

Research Repository

Groups of permutations that are even on subsets of a fixed size, and related monoids

Accepted for publication in the International Journal of Algebra and Computation.

Research Repository link: <https://repository.essex.ac.uk/41665/>

Please note:

Changes made as a result of publishing processes such as copy-editing, formatting and page numbers may not be reflected in this version. For the definitive version of this publication, please refer to the published source. You are advised to consult the published version if you wish to cite this paper.

<https://doi.org/10.1142/S0218196725500407>

Groups of permutations that are even on subsets of a fixed size, and related monoids

Vítor Fernandes*, Alexei Vernitski

September 29, 2025

Abstract

We study permutations on n elements that are even on every subset of size t . We describe all groups of these permutations. Unexpectedly, these groups (except for some special cases) are either trivial, cyclic or dihedral. In this context, we define and study monoids that generalize both monoids of order-preserving mappings and monoids of orientation-preserving mappings.

Keywords: semigroups, order-preserving mappings, orientation-preserving mappings, permutations, dihedral groups.

MSC classification numbers: 20M20, 20B35.

1 Introduction

For a positive integer n , notation $[n]$ will stand for $\{0, 1, \dots, n-1\}$. We will use letters a, b, c for elements of $[n]$ and lower-case Greek letters for mappings on $[n]$. The following three mappings will be used repeatedly.

- ι_n , which is the identity mapping on $[n]$;
- σ_n , which is the cyclic shift $a \mapsto (a+1) \pmod{n}$;
- ρ_n , which is the permutation $a \mapsto (n-1)-a$ reversing the order of elements in $[n]$.

We will denote the cyclic group generated by σ_n by Z_n and the dihedral group generated by σ_n and ρ_n by D_n . As usual, S_n is the symmetric group on $[n]$, and T_n is the symmetric semigroup on $[n]$.

The notation $\text{Dom } \alpha$ stands for the domain of α . A (partial or full) mapping α on $[n]$ is called *order-preserving* [respectively, *order-reversing*] if $x < y$ implies

*This work is funded by national funds through the FCT - Fundação para a Ciência e a Tecnologia, I.P., under the scope of the (Center for Mathematics and Applications) projects UIDB/00297/2020 (<https://doi.org/10.54499/UIDB/00297/2020>) and UIDP/00297/2020 (<https://doi.org/10.54499/UIDP/00297/2020>).

$\alpha(x) \leq \alpha(y)$ [respectively, $\alpha(x) \geq \alpha(y)$] for all $x, y \in \text{Dom } \alpha$. The set of full order-preserving mappings is closed with respect to composition and contains ι_n , therefore, it is a monoid, which is denoted by O_n . The monoid O_n is one of the most frequently studied semigroups in semigroup theory since the 1960s; its research originated in [Aiz62, How71, GH92, Hig93, VV95, McA00] and led to hundreds of other publications. Many natural generalizations of O_n have been considered, including the monoid of partial order-preserving mappings, see for example [Fer02, FGJ05]. One generalization we will use in this paper is M_n , the monoid of all full *monotone* mappings, that is, full mappings that are order-preserving or order-reversing.

Another important way in which semigroup theorists have generalized the monoid O_n is the following. Let $s = (a_1, a_2, \dots, a_t)$ be a sequence of t ($t \geq 0$) elements of $[n]$. We say that s is *cyclic* [respectively, *anti-cyclic*] if there exists no more than one index $i \in \{1, \dots, t\}$ such that $a_i > a_{i+1}$ [respectively, $a_i < a_{i+1}$], where a_{t+1} denotes a_1 . Notice that the sequence s is cyclic [respectively, anti-cyclic] if and only if s is empty or there exists $i \in \{1, \dots, t\}$ such that $a_{i+1} \leq a_{i+2} \leq \dots \leq a_t \leq a_1 \leq \dots \leq a_i$ [respectively, $a_{i+1} \geq a_{i+2} \geq \dots \geq a_t \geq a_1 \geq \dots \geq a_i$] (the index $i \in \{1, \dots, t\}$ is unique unless s is constant and $t \geq 2$). We say that s is *oriented* if s is cyclic or s is anti-cyclic. Given a (partial or full) mapping α on $[n]$ such that $\text{Dom } \alpha = \{a_1 < \dots < a_t\}$, we say that α is *orientation-preserving* [respectively, *orientation-reversing*, *oriented*] if the sequence of its images $(\alpha(a_1), \dots, \alpha(a_t))$ is cyclic [respectively, anti-cyclic, oriented]. The set of orientation-preserving full mappings is a monoid, which is denoted by OP_n . The monoid OP_n was introduced 25 years ago [CH99, Fer00] and has been repeatedly studied since then. Some relevant recent publications on OP_n are [HV22, FP23, Fer23]. One generalization of OP_n we will use in this paper is OR_n , the monoid of all full oriented mappings.

The *rank* of a mapping is the size of its image. Unless the rank of α is 2, being orientation-preserving can be expressed in a language of triples of elements of $[n]$. Let us stress this because this is an important theme in this paper. Instead of defining mappings in OP_n as being, in a certain sense, ‘cyclic’ (as per the definition above), one can redefine them as preserving the parity of subsets of size 3 (as generalized and discussed in more detail in the following sections). To be more specific, the following result is true; see [HV22, Fer23, HV24, Fer24, LM06].

Lemma 1. *A mapping $\alpha \in T_n$ whose rank is not 2 is orientation-preserving if and only if from $a < b < c$ it follows that $\alpha(a) \leq \alpha(b) \leq \alpha(c)$ or $\alpha(b) \leq \alpha(c) \leq \alpha(a)$ or $\alpha(c) \leq \alpha(a) \leq \alpha(b)$.*

As one compares the monoid of order-preserving mappings O_n and the monoid of orientation-preserving mappings OP_n , one can ask the following questions.

- O_n is defined using a condition on pairs of elements of $[n]$, and the mappings in OP_n can be described (with the exception of rank 2 mappings) using a condition on triples of elements of $[n]$. Are there natural generalizations of these conditions for t -tuples that would produce new interesting

monoids?

- The only permutation belonging to O_n is ι_n . In other words, the set of permutations belonging to O_n is the one-element group $\{\iota_n\}$. As to OP_n , the set of permutations belonging to it is the cyclic group Z_n . How will this generalize if we consider conditions on t -tuples of elements of $[n]$?

The aim of this article is to explore these questions. For one-to-one mappings, we introduce a property which generalizes the properties of being order-preserving and orientation-preserving. We successfully describe all groups of permutations satisfying this generalized property. Then we define and study monoids that can be seen as generalizations of monoids of order-preserving mappings and orientation-preserving mappings.

