Canad. Math. Bull. 2025, pp. 1–19
http://dx.doi.org/10.4153/S0008439525000396

3-manifold spine cyclic presentations with seldom seen Whitehead graphs

Gerald Williams

Abstract. We consider a family of cyclic presentations and show that, subject to certain conditions on the defining parameters, they are spines of closed 3-manifolds. These are new examples where the reduced Whitehead graphs are of the same type as those of the Fractional Fibonacci presentations; here the corresponding manifolds are often (but not always) hyperbolic. We also express a lens space construction in terms of a class of positive cyclic presentations that are spines of closed 3-manifolds. These presentations then furnish examples where the Whitehead graphs are of the same type as those of the positive cyclic presentations of type $\mathfrak{Z}$, as considered by McDermott.

1 Introduction

The *cyclically presented group* $G_n(w)$ is the group defined by the *cyclic presentation*

$$\mathcal{G}_n(w) = \langle x_0, \ldots, x_{n-1} \mid w, \theta(w), \ldots, \theta^{n-1}(w)\rangle,$$

where $w(x_0, \ldots, x_{n-1})$ is a word in the free group F_n with generators $x_0, \ldots, x_{n-1}$ and $\theta : F_n \to F_n$ is the *shift automorphism* of F_n given by $\theta(x_i) = x_{i+1}$ for each $0 \le i < n$ (subscripts mod n, $n > 0$). Cyclic presentations that are spines of 3-manifolds have been widely researched, with notable early studies by Dunwoody [18], Sieradski [50], Helling, Kim, Mennicke [22], and Cavicchioli and Spaggiari [16].

A necessary condition for a presentation to be a spine of a closed 3-manifold is that its Whitehead graph is planar (see, for example, [25, p. 33], [4, Documentation, Section 11]). The planar, reduced, Whitehead graphs of cyclic presentations were classified in [27], whose labeling (I.*j*),(II.*j*),(III.*j*) we now adopt. (Types (I.5) and (II.6) are precisely the graphs that correspond to positive or negative cyclic presentations.) In the overwhelming majority of studies of cyclic presentations as spines of 3-manifolds (for example [1, 2, 7–13, 15, 18, 20, 21, 26, 33–38, 42, 44, 48, 50–53]) the reduced Whitehead graph is of one of the types (I.1), (I.3) or (I.5), which correspond to the graphs given by Dunwoody in [18, Figure 1] and, as such, provide examples of so-called *Dunwoody manifolds*. By [8] and [21] the class of Dunwoody manifolds is exactly the class of strongly-cyclic branched covers of $(1,1)$-knots. Strictly

Received by the editors September 2, 2024; revised March 25, 2025; accepted March 31, 2025.
Published online on Cambridge Core April 4, 2025.
AMS subject classification: **20F05, 57M05, 57M50.**
Keywords: 3-manifold spine, Whitehead graph, cyclically presented group, Fractional Fibonacci group.

speaking, the hypothesis $ab \neq 0$ of [18], which is removed in many later references (including [8, 21]), excludes Whitehead graphs of type (I.5) and for this reason no positive presentations appear in [18, Table 1]. However, an explicit family of cyclic presentations that are spines of 3-manifolds and where the reduced Whitehead graph is of this type are given by the positive presentations of type $\mathfrak{Z}$ considered in [42, Theorem 8] (that is, those with $s < 0$). Cyclic presentations that are spines of 3-manifolds where the reduced Whitehead graph is of a different type to (I.1), (I.3), (I.5) are few and far between. We are only aware of the following families of such presentations. The non-cyclically reduced cyclic presentation $\mathcal{G}_n(x_0x_1x_1^{-1})$ is a spine of S^3 for all $n \geq 1$ [26, Lemma 9]. Its Whitehead graph contains loops, but once these are removed the graph is of type (I.4). Helling, Kim, and Mennicke [22] and Cavicchioli and Spaggiari [16, Theorem 3] showed that, for even n, the Fibonacci presentations $\mathcal{F}(n) = \mathcal{G}_n(x_0x_1x_2^{-1})$ are spines of closed, oriented 3-manifolds; here (for $n \geq 4$) the Whitehead graph is of type (II.11). A special case of [47, Theorem 6] gives that, for coprime integers $k, l \geq 1$ and even n, the cyclic presentations $\mathcal{G}_n(x_0^l x_1^k x_2^{-l})$ are spines of closed, oriented 3-manifolds. These presentations are the *Fractional Fibonacci presentations* $\mathcal{F}^{k/l}(n)$ of [55, 56] (or of [40, 41] in the case $l = 1$) and if $(k, l) \neq (1, 1)$ their reduced Whitehead graph is of type (II.7). Jeong and Wang [28, 29] showed that, for $l \geq 2$ and even n the cyclic presentations $\mathcal{G}_n(x_1(x_2^{-1}x_0)^l)$ are spines of closed, oriented 3-manifolds; here the reduced Whitehead graph is of type (II.14).

In Section 3, we present a family of cyclic presentations that are spines of closed 3-manifolds where the Whitehead graphs are of type (I.5) and show that the manifolds are cyclic branched covers of a lens space. This construction is essentially well known (see, for example [49, p. 217], [43, p. 4], or [39, Section 3]), but the connection to the planar Whitehead graph classification of [27] has not previously been observed. In Section 4, we present a new family of cyclic presentations that are spines of closed 3-manifolds where the reduced Whitehead graphs are of type (II.7) (i.e., the same type as those of the Fractional Fibonacci presentations) and show that many, but not all, of the manifolds are hyperbolic. Experiments in `Heegaard` [4] were used in formulating these results.

2 Presentation complexes as spines of closed 3-manifolds

The *presentation complex* (or *cellular model*) $K = K_{\mathcal{P}}$ of a group presentation $\mathcal{P} = \langle X \mid R \rangle$ is the 2-complex with one 0-cell O, a loop at O for each generator $x \in X$ and a 2-cell for each relator (the boundary of that 2-cell spelling the relator). If N is a regular neighborhood of O then $K \cap \partial N$ is a 1-dimensional cell complex called the *Whitehead graph* or *link graph* of $\mathcal{P}$. Thus the Whitehead graph of $\mathcal{P}$ is the graph with $2|X|$ vertices v_x, v'_x ($x \in X$) and an edge (v_x, v_y) (resp. (v'_x, v'_y), (v_x, v'_y)) for each occurrence of a cyclic subword xy^{-1} (resp. $x^{-1}y$, $(xy)^{\pm 1}$) in a relator $r \in R$. The *reduced Whitehead graph* is the graph obtained from the Whitehead graph by replacing all multiedges between two vertices by a single edge. We say that a group presentation $\mathcal{P}$ is the *spine* of a closed 3-manifold M if there exists a 3-ball $B^3 \subset M$ such that $M - \mathring{B}^3$ collapses onto $K_{\mathcal{P}}$ (where $\mathring{B}^3$ denotes the interior of B^3). Since $K_{\mathcal{P}}$ is connected, the manifold M is necessarily connected.

