsilon,\varepsilon')$ and $W(\varepsilon',\varepsilon)$ are different from those in Eq. (4). We will discuss this later, after further calculations of the above transition rates. The *e-e* collision integral in the form as given in Eq. (39) was used in Ref. [30], but without its derivation.

The next step is to apply the standard procedure to derive the differential form of the master equation. For this, we multiply both sides of Eq. (39) by $[g(\varepsilon)R(\varepsilon)]$, where $R(\varepsilon)$ is an arbitrary differentiable function, and integrate the obtained expression over ε. Note, that contrary to usual similar derivations for the linear scattering operators [29], here, for the case of the *e-e* scattering operator, we need to use the product of $R(\varepsilon)$ and the density of states $g(\varepsilon)$. Otherwise, we will not obtain the intended result. In the left-hand side of Eq. (39), the function $R(\varepsilon)$ is expanded in a Taylor series using the standard expression $R(\varepsilon) = R[\varepsilon' + (\varepsilon-\varepsilon')] = R(\varepsilon') + \sum_{m=1}^{\infty} (1/m!)(\varepsilon-\varepsilon')^m \partial^m R(\varepsilon)/\partial\varepsilon^m$, and the assumption that the energy transfer satisfies the condition $|\varepsilon-\varepsilon'| \ll \varepsilon, \varepsilon'$. With this, the first term in the integrand of Eq. (39) is canceled, and the master equation is transformed to

$$\int_0^{\infty} \frac{\partial F(\varepsilon)}{\partial t} g(\varepsilon)R(\varepsilon)d\varepsilon = \sum_{m=1}^{\infty} \int_0^{\infty} g(\varepsilon')F(\varepsilon')C_m(\varepsilon')\frac{\partial^m R(\varepsilon')}{\partial\varepsilon'^m}\, d\varepsilon', \tag{42}$$

where the coefficients $C_m(\varepsilon)$ are defined as

$$C_m(\varepsilon') = \frac{1}{m!}\int_0^{\infty} g(\varepsilon)(\varepsilon-\varepsilon')^m W(\varepsilon',\varepsilon)d\varepsilon. \tag{43}$$

We then integrate each term m in the right-hand side of Eq. (43) by-parts m times, and using that the product $[g(\varepsilon')F(\varepsilon')] = 0$ for $\varepsilon' = 0$ (due to $g(\varepsilon')$) and for $\varepsilon' = \infty$ (due to $F(\varepsilon')$), we obtain

$$\int_0^{\infty} \frac{\partial F(\varepsilon)}{\partial t} g(\varepsilon)R(\varepsilon)\, d\varepsilon = \int_0^{\infty} R(\varepsilon')\left[\sum_{m=1}^{\infty} (-1)^m \frac{\partial^m}{\partial\varepsilon'^m}[g(\varepsilon')F(\varepsilon')C_m(\varepsilon')]\right] d\varepsilon'. \tag{44}$$

Changing the variables in the right-hand side integrand $\varepsilon' \rightleftarrows \varepsilon$, and taking into account that $R(\varepsilon)$ is an arbitrary function, we obtain the final differential form of the master equation:

$$\frac{\partial F(\varepsilon)}{\partial t} = \frac{1}{g(\varepsilon)}\sum_{m=1}^{\infty}\frac{\partial^m}{\partial\varepsilon^m}[C_m(\varepsilon)g(\varepsilon)F(\varepsilon)]. \tag{45}$$

Here we redefined the coefficients $C_m(\varepsilon)$ in Eq. (43) in such a way that $(-1)^m$ is absorbed into it, and $C_m(\varepsilon)$ has a minus sign for all odd values of m (note, that this does not mean that the corresponding coefficients are negative):

$$C_m(\varepsilon) = \frac{1}{m!}\int_0^{\infty} g(\varepsilon')(\varepsilon-\varepsilon')^m W(\varepsilon,\varepsilon')d\varepsilon'. \tag{46}$$

The obtained Eqs. (45) and (46) represent the generalization of the Kramers-Moyal expansion (KME) [3,31,32] to the case of the *e-e* scattering operator in the energy space. If we retain in the KME only the first two terms with $m = 1$ and $m = 2$, the KME then will take the form of the LFP equation (we can use now the full derivative over ε):

$$\frac{\partial F(\varepsilon)}{\partial t} = \frac{1}{g(\varepsilon)}\frac{d}{d\varepsilon}\left[C_1(\varepsilon)g(\varepsilon)F(\varepsilon) + \frac{d}{d\varepsilon}[C_2(\varepsilon)g(\varepsilon)F(\varepsilon)]\right] = -\frac{1}{g(\varepsilon)}\frac{d}{d\varepsilon}[\tilde{J}_{ee}(\varepsilon)], \tag{47}$$

$$\tilde{J}_{ee}(\varepsilon) = -\left[C_1(\varepsilon)g(\varepsilon)F(\varepsilon) + \frac{d}{d\varepsilon}[C_2(\varepsilon)g(\varepsilon)F(\varepsilon)]\right], \tag{48}$$

where $\tilde{J}_{ee}(\varepsilon)$ is the carrier flux along the energy axis, and the kinetic coefficients are

$$C_1(\varepsilon) = \int_0^{\infty} g(\varepsilon')(\varepsilon-\varepsilon')W(\varepsilon,\varepsilon')d\varepsilon', \tag{49}$$

$$C_2(\varepsilon) = \frac{1}{2}\int_0^{\infty} g(\varepsilon')(\varepsilon-\varepsilon')^2 W(\varepsilon,\varepsilon')d\varepsilon'. \tag{50}$$

The flux in Eq. (48) actually describes the drift (the first term) and the diffusion (the second term) of the carrier density $n(\varepsilon) = g(\varepsilon)F(\varepsilon)$ in the energy space.

As can be seen from the derivation of the KME, although the approach used is of the most general form, at the same time, it was based on the key assumption that $|\varepsilon-\varepsilon'| \ll \varepsilon, \varepsilon'$. However, there is no general small parameter which ensures that we can truncate the KME at some value of $m = m_{\max}$, and neglect all higher order terms. In addition, as was pointed out in Ref. [29], retaining the terms of the order greater than two, creates serious challenges, besides maintaining the satisfactory accuracy. Since the KME is a differential operator, retaining the higher order terms means that we will obtain the higher order of the corresponding differential equations, which in turn will require to somehow specify the additional boundary conditions, which is a challenge. The analysis of Ref. [29] (known a Pawula Theorem) stipulates some generic conditions when the KME can be truncated at $m_{\max} = 2$ for case of the linear Boltzmann integral operator. Obviously, due to the nonlinearity of the *e-e* scattering operator, the Pawula Theorem cannot be directly applied to this case. As was already pointed out, in the latter case when the LFP approximation is used, we will need to check the accuracy of the obtained LFP solution by substituting it into the initial exact *e-e* scattering integral operator in Eq. (39), and evaluate in this way the accuracy of the LFP approximation. It seems, there is no alternative to this procedure in the case of the *e-e* scattering.

VI. ANALYSIS AND COMPARISON OF THE OBTAINED LFP EQUATIONS

From a direct comparison of the LFP expressions in Eqs. (15) and (16), which were obtained from the Boltzmann equation for the *e-e* scattering, with the corresponding expressions in Eqs. (47) and (48), obtained from the CKE, it can be seen that they differ. However, since these expressions represent the same quantity, namely, the rate of the change of the distribution function $(\partial F(\varepsilon)/\partial t)_{ee}$, they must be equal to each other. This is only possible, if the following relationships between the coefficients in these equations are satisfied:

$$A_{ee}(\varepsilon) = C_1(\varepsilon) + \frac{1}{g(\varepsilon)}\frac{d}{d\varepsilon}[g(\varepsilon)C_2(\varepsilon)], \qquad B_{ee}(\varepsilon) = C_2(\varepsilon). \tag{51}$$

These expressions have important practical implications, as they establish general relationships between the kinetic coefficients $A_{ee}(\varepsilon)$ and $B_{ee}(\varepsilon)$ in the FP differential form of the Boltzmann scattering operator, and the coefficients $C_1(\varepsilon)$ and $C_2(\varepsilon)$ in the FP form of the same scattering operator obtained from the CKE. Although, here our consideration was focussed on the *e-e* scattering, however, similar analysis can be carried for the other scattering mechanisms under the assumption that the scattering process satisfies the necessary condition $|\varepsilon - \varepsilon'| \ll \varepsilon,\ \varepsilon'$ given by Eq. (9). The relationships between the corresponding kinetic coefficients in the FP operator for any scattering mechanism will be the same as is shown in Eq. (51). The CKE approach is more universal in methodological and in computational senses in comparison with the Boltzmann scattering integral transformation, although, both methods lead to the same result. To calculate the coefficients $C_m(\varepsilon)$ using Eq. (46), what one needs is just to derive the transition probability rate $W(\varepsilon, \varepsilon')$ in the energy space using the general expressions of the form given in Eqs. (40) and (41) for the case of the *e-e* scattering. (Note, that one only needs to calculate the probability rate $W(\varepsilon, \varepsilon')$, the reverse transition rate $W(\varepsilon', \varepsilon)$ can be obtained from the DB conditions, however, this requires some additional analysis. We will demonstrate this later). We also want to point out that that there is a certain computational asymmetry between the above two approaches for deriving the differential FP form of the scattering operator. It is actually quite difficult to derive directly from the Boltzmann scattering operator the neat FP form, which is shown in Eqs. (45) and (46), while it is straightforward to obtain these equations using the CKE approach for the corresponding scattering mechanisms.