2 The generalized definition for groups

Consider a one-to-one partial mapping α on $[n]$ whose domain and image both have size t . By an inversion we mean a pair a, b such that $a < b$ and $\alpha(a) > \alpha(b)$. We will say that α is even (odd) if the number of its inversions is even (odd). It is useful to note that if $t = n$ then this definition of being even (odd) coincides with the usual definition of an even (odd) permutation.

Lemma 2. *Consider one-to-one partial mappings α and β on $[n]$ such that the domain of β coincides with the image of α . If α and β are both even or both odd then the composition $\beta(\alpha(\cdot))$ is even. If one of α and β is even and the other is odd then the composition $\beta(\alpha(\cdot))$ is odd.*

Proof. We will denote the size of the domain and the image by t , as in the definition above. Define a matrix M of a one-to-one partial mapping μ as follows. The size of M is $t \times t$. Suppose the domain of μ consists of elements $d_1 < \dots < d_t$ and the image of μ consists of elements $e_1 < \dots < e_t$. The entry in M at the position i, j is 1 if $\mu(d_i) = e_j$ and 0 otherwise.

It is a well-known fact that a permutation is odd (even) if and only if the corresponding permutation matrix has the determinant equal to -1 (equal to 1). This fact obviously generalizes to the matrices defined above; namely, μ is odd (even) if and only if the determinant of M is equal to -1 (equal to 1).

Now notice that the matrix of $\beta(\alpha(\cdot))$ is the product of the matrix of α and the matrix of β . Hence, the result follows. $\square$

When we speak of a restriction of a mapping $\alpha \in T_n$ to a t -element subset of $[n]$, we call it a restriction of width t . We define Γ_n^t , where $t = 1, \dots, n$, as the set consisting of those permutations on $[n]$ whose every restriction of width t is even. Due to Lemma 2, the set Γ_n^t is a group.

Before we explore groups Γ_n^t in detail in Sections 3 and 4, we present a brief overview of these groups. When $t = 1$, since partial mappings with one-element domains cannot have inversions, every permutation belongs to group Γ_n^1 ; therefore, Γ_n^1 coincides with the symmetric group S_n . At the opposite end

of the spectrum, if $t = n$, group Γ_n^n is the alternating group A_n . As we remarked in the previous section, the set of permutations belonging to O_n is $\Gamma_n^2 = \{\iota_n\}$, and the set of permutations belonging to OP_n is $\Gamma_n^3 = Z_n$.

As an illustration, the table below shows values of t in the top row and values of $|\Gamma_n^t|$ in the bottom row for $n = 11$.

1	2	3	4	5	6	7	8	9	10	11
$ S_n $	1	11	2	22	1	11	2	22	43,200	$ A_n $

As you can see, the values of $|\Gamma_n^t|$ oscillate between 1, n , 2 and $2n$ on the whole range $t = 2, \dots, n-2$; we will explain this behavior in Section 3. The values $t = 1$ and $t = n$ are special cases described above. The value $t = n-1$ is a special case which we will explore in Section 4.

When $n = 2$, the table above shrinks to just its first and last columns, with $\Gamma_2^1 = S_2$ and $\Gamma_2^2 = A_2$. The results in Sections 3 and 4 below assume that $n \geq 3$.

3 Groups Γ_n^t for $t = 2, \dots, n-2$

Let R be a subset of $[n]$ (the mnemonic for R is ‘restriction’). Let $\alpha \in S_n$. Denote the restriction of α to R by $\alpha \upharpoonright R$. Denote the number of inversions of the restriction of α to R by $\text{inv}(\alpha \upharpoonright R)$.

Let $\alpha \in S_n$, and let $a \in \{0, \dots, n-2\}$. We will split the set $[n] \setminus \{a, a+1\}$ into two parts, as follows. The sets $\Xi(\alpha, a)$ and $\Pi(\alpha, a)$ consist of those elements $b \in [n] \setminus \{a, a+1\}$ for which $\text{inv}(\alpha \upharpoonright \{a, b\}) \neq \text{inv}(\alpha \upharpoonright \{a+1, b\})$ and for which $\text{inv}(\alpha \upharpoonright \{a, b\}) = \text{inv}(\alpha \upharpoonright \{a+1, b\})$, respectively. In other words, $\Pi(\alpha, a)$ contains elements $b \in [n] \setminus \{a, a+1\}$ such that $\alpha(b) < \alpha(a)$ and $\alpha(b) < \alpha(a+1)$ or $\alpha(a) < \alpha(b)$ and $\alpha(a+1) < \alpha(b)$. As to $\Xi(\alpha, a)$, if $\alpha(a) < \alpha(a+1)$ then $\Xi(\alpha, a)$ consists of elements $b \in [n] \setminus \{a, a+1\}$ such that $\alpha(a) < \alpha(b) < \alpha(a+1)$, and if $\alpha(a+1) < \alpha(a)$ then $\Xi(\alpha, a)$ consists of elements $b \in [n] \setminus \{a, a+1\}$ such that $\alpha(a+1) < \alpha(b) < \alpha(a)$.

Lemma 3. *Let $a \in [n]$. Consider pairwise distinct $b_1, \dots, b_r \in \Xi(\alpha, a)$ and pairwise distinct $c_1, \dots, c_s \in \Pi(\alpha, a)$. Let $R = \{b_1, \dots, b_r; c_1, \dots, c_s\}$. Let $R' = S \cup \{a\}$ and $R'' = R \cup \{a+1\}$. Then $\text{inv}(\alpha \upharpoonright R') = \text{inv}(\alpha \upharpoonright R'') + r \pmod{2}$.*

Proof. Note that each of $\text{inv}(\alpha \upharpoonright \{a, b\})$ and $\text{inv}(\alpha \upharpoonright \{a+1, b\})$ has value 0 or 1, so the inequality in the definition of $\Xi(\alpha, a)$ can be rewritten as an equality

$$\text{inv}(\alpha \upharpoonright \{a, b\}) = \text{inv}(\alpha \upharpoonright \{a+1, b\}) + 1 \pmod{2}.$$

Hence, the lemma immediately follows. $\square$

Lemma 4. $\Gamma_n^t \subseteq D_n$ for every $t = 2, \dots, n-2$.

Proof. It is obvious that a permutation $\alpha \in S_n$ belongs to D_n if and only if for each $a \in \{0, \dots, n-2\}$ either $\alpha(a) = \alpha(a+1) + 1 \pmod{n}$ or $\alpha(a) = \alpha(a+1) - 1 \pmod{n}$. In terms of $\Xi(\alpha, a)$ and $\Pi(\alpha, a)$, this observation can be

reformulated as $\alpha \in D_n$ if and only if for each $a \in \{0, \dots, n-2\}$ either $\Xi(\alpha, a)$ is empty or $\Pi(\alpha, a)$ is empty.