Suppose that a group presentation $\mathcal{P} = \langle X \mid R \rangle$ with an equal number of generators and relators is a spine of a closed, oriented 3-manifold. Then (see [49, Chapter 9], [45], [50, p. 125]) there is a 3-complex C whose set of faces consists of precisely one pair of oppositely oriented faces F_r^+, F_r^- for each relator $r \in R$, whose boundaries spell r. Let M_0 denote the 3-complex obtained from C by identifying the faces F_r^+, F_r^- ($r \in R$). Then M_0 is a closed, connected, oriented pseudo-manifold. The *Seifert–Threlfall condition* for M_0 to be a manifold M is that its Euler characteristic is zero [49, Theorem I, Section 60]. In this case the cell structure on $M = M_0$ has one vertex, one 3-cell, and 2-skeleton homeomorphic to the presentation complex $K_{\mathcal{P}}$ of $\mathcal{P}$. The Whitehead graph Γ of $\mathcal{P}$ is the link of the single vertex of $K_{\mathcal{P}}$, and so embeds in the link of the single vertex of M. The link of the vertex of M is the 2-sphere, and so Γ has a planar embedding on this sphere. When Γ is connected the 3-complex C is a polyhedron π bounding a 3-ball. Given a planar embedding of Γ there is a one-to-one correspondence between the faces F of this embedding and the vertices u_F of π. This correspondence maps a face of degree d to a vertex of degree d of π. Moreover, if the vertices in the face read cyclically around the face are $w_1, \dots, w_d$ where $w_i \in \{v_x, v'_x\ (x \in X)\}$ then the arcs $e_1, \dots, e_d$ incident to u_F, read cyclically, are directed toward u_F if $w_i = v_x$ and away from u_F if $w_i = v'_x$, and the arc e_j, corresponding to vertex $w_j = v_x$ or v'_x, is labeled x.

3 Cyclic presentations $\mathcal{H}(r,n)$ as spines of type (I.5)

For $n > 1, r \geq 1$ let

$$\mathcal{H}(r,n) = \mathcal{G}_n(x_0 x_1 \dots x_{r-1})$$

and let $H(r,n)$ be the group it defines. By [54, Theorems 2 and 3], $H(r,n)$ is finite if and only if $(n,r) = 1$, in which case $H(r,n) \cong \mathbb{Z}_r$. The Whitehead graph of $\mathcal{H}(r,n)$ is connected if and only if $r > 1$ and $(n,r) = 1$, and in this case it is of type (I.5), as shown in Figure 1, where (as also in Figure 5) the edge labels denote their multiplicities, a vertex label i denotes v_{x_i}, and a vertex label $\bar{i}$ denotes v'_{x_i}.

As a warm up to our main result, Theorem A, in this section we show that, when the Whitehead graph is connected (that is, if $n, r > 1$ and $(n,r) = 1$), then $\mathcal{H}(r,n)$ is a spine of a closed, oriented 3-manifold M. The positive presentations of type $\mathfrak{Z}$, considered in [42], also have Whitehead graphs of type (I.5) and, subject to

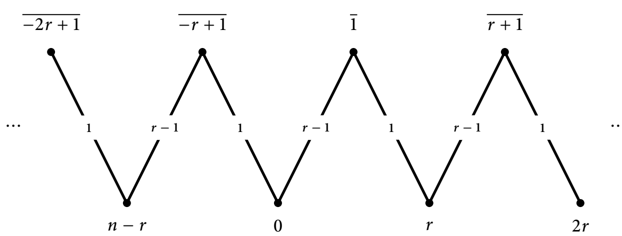

Figure 1: Whitehead graph for $\mathcal{H}(r,n)$ (where $(r,n) = 1$).

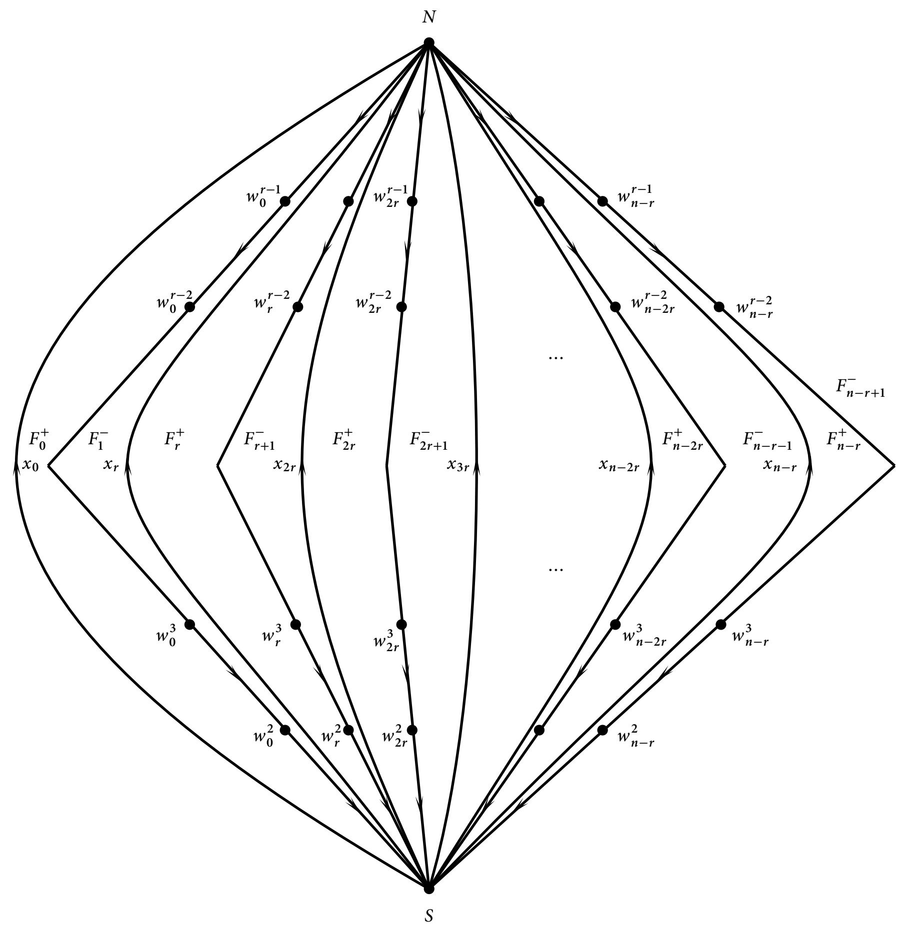

Figure 2: Face pairing polyhedron for $\mathcal{H}(r,n)$.

certain hypotheses, they are spines of closed, oriented manifolds N [42, Theorem 8]. The manifolds M can often be distinguished from the manifolds N by comparing their fundamental groups $\pi_1(M) = H(r,n) \cong \mathbb{Z}_r$ and $\pi_1(N)$. For example, by [42, Proposition 6], $\pi_1(N)$ is often infinite and [42, Table 1] provides computational evidence that many of the finite groups $\pi_1(N)$ are non-cyclic.

Consider the polyhedron in Figure 2 with face pairing given by identifying the faces F_i^+, F_i^- $(0 \leq i < n)$. We shall show that the identification of faces results in a 3-complex M whose 2-skeleton is the presentation complex of $\mathcal{H}(r,n)$, so M has one 0-cell, n 1-cells, n 2-cells, and one 3-cell, so is a manifold by [49, Theorem I, Section 60].

Let $[S,N]_i$ denote the arc of the face F_i^+ with initial vertex S and terminal vertex N. All the arcs labeled x_0 are contained in the following cycle (of length r):

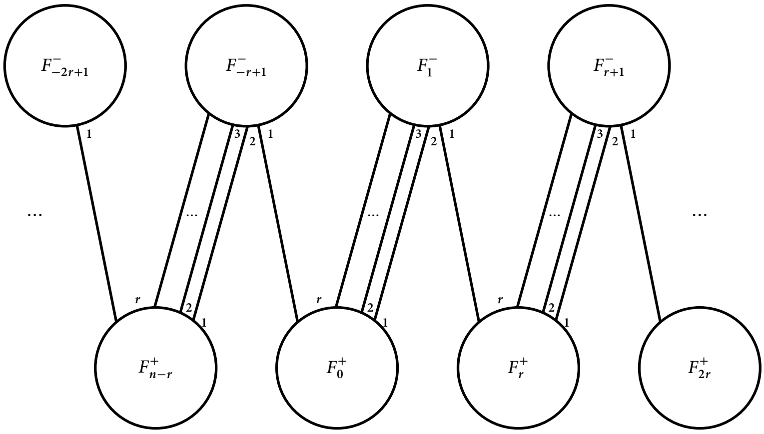

Figure 3: Heegaard diagram for the manifold $M(r,n)$.