To check that the relationships in Eq. (51) are valid, we need to carry out the explicit calculations of the coefficients $C_1(\varepsilon)$ and $C_2(\varepsilon)$, and then, using $A_{ee}(\varepsilon)$ and $B_{ee}(\varepsilon)$ from Eqs. (22) and (28), to check the validity of Eq. (51). For an explicit calculation of $C_1(\varepsilon)$ and $C_2(\varepsilon)$, we have two options. We could calculate first the scattering probability rate $W(\varepsilon, \varepsilon')$ using Eq. (40), and then substitute it into Eqs. (49) and (50) to obtain the coefficients $C_1(\varepsilon)$ and $C_2(\varepsilon)$. However, as we will see later, the exact expression for $W(\varepsilon, \varepsilon')$ is quite complex for a direct integration in Eqs. (49) and (50). In addition, for the overall consistency of the theory, we need to calculate the coefficients with the same accuracy as was used for derivation of the LFP equation. It turned out, that the latter can be achieved, if we substitute $W(\varepsilon, \varepsilon')$ into Eqs. (49) and (50) in its general form as given in Eq. (40). This gives for the coefficient $C_1(\varepsilon)$:

$$C_1(\varepsilon) = \frac{1}{g(\varepsilon)} \sum_{\vec{k}\vec{k}'\vec{p}\vec{p}'} \int_0^\infty (\varepsilon - \varepsilon') W(\vec{k}, \vec{p} \to \vec{k}', \vec{p}') F(\varepsilon_{\vec{p}}) \times \delta(\varepsilon_{\vec{k}'} - \varepsilon')\delta(\varepsilon_{\vec{k}} - \varepsilon) d\varepsilon'. \tag{52}$$

Using in the above integrand the expressions from Eqs. (1) and (7), and integrating over ε', we obtain the exact expression for $C_1(\varepsilon)$:

$$C_1(\varepsilon) = \frac{1}{g(\varepsilon)} \sum_{\vec{k}\vec{k}'\vec{p}\vec{p}'} (\varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'}) w(\vec{q}) F(\varepsilon_{\vec{p}}) \delta_{\vec{p}-\vec{p}',\vec{q}} \times \delta(\varepsilon_{\vec{k}} - \varepsilon)\delta(\varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'} - \varepsilon_{\vec{p}'} + \varepsilon_{\vec{p}}). \tag{53}$$

Replacing now $(\varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'})$ by $(\varepsilon_{\vec{p}'} - \varepsilon_{\vec{p}})$, which follows from the last δ function in Eq. (53), and expanding this δ function in a series:

$$\delta\left[\varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'} + (\varepsilon_{\vec{p}} - \varepsilon_{\vec{p}'})\right] = \delta(\varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'}) - (\varepsilon_{\vec{p}'} - \varepsilon_{\vec{p}}) \frac{d}{d\varepsilon_{\vec{k}}} \delta(\varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'}), \tag{54}$$

we obtain, after some algebra, the relationship for $C_1(\varepsilon)$:

$$\begin{aligned} C_1(\varepsilon) = {} & \frac{1}{g(\varepsilon)} \sum_{\vec{k}\vec{k}'\vec{p}\vec{p}'} w(\vec{q}) \delta_{\vec{p}'-\vec{p},\vec{q}} (\varepsilon_{\vec{p}'} - \varepsilon_{\vec{p}}) F(\varepsilon_{\vec{p}})\, \delta(\varepsilon_{\vec{k}'} - \varepsilon)\delta(\varepsilon_{\vec{k}} - \varepsilon) \\ & - \frac{1}{g(\varepsilon)} \frac{d}{d\varepsilon} \left[\frac{1}{2} \sum_{\vec{k}\vec{k}'\vec{p}\vec{p}'} w(\vec{q}) \delta_{\vec{p}'-\vec{p},\vec{q}} (\varepsilon_{\vec{p}'} - \varepsilon_{\vec{p}})^2 F(\varepsilon_{\vec{p}})\, \delta(\varepsilon_{\vec{k}'} - \varepsilon)\delta(\varepsilon_{\vec{k}} - \varepsilon) \right] \\ = {} & A_{ee}(\varepsilon) - \frac{1}{g(\varepsilon)} \frac{d}{d\varepsilon} [g(\varepsilon) B_{ee}(\varepsilon)], \end{aligned} \tag{55}$$

where in the last expression we used Eqs. (17) and (18).

To see that Eq. (55) agrees with the first Eq. (51), we need to prove that $B_{ee}(\varepsilon) = C_2(\varepsilon)$, i.e., to prove the second Eq. (51). This result follows straightforward after we substitute $W(\varepsilon, \varepsilon')$ from Eq. (40) into Eq. (50), and use only the zero-order term in the expansion of δ function in Eq. (54) (that only the zero-order term should be retained is clear, since the contribution to $C_2(\varepsilon)$ from the next term in the expansion will be $\sim (\varepsilon_{\vec{p}'} - \varepsilon_{\vec{p}})^3$, which is small). We obtain from Eqs. (40) and (50), taking into account Eq. (18):

$$C_2(\varepsilon) = \frac{1}{2} \frac{1}{g(\varepsilon)} \sum_{\vec{k}\vec{k}'\vec{p}\vec{p}'} (\varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'})^2 w(\vec{q}) F(\varepsilon_{\vec{p}}) \delta_{\vec{p}-\vec{p}',\vec{q}} \delta(\varepsilon_{\vec{k}'} - \varepsilon)\delta(\varepsilon_{\vec{k}} - \varepsilon) = B_{ee}(\varepsilon). \tag{56}$$

The results in Eqs. (55) and (56) prove both expressions in Eq. (51).

Since we have already calculated $A_{ee}(\varepsilon)$ and $B_{ee}(\varepsilon)$ in Eqs. (22) and (28), we can use these results to obtain $C_1(\varepsilon)$ and $C_2(\varepsilon)$ from Eq. (51). [Otherwise, one will simply need to complete calculations of $C_1(\varepsilon)$ and $C_2(\varepsilon)$ in Eqs. (55) and (56) using the same method as was used in Sec. IV for calculations of $A_{ee}(\varepsilon)$ and $B_{ee}(\varepsilon)$.] The physical meaning of these coefficients according to Eqs. (49) and (50), is that $C_1(\varepsilon)$ is a dynamic friction coefficient, which represents the power loss $C_1(\varepsilon) \equiv Q_{\text{loss}}^{ee}(\varepsilon)$ of the test electron with the energy ε due to scattering on the other field electrons. The coefficient $C_2(\varepsilon) \equiv D_{ee}(\varepsilon)$ is the diffusion coefficient of the electron with the energy ε in the energy space due to the *e-e* scattering.

Using Eq. (51) and taking into account Eq. (30), we obtain for $C_1(\varepsilon)$ (the power loss):

$$C_1(\varepsilon) = Q_{\text{loss}}^{ee}(\varepsilon) = \frac{3}{2}\frac{D_{ee}(\varepsilon)}{\bar{\varepsilon}} - \frac{1}{g(\varepsilon)}\frac{d}{d\varepsilon}[g(\varepsilon)D_{ee}(\varepsilon)]$$
$$= \frac{\varepsilon}{\tau_{ee}(\varepsilon)} - \frac{2}{3}\bar{\varepsilon}\frac{1}{g(\varepsilon)}\frac{d}{d\varepsilon}\left[\frac{\varepsilon g(\varepsilon)}{\tau_{ee}(\varepsilon)}\right]. \tag{57}$$

In the case of the *e-e* scattering, the product in the square brackets does not depend on ε, it is a constant, as it follows from Eq. (29) if we ignore a weak energy dependence of $\Lambda(\varepsilon)$. This finally gives from Eqs. (51), (57), and (30):

$$A_{ee}(\varepsilon) = C_1(\varepsilon) \equiv Q_{\text{loss}}^{ee}(\varepsilon) \approx \frac{\varepsilon}{\tau_{ee}(\varepsilon)},$$
$$B_{ee}(\varepsilon) = C_2(\varepsilon) \equiv D_{ee}(\varepsilon) = \frac{2}{3}\bar{\varepsilon}\left[\frac{\varepsilon}{\tau_{ee}(\varepsilon)}\right]. \tag{58}$$

In addition to Eq. (51), we can also derive another important general relationship between the kinetic coefficients $C_1(\varepsilon)$ and $C_2(\varepsilon)$ [or, equivalently, between $Q_{\text{loss}}^{ee}(\varepsilon)$ and $D_{ee}(\varepsilon)$]. Using the Maxwellian function with an arbitrary temperature T (including the equilibrium case, when $T = T_0$), we obtain from Eq. (48) the condition $\tilde{J}_{ee}(\varepsilon) = 0$, which gives

$$Q_{\text{loss}}^{ee}(\varepsilon) = -D_{ee}(\varepsilon)\frac{d}{d\varepsilon}\left[\ln\left(D_{ee}(\varepsilon)g(\varepsilon)\exp\left(-\frac{\varepsilon}{k_BT}\right)\right)\right]. \tag{59}$$

This general relationship between the dynamic friction coefficient $C_1(\varepsilon) = Q_{\text{loss}}^{ee}(\varepsilon)$ and the diffusion coefficient $C_2(\varepsilon) = D_{ee}(\varepsilon)$ is an analogue of the fluctuation-dissipation theorem for the *e-e* scattering. For the case of a quasielastic electron scattering on acoustic phonons it was derived in Ref. [22] [in this case $T = T_0$ in Eq. (59)]. However, to apply this relationship to a nonequilibrium system with the *e-e* scattering, it is necessary to make an additional assumption that the high-energy hot electrons scatter mainly on the ensemble of thermalized field electrons with the distribution function $F(\varepsilon_{\vec{p}}) = A\exp(-\varepsilon_{\vec{p}}/k_BT)$ [the function $F(\varepsilon_{\vec{p}})$ enters the expressions for the coefficients $C_1(\varepsilon)$ and $C_2(\varepsilon)$, see Eqs. (53) and (56)]. This is also equivalent to imposing the condition $\Delta\varepsilon \equiv \varepsilon_{\vec{k}'} - \varepsilon_{\vec{k}} = \varepsilon_{\vec{p}} - \varepsilon_{\vec{p}'} \ll T$ on the energy exchange between the scattering electrons. The latter condition is stronger in comparison with the condition in Eq. (9), which we used in all previous derivations. As a result, in this case we obtain $\bar{\varepsilon} = (3/2)k_BT$, which in turn gives the following relationships between $C_1(\varepsilon)$ and $C_2(\varepsilon)$, and between $Q_{\text{loss}}^{ee}(\varepsilon)$ and $D_{ee}(\varepsilon)$:

$$C_1(\varepsilon) = C_2(\varepsilon)/k_BT,$$
$$Q_{\text{loss}}^{ee}(\varepsilon) = D_{ee}(\varepsilon)/k_BT, \tag{60}$$

which agree with Eqs. (51) and (59).

It is necessary to stress that the relationships in Eq. (60) are valid only for the case of the *e-e* scattering, since only in this case the second term in Eq. (57) is equal zero. However, all general equations and the corresponding expressions for the kinetic coefficients derived from the CKE remain valid for any other scattering mechanism under the condition that we can utilize the assumption in Eq. (9) for diffusivity of the electron scattering in the energy space. To demonstrate general character of the relationships obtained here from the CKE, we considered in Appendix their application to the case of a quasielastic electron scattering on acoustic phonons. The main reason for this consideration is that this will demonstrate some modifications of the fluctuation-dissipation relationships between $D(\varepsilon)$ and $Q_{\text{loss}}^{ee}(\varepsilon)$ in comparison with the case of the *e-e* scattering. The latter can be of interest for experimental investigation of these parameters for different scattering mechanisms.