Suppose a permutation $\alpha \in S_n$ does not belong to D_n . Then there is $a \neq n-1$ such that both $\Xi(\alpha, a)$ and $\Pi(\alpha, a)$ are nonempty. Denote $|\Xi(\alpha, a)|$ by p and $|\Pi(\alpha, a)|$ by q . Note that $p+q = n-2$ and $p, q \geq 1$. The number $t-1$ is in the range $1, \dots, n-3$. Represent $t-1$ as a sum of an odd integer r in the range $1, \dots, p$ and an integer s in the range $0, \dots, q$. Choose r elements $b_1, \dots, b_r \in \Xi(\alpha, a)$. Choose s elements $c_1, \dots, c_s \in \Pi(\alpha, a)$. Like in Lemma 3, consider $R = \{b_1, \dots, b_r; c_1, \dots, c_s\}$ and define $R' = S \cup \{a\}$ and $R'' = S \cup \{a+1\}$. Since r is odd, by Lemma 3, we have $\text{inv}(\alpha \upharpoonright R') \neq \text{inv}(\alpha \upharpoonright R'') \pmod{2}$. That is, the restriction of α to R' is even if and only if the restriction of α to R'' is odd. Thus, $\alpha \notin \Gamma_n^t$. $\square$

Lemma 5. *Let $t = 2, \dots, n-1$. If t is odd then $Z_n \subseteq \Gamma_n^t$. If t is even then $\Gamma_n^t \cap Z_n = \{\iota_n\}$*

Proof. Let k be a number in the range $1, \dots, n-1$. Note that σ_n^k acts on $[n]$ by mapping elements of $P = \{0, \dots, n-k-1\}$ to $\{k, \dots, n-1\}$, respectively, and elements of $Q = \{n-k, \dots, n-1\}$ to $\{0, \dots, k-1\}$, respectively. Consider a t -element subset $\{a_1, \dots, a_t\} \subseteq [n]$. Denote by p and q the number of elements of $\{a_1, \dots, a_t\}$ that belong to P and Q , respectively. Then $t = p+q$. The number of inversions of the restriction of α to $\{a_1, \dots, a_t\}$ is pq .

If t is odd then for every t -element subset either p or q is even, hence pq is even. Therefore, $\sigma_n^k \in \Gamma_n^t$.

If t is even then it can be presented as a sum of two odd numbers $t = p+q$, where $p \leq |P|$ and $q \leq |Q|$. Choose a t -element subset of $[n]$ containing p elements of P and q elements of Q . Then pq is odd, therefore, $\sigma_n^k \notin \Gamma_n^t$. $\square$

Lemma 6. $\rho_n \in \Gamma_n^t$ if and only if $t \equiv 0 \pmod{4}$ or $t \equiv 1 \pmod{4}$.

Proof. As ρ_n acts on a t -element subset of $[n]$ consisting of elements $a_1 < \dots < a_t$, it reverses their order, $\alpha(a_t) < \dots < \alpha(a_1)$. Thus, $\text{inv}(\rho_n \upharpoonright \{a_1, \dots, a_t\}) = \frac{t(t-1)}{2}$. The number expressed by this fraction is even if and only if $t \equiv 0 \pmod{4}$ or $t \equiv 1 \pmod{4}$. $\square$

From Lemmas 4, 5 and 6 our main result follows.

Theorem 7. *For every $t = 2, \dots, n-2$,*

- if $t \equiv 2 \pmod{4}$ then Γ_n^t is the one-element group $\{\iota_n\}$;
- if $t \equiv 3 \pmod{4}$ then Γ_n^t is the cyclic group Z_n ;
- if $t \equiv 0 \pmod{4}$ then Γ_n^t is the two-element group $\{\rho_n, \iota_n\}$;
- if $t \equiv 1 \pmod{4}$ then Γ_n^t is the dihedral group D_n .

4 Group Γ_n^{n-1}

In this section we will use a type of permutations called parity-alternating permutations [Tan10b, Tan10a]. A permutation $\alpha \in S_n$ is called *parity-alternating* if for every $a = 0, \dots, n-2$ (we do not consider $a = n-1$) we have $\alpha(a) \neq \alpha(a+1) \pmod{2}$. Denote the set of all parity-alternating permutations in S_n by PAP_n . It is not difficult to find a formula for the size of PAP_n , and that has been done in [Tan10b]. The size $|PAP_n|$ is $2 \cdot (\frac{n}{2})!^2$ if n is even or $(\frac{n-1}{2})! \cdot (\frac{n+1}{2})!$ if n is odd. We will prove that every permutation in Γ_n^{n-1} is parity-alternating. To be more precise, the permutations belonging to Γ_n^{n-1} constitute exactly a half of PAP_n ; therefore, $|\Gamma_n^{n-1}| = \frac{1}{2} \cdot |PAP_n|$.

Each $(n-1)$ -element subset of $[n]$ can be expressed as $[n] \setminus \{a\}$ for some $a \in [n]$. In this section we will denote this set by R_a .

Lemma 8. *For each $a \in \{0, \dots, n-2\}$ we have $\text{inv}(\alpha \upharpoonright R_a) = \text{inv}(\alpha \upharpoonright R_{a+1}) + \alpha(a) + \alpha(a+1) + 1 \pmod{2}$.*

Proof. Consider a permutation $\alpha \in S_n$. Fix $a \in \{0, \dots, n-2\}$. Refer to Lemma 3, using $\{b_1, \dots, b_r\} = \Xi(\alpha, a)$ and $\{c_1, \dots, c_s\} = \Pi(\alpha, a)$. We conclude that $\text{inv}(\alpha \upharpoonright R_a) = \text{inv}(\alpha \upharpoonright R_{a+1}) + |\Xi(\alpha, a)| \pmod{2}$. Recall from the definition of $\Xi(\alpha, a)$ that $\alpha \upharpoonright \Xi(\alpha, a)$ is a one-to-one correspondence between $\Xi(\alpha, a)$ and the set of elements of $[n]$ that are greater than $\alpha(a)$ and less than $\alpha(a+1)$ or greater than $\alpha(a+1)$ and less than $\alpha(a)$. Hence, $|\Xi(\alpha, a)| = |\alpha(a) - \alpha(a+1)| - 1$. Therefore, $\text{inv}(\alpha \upharpoonright R_a) = \text{inv}(\alpha \upharpoonright R_{a+1}) + \alpha(a) + \alpha(a+1) + 1 \pmod{2}$. $\square$

Lemma 9. *If $\alpha \in PAP_n$ then $\text{inv}(\alpha \upharpoonright R_a) = \text{inv}(\alpha \upharpoonright R_b) \pmod{2}$ for any $a, b \in [n]$.*

Proof. Since $\alpha \in PAP_n$, the difference between $\alpha(a)$ and $\alpha(a+1)$ is odd; therefore, by Lemma 8, $\text{inv}(\alpha \upharpoonright R_a) = \text{inv}(\alpha \upharpoonright R_{a+1}) \pmod{2}$. This equality is true for every $a \in \{0, \dots, n-2\}$. Hence, either $\text{inv}(\alpha \upharpoonright R_a)$ is even for every $a \in [n]$, or $\text{inv}(\alpha \upharpoonright R_a)$ is odd for every $a \in [n]$. $\square$

Recall that Γ_n^{n-1} consists of all permutations on $[n]$ such that for every $a \in [n]$ the size of $\text{inv}(\alpha \upharpoonright R_a)$ is even. It will be useful to consider a set of all permutations on $[n]$ such that for every $a \in [n]$ the number $\text{inv}(\alpha \upharpoonright R_a)$ is odd (instead of even). We will denote this set by $-\Gamma_n^{n-1}$.