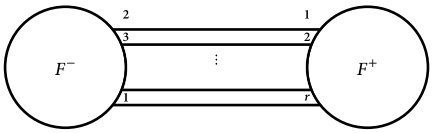

Figure 4: Heegaard diagram for the manifold $M(r,n)/\rho$.

$$[S,N]_0 \xrightarrow{F_{n-(r-1)}} [w^2_{n-(r-1)}, w^1_{n-(r-1)}] \xrightarrow{F_{n-(r-2)}} [w^3_{n-(r-2)}, w^2_{n-(r-2)}]$$
$$\xrightarrow{F_{n-(r-3)}} [w^4_{n-(r-3)}, w^3_{n-(r-3)}] \xrightarrow{F_{n-(r-4)}} \cdots \xrightarrow{F_{n-2}} [w^{r-1}_{n-2}, w^{r-2}_{n-2}]$$
$$\xrightarrow{F_{n-1}} [N, w^{r-1}_{n-1}] \xrightarrow{F_0} [S,N]_0.$$

All the vertices that are a vertex (either initial or terminal) of an arc labeled x_0 are contained in the (induced) cycle of the initial vertices in the arcs above, so in the resulting complex M these vertices are identified. In particular, vertices N, S are identified, and (by comparing initial vertices) for $1 < j < r$ vertices w^{r-j+1}_{n-j} are identified with S. Therefore, for each $0 \leq i < n, 1 < j < r$ the vertex $w^j_i = \theta^{i+r-j+1}(w^j_{n-(r-j+1)})$ is identified with S. Therefore all the vertices of the polyhedron are identified. Thus the quotient M has one 3-cell, n 2-cells, n 1-cells, and one 0-cell, and since the boundaries of the 2-cells spell the relators of $\mathcal{H}(r,n)$, it follows that M is a closed, oriented 3-manifold, and $\mathcal{H}(r,n)$ is a spine of M.

As in (for example) [57, proof of Proposition 2], a Heegaard diagram for M arising from the face pairing polyhedron is given in Figure 3. This diagram has a rotational symmetry ρ of order n cyclically permuting the faces $F^+_i \to F^+_{i+r}$ and $F^-_i \to F^-_{i+r}$. The quotient of M by ρ is a 3-orbifold in which the image of the rotation axis is a singular set. This yields the Heegaard diagram for M/ρ in Figure 4, which is the canonical diagram of the lens space $L(r,1)$. Hence the manifold M is an n-fold cyclic branched cover of $L(r,1)$.

4 Cyclic presentations $\mathcal{G}^{k/l}(n,f)$ as spines of type (II.7)

For $n \geq 2$, $k,l \geq 1$, $0 \leq f < n$, let $\mathcal{G}^{k/l}(n,f)$ be the cyclic presentation

$$\mathcal{G}_n((y_0 y_f \dots y_{(l-1)f})(y_{lf+1} y_{(l+1)f+1} \dots y_{(l+(k-1))f+1})(y_2 y_{2+f} \dots y_{2+(l-1)f})^{-1})$$

and let $G^{k/l}(n,f)$ be the group it defines. When $n \geq 4$, the Whitehead graph of $\mathcal{G}^{k/l}(n,f)$ is planar if and only if n is even and either $fk \equiv 0 \bmod n$ or $fk \equiv 2 \bmod n$, in which case it is of type (II.7) if $(k,l) \neq (1,1)$ and is of type (II.11) if $k = l = 1$; see Figure 5. Our main result is the following.

Theorem A *Let $n \geq 4$, $fk \not\equiv 2 \bmod n$, and $(k,l) \in \{(k,1),(1,l),(5,2),(2,5)\}$. Then $\mathcal{G}^{k/l}(n,f)$ is a spine of a closed, oriented 3-manifold $M^{k/l}(n,f)$ if and only if n and f are even and $fk \equiv 0 \bmod n$.*

Our motivation for studying these presentations (and, in particular, under the condition $fk \equiv 0 \bmod n$ but not under the condition $fk \equiv 2 \bmod n$) stems from the following connection to the Fractional Fibonacci groups. The presentations $\mathcal{G}^{k/l}(n,0)$ are the Fractional Fibonacci group presentations

$$\mathcal{F}^{k/l}(n) = \mathcal{G}_n(x_0^l x_1^k x_2^{-l})$$

introduced in [40, 41, 55, 56], generalizing the Fibonacci group presentations $\mathcal{F}^{1/1}(n)$. For even n and coprime integers $k,l \geq 1$, the Fractional Fibonacci group $F^{k/l}(n)$ is a 3-manifold group [40, Theorem 4.1], [41, Section 4], [55, Section 2], [56, Section 2]. If $n \geq 2$ is even then $\mathcal{F}^{1/1}(n)$ was shown to be a spine of a closed, oriented, 3-manifold in [16],[22],[23],[24]. More generally, if $k,l \geq 1$ are coprime and n is even, then setting $s_i = q_i = l$, $p_i = -r_i = k$ in [47, Theorem 6] gives that $\mathcal{F}^{k/l}(n)$ is a spine of a closed, oriented, 3-manifold. If n is odd then $\mathcal{F}^{k/l}(n)$ is not a spine, since its Whitehead graph is non-planar. The question as to when, for odd n, $F^{k/l}(n)$ is a 3-manifold group is considered in [26, Theorem 3] for the case $k = l = 1$, and in [17, Theorem 6.2] for the case $l = 1$ and the case $n = 3$. The abelianization $F^{k/1}(n)^{\mathrm{ab}}$ is obtained in [41, Lemma 1], [46, Corollary 4.3].

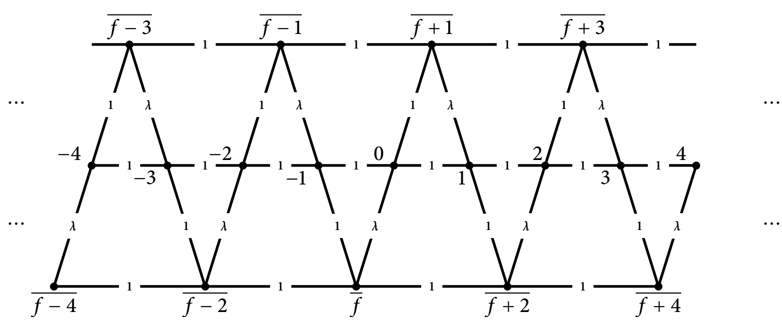

Figure 5: Whitehead graph for $\mathcal{G}^{k/l}(n,f)$ (where $\lambda = 2l + k - 3$, n even and $fk \equiv 0$ or $2 \bmod n$).

The *shift extension* $E^{k/l}(n) = F^{k/l}(n) \rtimes_\theta \mathbb{Z}_n$ of $F^{k/l}(n)$ by $\mathbb{Z}_n = \langle t \mid t^n \rangle$ has the presentation

$$E^{k/l}(n) = \langle x, t \mid t^n, x^l t x^k t x^{-l} t^{-2} \rangle, \tag{1}$$

where the second relator is obtained by rewriting the relators of $F^{k/l}(n)$ in terms of the substitutions $x_i = t^i x t^{-i}$ (see, for example, [31, Theorem 4]). As set out in [5, Section 2], for $0 \le f < n$ such that $fk \equiv 0 \bmod n$, there is a retraction $\nu^f : E^{k/l}(n) \to \mathbb{Z}_n = \langle t \mid t^n \rangle$ given by $\nu^f(t) = t$, $\nu^f(x) = t^f$. The kernel, $\ker(\nu^f)$, has a presentation with generators $y_i = t^i x t^{-(i+f)}$ $(0 \le i < n)$ and relators that are rewrites of conjugates of the second relator of $E^{k/l}(n)$ by powers of t, and so has the cyclic presentation $\mathcal{G}^{k/l}(n,f)$. In particular (under the condition $fk \equiv 0 \bmod n$),

$$\begin{aligned}
\mathcal{G}^{k/1}(n,0) &= \mathcal{G}_n(y_0 y_1^k y_2^{-1}) = \mathcal{F}^{k/1}(n),\\
\mathcal{G}^{1/l}(n,f) &= \mathcal{G}^{1/l}(n,0) = \mathcal{G}_n(y_0^l y_1 y_2^{-l}) = \mathcal{F}^{1/l}(n),\\
\mathcal{G}^{5/2}(n,f) &= \mathcal{G}_n((y_0 y_f)(y_{2f+1} y_{3f+1} y_{4f+1} y_1 y_{f+1})(y_2 y_{2+f})^{-1}),\\
\mathcal{G}^{2/5}(n,f) &= \mathcal{G}_n((y_0 y_f y_0 y_f y_0)(y_{f+1} y_1)(y_2 y_{2+f} y_2 y_{2+f} y_2)^{-1}).
\end{aligned}$$