VII. THE EXPLICIT EXPRESSION FOR THE *e-e* SCATTERING RATE $W(\varepsilon, \varepsilon')$

In the previous part, we have transformed the integral *e-e* scattering operator in Eq. (3) to the LFP differential form using different approaches. However, this transformation depends on the condition of a small energy transfer in the *e-e* scattering events, as required by Eq. (9). In the absence of screening of the *e-e* interaction, it is straightforward to show from kinematics of the electron motion that the energy exchange of the scattered electron can approximately be estimated as [22] $\Delta\varepsilon \approx 2\varepsilon(1-\cos\theta) + \varepsilon\sin^2\theta$, which for a small-angle scattering ($\theta \ll 1$) gives $\Delta\varepsilon = 2\varepsilon\theta^2 \ll \varepsilon$, i.e., in this case the small-angle scattering also corresponds to the small-energy exchange. The Landau's conjecture about the dominance of the small-angle scattering for an unscreened Coulomb interaction will also support the assumption of the small-energy transfer, and thus, it justifies in this case the LFP differential form of the *e-e* operator in the energy space. However, to self-consistently resolve the issue of divergence of the scattering rate in the LFP operator, we have to include screening into the *e-e* interaction potential. Then, the question is whether the above estimate for $\Delta\varepsilon$ will still remain valid in the presence of screening. As was discussed in Ref. [30], the answer depends on the strength of screening. Some insight into this can be obtained from the explicit expression for the scattering rate $W(\varepsilon, \varepsilon')$. However, instead of using the approximate expression from Ref. [30], we will further transform the integral operator $\hat{I}_{ee}F(\varepsilon)$ in Eq. (39) with the scattering probability rates $W(\varepsilon, \varepsilon')$ in the energy space without making any approximations. Substituting $w(\vec{q})$ from Eq. (7) into Eqs. (40) and (41) and carrying out summation over $\vec{k}$, $\vec{k}'$, and over $\vec{p}$, $\vec{p}'$, we obtain for these scattering probability rates the following expressions:

$$W(\varepsilon, \varepsilon', \theta) = \frac{\pi}{4}\left(\frac{L}{2\pi\hbar}\right)^3 \frac{w_0}{\sqrt{2m}} \frac{1}{\varepsilon_{\vec{q}}^{1/2}(\varepsilon_{\vec{q}} + \varepsilon_0)^2}$$
$$\times \int_{\varepsilon'_{\parallel}}^{\infty} F(\varepsilon_{\vec{p}})d\varepsilon_{\vec{p}}, \tag{61}$$

$$W(\varepsilon', \varepsilon, \theta) = \frac{\pi}{4}\left(\frac{L}{2\pi\hbar}\right)^3 \frac{w_0}{\sqrt{2m}} \frac{1}{\varepsilon_{\vec{q}}^{1/2}(\varepsilon_{\vec{q}} + \varepsilon_0)^2}$$
$$\times \int_{\varepsilon_{\parallel}}^{\infty} F(\varepsilon_{\vec{p}'})d\varepsilon_{\vec{p}'}, \tag{62}$$

$$\varepsilon_{\vec{q}} = \varepsilon + \varepsilon' - 2\sqrt{\varepsilon\varepsilon'}\cos\theta$$
$$= (\sqrt{\varepsilon'} - \sqrt{\varepsilon})^2 + 4\sqrt{\varepsilon\varepsilon'}\sin^2(\theta/2), \tag{63}$$

where θ is the scattering angle, and the energies $\varepsilon_\parallel$ and $\varepsilon_\parallel^{'}$ are defined by the projections of the momenta $\vec{k}$ and $\vec{k}'$, respectively, on the direction of the scattering momentum $\vec{q} = \vec{k}' - \vec{k}$:

$$\varepsilon_\parallel = \frac{k_\parallel^2}{2m} = \frac{1}{2m}\left(\frac{\vec{k}\cdot\vec{q}}{q}\right)^2 = \frac{[\varepsilon - \sqrt{\varepsilon\varepsilon'}\cos\theta]^2}{\varepsilon + \varepsilon' - 2\sqrt{\varepsilon\varepsilon'}\cos\theta}, \tag{64}$$

$$\varepsilon_\parallel^{'} = \frac{(k_\parallel^{'})^2}{2m} = \frac{1}{2m}\left(\frac{\vec{k}'\cdot\vec{q}}{q}\right)^2 = \frac{[\varepsilon' - \sqrt{\varepsilon\varepsilon'}\cos\theta]^2}{\varepsilon + \varepsilon' - 2\sqrt{\varepsilon\varepsilon'}\cos\theta}. \tag{65}$$

The scattering rates in the energy space are then obtained from the angle averaging:

$$W(\varepsilon, \varepsilon') = \frac{1}{2}\int_{-1}^{+1} W(\varepsilon, \varepsilon', \theta) d(\cos\theta),$$
$$W(\varepsilon', \varepsilon) = \frac{1}{2}\int_{-1}^{+1} W(\varepsilon', \varepsilon, \theta) d(\cos\theta). \tag{66}$$

As is seen from Eqs. (61), (62), and (66), these probabilities in general do not satisfy the DB condition. Therefore, in an integral form of the *e-e* scattering operator $\hat{I}_{ee}F(\varepsilon)$ we should use both probabilities. It is only under some additional conditions, which we will discuss later, the probabilities $W(\varepsilon, \varepsilon')$ and $W(\varepsilon', \varepsilon)$ will satisfy the DB condition (actually, in this case the probabilities $W(\varepsilon, \varepsilon', \theta)$ and $W(\varepsilon', \varepsilon, \theta)$ will also satisfy the same DB conditions).

Using in Eq. (39) the results from Eqs. (61) and (62), we then obtain the exact expression for the integral operator $\hat{I}_{ee}F(\varepsilon)$:

$$\hat{I}_{ee}F(\varepsilon) = \frac{\pi}{8}\left(\frac{L}{2\pi\hbar}\right)^3 \frac{w_0}{\sqrt{2m}}$$
$$\times \int_0^\infty d\varepsilon' g(\varepsilon') \int_{-1}^{+1} d(\cos\theta) \frac{1}{\varepsilon_{\vec{q}}^{1/2}(\varepsilon_{\vec{q}} + \varepsilon_0)^2}$$
$$\times \left[F(\varepsilon') \int_{\varepsilon_\parallel^\infty} F(\varepsilon_{\vec{p}'}) d\varepsilon_{\vec{p}'} - F(\varepsilon) \int_{\varepsilon_\parallel^{'}}^\infty F(\varepsilon_{\vec{p}}) d\varepsilon_{\vec{p}}\right]. \tag{67}$$

This is how far one can go in transforming the integral *e-e* scattering operator without making any approximations. Note, that this operator does satisfy the equilibrium condition $\hat{I}_{ee}F(\varepsilon) = 0$, as it should be. Using for all distribution functions in Eq. (67) the Maxwellian functions with the corresponding energy variables, we obtain after the integrations the following equilibrium condition:

$$\exp\left[-\frac{\varepsilon' + \varepsilon_\parallel}{k_B T_0}\right] = \exp\left[-\frac{\varepsilon + \varepsilon_\parallel^{'}}{k_B T_0}\right]. \tag{68}$$

Using the expressions for energies $\varepsilon_\parallel$ and $\varepsilon_\parallel^{'}$ in Eqs. (64) and (65), it can be shown straightforward that these energies satisfy the following relationship:

$$\varepsilon_\parallel^{'} - \varepsilon_\parallel = \varepsilon' - \varepsilon, \tag{69}$$

which then proves that the Eq. (68) is valid, i.e., that the integral *e-e* operator $\hat{I}_{ee}F(\varepsilon)$ satisfies the DB condition at the equilibrium.

If the screening is neglected in Eq. (67), then both integrals over the energy ε' and the scattering angle θ will diverge, when $\theta \to 0$ and $\varepsilon' \to \varepsilon$, (i.e., for a small-angle scattering and a small-energy transfer), since in this case $\varepsilon_{\vec{q}} \to 0$ in the denominator. The screening removes this divergence. However, if the approximation of the small-energy transfer given by Eq. (9) cannot be used for the screened *e-e* interaction, then one will need to use the integral equation for $\hat{I}_{ee}F(\varepsilon)$ given by Eq. (67). This is a very challenging task even for a numerical treatment. Thus, having even an approximate solution for using the LFP differential operator, as we have considered in this paper, will at least provide a good physical insight into the behavior of a nonequilibrium system with strong *e-e* scattering, provided that the conditions in Eq. (9) are satisfied.

There exists an important physical situation which allows to transform further the integral operator $\hat{I}_{ee}F(\varepsilon)$. This is the case when one is interested in the distribution function of the hot electrons at high energies ε, which are considerably bigger than the mean energy of the electron gas $\bar{\varepsilon}$, $\varepsilon \gg \bar{\varepsilon}$, at the nonequilibrium conditions. Such situations were discussed for plasmas [16,20] and for semiconductors [30]. In both cases it is assumed that there is another interaction process, with some characteristic threshold energy, which is physically important at high energies. For plasmas, this may be, e.g., an excitation of atoms into the higher energy states, or their impact ionization by the high-energy electrons; for semiconductors this also can be an impact ionization, or, more realistically, the interaction of hot electrons with the high-energy optical phonons, or the intervalley phonons. Both these mechanisms are strongly inelastic, and, importantly, they usually dominate at high energies (above the threshold), in a sense that the related inelastic interaction rate is considerably higher than the *e-e* scattering rate. This allows to assume that at the lower energies (below the threshold), where the *e-e* scattering dominates, the distribution function is given by the equation $\hat{I}_{ee}F(\varepsilon) \approx 0$, which gives the Maxwellian distribution function with the electron temperature T. At the high energies this solution is not applicable, as one has to take into account both, the inelastic scattering (e.g., due to interaction with the optical phonons) and the *e-e* scattering of the high-energy electrons on the lower-energy field electrons. The *e-e* scattering provides the additional channel of the electron flux from the sub-threshold energies to the energies at or above the threshold. Both these scattering mechanisms must be considered in the kinetic equation for the distribution function $F(\varepsilon)$ of the high-energy electrons. However, specifically in this case for the *e-e* scattering it is sufficient to assume that the high-energy hot electrons [with an unknown distribution function $F(\varepsilon)$] interact mainly with the thermalized electron gas with the energies below the threshold energy, i.e., in this case the distribution functions $F(\varepsilon_{\vec{p}})$ and $F(\varepsilon_{\vec{p}'})$ are the Maxwellian functions with the electron temperature T. The *e-e* interaction between the high-energy electrons can be ignored at not very high carrier densities, since the scattering rate $\nu_{ee}(\varepsilon) \sim \varepsilon^{-3/2}$, and it decreases when the electron energy increases, according to Eq. (29). Here we discuss this case to obtain some additional physical insight into the conditions, which are necessary for derivation of differential LFP operator for the hot electrons. Under these conditions, we have a considerable simplification of the initial problem of the *e-e* scattering, since the assumption about the Maxwellian functions $F(\varepsilon_{\vec{p}})$ and $F(\varepsilon_{\vec{p}'})$ means that we actually obtain the linearized *e-e* operator $\hat{I}_{ee}F(\varepsilon)$ in Eq. (67) for the hot electrons:

$$\hat{I}_{ee}F(\varepsilon) = \frac{\pi}{8}\left(\frac{L}{2\pi\hbar}\right)^3 \frac{w_0 k_B T A}{\sqrt{2m}} \times \int_0^\infty d\varepsilon' g(\varepsilon') \int_{-1}^{+1} d(\cos\theta) \frac{\exp\left(-\frac{\varepsilon'_\parallel}{k_B T}\right)}{\varepsilon_{\vec{q}}^{1/2}(\varepsilon_{\vec{q}}+\varepsilon_0)^2} \times \left[\exp\left(\frac{\varepsilon'-\varepsilon}{k_B T}\right)F(\varepsilon') - F(\varepsilon)\right], \tag{70}$$

where Eq. (69) was used. The normalization constant A can be obtained from Eq. (20), which gives $A = (n/2)(2\pi\hbar^2/mk_BT)^{3/2}$. From the comparison in Eq. (70) the terms with $F(\varepsilon)$ and $F(\varepsilon')$, we obtain the DB condition for the transition rates:

$$W(\varepsilon', \varepsilon) = W(\varepsilon, \varepsilon') \exp\left(\frac{\varepsilon'-\varepsilon}{k_B T}\right). \tag{71}$$