Lemma 10. *PAP_n is a disjoint union of Γ_n^{n-1} and $-\Gamma_n^{n-1}$.*

Proof. Consider $\alpha \in PAP_n$. By Lemma 9, either for every $a \in [n]$ the number $\text{inv}(\alpha \upharpoonright R_a)$ is even, or for every $a \in [n]$ the number $\text{inv}(\alpha \upharpoonright R_a)$ is odd. In the former case, α belongs to Γ_n^{n-1} , and in the latter case, α belongs to $-\Gamma_n^{n-1}$.

Conversely, consider a permutation $\alpha \in \Gamma_n^{n-1}$ or $\alpha \in -\Gamma_n^{n-1}$. Consider $a \in \{0, \dots, n-2\}$. If $\alpha \in \Gamma_n^{n-1}$, both $\text{inv}(\alpha \upharpoonright R_a)$ and $\text{inv}(\alpha \upharpoonright R_{a+1})$ are even. If $\alpha \in -\Gamma_n^{n-1}$, both $\text{inv}(\alpha \upharpoonright R_a)$ and $\text{inv}(\alpha \upharpoonright R_{a+1})$ are odd. Hence, in both cases, by Lemma 8, the difference between $\alpha(a)$ and $\alpha(a+1)$ is odd, and α is parity-alternating. $\square$

Theorem 11. $|\Gamma_n^{n-1}| = \frac{1}{2} \cdot |PAP_n|$ for $n \geq 3$.

Proof. On the set PAP_n , consider an involution Θ induced by applying the cycle $(0\ 2)$ before the permutation. That is, let θ be a permutation on $[n]$ swapping 0 and 2. For each $\alpha \in PAP_n$, the mapping Θ maps α to $\alpha(\theta(\cdot))$. Since $\theta \in PAP_n$ and PAP_n is a group, it is clear that if $\alpha \in PAP_n$ then $\Theta(\alpha) \in PAP_n$. Thus, Θ is, indeed, an involution on PAP_n . Note that since $\alpha(0) \neq \alpha(2)$, the involution has no fixed points, that is, for every permutation α we have $\Theta(\alpha) \neq \alpha$.

Now we will prove that Θ always maps an element of Γ_n^{n-1} to an element of $-\Gamma_n^{n-1}$ and vice versa. Compare the set I' of inversions of the restriction of α to $[n] \setminus \{0\}$ and the set I'' of inversions of the restriction of $\alpha(\theta(\cdot))$ to $[n] \setminus \{2\}$. Since θ swaps 0 and 2, the set I' contains an inversion 1, 2 if and only if the set I'' does not contain an inversion 0, 1. All other inversions in I' and I'' are the same (subject to changing notation from 2 to 0 when comparing I' and I''). This means that either $|I'|$ is odd and $|I''|$ is even, or $|I'|$ is even and $|I''|$ is odd. By Lemma 9, we conclude that either $\alpha \in -\Gamma_n^{n-1}$ and $\Theta(\alpha) \in \Gamma_n^{n-1}$, or $\alpha \in \Gamma_n^{n-1}$ and $\Theta(\alpha) \in -\Gamma_n^{n-1}$.

The existence of the involution Θ proves the result. $\square$

As one compares the sizes of Γ_n^{n-1} and $\Gamma_n^n = A_n$, one can notice that $|\Gamma_n^{n-1}|$ is much smaller than $|\Gamma_n^n|$, and one can be tempted to conjecture that $\Gamma_n^{n-1} \subseteq \Gamma_n^n$. This is not true if n is even; indeed, we have $\sigma_n \in \Gamma_n^{n-1}$ and $\sigma_n \notin \Gamma_n^n$. However, it is true that $\Gamma_n^{n-1} \subseteq \Gamma_n^n$ if n is odd, as the following result shows.

Proposition 12. *If n is odd then $\Gamma_n^{n-1} = PAP_n \cap \Gamma_n^n$.*

Proof. For any permutation α on $[n]$, obviously there are exactly $\alpha(0)$ elements i in $[n]$ such that $\alpha(i) < \alpha(0)$. Hence, by the definition of an inversion, we have $\text{inv}(\alpha) = \text{inv}(\alpha \upharpoonright R_0) + \alpha(0)$.

Suppose $\alpha \in \Gamma_n^{n-1}$. Hence, by Lemma 10, $\alpha \in PAP_n$. Since n is odd and $\alpha \in PAP_n$, we conclude that $\alpha(0)$ is even. By the definition of Γ_n^{n-1} , $\text{inv}(\alpha \upharpoonright R_0)$ is even; therefore, using $\text{inv}(\alpha) = \text{inv}(\alpha \upharpoonright R_0) + \alpha(0)$, we conclude that $\text{inv}(\alpha)$ is even, that is, $\alpha \in \Gamma_n^n$.

Conversely, suppose $\alpha \in PAP_n \cap \Gamma_n^n$. Since n is odd and $\alpha \in PAP_n$, we conclude that $\alpha(0)$ is even. Since $\alpha \in \Gamma_n^n$, $\text{inv}(\alpha)$ is even. Applying the argument in the previous paragraph in the opposite direction, we conclude that $\text{inv}(\alpha \upharpoonright R_0)$ is even, therefore, $\alpha \in \Gamma_n^{n-1}$. $\square$

5 A generalized definition for monoids

Now we proceed to considering monoids of mappings which generalize monoids of order-preserving mappings and monoids of orientation-preserving mappings. For $t = 1, 2, \dots, n$, we define Σ_n^t as the monoid consisting of those mappings on $[n]$ whose all injective restrictions of width t are even.

Suppose S is a set of mappings on $[n]$, and let $k = 1, \dots, n$. We will denote by $S(\mathbf{r} = k)$ (by $S(\mathbf{r} \leq k)$, by $S(\mathbf{r} \geq k)$) the subset of S consisting of those

mappings whose rank is equal to k (is less than or equal to k , is greater than or equal to k).