The shift extension

$$\begin{aligned}
G^{k/l}(n,f) \rtimes_\theta \mathbb{Z}_n &= \langle y, t \mid t^n, (yt^f)^l t (yt^f)^k t^{-fk+1} (yt^f)^{-l} t^{-2} \rangle\\
&= \langle x, t \mid t^n, x^l t x^k t^{-fk+1} x^{-l} t^{-2} \rangle
\end{aligned}$$

(by setting $x = yt^f$ and eliminating y). In the case $fk \equiv 0 \bmod n$, this coincides with $E^{k/l}(n)$, which is independent of the value of f. This implies, for example, that the order of $G^{k/l}(n,f)$ is independent of f, the shift dynamics of $G^{k/l}(n,f)$ are identical for all values of f [5, Lemma 2.2], and that $G^{k/l}(n,f)$ is (non-elementary) word hyperbolic if and only if $F^{k/l}(n)$ is (non-elementary) word hyperbolic. In Theorem 4.5 we similarly show that, for fixed k, l, n, if two groups $G^{k/l}(n,f_1), G^{k/l}(n,f_2)$ sharing a shift extension are fundamental groups of closed, connected, orientable 3-manifolds then either both manifolds are hyperbolic, or neither are. Observe that (under the hypothesis $fk \equiv 0 \bmod n$), as in Figure 5, the Whitehead graph of $\mathcal{G}^{k/l}(n,f)$ has vertices v_{y_i}, v'_{y_i} and edges $(v_{y_i}, v_{y_{i+1}})$, $(v'_{y_i}, v'_{y_{i+2}})$, $(v_{y_i}, v'_{y_{i+f}})$ (of multiplicity $2l + k - 3$), and so the Whitehead graph of $\mathcal{G}^{k/l}(n,f)$ is obtained from that of $\mathcal{F}^{k/l}(n) = \mathcal{G}^{k/l}(n,0)$ by replacing each edge (v_{y_i}, v'_{y_j}) by the edge $(v_{y_i}, v'_{y_{j+f}})$.

Remark 4.1 In the general setting of [5], if $\nu^f : \langle x, t \mid t^n, W(x,t) \rangle \to \langle t \mid t^n \rangle$ $(n \ge 2)$ is a retraction given by $\nu^f(t) = t$ and $\nu^f(x) = t^f$ then $\ker(\nu^f)$ has a cyclic presentation $\mathcal{G}_n(\rho^f(W(x,t)))$ where $\rho^f(W(x,t))$ is as defined in [5, p. 159]. Analysis of the length two subwords $x_{u(\iota)}^{\varepsilon_\iota} x_{u(\iota+1)}^{\varepsilon_{\iota+1}}$ of $\rho^f(W(x,t))$ (where $\varepsilon_\iota = \pm 1, \varepsilon_{\iota+1} = \pm 1$) provides a description of the edge set of the Whitehead graph of $\mathcal{G}_n(\rho^f(W(x,t)))$. Given two such retractions ν^{f_1}, ν^{f_2} this description yields that the Whitehead graph of $\mathcal{P}_2 = \mathcal{G}_n(\rho^{f_2}(W(x,t)))$ is obtained from that of $\mathcal{P}_1 = \mathcal{G}_n(\rho^{f_1}(W(x,t)))$ by replacing each edge (v_{x_i}, v'_{x_j}) by the edge $(v_{x_i}, v'_{x_{j+(f_2-f_1)}})$, leaving all other edges unchanged. In particular, if the Whitehead graph of $\mathcal{P}_1$ is one of the types in the planarity classification of [27] then the Whitehead graph of $\mathcal{P}_2$ is of the same type.

In contrast, for fixed k, l, n, two groups $G^{k/l}(n, f_1), G^{k/l}(n, f_2)$ sharing a shift extension will typically be non-isomorphic. We give examples of this in Example 4.2 and Lemma 4.3. Further, in the case $fk \equiv 0 \bmod n$, we also have that the presentation $\mathcal{G}^{k/l}(n, f)$ has a planar Whitehead graph. Since the group $G^{k/l}(n, f)$ shares its shift extension with that of $F^{k/l}(n)$, these properties combined suggest that the geometric properties of $F^{k/l}(n)$ and $\mathcal{F}^{k/l}(n)$ may be inherited by $G^{k/l}(n, f)$ and $\mathcal{G}^{k/l}(n, f)$. However, while [47, Theorem 6] implies that $\mathcal{F}^{k/l}(n)$ is a spine for all coprime k, l and even $n \geq 2$, Lemma 4.4 will show that, additionally, f must be even for $\mathcal{G}^{k/l}(n, f)$ to be a spine of a closed, oriented, 3-manifold. This demonstrates that, generally, given two retractions $\nu^{f_1}, \nu^{f_2} : \langle x, t \mid t^n, W(x, t)\rangle \to \langle t \mid t^n\rangle$ whose kernels have cyclic presentations $\mathcal{P}_1, \mathcal{P}_2$, as in Remark 4.1, it is not the case that $\mathcal{P}_1$ is a spine of a closed, oriented 3-manifold if and only if $\mathcal{P}_2$ is such a spine. In the other planar case, $fk \equiv 2 \bmod n$, the groups $G^{k/l}(n, f)$ do not share a shift extension with $F^{k/l}(n)$, so there is no apriori reason to expect similar geometric properties to hold.

Example 4.2 (Non-cyclic finite groups.) The groups $G^{3/1}(3, 0), G^{3/1}(3, 1)$, are solvable of order 3528, and derived lengths 3, and 4, respectively ([30, p. 282], [41, Remark 1]) and hence are non-isomorphic. The groups $G^{3/2}(3, 0), G^{3/2}(3, 1)$, are solvable of order 504, and derived lengths 2 and 3, respectively (using [19]) and hence are non-isomorphic.

In Lemma 4.3, we use the fact that the order of the abelianization $G^{k/l}(n, f)^{\mathrm{ab}}$ is given by $|\mathrm{Res}(p_f(t), t^n - 1)|$, if this is non-zero, and is infinite otherwise, where

$$p_f(t) = (1 - t^2)(1 + t^f + \cdots + t^{(l-1)f}) + t^{lf+1}(1 + t^f + \cdots + t^{(k-1)f})$$

is the *representer polynomial* of $G^{k/l}(n, f)$, and $\mathrm{Res}(\cdot, \cdot)$ denotes the resultant [32, p. 82]. As we will only be interested in the absolute values of resultants (and not the sign), to avoid repetitive use of modulus signs we will take $\mathrm{Res}(\cdot, \cdot)$ to mean $|\mathrm{Res}(\cdot, \cdot)|$.

Lemma 4.3 *Let $n, k, l \geq 1$ where n, k are even and* $\gcd(k, l) = 1$ *then* $F^{k/l}(n) = G^{k/l}(n, 0) \not\cong G^{k/l}(n, n/2)$.