The anisotropic transition rates $W(\varepsilon', \varepsilon, \theta)$ and $W(\varepsilon, \varepsilon', \theta)$ in Eqs. (61) and (62) also satisfy this condition. From Eqs. (61) and (66) [or directly from Eq. (70)] we obtain the scattering probability rate $W(\varepsilon, \varepsilon')$:

$$W(\varepsilon, \varepsilon') = \frac{1}{2}\left(\frac{\hbar}{L}\right)^3 \frac{\pi^{5/2} e^4 n}{\kappa_0^2 m^2 (k_B T)^{1/2}} \times \int_{-1}^{+1} d(\cos\theta) \frac{\exp\left(-\frac{\varepsilon'_\parallel}{k_B T}\right)}{\varepsilon_{\vec{q}}^{1/2}(\varepsilon_{\vec{q}}+\varepsilon_0)^2}. \tag{72}$$

The integral over the scattering angles can be calculated by switching to the new integration variable $\varepsilon_{\vec{q}}$ using Eq. (63): $d(\cos\theta) = -d\varepsilon_{\vec{q}}/2\sqrt{\varepsilon\varepsilon'}$, which gives

$$W(\varepsilon, \varepsilon') = \frac{1}{4}\left(\frac{\hbar}{L}\right)^3 \frac{\pi^{5/2} e^4 n}{\kappa_0^2 m^2 (k_B T)^{1/2}} \frac{1}{\sqrt{\varepsilon\varepsilon'}} \exp\left(-\frac{\Delta + |\Delta|}{2k_B T}\right) \times \int_{(\sqrt{\varepsilon'}-\sqrt{\varepsilon})^2}^{(\sqrt{\varepsilon'}+\sqrt{\varepsilon})^2} d\varepsilon_{\vec{q}} \frac{\exp\left(-\frac{(\varepsilon_{\vec{q}}-|\Delta|)^2}{4k_B T\varepsilon_{\vec{q}}}\right)}{\varepsilon_{\vec{q}}^{1/2}(\varepsilon_{\vec{q}}+\varepsilon_0)^2}, \tag{73}$$

where $\Delta = \varepsilon' - \varepsilon$ is the energy transfer. This is the exact expression for the scattering probability rate $W(\varepsilon, \varepsilon')$. The only assumption, which was used for its derivation, is that the test electron with the energy ε (and an unknown distribution function) is scattered by the other field electrons, which are described by the known (Maxwellian) distribution function. No other assumptions, like the small energy transfer specified by Eq. (9), were used. This observation regarding Eq. (73) has an important implication for correct calculation of the kinetic coefficients $C_1(\varepsilon)$ and $C_2(\varepsilon)$ using Eqs. (49) and (50). As was discussed in Sec. VI, for overall consistency of the theory, these coefficients in the LFP *e-e* scattering operator in Eq. (48) should be calculated with the same accuracy and approximations as it was used in derivation of the LFP operator. Note also, that Δ in Eq. (73) can be either positive or negative. As a result of this, the scattering rate $W(\varepsilon, \varepsilon')$ displays an asymmetric dependence on ε' with respect to the line $\varepsilon' = \varepsilon$, i.e., for $\Delta = 0$. However, $W(\varepsilon, \varepsilon')$ for different signs of Δ are related to each other by the DB condition, as it follows from considering the $\varepsilon \leftrightarrow \varepsilon'$ transition rates for the upward, $\Delta > 0$, and for the downward, $\Delta < 0$, transitions. Therefore, one only needs to obtain $W(\varepsilon, \varepsilon')$ for one sign of Δ, and then use the DB condition to obtain it for the opposite sign of Δ.

The new integration variable $\varepsilon_{\vec{q}}$ contains information about the scattering angle θ and the energy transfer Δ in the scattering event. Using Eq. (63), $\varepsilon_{\vec{q}}$ can be presented as $\varepsilon_{\vec{q}} = (\sqrt{\varepsilon+\Delta} - \sqrt{\varepsilon})^2 + 4\sqrt{\varepsilon(\varepsilon+\Delta)}\sin^2(\theta/2)$. This dependence affects the order of the integration in calculating the energy dependence of the *e-e* scattering rate for hot electrons, which, after using Eq. (73) with some transformations, is given by the expression:

$$\frac{1}{\tau_{ee}(\varepsilon)} = \int_0^\infty g(\varepsilon') W(\varepsilon, \varepsilon') d\varepsilon' = \frac{\sqrt{\pi}}{2} \frac{e^4 n}{\kappa_0^2} \frac{1}{\sqrt{2mk_B T\varepsilon}} \times \int_0^\infty \exp\left(-\frac{\Delta + |\Delta|}{2k_B T}\right) d\varepsilon' \times \int_{(\sqrt{\varepsilon'}-\sqrt{\varepsilon})^2}^{(\sqrt{\varepsilon'}+\sqrt{\varepsilon})^2} \frac{\exp\left(-\frac{(\varepsilon_{\vec{q}}-|\Delta|)^2}{4k_B T\varepsilon_{\vec{q}}}\right)}{\varepsilon_{\vec{q}}^{1/2}(\varepsilon_{\vec{q}}+\varepsilon_0)^2} d\varepsilon_{\vec{q}}. \tag{74}$$

As is seen, due to a quite complex dependence of the integrand on the energy ε', which is present in Δ, and it also enters the integration limits of the internal integral, one has to integrate first over $d\varepsilon_{\vec{q}}$, i.e., over the scattering angles, and only after this one can complete integration over $d\varepsilon'$ (or, over the energy transfer $d\Delta = d\varepsilon'$, with the corresponding change of the integration limits). This also points out to difficulties in evaluating the necessary conditions for a small-angle scattering and for a small-energy transfer.

The above expressions also allow to present the operator $\hat{I}_{ee}F(\varepsilon)$ in Eq. (70) in the final (integral) form:

$$\hat{I}_{ee}F(\varepsilon) = \frac{\sqrt{\pi}}{2} \frac{e^4 n}{\kappa_0^2} \frac{1}{\sqrt{2mk_B T\varepsilon}} \times \int_0^\infty \exp\left(-\frac{\Delta + |\Delta|}{2k_B T}\right) d\varepsilon' \int_{(\sqrt{\varepsilon'}-\sqrt{\varepsilon})^2}^{(\sqrt{\varepsilon'}+\sqrt{\varepsilon})^2} d\varepsilon_{\vec{q}} \frac{\exp\left(-\frac{(\varepsilon_{\vec{q}}-|\Delta|)^2}{4k_B T\varepsilon_{\vec{q}}}\right)}{\varepsilon_{\vec{q}}^{1/2}(\varepsilon_{\vec{q}}+\varepsilon_0)^2} \times \left[\exp\left(\frac{\varepsilon'-\varepsilon}{k_B T}\right)F(\varepsilon') - F(\varepsilon)\right] = \int_0^\infty d\varepsilon' g(\varepsilon') W(\varepsilon, \varepsilon') \exp\left(\frac{\varepsilon'-\varepsilon}{k_B T}\right) F(\varepsilon') - \frac{F(\varepsilon)}{\tau_{ee}(\varepsilon)}. \tag{75}$$

This exact integral form of the *e-e* scattering operator $\hat{I}_{ee}F(\varepsilon)$ for high-energy electrons should be used in all cases when its transformation to the differential LFP form is not possible [i.e., when the conditions in Eq. (9) are not satisfied]. For example, as will be shown later, this takes place if the screening effect is strong. In these cases, due to complexity of $W(\varepsilon, \varepsilon')$, the numerical solution for $\hat{I}_{ee}F(\varepsilon)$ will be necessary.

The exact expression for $W(\varepsilon, \varepsilon')$ given by Eq. (73) is very convenient for general analysis. It is clear that for the scattering with very large energy transfer, when $\varepsilon' \gg \varepsilon$ or $\varepsilon' \ll \varepsilon$ (i.e., when $|\Delta| \gg \varepsilon,\ \varepsilon'$) we have $W(\varepsilon, \varepsilon') \to 0$, since both integration limits become equal. However, this still does not mean that the *e-e* scattering events with small energy transfer, when $\varepsilon' \sim \varepsilon$ or $|\Delta| \ll \varepsilon, \varepsilon'$, or with a moderate energy transfer, when $|\Delta| \sim \varepsilon,\ \varepsilon'$, are dominant, particularly, in presence of screening. The function $W(\varepsilon, \varepsilon')$ has a peak at $\Delta = 0$ when the integration range is maximal:

$$W_{\max} = \frac{1}{4}\left(\frac{\hbar}{L}\right)^3 \frac{\pi^{5/2} e^4 n}{\kappa_0^2 m^2 (k_B T)^{1/2}} \frac{1}{\varepsilon} \times \int_0^{4\varepsilon} d\varepsilon_{\bar{q}} \frac{\exp\left(-\frac{\varepsilon_{\bar{q}}}{4k_B T}\right)}{\varepsilon_{\bar{q}}^{1/2} (\varepsilon_{\bar{q}} + \varepsilon_0)^2}. \tag{76}$$

For scattering of the high-energy electrons, for which $\varepsilon \gg T$, the upper limit in Eq. (76) can be replaced by ∞, and the integral can be calculated analytically:

$$W_{\max} = \frac{1}{8}\left(\frac{\hbar}{L}\right)^3 \frac{\pi^{7/2} e^4 n}{\kappa_0^2 m^2 (k_B T)^2} \frac{1}{\varepsilon} \times \frac{k_B T}{\varepsilon_0}\left[\frac{1}{\sqrt{\pi}} + \frac{2 - \varepsilon_0/k_B T}{2\sqrt{\varepsilon_0/k_B T}} \times \exp\left(\frac{\varepsilon_0}{4k_B T}\right) \operatorname{erfc}\left(\sqrt{\frac{\varepsilon_0}{4k_B T}}\right)\right]. \tag{77}$$

As is seen, the maximum probability rate is defined by the ratio $\varepsilon_0/k_B T$. In the limit of a weak ($\varepsilon_0 \ll k_B T$) or a strong ($\varepsilon_0 \gg k_B T$) screening we obtain, respectively:

$$W_{\max} = \frac{1}{8}\left(\frac{\hbar}{L}\right)^3 \frac{\pi^{7/2} e^4 n}{\kappa_0^2 m^2 (k_B T)^2} \frac{1}{\varepsilon} \times \begin{cases} \left(\frac{\varepsilon_0}{k_B T}\right)^{-3/2} & (\text{For } \varepsilon_0 \ll k_B T), \\ \frac{4}{\sqrt{\pi}}\left(\frac{\varepsilon_0}{k_B T}\right)^{-2} & (\text{For } \varepsilon_0 \gg k_B T). \end{cases} \tag{78}$$

For any given electron energy ε, the peak scattering rate $W_{\max}$ is a monotonically decreasing function of the parameter $\varepsilon_0/k_B T$. This means that with increasing of the screening parameter, the probability of the scattering events with small energy transfer $|\Delta| \ll \varepsilon,\ \varepsilon'$ decreases, and at a sufficiently strong screening this probability will have a very weak maximum, which in turn means that the conditions of Eq. (9), which are required for the validity of the LFP approximation in the energy space, will be violated (since the probability of such transitions becomes too small). Therefore, although the inclusion of the screening effect into the *e-e* scattering resolves the singularity problem for the scattering probability rates, however, at the same time, the same screening phenomenon brings into consideration another problem–for sufficiently strong screening it will violate the conditions of a small energy transfer which are necessary for the LFP approximation. It seems, that the only physically safe situation takes place at a week screening, when the screening does not lead to violation of the conditions of Eq. (9), and at the same time it also allows to remove the singularity of $W(\varepsilon, \varepsilon')$ for the *e-e* scattering with small-energy transfer. In this case, the differential form of the LFP operator for the *e-e* scattering is physically and mathematically justified.