We can immediately make the following easy observations.

Proposition 13. 1) $\Sigma_n^1 = T_n$.

2) $\Sigma_n^n = T_n(\mathbf{r} \leq n-1) \cup A_n$.

3) $\Gamma_n^t = \Sigma_n^t \cap S_n$, that is, Γ_n^t is the group of units of Σ_n^t for $t = 1, 2, \dots, n$.

4) $\Sigma_n^t(\mathbf{r} \leq t-1) = T_n(\mathbf{r} \leq t-1)$ for $t = 2, \dots, n$.

The definition of O_n and the description of OP_n given in Lemma 1 use restrictions of width 2 and 3, respectively. The following statement redefines O_n and OP_n using only injective restrictions of width 2 and 3.

Proposition 14. 1) A mapping $\alpha \in T_n$ belongs to O_n if and only if each injective restriction of α of width 2 is order-preserving. Therefore, $\Sigma_n^2 = O_n$.

2) A mapping $\alpha \in T_n$ with rank not equal to 2 belongs to OP_n if and only if each injective restriction of α of width 3 is orientation-preserving.

Proof. 1) is obvious. As to 2), recall from Lemma 1 that a mapping $\alpha \in T_n$ with rank not equal to 2 belongs to OP_n if and only if each restriction of α of width 3 is orientation-preserving. Notice that if a restriction of width 3 is not injective then it is orientation-preserving. Hence, one only needs to consider injective restrictions of width 3. $\square$

The following statements redefines OR_n and M_n via injective restrictions. In comparison with the previous statement, the proof is unexpectedly fiddly.

Theorem 15. Consider $\alpha \in T_n$ with rank at least 4.

1) $\alpha \in OR_n$ if and only if each injective restriction of α of width 4 is oriented.

2) $\alpha \in M_n$ if and only if each injective restriction of α of width 4 is monotone.

Proof. Part 2 follows from Part 1 if we notice that monotone mappings are oriented mappings. Now we will concentrate on proving Part 1.

A restriction of an oriented mapping is an oriented partial mapping. So, it remains to prove the converse implication.

From now on, we assume that each injective restriction of α of width 4 is oriented. Let X be any transversal of $\text{Ker } \alpha$. Then $\alpha \upharpoonright X$ is an injective partial mapping on $[n]$ and so any restriction of $\alpha \upharpoonright X$ is injective and also a restriction of α . Hence, each restriction of $\alpha \upharpoonright X$ of width 4 is oriented. Since it is known [Fer23, Theorem 2.6] that a partial mapping on $[n]$ is oriented if and only if any of its restriction of width 4 is oriented, it follows that $\alpha \upharpoonright X$ is an oriented partial mapping on $[n]$.

Denote the rank of α by k ; recall that $k \geq 4$. We will construct recursively a transversal $\{c_1, \dots, c_k\}$ of $\text{Ker } \alpha$. For each c_i , we will denote $\alpha(c_i)$ by d_i and $\alpha^{-1}(d_i)$ by C_i . Let $c_1 = 0$. Then for $i \in \{2, \dots, k\}$, let

$$c_i = \min([n] \setminus (C_1 \cup \dots \cup C_{i-1})).$$

It is easy to verify that $c_1 < c_2 < \dots < c_k$, $c_i = \min C_i$ for $1 \leq i \leq k$, and $X = \{c_1, \dots, c_k\}$ is a transversal of $\text{Ker } \alpha$. Since $\alpha \upharpoonright X$ is oriented, then the sequence $(d_1, d_2, \dots, d_k)$, consisting of k distinct elements of $[n]$, is oriented. Let

$$c_{k+1} = \begin{cases} \min\{i \in [n] \mid \{i, \dots, n-1\} \subseteq C_1\} & \text{if } n-1 \in C_1 \\ n & \text{otherwise.} \end{cases}$$

Observe that by this definition, c_{k+1} is greater than every element in C_k .

Consider one of classes C_i and an element $b \in C_i$. Note that $c_i \leq b$. If $i = k$ then $c_k \leq b < c_{k+1}$. Now suppose $i < k$. Then $c_{k+1} \leq b \leq n-1$ (and, necessarily, $b \in C_1$) or $b < c_{i+1}$. In fact, if $c_{i+1} \leq b < c_{k+1}$, then $c_1 < \dots < c_{i-1} < c_{i+1} < \dots < c_j < b < c_{j+1} < \dots < c_k < c_{k+1}$ for some j and $\{c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_j, b, c_{j+1}, \dots, c_k\}$ is a transversal of $\text{Ker } \alpha$, whence the sequence $(d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_j, d_i, d_{j+1}, \dots, d_k)$ is oriented, which is a contradiction. Therefore,

$$C_1 = \{0, \dots, c_2 - 1\} \cup \{c_{k+1}, \dots, n-1\}$$

(here, if $c_{k+1} = n$ then $\{c_{k+1}, \dots, n-1\}$ stands for the empty set) and

$$C_i = \{c_i, \dots, c_{i+1} - 1\} \text{ for } 2 \leq i \leq k.$$

Hence it is easy to conclude that $\alpha \in OR_n$. $\square$

Proposition 16. *For all $t = 2, \dots, n$, $O_n \subseteq \Sigma_n^t$. Also, for all $t = 2, \dots, n-2$,*

- *if $t = 3 \pmod{4}$ then $OP_n \subseteq \Sigma_n^t$;*
- *if $t = 0 \pmod{4}$ then $M_n \subseteq \Sigma_n^t$;*
- *if $t = 1 \pmod{4}$ then $OR_n \subseteq \Sigma_n^t$.*

Proof. It follows from the definitions of O_n and Σ_n^t that $O_n \subseteq \Sigma_n^t$. It is well known that O_n is generated by $O_n(\mathbf{r} = n-1)$ and ι_n , whereas M_n , OP_n and OR_n are generated by $O_n \cup \{\rho_n\}$, $O_n \cup Z_n$ and $O_n \cup D_n$, respectively; see [Aiz62, FGJ05, CH99, McA00]. Combining these facts and Theorem 7, we obtain the results in the bullet points. $\square$

We might add that if one wishes, it is possible to reword Proposition 16 in a form which looks parallel to Theorem 7, as follows.

- if $t = 2 \pmod{4}$ then $O_n = M_n \cap OP_n \subseteq \Sigma_n^t$;
- if $t = 3 \pmod{4}$ then $OP_n \subseteq \Sigma_n^t$;
- if $t = 0 \pmod{4}$ then $M_n \subseteq \Sigma_n^t$;
- if $t = 1 \pmod{4}$ then $OR_n = \langle M_n \cup OP_n \rangle \subseteq \Sigma_n^t$.