Proof For any $0 \leq f < n$

(2) $$\mathrm{Res}(p_f(t), t^n - 1) = \mathrm{Res}(p_f(t), t^{n/2} - 1) \cdot \mathrm{Res}(p_f(t), t^{n/2} + 1).$$

The hypotheses imply that l is odd so we have

$$\begin{aligned}\mathrm{Res}(p_{n/2}(t), t^{n/2} - 1) &= \mathrm{Res}(l(1 - t^2) + kt, t^{n/2} - 1)\\ &= \mathrm{Res}(p_0(t), t^{n/2} - 1),\end{aligned}$$

and, setting $\alpha, \bar{\alpha} = (k \pm \sqrt{k^2 + 4l^2})/(2l)$,

$$\begin{aligned}\mathrm{Res}(p_0(t), t^{n/2} + 1) &= \mathrm{Res}\left(l(t - \alpha)(t - \bar{\alpha}), t^{n/2} + 1\right)\\ &= l^{n/2}(\alpha^{n/2} + 1)(\bar{\alpha}^{n/2} + 1)\\ &= l^{n/2}\left((-1)^{n/2} + 1 + (\alpha^{n/2} + \bar{\alpha}^{n/2})\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathrm{Res}(p_{n/2}(t), t^{n/2}+1) &= \mathrm{Res}\left(l(1-t^2)+kt(1+t^{n/2})/2, t^{n/2}+1\right)\\ &= l^{n/2}\cdot 2(1+(-1)^{n/2}).\end{aligned}$$

Therefore $\mathrm{Res}(p_0(t), t^{n/2}+1) \neq \mathrm{Res}(p_{n/2}(t), t^{n/2}+1)$ so, by equation (2), $\mathrm{Res}(p_0(t), t^n-1) \neq \mathrm{Res}(p_{n/2}(t), t^n-1)$. Hence $|G^{k/l}(n,0)^{\mathrm{ab}}| \neq |G^{k/l}(n,n/2)^{\mathrm{ab}}|$, and the result follows. ■

Lemma 4.4 *Suppose $n \geq 4$, $k, l \geq 1$, $0 \leq f < n$ where n is even, $fk \equiv 0 \bmod n$, and $\gcd(k,l) = 1$. If f is odd then $\mathcal{G}^{k/l}(n,f)$ is not a spine of a closed, oriented 3-manifold.*

Proof First note that if f is odd, then the hypotheses imply that l is odd. The Whitehead graph of $\mathcal{G}^{k/l}(n,f)$ has a planar embedding, as shown in Figure 5. It has two $n/2$-gons $v'_{y_0} - v'_{y_2} - \cdots - v'_{y_{n-2}} - v'_{y_0}$ and $v'_{y_1} - v'_{y_3} - \cdots - v'_{y_{n-1}} - v'_{y_1}$, n 3-gons $v_{y_i} - v_{y_{i+1}} - v'_{y_{i+f+1}} - v_{y_i}$, n 4-gons $v_{y_i} - v_{y_{i+1}} - v'_{y_{i+f+2}} - v'_{y_{i+f}} - v_{y_i}$, and λn 2-gons $v_{y_i} - v'_{y_{i+f+1}} - v_{y_i}$, where $\lambda = 2l + k - 3$. It follows that the reduced Whitehead graph is unique up to self homeomorphism of S^2.

If $\mathcal{G}^{k/l}(n,f)$ is a spine of a closed, oriented, 3-manifold then in a putative corresponding face-pairing polyhedron, there are two degree $n/2$ source vertices, N, S, say, where N has outgoing arcs in cyclic order $y_0, y_2, y_4, \ldots, y_{n-2}$ and S has outgoing arcs in cyclic order $y_1, y_3, y_5, \ldots, y_{n-1}$. The 2-cells incident to N are therefore, in cyclic order, $F_0^-, F_2^-, F_4^-, \ldots, F_{n-2}^-$, where the boundary of F_i^- reads, anticlockwise, the i'th relator of $\mathcal{G}^{k/l}(n,f)$. The arc labeled y_2, with initial vertex N, is the first of a path $y_2 - y_{2+f} - \cdots - y_{2+(l-1)f}$. Let v denote the terminal vertex of the last arc in this path. Then v has two incoming arcs labeled $y_{(l-1)f+1}, y_{(l-1)f+2}$, and an outgoing arc labeled y_{lf+3}. Therefore v is a degree 4 vertex, and its remaining arc is outgoing, and labeled y_{lf+1}. The outgoing arcs y_{lf+1}, y_{lf+3} then bound the F_{lf+1}^- face. Since l, f are odd, $lf+1$ is even, so the face pairing contains two F_{lf+1}^- faces, a contradiction. ■

In Figures 6, 7, 8, and 9, we present face-pairing polyhedra with n pairs of faces F_i^+, F_i^- $(0 \leq i < n)$ of opposite orientation whose boundaries spell the defining relators of the relevant presentation $\mathcal{G}^{k/l}(n,f)$. In the proof of Theorem A, we show that the identification of faces F_i^+, F_i^- results in a 3-complex that satisfies the Seifert–Threlfall condition, and so is a manifold whose spine is the presentation complex of $\mathcal{G}^{k/l}(n,f)$. (Note that, in these figures, the condition that f is even is necessary for each relator to appear as the label of a pair of oppositely oriented faces.) The diagrams have different forms depending on whether $l > k$ or $k > l$. The figures should provide the enthusiastic reader with sufficient information to construct face-pairing polyhedra in the general cases. The values $\{k,l\} = \{2,5\}$ were selected so that $k, l, |k-l|$ are distinct, to make it straightforward to infer paths of lengths k, l, and $|k-l|$. (For example, in Figure 7 the path from v_{2f} to u_2 has length $3 = k - l$, and in Figure 9 the path from w_f to v_{f-1} has length $3 = l - k$.) We expect the Seifert–Threlfall condition to hold in the general case for coprime k, l; however, the corresponding analysis to check this would be highly technical. Since our goal is to exhibit new cyclic presentations that arise as spines of closed 3-manifolds, whose Whitehead graphs are of type (II.7), rather than to be exhaustive, we have not sought to do this.

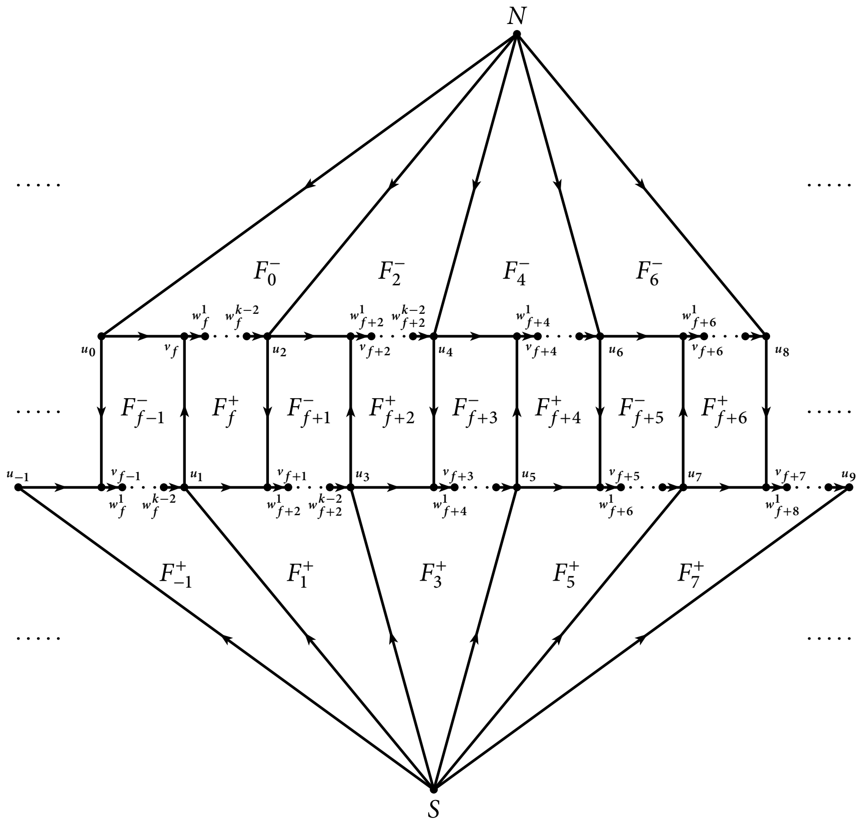

Figure 6: Face pairing polyhedron for $\mathcal{G}^{k/1}(n, f)$.