For the unscreened *e-e* interaction, the situation looks completely different. In this case we can put $\varepsilon_0 = 0$ in Eq. (73). The analytical integration in Eq. (73) can be carried out without any approximations, i.e., with the exact integration limits, and we obtain

$$W(\varepsilon, \varepsilon') = \left(\frac{\hbar}{L}\right)^3 \frac{\pi^{5/2} e^4 n}{\kappa_0^2 m^2 (k_B T)^{1/2}} \frac{1}{\sqrt{\varepsilon\varepsilon'}} \frac{\exp\left(-\frac{\Delta + |\Delta|}{2k_B T}\right)}{|\Delta|^3} \times \begin{cases} \Psi(\varepsilon,\ |\Delta|) & (\text{if } \Delta > 0), \\ \Psi(\varepsilon',\ |\Delta|) & (\text{if } \Delta < 0), \end{cases}$$

$$\Psi(x, |\Delta|) = |\Delta| x^{1/2} \exp\left(-\frac{x}{k_B T}\right) + (k_B T)^{3/2}\left(2 + \frac{|\Delta|}{k_B T}\right) \times \int_0^{x/k_B T} u^{1/2} \exp(-u)\, du. \tag{79}$$

In the limit of large carrier energies, $\varepsilon,\ \varepsilon' \gg k_B T$ (very hot electrons), the second term in $\Psi(x, |\Delta|)$ dominates, and this gives the scattering rate for the no screening case:

$$W(\varepsilon, \varepsilon') = \frac{1}{2}\left(\frac{\hbar}{L}\right)^3 \frac{\pi^3 e^4 n}{\kappa_0^2 m^2} \frac{k_B T}{\sqrt{\varepsilon\varepsilon'}} \times \frac{\exp\left(-\frac{\Delta + |\Delta|}{2k_B T}\right)}{|\Delta|^3} \times \left(2 + \frac{|\Delta|}{k_B T}\right). \tag{80}$$

As is seen from Eqs. (79) and (80), the scattering rate $W(\varepsilon, \varepsilon')$ has an asymmetric dependence with respect to the line $\varepsilon' = \varepsilon$, and it has a singularity $W(\varepsilon, \varepsilon') \sim 1/|\Delta|^3$ for small energy transfer $\Delta \to 0$. The latter indicates that the small-energy transfer strongly dominates the *e-e* scattering in the absence of screening, and thus the condition in Eq. (9) is well satisfied in this case. Also, since according to Eq. (66), the scattering rate $W(\varepsilon, \varepsilon')$ is obtained after integration over all scattering angles, the expressions in Eqs. (79) and (80) show that in the absence of screening, the small-angle scattering singularity is now "transferred" to the small-energy change singularity. Similarly, as this was necessary for the small-angle scattering, for the unscreened interaction potential one will also need to introduce some ad hoc cutoff parameter, the minimum energy transfer, which will remove the small-energy transfer singularity from the scattering probability and the related kinetic coefficients, such as the dynamic friction coefficient and the diffusion coefficient in the energy space. This reveals the fundamental physical importance of inclusion of screening into the *e-e* interaction, which thus allows to tackle both the small-angle scattering singularity and the small-energy change singularity on the same footing.

However, we want to stress that the introduction of an ad hoc cutoff parameter only partially (and spuriously) resolves the problem. One of the widely used physical models, which provides such parameter is the model of the screened interaction potential, and the corresponding ad hoc parameter is the Debye's screening radius R_D (or the related screening momentum $q_0 = \hbar/R_D$). The use of the Debye's screening momentum as a cutoff parameter in the model which does not include screening explicitly, resolves the singularity problem mathematically, but it also brings physical inconsistency into the situation. The problem is that with the ad hoc introduction and use of the cutoff parameter to resolve the singularity problem in the scattering rate $W(\varepsilon, \varepsilon')$, we at the same time are excluding the explicit effect of the screening on the energy exchange in the *e-e* collisions. As we show here, when the strength of the screening increases (e.g., when the carrier density increases), this also modifies the energy exchange between the scattering electrons, the latter changes from a quasielastic scattering (when $|\Delta| \ll k_BT,\ \varepsilon$) to an inelastic scattering (when $|\Delta| \sim k_BT,\ \varepsilon$). As a result, in such situations if the screening is not included explicitly into the *e-e* operator, then it is impossible to correctly take into account the effect of screening on the energy exchange in the *e-e* scattering and its impact on the *e-e* kinetics. The screening effect must be explicitly included into the theory, and the cases of a weak and a strong screening need to be analyzed in detail.

For a weak screening, when $\varepsilon_0/k_BT \ll 1$, its effect on the energy exchange in the *e-e* scattering can be estimated by the direct comparison of the expression in Eq. (80) with the first expression in Eq. (78), assuming that both these expressions are of the same order of magnitude. This gives the estimate for the energy exchange as $|\Delta| \sim 2(\varepsilon_0 k_BT)^{1/2}$, or $|\Delta| \sim \sqrt{2}\hbar\omega_p$, where $\omega_p = (4\pi ne^2/\kappa_0 k_BT)^{1/2}$ is the plasma frequency [30]. It is important that for a weak screening we also obtain the small energy transfer $|\Delta| \ll k_BT,\ \varepsilon$.

For a strong screening, when $\varepsilon_0/k_BT \gg 1$, the analytical calculation of $W(\varepsilon, \varepsilon')$ in Eq. (73) is not possible, if the exact integration limits are used. However, the exact Eq. (73) can be transformed to the form which then allows further approximation:

$$W(\varepsilon, \varepsilon') = \left(\frac{\hbar}{L}\right)^3 \frac{\pi^{5/2}e^4n}{\kappa_0^2m^2} \frac{\exp\left(-\frac{\Delta+|\Delta|}{2k_BT}\right)}{\sqrt{\varepsilon\varepsilon'}} \times \begin{cases} \Phi\,(\varepsilon,\ |\Delta|) & (\text{if } \Delta> 0), \\ \Phi\,(\varepsilon',\ |\Delta|) & (\text{if } \Delta< 0), \end{cases}$$

$$\Phi(x, |\Delta|) = \frac{1}{(|\Delta| + \varepsilon_0)^2} \int_0^{\sqrt{x/k_BT}} \frac{\left[\frac{4k_BT|\Delta|}{(|\Delta|+\varepsilon_0)^2}u^2 + 1\right]}{\left[\frac{4k_BT\varepsilon_0}{(|\Delta|+\varepsilon_0)^2}u^2 + 1\right]^2} \exp(-u^2)\,du. \tag{81}$$

The integral $\Phi(x, |\Delta|)$ can be calculated analytically only if $\varepsilon,\ \varepsilon' \gg k_BT$, i.e., for a very high-energy electrons. In this case we can replace the upper integration limit by ∞ and carry out the integration. The result for $W(\varepsilon, \varepsilon')$ is

$$W(\varepsilon, \varepsilon') = \frac{1}{8}\left(\frac{\hbar}{L}\right)^3 \frac{\pi^{7/2}e^4n}{\kappa_0^2m^2(k_BT)^2} \frac{\exp\left(-\frac{\Delta+|\Delta|}{2k_BT}\right)}{\sqrt{\varepsilon\varepsilon'}}$$
$$\times \frac{k_BT}{\varepsilon_0} \times \left[\frac{1 - |\Delta|/\varepsilon_0}{\sqrt{\pi}} + \frac{2 - \varepsilon_0/k_BT + \Delta^2/\varepsilon_0\, k_BT}{2\sqrt{\varepsilon_0/k_BT}} \exp\left[\frac{(|\Delta| + \varepsilon_0)^2}{4\varepsilon_0 k_BT}\right] \text{erfc}\left(\frac{|\Delta| + \varepsilon_0}{\sqrt{4\varepsilon_0 k_BT}}\right)\right]. \tag{82}$$

This is the general expression for $W(\varepsilon, \varepsilon')$, which is valid for any strength of the screening under the above approximations. [For a weak screening, Eq. (80) can also be obtained from Eq. (82).] For the case of strong screening $\varepsilon_0/k_BT \gg 1$ we can obtain from Eq. (82) a suitable expression for $W(\varepsilon, \varepsilon')$ which is applicable for any value of the energy transfer Δ:

$$W(\varepsilon, \varepsilon') = \frac{1}{2}\left(\frac{\hbar}{L}\right)^3 \frac{\pi^3e^4n}{\kappa_0^2m^2} \frac{1}{\sqrt{\varepsilon\varepsilon'}} \frac{\exp\left(-\frac{\Delta+|\Delta|}{2k_BT}\right)}{(|\Delta| + \varepsilon_0)^3} \times (|\Delta| + 2k_BT + \varepsilon_0). \tag{83}$$

This expression shows that at strong screening the large energy transfer is of the order of $|\Delta| \sim \varepsilon_0$. The obtained estimates demonstrate that with increasing of the screening, the energy change Δ of the electron in the scattering event also increases, initially as $|\Delta| \sim 2(\varepsilon_0 k_BT)^{1/2}$ for a weak screening, and then as $|\Delta| \sim \varepsilon_0$ for strong screening. These results for dependence of $|\Delta|$ on the screening strength can also be deduced using simpler qualitative arguments by considering kinematics of the *e-e* scattering, which gives a clear physical insight into the issue. Due to the momentum and the energy conservation, the electron energy change in the scattering event can be presented as $\Delta\varepsilon_{\vec{k}} = \varepsilon_{\vec{k}'} - \varepsilon_{\vec{k}} = \varepsilon_{\vec{k}+\vec{q}} - \varepsilon_{\vec{k}} = \varepsilon_{\vec{q}} + 2\sqrt{\varepsilon_{\vec{k}}\varepsilon_{\vec{q}}}\cos\varphi$, where φ is the scattering angle between $\vec{k}$ and $\vec{q}$. Using Eq. (2), which gives the cutoff scattering momentum $q \sim q_0$ (or for the energy $\varepsilon_{\vec{q}} \sim \varepsilon_0$), we can estimate the energy change as $|\Delta| \sim \varepsilon_0 + 2\sqrt{\varepsilon\varepsilon_0}$ (here, for consistency with the previous formulas, we made an obvious change of the notations). For a weak screening, when the second term dominates (i.e., for $(\varepsilon_0/\varepsilon)^{1/2}/2 \ll 1$), we obtain $|\Delta| \sim 2\sqrt{\varepsilon\varepsilon_0}$. For a strong screening (i.e., for $(\varepsilon_0/\varepsilon)^{1/2}/2 \gg 1$), the first term dominates, and we obtain $|\Delta| \sim \varepsilon_0$. These rough estimates are in agreement with the previous accurate results.