Proposition 17. $\Sigma_n^3 = T_n(\mathbf{r} \leq 2) \cup OP_n$.

Proof. We have $\Sigma_n^3 \supseteq T_n(\mathbf{r} \leq 2)$ by Proposition 13, part 4). We have $\Sigma_n^3 \supseteq OP_n$ by Proposition 16.

Now consider $\alpha \in \Sigma_n^3$. If the rank of α is 1 or 2 then $\alpha \in T_n(\mathbf{r} \leq 2)$. Now suppose α has rank greater than or equal to 3. Let $T = \{i < j < k\} \subseteq [n]$ be such that $\alpha \upharpoonright T$ is injective and suppose that $\alpha(T) = \{a < b < c\}$. Then $\alpha \upharpoonright T$ is presented by one of the following 6 cases.

$$\begin{pmatrix} i & j & k \\ a & b & c \end{pmatrix}, \begin{pmatrix} i & j & k \\ a & c & b \end{pmatrix}, \begin{pmatrix} i & j & k \\ b & a & c \end{pmatrix}, \begin{pmatrix} i & j & k \\ c & a & b \end{pmatrix}, \begin{pmatrix} i & j & k \\ b & c & a \end{pmatrix}, \begin{pmatrix} i & j & k \\ c & b & a \end{pmatrix}$$

Inspecting these cases, we notice that $\alpha \upharpoonright T$ is orientation-preserving if and only if $\alpha \upharpoonright T$ is even, namely, in the following 3 cases.

$$\begin{pmatrix} i & j & k \\ a & b & c \end{pmatrix}, \begin{pmatrix} i & j & k \\ c & a & b \end{pmatrix}, \begin{pmatrix} i & j & k \\ b & c & a \end{pmatrix}$$

Using part 2 of Proposition 14, we can conclude that if $\alpha \in \Sigma_n^3$ and α has rank greater than or equal to 3 then $\alpha \in OP_n$. $\square$

Comparing the equality $\Sigma_n^2 = O_n$ in Proposition 14 and a description of Σ_n^3 in Proposition 17, one can ask whether $\Sigma_n^3 = OP_n$. The answer is that for $n \geq 4$, $OP_n \subsetneq \Sigma_n^3$; for example,

$$\begin{pmatrix} 0 & 1 & 2 & 3 & \cdots & n-1 \\ 0 & 1 & 0 & 1 & \cdots & 1 \end{pmatrix} \in \Sigma_n^3 \setminus OP_n.$$

This example frequently features in the study of OP_n , including [HV22, Fer23, Fer24, HV24].

Comparing Propositions 17 and 16, one might conjecture that $\Sigma_n^4 = T_n(\mathbf{r} \leq 3) \cup M_n$ and $\Sigma_n^5 = T_n(\mathbf{r} \leq 4) \cup OR_n$. Both these equalities are false. For instance, for $n \geq 4$,

$$\alpha = \begin{pmatrix} 0 & \cdots & n-4 & n-3 & n-2 & n-1 \\ 2 & \cdots & 2 & 3 & 0 & 1 \end{pmatrix} \in \Sigma_n^4$$

but $\alpha \notin T_n(\mathbf{r} \leq 3)$ and $\alpha \notin M_n$; and, for $n \geq 5$,

$$\beta = \begin{pmatrix} 0 & \cdots & n-5 & n-4 & n-3 & n-2 & n-1 \\ 3 & \cdots & 3 & 4 & 1 & 2 & 0 \end{pmatrix} \in \Sigma_n^5$$

but $\beta \notin T_n(\mathbf{r} \leq 4)$ and $\beta \notin OR_n$.

6 How the monoids embed into one another

It is natural to ask if some of the monoids $\Sigma_n^2, \dots, \Sigma_n^n$ are equal to one another, like some of the groups $\Gamma_n^2, \dots, \Gamma_n^n$ are equal to one other, as described in Theorem 7. In this section we will show that the monoids $\Sigma_n^2, \dots, \Sigma_n^n$ are not equal to one another, but some of them are submonoids of others, in a periodic fashion somewhat similar to Theorem 7.

Consider $\alpha \in T_t(\mathbf{r} = k)$. Let X be a transversal of $\text{Ker } \alpha$. Let α_L be the unique order-preserving bijection from $[k]$ to X . Let α_R be the unique order-preserving bijection from $\text{Im } \alpha$ to $[k]$. Consider a mapping $\alpha_L \alpha \alpha_R$, that is, the

composition $\alpha_R(\alpha(\alpha_L(\cdot)))$; we will denote this mapping by $\alpha \nabla X$. Note that $\alpha \nabla X$ is a permutation on $[k]$.

The following result enables us to exploit our knowledge about groups Γ_k^t when we explore semigroups Σ_n^t .

Lemma 18. $\alpha \in \Sigma_n^t(\mathbf{r} = k)$ if and only if $\alpha \nabla X \in \Gamma_k^t$ for every transversal X of $\text{Ker } \alpha$.

Proof. Suppose $\alpha \in \Sigma_n^t(\mathbf{r} = k)$. Let Y be a t -element subset of $[k]$. Then we have

$$(\alpha \nabla X) \upharpoonright Y = (\alpha_L \upharpoonright Y)(\alpha \upharpoonright \alpha_L(Y))(\alpha_R \upharpoonright \alpha(\alpha_L(Y))).$$

Since $\alpha_L(Y)$ is a t -element subset of $[n]$ and $\alpha \in \Sigma_n^t$, we conclude that $\alpha \upharpoonright \alpha_L(Y)$ is even. At the same time, both $\alpha_L \upharpoonright Y$ and $\alpha_R \upharpoonright \alpha(\alpha_L(Y))$ are order-preserving injective mappings, thus, have no inversions. Therefore, by Lemma 2, $(\alpha \nabla X) \upharpoonright Y$ is even. Applying this argument to Y being every t -element subset of $[k]$, we conclude that $\alpha \nabla X \in \Gamma_k^t$.

Now suppose $\alpha \notin \Sigma_n^t(\mathbf{r} = k)$. Then an injective restriction $\alpha \upharpoonright Y$ is odd for a certain t -element set Y . Consider a transversal X of $\text{Ker } \alpha$ such that $Y \subseteq X$. Then proceeding like in the previous paragraph, one can conclude that $\alpha \nabla X \notin \Gamma_k^t$. $\square$

Lemma 19. Let $p, q \in \{2, \dots, n\}$. If $p, q \leq n - 2$ then $\Gamma_n^p \subseteq \Gamma_n^q$ if and only if $p = 2 \pmod{4}$ or $q = 1 \pmod{4}$ or $p = q \pmod{4}$. If $p \geq n - 1$ or $q \geq n - 1$ then $\Gamma_n^p \subseteq \Gamma_n^q$ if and only if $p \leq q$ and at least one of the following conditions is true, $p = 2 \pmod{4}$ or $q = 1 \pmod{4}$ or $p = q \pmod{4}$.