Proof of Theorem A We may assume n is even and $fk \equiv 0$, for otherwise the Whitehead graph of $\mathcal{G}^{k/l}(n, f)$ is not planar [27] (noting that $fk \not\equiv 2 \bmod n$, by hypothesis). Moreover, by Lemma 4.4 we may assume f is even.

Case 1: $(k, l) = (k, 1)$. Consider the face pairing polyhedron depicted in Figure 6 (where the arc labels can be deduced from the position of the unique source in the boundary of each face) with face pairing given by identifying the faces F_i^+, F_i^- $(0 \leq i < n)$. We shall show that the identification of faces results in a 3-complex M whose 2-skeleton is the presentation complex of $\mathcal{G}^{k/l}(n, f)$, so M has one 0-cell, n 1-cells, n 2-cells, and one 3-cell, so is a manifold by [49, Theorem I, Section 60].

All the arcs labeled x_0 are contained in the following cycle:

$$[N, u_0] \xrightarrow{F_0} [u_{1-f}, v_0] \xrightarrow{F_{-1}} [w_f^{k-2}, u_1] \xrightarrow{F_{f-1}} [w_{2f}^{k-3}, w_{2f}^{k-2}]$$
$$\xrightarrow{F_{2f-1}} [w_{3f}^{k-4}, w_{3f}^{k-3}] \xrightarrow{F_{3f-1}} \cdots \xrightarrow{F_{(k-3)f-1}} [w_{(k-2)f}^1, w_{(k-2)f}^2]$$
$$\xrightarrow{F_{(k-2)f-1}} [v_{(k-1)f-2}, w_{(k-1)f}^1] \xrightarrow{F_{(k-1)f-1}} [v_{(k-1)f-3}, w_0^1] \xrightarrow{F_{-2}} [N, u_0].$$

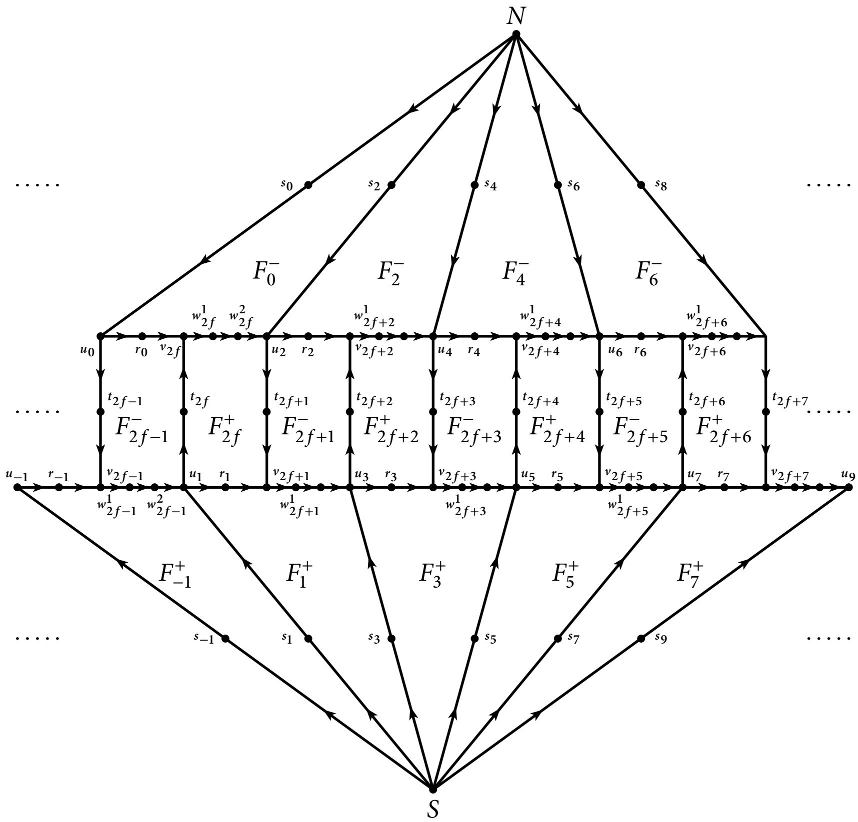

Figure 7: Face pairing polyhedron for $\mathcal{G}^{5/2}(n, f)$.

Therefore, in the resulting complex M all the arcs labeled x_0 are identified. Moreover, all the vertices that are a vertex (either initial or terminal) of an arc labeled x_0 are contained in the (induced) cycle of the initial vertices in the arcs above, so these vertices are identified in M. By applying the shift θ^{2j} $(0 \le j < n/2)$ to the above, all the arcs labeled x_{2j} are identified and all the vertices of such arcs are identified. That is, the vertices $N, u_{2j+(1-f)}, w^{k-2}_{2j+f}, w^{k-3}_{2j+2f}, \dots, w^1_{2j+(k-2)f}, v_{2j+(k-1)f-2}$ are identified; equivalently, N, $u_{2j+1}, w^1_{2j}, \dots w^{k-2}_{2j}, v_{2j}$ $(0 \le j < n/2)$ are identified.

All the arcs labeled x_1 are contained in the following cycle:

$$[S, u_1] \xrightarrow{F_1} [u_{2-f}, v_1] \xrightarrow{F_0} [w^{k-2}_{f+1}, u_2] \xrightarrow{F_f} [w^{k-3}_{2f+1}, w^{k-2}_{2f+1}]$$
$$\xrightarrow{F_{2f}} [w^{k-4}_{3f+1}, w^{k-3}_{3f+1}] \xrightarrow{F_{3f}} \cdots \xrightarrow{F_{(k-3)f}} [w^1_{(k-2)f+1}, w^2_{(k-2)f+1}]$$
$$\xrightarrow{F_{(k-2)f}} [v_{(k-1)f-1}, w^1_{(k-1)f+1}] \xrightarrow{F_{(k-1)f}} [v_{(k-1)f-2}, w^1_1] \xrightarrow{F_{-2}} [S, u_1].$$

Therefore, in M all the arcs labeled x_1 are identified. Moreover, all the vertices that are a vertex (either initial or terminal) of an arc labeled x_1 are contained in the