It is obvious that with increasing of the strength of screening, the necessary conditions in Eq. (9) for the LFP approximation of the $\hat{I}_{ee}F(\varepsilon)$ scattering operator can be violated at sufficiently strong screening, and thus the diffusion form of the LFP operator given by Eqs. (47)–(50) will cease to be valid. In this case, one has to use the integral form of the $\hat{I}_{ee}F(\varepsilon)$ operator given by Eq. (75) [and use $W(\varepsilon, \varepsilon')$ from Eq. (73)]. However, taking into account that $\varepsilon_0 = q_0^2/2m = 2\pi\hbar^2e^2n/m\kappa_0 k_BT$, such situations occur at quite high electron densities, at or above of $n \sim 10^{18}\,\text{cm}^{-3}$. For example, for GaAs at $n = 10^{18}\,\text{cm}^{-3}$ and a moderately heated electron gas at $T = 400\ K$, we obtain $\varepsilon_0 = 24$ meV. For high-energy electrons with $\varepsilon > T$ and at the densities below this estimate, one can safely use the LFP form of the $\hat{I}_{ee}F(\varepsilon)$ operator with the screened interaction potential and all corresponding expressions, which were obtained in this paper. The screened Coulomb potential self-consistently resolves the problem of convergence of the scattering rates $W(\varepsilon, \varepsilon')$ and it also supports the required condition for diffusivity of the *e-e* scattering in the energy space with small energy transfer.

So far in our consideration of the *e-e* interaction/scattering, we used the static dielectric function $\kappa_e(\vec{q})$ for the free electron gas. The direct inclusion of dynamic screening effects in the dielectric function $\kappa_e(\vec{q}, \omega)$ will bring some computational difficulties into calculation of the LFP operator, but it will not change the theory. However, we can evaluate qualitatively the effect of a dynamic screening. According to Ref. [33], the origin of this effect for the free interacting carriers is in the mobile nature of the Coulomb scattering centers (electrons, in our case). As a result, the dynamic dielectric function $\kappa_e(\vec{q}, \omega)$ should be evaluated at a particular frequency $\omega = \vec{q} \cdot \vec{P}_{\text{c.m.}}/\hbar m$ (instead of at $\omega = 0$, as it is in the case of a static screening), where $\vec{P}_{\text{c.m.}} = (\vec{k} + \vec{p})/2$ is the center of mass momentum of the two colliding electrons with the momenta $\vec{k}$ and $\vec{p}$, respectively. The expression for the dielectric function $\kappa_e(\vec{q}, \omega)$ for free electrons is well-known and can be obtained using the Lindhard's formula or the other methods (see, e.g., Refs. [34,35]). For scattering of a test hot electron with the momentum $\vec{k}$ on one of the field electrons with the momentum $\vec{p}$, the field electrons are described by a nonequilibrium Maxwellian distribution function with the electron temperature T. The distribution of the field electrons is responsible for formation of the dielectric function $\kappa_e(\vec{q}, \omega)$. This allows to decouple calculation of $\kappa_e(\vec{q}, \omega)$ and the *e-e* scattering probability rates $W(\vec{k}, \vec{p} \to \vec{k}', \vec{p}')$ from solution of the kinetic equation with the *e-e* scattering operator $\hat{I}_{ee}(\varepsilon)$ for the unknown nonequilibrium distribution function $F(\varepsilon)$ of the hot test electrons. The important for our theory observation is that taking into account the mobile nature of the scattering electrons in $\kappa_e(\vec{q}, \omega)$ makes the screening effect weaker, i.e., the carrier-carrier scattering rates increase in comparison with the case of static screening. As was shown in Ref. [33] and in the other subsequent publications, typical change in the scattering rate due to dynamic screening is of order of 1.5–5. In Refs. [33,36] it was also suggested to use a physically elegant way to incorporate dynamic screening effects into the static screening theory. One needs to analyze the relevant kinetic parameter [in our case, it is the *e-e* scattering rate $\nu_{ee}(\varepsilon)$ in Eq. (29)] which should be calculated using the rigorous dynamic screening theory, and then introduce the "effective" static screening momentum q_0^{eff} in the static screening function $\kappa_e(\vec{q})$, which gives the same value of $\nu_{ee}(\varepsilon)$. (This essentially means that the screening momentum q_0 in Eq. (2) is replaced by the "effective" momentum q_0^{eff}, which in this case can be considered as a fitting parameter). Although, the reported contributions of the dynamic screening to kinetic coefficients, like increase of the scattering rate or the energy loss rate, are typically of a factor of 2, however, at the same time, this means that the corresponding decrease of the "effective" static screening momentum q_0^{eff} is considerably bigger. This can be understood from Eq. (23), in which the related "effective" screening energy $\varepsilon_0^{\text{eff}} = (q_0^{\text{eff}})^2/2m$ is present under the Coulomb Logarithm [the second term in Eq. (23) is usually smaller]. Obviously, if the logarithm changes (increases) by a factor of 2, the change of the ratio $\varepsilon_0/\varepsilon_0^{\text{eff}}$ is considerably bigger. This means that using the dynamic dielectric function $\kappa_e(\vec{q}, \omega)$ makes the screening effect weaker in comparison with the purely static case. Or, in the other words, contribution of the dynamic screening expands the region of validity of the necessary condition $|\Delta| \ll k_BT,\ \varepsilon$ for diffusive *e-e* scattering in the energy space, and thus the validity of the differential form of the LFP operator derived here.

It is also necessary to note that in the case of a weak screening ($\varepsilon_0/k_BT \ll 1$), when the problem of singularity of $W(\varepsilon, \varepsilon')$ and diffusivity of the *e-e* scattering ($|\Delta| \ll \varepsilon,\ \varepsilon'$) are resolved consistently, still the derivation of the LFP equation and, particularly, the calculation of the corresponding kinetic coefficients of the dynamic friction and the diffusion coefficient presents a challenging nontrivial task. As was pointed out in Sec. IV, our approach and the methods which we used allow to circumvent the problems in computing the kinetic coefficients. The aforementioned challenge can be seen explicitly, if one will try to calculate the kinetic coefficients given by their general formulas in Eqs. (49) and (50) using the expression for $W(\varepsilon, \varepsilon')$ from Eq. (82). It is not easy to see, how one can obtain in this approach our neat expressions for these coefficients given by Eqs. (57) and (58).

We want now to comment on possible applications of our results. The most straightforward case is the optically excited semiconductors, e.g., the hot-electron solar cells. The key question for such devices is the power loss which defines the solar cell efficiency. Typically, the electrons are excited to high energies, around 2.0–2.5 eV with a sufficiently wide energy distribution in the conduction band. It is usually assumed, that the *e-e* scattering is strong enough to maintain the Maxwellian (or the Fermi-Dirac distribution) with the electron temperature $T \gg T_0$ for all electron energies [37–39]. This function is used for calculation of the electron power loss due to interaction with various phonons, particularly, with the high-energy optical phonons. However, in the high-energy region interaction with optical phonons is faster (typically, ~0.1 ps) than the *e-e* interaction (typically, ~1 ps). This is true even for a moderately high generated carrier densities, since the rate of the *e-e* scattering decreases when the energy increases [see Eq. (29)]. Therefore, the assumption about the Maxwellian or Fermi-Dirac distribution function for the high-energy (hot) electrons is not justified. The obtaining of the correct distribution function requires solution of the kinetic equation which includes all relevant scattering mechanisms. The high-energy excited electrons emit a cascade of the optical phonons and eventually they arrive at the energy region around the optical phonon energy. The electrons which end up below the optical phonon energy will mainly interact via the *e-e* scattering and will establish the Maxwellian distribution with the electron temperature. The electrons above the optical phonon energy will mainly interact with the optical phonons. The *e-e* scattering plays important role in establishing the hot-electron distribution at the energies in the vicinity of the threshold energy, and it will affect the power loss of these electrons. The *e-e* interaction opens the new channel for some of the electrons below the phonon energy to be pushed to the higher energies, and thus they will be able to emit the optical phonons. To calculate these losses, one needs to calculate the corresponding distribution function at these energies by solving the corresponding kinetic equation, which includes both the *e-e* and the electron phonon scattering. The *e-e* interaction in this region can be described by the LFP equations as derived in this paper. The presence of this new channel of the energy relaxation will contribute to thermalization of the electrons in solar cells, which in turn can strongly affect the conversion efficiency [40].

Even more physically interesting situation takes place in the hot-electron solar cells based on perovskite materials [41–44]. These materials and the related solar cell devices are currently at an intensive investigation worldwide, since their show a superior performance in terms of the conversion efficiency. In spite of the considerable research efforts, the physical mechanism behind the enhanced conversion efficiency is not fully understood yet [45]. Practically in all published works it is postulated that the Maxwellian (or Fermi-Dirac) distribution function describes the electron gas, without actually solving the kinetic equation for correct evaluation of the power loss. The most interesting feature of these materials from a point of view of the electron kinetics is that they have very close energies of the optical phonons (∼8 meV) and the acoustic phonons (∼2.5 meV), and both these energies are considerably lower than the room temperature energy (∼25 meV). This means that at the room temperature all three interaction mechanisms, with the optical and acoustic phonons, and the *e-e* scattering, must be considered together on the same footing. The above numerical parameters indicate that one can use in the kinetic equation the FP approximation for the first two interactions, and the obtained LFP operator for the *e-e* scattering. This approach will produce the distribution function which will be different from the Maxwellian function for all energies. This function should then be used for correct calculation of the power loss. It was also recently reported [46,47] that the effect of screening of the electron-phonon interaction in perovskites plays important role in cooling of the photoexcited carriers. As we shown here, the screening also plays a crucial role in the *e-e* scattering, and thus will also affect the power loss. There are other physically interesting and practically important systems where our equations can be applied.

VIII. CONCLUSION

In this paper we presented a comprehensive general analysis of the *e-e* scattering operator $\hat{I}_{ee}F(\varepsilon)$, using the screened Coulomb potential with the static dielectric function and a quantum-mechanical calculation of the transition matrix elements (in the first Born approximation). We have considered the case of the electron transitions between different states in the energy space for an isotropic distribution function, and obtained differential LFP form of the *e-e* scattering operator with the explicit expressions for the dynamic friction coefficient and the diffusion coefficient. Two different approaches were used in the paper: the direct transformation of the Boltzmann scattering integral to the LFP form, and the derivation of the LFP operator using a nonlinear CKE. Since for each case the form of the LFP operator and the kinetic coefficients have different expressions, we have established the relationship between the kinetic coefficients for both cases. Incorporation of the electronic screening of the interaction potential allows to avoid divergence of the transition probability rates $W(\varepsilon,\varepsilon')$ for a small energy transfer. At the same time, the screening affects the character of the *e-e* interaction, when at strong screening the *e-e* scattering becomes strongly inelastic. When this happens, the LFP approximation cannot be used in principle.