Proof. For values of $p, q \leq n - 2$ the result follows from Theorem 7. If one or both of p, q is $n - 1$ or n , refer also to the following facts:

$\sigma_n \in A_n$ if and only if n is odd;

$\rho_n \in A_n$ if and only if $n = 0 \pmod{4}$ or $n = 1 \pmod{4}$;

By Lemma 5, $\sigma_n \in \Gamma_n^{n-1}$ if and only if n is even;

By Lemma 6, $\rho_n \in \Gamma_n^{n-1}$ if and only if $n = 1 \pmod{4}$ or $n = 2 \pmod{4}$. $\square$

Theorem 20. Let $p, q \in \{2, \dots, n\}$. Then $\Sigma_n^p \subseteq \Sigma_n^q$ if and only if $p \leq q$ and at least one of the following conditions is true, $p = 2 \pmod{4}$ or $q = 1 \pmod{4}$ or $p = q \pmod{4}$.

Proof. Suppose $\Sigma_n^p \subseteq \Sigma_n^q$. Assume that $q < p$ (this assumption will lead to a contradiction). Consider

$$\alpha = \begin{pmatrix} 0 & 1 & 2 & \dots & p-2 & p-1 & \dots & n-1 \\ 1 & 0 & 2 & \dots & p-2 & p-2 & \dots & p-2 \end{pmatrix}.$$

As to $\alpha \upharpoonright [q]$, it is injective (recall that $q \leq p - 1$) and odd (it has only one inversion). Hence, $\alpha \notin \Sigma_n^q$. At the same time, $\alpha \in T_n(\mathbf{r} \leq p - 1) \subseteq \Sigma_n^p \subseteq \Sigma_n^q$. This contradiction proves that $p \leq q$.

Also, from $\Sigma_n^p \subseteq \Sigma_n^q$ it follows that $\Gamma_n^p \subseteq \Gamma_n^q$; hence, referring to Lemma 19, we have $p = 2 \pmod{4}$, or $q = 1 \pmod{4}$, or $p = q \pmod{4}$.

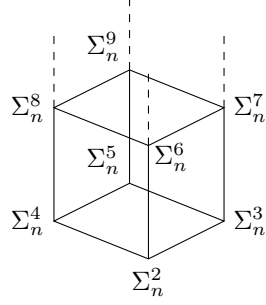


Figure 1: Illustrating Theorem 20

To prove the reverse implication of the theorem, consider $\alpha \in \Sigma_n^p$, then by Lemma 18, $\alpha \nabla X \in \Gamma_k^p$, where $k = |X|$, for every transversal X of $\text{Ker } \alpha$. Hence, by Lemma 19, we have $\alpha \nabla X \in \Gamma_k^q$ for every transversal X of $\text{Ker } \alpha$. Hence, by Lemma 18, we have $\alpha \in \Sigma_n^q$. $\square$

The following result describes elements of Σ_n^t with ranks greater than or equal to $t + 2$. This complements part 4) of Proposition 13, which described elements of Σ_n^t with ranks less than or equal to $t - 1$. Describing elements of Σ_n^t with ranks equal to t or $t + 1$ is a problem for future research.

Proposition 21. *For $t \leq n - 2$,*

- *if $t \equiv 2 \pmod{4}$ then $\Sigma_n^t(\mathbf{r} \geq t + 2) = O_n(\mathbf{r} \geq t + 2)$;*
- *if $t \equiv 3 \pmod{4}$ then $\Sigma_n^t(\mathbf{r} \geq t + 2) = OP_n(\mathbf{r} \geq t + 2)$;*
- *if $t \equiv 0 \pmod{4}$ then $\Sigma_n^t(\mathbf{r} \geq t + 2) = M_n(\mathbf{r} \geq t + 2)$;*
- *if $t \equiv 1 \pmod{4}$ then $\Sigma_n^t(\mathbf{r} \geq t + 2) = OR_n(\mathbf{r} \geq t + 2)$.*

Proof. The reverse inclusions ($\supseteq$) follow from Proposition 16. To prove the direct inclusions ($\subseteq$), consider α belonging to $\Sigma_n^t(\mathbf{r} = k)$, where $k \geq t + 2$. Consider a transversal X of $\text{Ker } \alpha$.

Then by Lemma 18 and Theorem 7, we have the following cases.

- if $t \equiv 2 \pmod{4}$ then $\alpha \nabla X = \iota_k$;
- if $t \equiv 3 \pmod{4}$ then $\alpha \nabla X \in Z_k$;
- if $t \equiv 0 \pmod{4}$ then $\alpha \nabla X \in \{\rho_k, \iota_k\}$;
- if $t \equiv 1 \pmod{4}$ then $\alpha \nabla X \in D_k$.

Recalling the definition of $\alpha \nabla X$, the four cases above can be reworded as shown below.

- if $t \equiv 2 \pmod{4}$ then $\alpha \upharpoonright X$ is order-preserving;

- if $t = 3 \pmod{4}$ then $\alpha \upharpoonright X$ is orientation-preserving;
- if $t = 0 \pmod{4}$ then $\alpha \upharpoonright X$ is monotone;
- if $t = 1 \pmod{4}$ then $\alpha \upharpoonright X$ is oriented.

Combining these observations with Proposition 14 and Theorem 15, we obtain the result. $\square$

One might ask if it is possible to generalize the above result so it also includes $\Sigma_n^t(\mathbf{r} = t)$ and $\Sigma_n^t(\mathbf{r} = t + 1)$. As to $\Sigma_n^t(\mathbf{r} = t)$, mappings α and β from the counterexamples at the end of Section 5 serve as counterexamples here. As to $\Sigma_n^t(\mathbf{r} = t + 1)$, consider a mapping

$$\gamma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots & n-1 \\ 2 & 1 & 0 & 5 & 4 & 3 & 6 & \cdots & 6 \end{pmatrix}.$$

We have $\gamma \in \Sigma_n^6(\mathbf{r} = 7)$ but $\gamma \notin O_n$.

7 Another generalized definition for monoids

As we reflect on Proposition 16 and Theorem 20, this motivates us to consider the following alternative family of monoids. For $t = 1, 2, \dots, n$, we define Δ_n^t as the monoid generated by all mappings on $[n]$ whose rank is at least $n - 1$ and whose injective restrictions of width t is even. In other words, Δ_n^t is generated by $\Sigma_n^t(\mathbf{r} \geq n - 1)$.