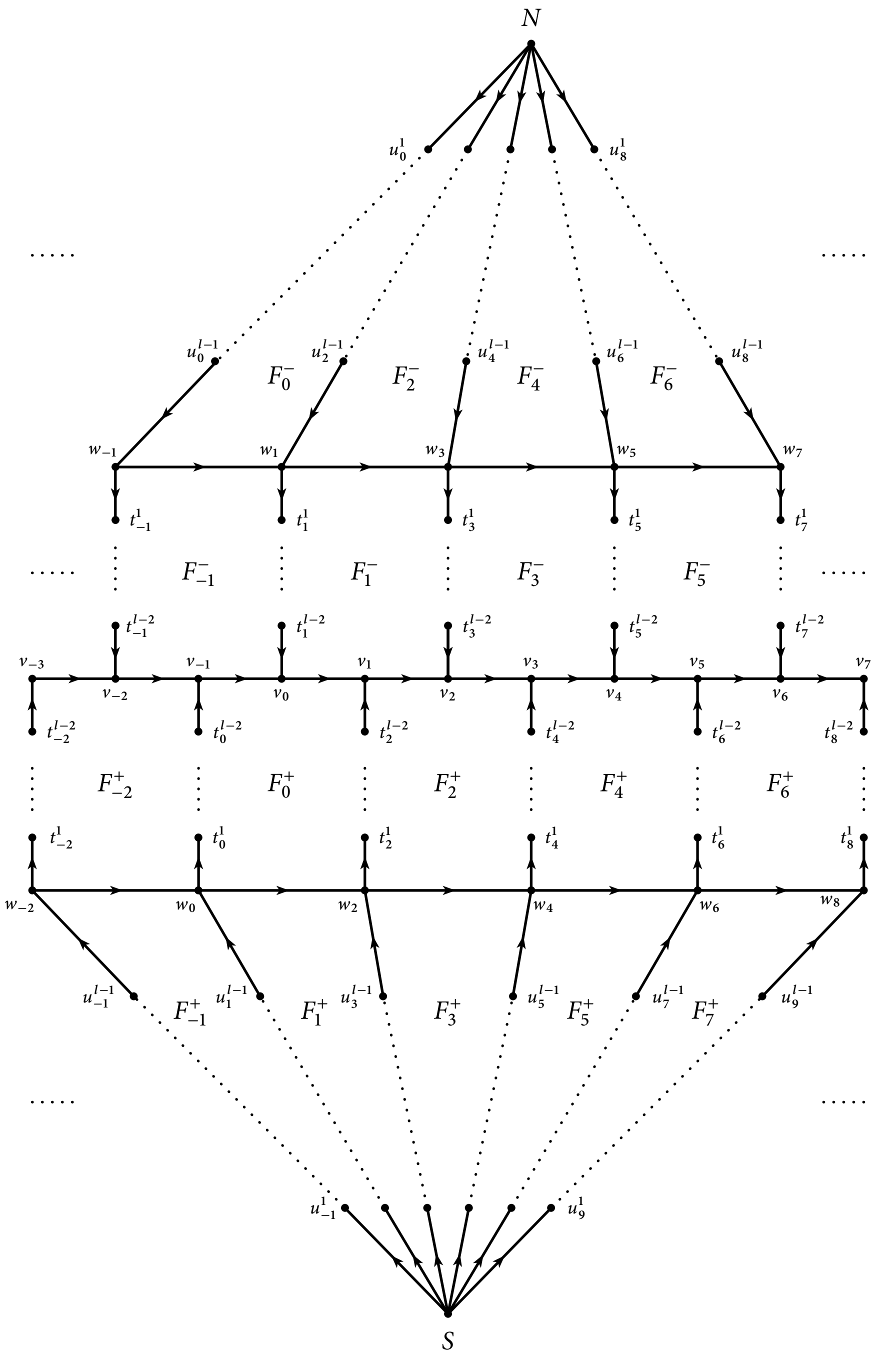

Figure 8: Face pairing polyhedron for $\mathcal{G}^{1/l}(n,0) = \mathcal{F}^{1/l}(n)$.

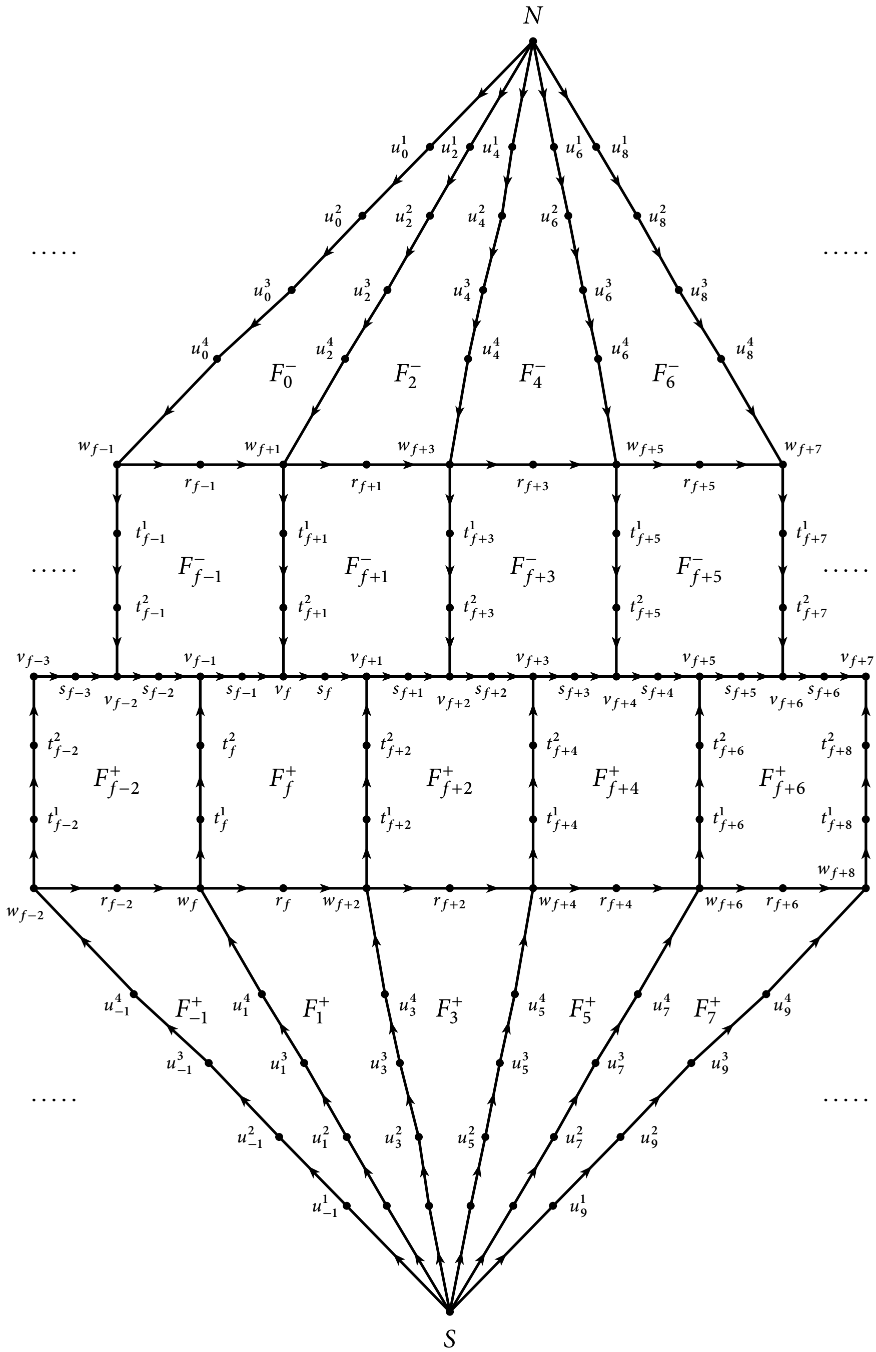

Figure 9: Face pairing polyhedron for $\mathcal{G}^{2/5}(n,f)$.

(induced) cycle of the initial vertices in the arcs above, so these vertices are identified in M. By applying the shift θ^{2j} $(0 \leq j < n/2)$ to the above, all the arcs labeled x_{2j+1} are identified and all the vertices of such arcs are identified. That is, the vertices $S, u_{2j+(2-f)}, w^{k-2}_{2j+f+1}, w^{k-3}_{2j+2f+1}, \ldots, w^{1}_{2j+(k-2)f+1}, v_{2j+(k-1)f-1}$ are identified; that is S, $u_{2j}, w^1_{2j+1}, \ldots w^{k-2}_{2j+1}, v_{2j+1}$ $(0 \leq j < n/2)$ are identified. Moreover, examining the terminal vertices of the first cycle above we see that u_0 and u_1 are identified. Therefore the vertices induced from the first cycle are identified with the vertices from the second cycle, so all vertices of the polyhedron are identified. Thus M has one 3-cell, n 2-cells, n 1-cells, and one 0-cell, and since the boundaries of the 2-cells spell the relators of $\mathcal{G}^{1/l}(n, 0)$, it follows that $\mathcal{G}^{1/l}(n, 0)$ is a spine of a closed 3-manifold, as required.