We argue that the CKE approach is more universal, as the obtained equations and the expressions for the kinetic coefficients have a universal form which can be used for various scattering mechanisms under condition of a small energy transfer in the scattering process. This was demonstrated by considering the cases of the *e-e* scattering and the electron-acoustic-phonon scattering. The obtained LFP operator has a Kramers-Moyal form with an infinite number of terms. However, if only two terms are retained in the KME, then there are two coefficients, the dynamic friction coefficient and the diffusion coefficient for description of the nonequilibrium electron kinetics. In general, the LFP operator is still a nonlinear integrodifferential operator, however, only the diffusion coefficient contains the additional nonequilibrium parameter (the mean energy $\bar{\varepsilon}$ of the electrons) which depends on the unknown distribution function $F(\varepsilon)$. The dynamic friction coefficient does not depend on the unknown distribution function. The dependence of the diffusion coefficient on the parameter $\bar{\varepsilon}$ can be very useful for a numerical solution of the LFP equation, where $\bar{\varepsilon}$ can be used as a fitting parameter in the numerical simulations.

It is also shown that the LFP operator can be linearized in the physically important case, when one is interested in a correct asymptotic of the distribution function for high-energy (hot) electrons, where the *e-e* scattering compete with another strongly inelastic scattering process with some threshold energy, e.g., the interaction with high-energy optical phonons or the intervalley phonons, or it takes part in the impact ionization process. In this case, for the *e-e* scattering we can consider the interaction of the high-energy electrons (with an unknown distribution function) with the rest (majority) of the field electrons with the energies below the threshold energy. The latter electrons are described by the known Maxwellian distribution function with the electron temperature T, which linearizes the LFP operator.

We have analyzed the required conditions for the validity of the LFP differential forms. It is shown that for a weak and an intermediate screening, i.e., for the electron densities below $10^{18}\,\mathrm{cm}^{-3}$ the necessary conditions are satisfied, and one can safely use the obtained differential LFP forms. However, at higher carrier density the minimum energy transfer in the *e-e* scattering increases, the scattering becomes inelastic, and the necessary conditions of a small energy transfer are not satisfied. In this case, one has to use the integral form of the $\hat{I}_{ee}F(\varepsilon)$ operator. We have obtained general form for this operator suitable for numerical analysis. The influence of the screening on the *e-e* interaction is weakened when one considers the dynamic dielectric function in the transition matrix element. It follows from a qualitative physical analysis that although typical effect of dynamic dielectric function on the *e-e* scattering rate increase is substantially less than an order of magnitude (about a factor of 2), however, since the *e-e* scattering rate depends on the screening energy ε_0

logarithmically, thus the decrease in ε_0 is large. The strength of the screening decreases, and thus it expands the range of the validity of the LFP theory for hot electrons in semiconductors.

We also suggested few physically interesting and practically important systems where our derived equations and the other relationships can be used for investigation of the power loss. Particularly interesting case are the hot-electron solar cells based on perovskite materials. Due to comparable values of the energies of the optical phonons and acoustic phonons in these materials (which are also small in comparison with the room temperature energy), and the relatively high photoexcitation densities $\sim 10^{18}\,\mathrm{cm}^{-3}$, it is physically clear that the electron interactions with the optical and acoustic phonons, as well as the *e-e* scattering can be described by the same set of differential FP operators (for interaction with the phonons) and the LFP operator for the *e-e* interaction, all of which were derived in the paper. Obviously, the nonequilibrium distribution function obtained from this kinetic equation will not be of a Maxwellian form. This function should be used for calculation of the related kinetic coefficients, in particular, for calculation of the hot-electron power loss. There are many other physical systems where the obtained equations can be used for physical analysis of a nonequilibrium electron kinetics.

It is necessary to note, that usually in an absolute majority of publications which consider the *e-e* scattering, the operator $\hat{I}_{ee}F(\varepsilon)$ is used in a very limited way. If the contribution from the *e-e* scattering dominates in comparison with contributions from the other scattering mechanisms, then, as a zero-order approximation to the solution of the Boltzmann kinetic equation, the equation $\hat{I}_{ee}F(\varepsilon) \approx 0$ is used. The solution of this equation is either the Maxwellian, or the Fermi-Dirac distribution functions with the electron temperature T as the only unknown parameter. To obtain this solution, one does not need to do any transformation of the initial integral nonlinear scattering operator $\hat{I}_{ee}F(\varepsilon)$ to the LFP form, as was discussed in this paper. One simply needs to solve a functional equation, which stipulates that the integrand in the $\hat{I}_{ee}F(\varepsilon)$ operator is equal zero. This functional equation gives the above Maxwellian or Fermi-Dirac solutions with the electron temperature T. To find the parameter T in the obtained distribution function, the next order approximation of the kinetic equation is used. This energy balance equation includes all other scattering operators, except the *e-e* operator. With the known distribution function, which is substituted into the scattering operators, the problem of finding T is reduced to solution of the energy balance equation. This situation is completely different from the physical situations for which our LFP differential operator was derived. In the discussed cases we are interested in the distribution function of the hot electrons at high energies where these electrons are subject to the other scattering mechanisms which compete with each other, so they all must be considered on an equal physical footing in the kinetic equation. We have shown that in this case the main contribution to the *e-e* scattering operator is the interaction of the high-energy electrons with the majority of the thermalized field electrons which occupy the lower energy region. The involvement in this interaction of the electrons with high-energy ε means that the necessary condition for the validity of the LFP differential operator integral, $\Delta = |\varepsilon' - \varepsilon| \ll \varepsilon$, is well justified. The involvement of the lower energy thermalized field electrons means that in this particular case we can linearize the $\hat{I}_{ee}F(\varepsilon)$ operator. The latter means that we can use for calculation of the kinetic coefficients (the dynamic friction coefficient and the diffusion coefficient) the known Maxwellian distribution function with the electron temperature, thus this results in a physical linearization of the *e-e* scattering operator for the hot electrons. Although, our derivation of the LFP differential operator is more general, and it does not rely on this linearization (until we need to calculate explicitly the dynamic friction coefficient and the diffusion coefficient), however, the actual value of LFP operator is that it can be used for description of the hot electron kinetics in semiconductors at high energies, where all actual scattering mechanisms, particularly, the *e-e* scattering and electron-optical-phonon scattering, equally compete with each other, and thus define the distribution function of hot electrons.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX

As an example of utilization of the general relationships obtained in Sec. V from the CKE approach, we consider the quasielastic scattering of electrons on acoustic phonons. In this case we need to remove from Eq. (40) the summations over $\vec{p}$, $\vec{p}'$, since there is only a single electron in the state $\vec{k}$, which scatters on the phonon with the momentum $\vec{q}$ into the new state $\vec{k}'$. We also should replace in Eq. (40) the term $W(\vec{k}, \vec{p} \to \vec{k}', \vec{p}')F(\varepsilon_{\vec{p}})$ by the corresponding acoustic phonon scattering rate $W_{\mathrm{ac}}(\vec{k} \to \vec{k}')$ for the electron-phonon interaction (see, e.g., Refs. [22,23]). This gives for the scattering probability rate

$$W_{ac}(\vec{k} \to \vec{k}') = \sum_{\vec{q}} w_{ac}(\vec{q})[(N_{\vec{q}} + 1)\delta_{\vec{k}',\vec{k}-\vec{q}}\delta(\varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'} - \hbar\omega_{\vec{q}}) + N_{\vec{q}}\delta_{\vec{k}',\vec{k}+\vec{q}}\delta(\varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'} + \hbar\omega_{\vec{q}})],$$

$$w_{ac}(\vec{q}) = \frac{2\pi}{\hbar}\frac{\Sigma^2 q^2}{2\rho\hbar\omega_{\vec{q}}L^3}, \qquad N_{\vec{q}} = [\exp(\hbar\omega_{\vec{q}}/k_B T_0) - 1]^{-1}, \tag{A1}$$

where $N_{\vec{q}}$ is the average number of phonons with the energy $\hbar\omega_{\vec{q}} = sq$, s is longitudinal sound velocity in semiconductor material, Σ is deformation potential, ρ is the material density. After this, it is straightforward to calculate the scattering rate $W_{ac}(\varepsilon, \varepsilon')$ in the energy space using Eq. (40) with the above modifications:

$$W_{ac}(\varepsilon,\varepsilon') = \frac{\pi\,\Sigma^2}{\rho s^2\hbar L^3}\frac{1}{8ms^2}\frac{(\varepsilon-\varepsilon')^2}{\sqrt{\varepsilon\varepsilon'}}\{N(|\varepsilon-\varepsilon'|)\,\Theta[(\sqrt{\varepsilon}+\sqrt{2ms^2})^2-\varepsilon']\Theta\,(\varepsilon'-\varepsilon)$$

$$+\,[N(|\varepsilon-\varepsilon'|)+1]\,\Theta[\varepsilon'-(\sqrt{\varepsilon}-\sqrt{2ms^2})^2]\,\Theta\,(\varepsilon-\varepsilon')\}. \tag{A2}$$

Here $\Theta(x)$ is the Heaviside step function [it essentially defines the integration limits in the integrals over ε' in Eqs. (49) and (50) from the main text]. Since the expression for $W_{ac}(\varepsilon,\varepsilon')$ in Eq. (A2) is relatively simple in comparison with the corresponding expressions for $W_{ee}(\varepsilon,\varepsilon')$ in Eq. (73) and Eqs. (79)–(82) for the *e-e* scattering, we can use it for the direct calculation of the kinetic coefficients $C_1(\varepsilon)$ and $C_2(\varepsilon)$ from general Eqs. (49)–(50). We have carried out such calculations (not shown here), however, it is instructive to deploy here another approach to calculation $C_1(\varepsilon)$ and $C_2(\varepsilon)$, namely, using Eqs. (52) and (53), as we did for the *e-e* scattering, with the above modifications to the case of the acoustic phonon scattering. This will also prove that both ways of the calculation of the coefficients lead to the same result for each scattering mechanism, respectively. An interesting feature of the electron-phonon kinetics is that due to the spontaneous (sp) and the stimulated (st) phonon scattering processes, there are two distinctive contributions to the coefficient $C_1(\varepsilon)$ and the power loss $Q^{\text{ac}}_{\text{loss}}(\varepsilon)$ from each of these processes. This is different from the *e-e* scattering case. Substituting Eq. (A1) into Eq. (53) we obtain

$$C_1(\varepsilon) = Q^{\text{ac}}_{\text{loss}}(\varepsilon) = Q^{\text{ac(sp)}}_{\text{loss}}(\varepsilon) + Q^{\text{ac(st)}}_{\text{loss}}(\varepsilon), \tag{A3}$$

$$Q^{\text{ac(sp)}}_{\text{loss}}(\varepsilon) = \frac{1}{g(\varepsilon)}\sum_{\vec k\vec k'\vec q} w(\vec q)\,\hbar\omega_{\vec q}\delta_{\vec k',\vec k+\vec q}\delta(\varepsilon_{\vec k'}-\varepsilon_{\vec k}-\hbar\omega_{\vec q})\delta(\varepsilon_{\vec k'}-\varepsilon), \tag{A4}$$

$$Q^{\text{ac(st)}}_{\text{loss}}(\varepsilon) = \frac{1}{g(\varepsilon)}\sum_{\vec k\vec k'\vec q} w(\vec q)N_{\vec q}\hbar\omega_{\vec q}\delta_{\vec k',\vec k+\vec q}\delta(\varepsilon_{\vec k'}-\varepsilon_{\vec k}-\hbar\omega_{\vec q})[\delta(\varepsilon_{\vec k'}-\varepsilon)-\delta(\varepsilon_{\vec k}-\varepsilon)]. \tag{A5}$$