The groups of units of the new monoids Δ_n^t are the groups Γ_n^t , for $t = 1, \dots, n$, in the same way as it was for the family of monoids Σ_n^t considered above. We have $\Delta_n^1 = \Sigma_n^1 = T_n$ (since T_n is generated by its mappings of rank greater than or equal to $n - 1$) and $\Delta_n^n = \Sigma_n^n = T_n(\mathbf{r} \leq n - 1) \cup A_n$. More interesting is the fact that the monoids $\Delta_n^2, \dots, \Delta_n^{n-3}$ demonstrate a periodic behavior similar to the one described by Theorem 7 for the groups $\Gamma_n^2, \dots, \Gamma_n^{n-2}$, as described in the following theorem.

Theorem 22. *For every $t = 2, \dots, n - 3$,*

- if $t = 2 \pmod{4}$ then $\Delta_n^t = O_n$;
- if $t = 3 \pmod{4}$ then $\Delta_n^t = OP_n$;
- if $t = 0 \pmod{4}$ then $\Delta_n^t = M_n$;
- if $t = 1 \pmod{4}$ then $\Delta_n^t = OR_n$.

Proof. The theorem follows from Propositions 16 and 21. $\square$

One can ask if the above theorem generalizes to the case $t = n - 2$. For $n = 4, 5$ we have $\Delta_4^2 = O_4$ and $\Delta_5^3 = OP_5$, in accordance with the above theorem. However, this is not true for $n \geq 6$; for example, we have

$$\alpha = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & \cdots & n-2 & n-1 \\ 2 & 3 & 0 & 1 & 4 & \cdots & n-2 & n-2 \end{pmatrix} \in \Delta_n^{n-2}$$

but $\alpha \notin OR_n$.

Similarly, the theorem does not generalize to the case $t = n - 1$. For example,

$$\beta = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & \cdots & n-1 \\ 2 & 2 & 0 & 4 & 3 & 5 & \cdots & n-1 \end{pmatrix} \in \Delta_n^{n-1}$$

but $\beta \notin OR_n$.

8 Possible future research

In Sections 5 and 7 we introduce monoids of full mappings on $[n]$ that extend the groups $\Gamma_n^1, \dots, \Gamma_n^n$. One can conduct further research by introducing either monoids of partial mappings on $[n]$ or inverse monoids of partial one-to-one mappings on $[n]$, instead of full mappings.

As discussed in Section 6, describing sets $\Sigma_n^t(\mathbf{r} = t)$ and $\Sigma_n^t(\mathbf{r} = t + 1)$ remains an open problem, and this is a missing part in describing the monoids Σ_n^t .

As discussed in Section 7, describing monoids Δ_n^{n-2} and Δ_n^{n-1} remains an open problem, along with obtaining a description of Σ_n^{n-2} and Σ_n^{n-1} .

Acknowledgements

The authors are grateful to the anonymous referee who wrote detailed and incisive comments on several versions of this paper and suggested ways to expand the results and to simplify proofs.

References

- [Aiz62] A Ya Aizenstat. The defining relations of the endomorphism semi-group of a finite linearly ordered set. In *Sibirsk. Mat.*, volume 3, pages 161–169, 1962.
- [CH99] P M Catarino and P M Higgins. The monoid of orientation-preserving mappings on a chain. In *Semigroup Forum*, volume 2, pages 190–206, 1999.
- [Fer00] Vitor H Fernandes. The monoid of all injective orientation preserving partial transformations on a finite chain. *Communications in Algebra*, 28(7):3401–3426, 2000.
- [Fer02] Vítor H. Fernandes. Presentations for some monoids of partial transformations on a finite chain: a survey. In *Semigroups, algorithms, automata and languages. Proceedings of workshops held at the International Centre of Mathematics, CIM, Coimbra, Portugal, May, June and July 2001*, pages 363–378. Singapore: World Scientific, 2002.

- [Fer23] Vítor Hugo Fernandes. Oriented transformations on a finite chain: another description. *Commun. Korean Math. Soc.*, 38(3):725–731, 2023.
- [Fer24] Vítor Hugo Fernandes. Corrigendum on “oriented transformations on a finite chain: another description” [commun. korean math. soc. 38 (2023), no. 3, pp. 725–731]. *Commun. Korean Math. Soc.*, 39(3):643–645, 2024.
- [FGJ05] Vítor H. Fernandes, Gracinda M. S. Gomes, and Manuel M. Jesus. Presentations for some monoids of partial transformations on a finite chain. *Commun. Algebra*, 33(2):587–604, 2005.
- [FP23] Vítor H Fernandes and Tânia Paulista. On the monoid of partial isometries of a cycle graph. *Turkish Journal of Mathematics*, 47(6):1746–1760, 2023.
- [GH92] Gracinda MS Gomes and John M Howie. On the ranks of certain semigroups of order-preserving transformations. In *Semigroup Forum*, volume 45, pages 272–282. Springer, 1992.
- [Hig93] Peter M Higgins. Combinatorial results for semigroups of order-preserving mappings. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 113, pages 281–296. Cambridge University Press, 1993.
- [How71] JM Howie. Products of idempotents in certain semigroups of transformations. *Proceedings of the Edinburgh Mathematical Society*, 17(3), 1971.
- [HV22] Peter M Higgins and Alexei Vernitski. Orientation-preserving and orientation-reversing mappings: a new description. In *Semigroup Forum*, volume 104, pages 509–514. Springer, 2022.
- [HV24] Peter M Higgins and Alexei Vernitski. Correction to: Orientation preserving and orientation reversing mappings: a new description. In *Semigroup Forum*, pages 1–2. Springer, 2024.
- [LM06] I. Levi and J. D. Mitchell. On rank properties of endomorphisms of finite circular orders. *Commun. Algebra*, 34(4):1237–1250, 2006.
- [McA00] D McAlister. Semigroups generated by a group and an idempotent. *Communications in Algebra*, 26(2):515–547, 2000.
- [Tan10a] Shinji Tanimoto. Combinatorics of the group of parity alternating permutations. *Advances in Applied Mathematics*, 44(3):225–230, 2010.
- [Tan10b] Shinji Tanimoto. Parity alternating permutations and signed eulerian numbers. *Annals of Combinatorics*, 14(3):355–366, 2010.

- [VV95] AS Vernitskii and MV Volkov. The proof and the generalization of Higgins' theorem on divisors of semigroups of order-preserving mappings. *Russian Mathematics-New York*, 39(1):34–39, 1995.

VÍTOR H. FERNANDES, Center for Mathematics and Applications (NOVA Math) and Department of Mathematics, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Monte da Caparica, 2829-516 Caparica, Portugal; e-mail: vhf@fct.unl.pt.

ALEXEI VERNITSKI, School of Mathematics, Statistics and Actuarial Science, University of Essex, Colchester, UK; e-mail: asvern@essex.ac.uk.