These are the exact expressions. We now need to use for the calculations the same approximation and the accuracy as it was used for derivation of the differential FP equation from the CKE. The FP differential form of the electron-phonon scattering operator has a strong justification, as in this case there is small parameter $\Delta\varepsilon/\varepsilon = \hbar\omega_{\vec q}/\varepsilon \approx \sqrt{8ms^2/\varepsilon} \ll 1$ which ensures diffusivity of the electron motion in the energy space with the electron-acoustic-phonon scattering practically for all electron energies above 1° K. We will now use the expansion in Eq. (54), where we also need to replace $(\varepsilon_{\vec p'}-\varepsilon_{\vec p})$ by the acoustic phonon energy $\hbar\omega_{\vec q}$. For calculation of the spontaneous power loss $Q^{\text{ac(sp)}}_{\text{loss}}(\varepsilon)$, it is sufficient to keep only the zero-order term in Eq. (54). This gives

$$Q^{\text{ac(sp)}}_{\text{loss}}(\varepsilon) = \frac{1}{g(\varepsilon)}\sum_{\vec k\vec k'\vec q} w(\vec q)\,\hbar\omega_{\vec q}\,\delta_{\vec k',\vec k+\vec q}\delta(\varepsilon_{\vec k'}-\varepsilon)\delta(\varepsilon_{\vec k}-\varepsilon). \tag{A6}$$

In calculating the power loss $Q^{\text{ac(st)}}_{\text{loss}}(\varepsilon)$ due to the stimulated scattering, we found that the zero-order term in the expansion in Eq. (54) gives exactly zero. Physically, this is explained by a complete compensation effect [22] of the stimulated scattering with the emission of the acoustic phonon, which gives a positive contribution to $Q^{\text{ac(st)}}_{\text{loss}}(\varepsilon)$, and the scattering with the absorption of the acoustic phonon, which gives a negative contribution to $Q^{\text{ac(st)}}_{\text{loss}}(\varepsilon)$. Since these contributions are equal in magnitude but have opposite signs, the total result is zero. As a result, we need to use the next order term in the expansion in Eq. (54). After some algebra, this gives a neat result:

$$Q^{\text{ac(st)}}_{\text{loss}}(\varepsilon) = -\frac{1}{g(\varepsilon)}\frac{d}{d\varepsilon}\left[\sum_{\vec k\vec k'\vec q} w(\vec q)N_{\vec q}(\hbar\omega_{\vec q})^2\delta_{\vec k,\vec k+\vec q}\delta(\varepsilon_{\vec k'}-\varepsilon)\delta(\varepsilon_{\vec k}-\varepsilon)\right]. \tag{A7}$$

Note, that for the *e-e* scattering case the situation with the calculation of $C_1(\varepsilon) = Q^{ee}_{\text{loss}}(\varepsilon)$ was very different. As it can be seen from Eqs. (53) and (57), the contribution from the second term in the expansion in Eq. (54) turned out to be negligible in comparison with the zero-order term contribution and it was neglected.

For calculation of the diffusion coefficient $C_2(\varepsilon) = D_{\text{ac}}(\varepsilon)$ in the energy space due to the electron-phonon scattering using Eqs. (50) and (56) with the corresponding modifications, we need to retain only the zero-order term in the expansion in Eq. (54). This is because the compensation effect in the diffusion coefficient is absent. This gives after some algebra

$$C_2(\varepsilon) = D_{\text{ac}}(\varepsilon) = \frac{1}{g(\varepsilon)}\sum_{\vec k\vec k'\vec q} w(\vec q)(N_{\vec q}+1/2)\,(\hbar\omega_{\vec q})^2$$

$$\times\,\delta_{\vec k',\vec k+\vec q}\delta(\varepsilon_{\vec k'}-\varepsilon)\delta(\varepsilon_{\vec k}-\varepsilon). \tag{A8}$$

To complete calculations in Eqs. (A6)–(A8), we assume that $\hbar\omega_{\vec q} \ll k_BT_0$, i.e., that $\sqrt{8ms^2\varepsilon} \ll k_BT_0$. Although, for hot electrons this condition is stronger than our main condition for derivation of the differential FP equation form from the CKE, however, taking into account that the energy ms^2 is very small (in typical semiconductors, $ms^2/k_B \leqslant 1^\circ K$), this condition holds well for hot electrons. This gives for the number of active phonons $N_{\vec q} \approx k_BT_0/\hbar\omega_{\vec q} \gg 1$. Using this result in Eq. (A7) and Eq. (A8), we obtain the following relationships:

$$\begin{aligned} Q^{\mathrm{ac}}_{\mathrm{loss}}(\varepsilon) &= Q^{\mathrm{ac(sp)}}_{\mathrm{loss}}(\varepsilon) + Q^{\mathrm{ac(st)}}_{\mathrm{loss}}(\varepsilon)\\ &= \frac{D_{\mathrm{ac}}(\varepsilon)}{k_BT_0} - \frac{1}{g(\varepsilon)}\frac{d}{d\varepsilon}[g(\varepsilon)D_{\mathrm{ac}}(\varepsilon)], \end{aligned} \tag{A9}$$

$$\begin{aligned} Q^{\mathrm{ac(sp)}}_{\mathrm{loss}}(\varepsilon) &= \frac{1}{g(\varepsilon)}\sum_{\vec k\vec k'\vec q} w(\vec q)\hbar\omega_{\vec q}\,\delta_{\vec k',\vec k+\vec q}\,\delta(\varepsilon_{\vec k'}-\varepsilon)\delta(\varepsilon_{\vec k}-\varepsilon)\\ &= \frac{D_{\mathrm{ac}}(\varepsilon)}{k_BT_0}, \end{aligned} \tag{A10}$$

$$Q^{\mathrm{ac(st)}}_{\mathrm{loss}}(\varepsilon) = -\frac{1}{g(\varepsilon)}\frac{d}{d\varepsilon}[g(\varepsilon)D_{\mathrm{ac}}(\varepsilon)]. \tag{A11}$$

These relationships essentially show that one only needs to calculate the diffusion coefficient $D_{\mathrm{ac}}(\varepsilon) = C_2(\varepsilon)$, then the coefficient $C_1(\varepsilon) = Q^{\mathrm{ac}}_{\mathrm{loss}}(\varepsilon)$ can be obtained using Eq. (A9). The coefficients $D_{\mathrm{ac}}(\varepsilon)$ and $Q^{\mathrm{ac}}_{\mathrm{loss}}(\varepsilon)$ also satisfy the general relationship in Eq. (59), where we should put $T = T_0$. All the above relationships can be useful for experimental investigation of the power loss and the energy diffusion of the hot electrons in semiconductors due to the acoustic phonon scattering.

Here we derived these relationships using the FP differential form of the CKE. In Ref. [22] similar relationships were proven using analysis of the electron-phonon scattering rates. However, if for the electron-phonon scattering one derives the FP form of the Boltzmann equation as is given in Eqs. (15) and (16), then the corresponding coefficients $A_{\mathrm{ac}}(\varepsilon)$ and $B_{\mathrm{ac}}(\varepsilon)$ can be obtained from the general relationships given in Eq. (51) using $C_1(\varepsilon)$ and $C_2(\varepsilon)$ from Eqs. (A9)–(A11):

$$\begin{aligned} A_{\mathrm{ac}}(\varepsilon) &= Q^{\mathrm{ac(sp)}}_{\mathrm{loss}}(\varepsilon),\\ B_{\mathrm{ac}}(\varepsilon) &= D_{\mathrm{ac}}(\varepsilon). \end{aligned} \tag{A12}$$

We can observe the difference between the first relationship in Eq. (A12) and the equivalent relationship in Eq. (58) for the *e-e* scattering. The dynamic friction coefficient $A_{\mathrm{ac}}(\varepsilon)$ is defined by the power loss $Q^{\mathrm{ac(sp)}}_{\mathrm{loss}}(\varepsilon)$ due to spontaneous emission of the acoustic phonons, while the dynamic friction coefficient $A_{ee}(\varepsilon)$ is defined by the total power loss $Q^{ee}_{\mathrm{loss}}(\varepsilon)$ due to the *e-e* scattering. The physical reason for this is that for the *e-e* scattering the second term in Eq. (51) is exactly zero [see Eqs. (57) and (58)], while for the acoustic-phonon scattering the corresponding term [which in this case is equal to $Q^{\mathrm{ac(st)}}_{\mathrm{loss}}(\varepsilon)$, see Eq. (A9)] is not zero, as can be seen from the direct calculations. Using in Eq. (A10) $w(\vec q)$ from Eq. (A1), we obtain the final result for the diffusion coefficient:

$$D_{\mathrm{ac}}(\varepsilon) = C_2(\varepsilon) = \frac{2ms^2}{k_BT_0}\frac{1}{\tau_{\mathrm{ac}}(\varepsilon)}\varepsilon k_BT_0 = \frac{\varepsilon}{\tilde\tau_{\mathrm{ac}}(\varepsilon)}k_BT_0, \tag{A13}$$

$$\frac{1}{\tilde\tau_{\mathrm{ac}}(\varepsilon)} = \frac{2ms^2}{k_BT_0}\frac{1}{\tau_{\mathrm{ac}}(\varepsilon)},\qquad \frac{1}{\tau_{\mathrm{ac}}(\varepsilon)} = \frac{\sqrt{2}C^2m^{3/2}k_BT_0}{\pi\rho\hbar^4s^2}\varepsilon^{1/2}, \tag{A14}$$

where $\tilde\tau_{\mathrm{ac}}(\varepsilon)$ is the energy relaxation time, and $\tau_{\mathrm{ac}}(\varepsilon)$ is the momentum scattering time for the electron-phonon interaction.

Substituting the obtained $D_{\mathrm{ac}}(\varepsilon)$ into Eq. (A11), we see that $Q^{\mathrm{ac(st)}}_{\mathrm{loss}}(\varepsilon) \neq 0$. The result in Eq. (A13) allows also to calculate explicitly the coefficients $C_1(\varepsilon)$. and $C_2(\varepsilon)$. Using them in Eq. (48), we can transform the carrier flux in the energy space for the electron-acoustic-phonon scattering to the standard FP form:

$$\tilde J_{\mathrm{ac}}(\varepsilon) = -\frac{\varepsilon g(\varepsilon)}{\tilde\tau_{\mathrm{ac}}(\varepsilon)}\left[F(\varepsilon) + k_BT_0\frac{dF(\varepsilon)}{d\varepsilon}\right]. \tag{A15}$$

This completes the analysis of the FP equation for the electron-acoustic phonon scattering.

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