Case 2: $(k, l) = (5, 2)$. Consider the face pairing polyhedron depicted in Figure 7 (where the arc labels can be deduced from the position of the unique source in the boundary of each face) with face pairing given by identifying the faces F^+_i, F^-_i $(0 \leq i < n)$. All the arcs labeled x_0 are contained in the following cycle:

$$\begin{aligned}[N, s_0] &\xrightarrow{F_0} [u_{1-2f}, t_0] \xrightarrow{F_{-1}} [w^1_{2f-1}, w^2_{2f-1}] \xrightarrow{F_{2f-1}} [r_{2f-1}, v_{4f-1}] \\ &\xrightarrow{F_{4f-2}} [s_{4f}, u_{4f}] \xrightarrow{F_{4f}} [t_{4f}, v_{4f}] \xrightarrow{F_{4f-1}} [w^2_{f-1}, u_{-f+1}] \\ &\xrightarrow{F_{f-1}} [v_{3f-1}, w^1_{3f-1}] \xrightarrow{F_{3f-1}} [u_{3f-1}, r_{3f-1}] \xrightarrow{F_{-2}} [N, s_0],\end{aligned}$$

and all the arcs labeled x_1 are contained in the following cycle:

$$\begin{aligned}[S, s_1] &\xrightarrow{F_1} [u_{2-2f}, t_1] \xrightarrow{F_0} [w^1_{2f}, w^2_{2f}] \xrightarrow{F_{2f}} [r_{2f}, v_{4f}] \\ &\xrightarrow{F_{4f-1}} [s_{4f+1}, u_{4f+1}] \xrightarrow{F_{4f+1}} [t_{4f+1}, v_{4f+1}] \xrightarrow{F_{4f}} [w^2_f, u_{-f+2}] \\ &\xrightarrow{F_f} [v_{3f}, w^1_{3f}] \xrightarrow{F_{3f}} [u_{3f}, r_{3f}] \xrightarrow{F_{-1}} [S, s_1].\end{aligned}$$

The proof then proceeds as in Case 1.

Case 3: $(k, l) = (1, l)$. Consider the face pairing polyhedron depicted in Figure 8. All the arcs labeled x_0 are contained in the following cycle:

$$\begin{aligned}[N, u^1_0] &\xrightarrow{F_0} [w_0, t^1_0] \xrightarrow{F_{-2}} [u^1_0, u^2_0] \xrightarrow{F_0} [t^1_0, t^2_0] \xrightarrow{F_{-2}} [u^2_0, u^3_0] \\ &\xrightarrow{F_0} [t^2_0, t^3_0] \longrightarrow \cdots \xrightarrow{F_{-2}} [u^{l-2}_0, u^{l-1}_0] \xrightarrow{F_0} [t^{l-2}_0, v_{-1}] \\ &\xrightarrow{F_{-2}} [u^{l-1}_0, w_{-1}] \xrightarrow{F_0} [v_{-1}, v_0] \xrightarrow{F_{-1}} [w_{-2}, w_0] \xrightarrow{F_{-2}} [N, u^1_0],\end{aligned}$$

and all the arcs labeled x_1 are contained in the following cycle:

$$\begin{aligned}[S, u^1_1] &\xrightarrow{F_1} [w_1, t^1_1] \xrightarrow{F_{-1}} [u^1_1, u^2_1] \xrightarrow{F_1} [t^1_1, t^2_1] \xrightarrow{F_{-1}} [u^2_1, u^3_1] \\ &\xrightarrow{F_1} [t^2_1, t^3_1] \longrightarrow \cdots \xrightarrow{F_{-1}} [u^{l-2}_1, u^{l-1}_1] \xrightarrow{F_1} [t^{l-2}_1, v_0] \\ &\xrightarrow{F_{-1}} [u^{l-1}_1, w_0] \xrightarrow{F_1} [v_0, v_1] \xrightarrow{F_0} [w_{-1}, w_1] \xrightarrow{F_{-1}} [S, u^1_1].\end{aligned}$$

The proof then proceeds as in Case 1.

Case 4: $(k,l) = (2,5)$. Consider the face pairing polyhedron depicted in Figure 9. All the arcs labeled x_0 are contained in the following cycle:

$$\begin{aligned}[N,u_0^1] &\xrightarrow{F_0} [w_0,t_0^1] \xrightarrow{F_{-2}} [u_0^2,u_0^3] \xrightarrow{F_0} [t_0^2,v_{-1}] \xrightarrow{F_{-2}} [u_0^4,w_{f-1}] \\ &\xrightarrow{F_0} [s_{-1},v_0] \xrightarrow{F_{-1}} [r_{f-2},w_f] \xrightarrow{F_{f-2}} [u_f^1,u_f^2] \xrightarrow{F_f} [t_f^1,t_f^2] \\ &\xrightarrow{F_{f-2}} [u_f^3,u_f^4] \xrightarrow{F_f} [v_{f-1},s_{f-1}] \xrightarrow{F_{f-1}} [w_{2f-2},r_{2f-2}] \xrightarrow{F_{2f-2}} [N,u_{2f}^1],\end{aligned}$$

and all the arcs labeled x_1 are contained in the following cycle:

$$\begin{aligned}[S,u_1^1] &\xrightarrow{F_1} [w_1,t_1^1] \xrightarrow{F_{-1}} [u_1^2,u_1^3] \xrightarrow{F_1} [t_1^2,v_0] \xrightarrow{F_{-1}} [u_1^4,w_f] \\ &\xrightarrow{F_1} [s_0,v_1] \xrightarrow{F_0} [r_{f-1},w_{f+1}] \xrightarrow{F_{f-1}} [u_{f+1}^1,u_{f+1}^2] \xrightarrow{F_{f+1}} [t_{f+1}^1,t_{f+1}^2] \\ &\xrightarrow{F_{f-1}} [u_{f+1}^3,u_{f+1}^4] \xrightarrow{F_{f+1}} [v_f,s_f] \xrightarrow{F_f} [w_{2f-1},r_{2f-1}] \xrightarrow{F_{2f-1}} [S,u_{2f+1}^1].\end{aligned}$$

The proof then proceeds as in Case 1. ■

We now consider the structures of the manifolds $M^{k/l}(n,f)$ of Theorem A.

Theorem 4.5 *Let $n \geq 2$, $k,l \geq 1$, $0 \leq f < n$, and $fk \equiv 0 \bmod n$, where n,f are even, and suppose $(k,l) \in \{(k,1),(1,l),(5,2),(2,5)\}$. Then $M^{k/l}(n,f)$ is hyperbolic if and only if $M^{k/l}(n,0)$ is hyperbolic.*

Proof By [17, Corollary 3.3] the shift $\theta_{G^{k/l}(n,0)}$ has order n, and hence the shift $\theta_{G^{k/l}(n,f)}$ also has order n. Suppose $M^{k/l}(n,0)$ is hyperbolic. Then since, by Theorem A, $M^{k/l}(n,0)$ is a closed, connected, orientable 3-manifold, $G^{k/l}(n,0)$ is a subgroup of $\mathrm{Isom}^+(H^3) \cong PSL(2,\mathbb{C})$. As in the proof of [40, Theorem 3.1] (see also [12, Theorem 3.1], [14, Theorem 3.1], [3, Theorem 3.1]) Mostow rigidity implies that $E^{k/l}(n)$ is a subgroup of $PSL(2,\mathbb{C})$. Hence $G^{k/l}(n,f)$ is a subgroup of $PSL(2,\mathbb{C})$, and so $M^{k/l}(n,f)$ is hyperbolic. Repeating the argument with the roles of $G^{k/l}(n,0)$ and $G^{k/l}(n,f)$ interchanged and the roles of $M^{k/l}(n,0)$ and $M^{k/l}(n,f)$ interchanged proves the converse. ■

Remark 4.6 The argument of the proof of Theorem 4.5 holds in the more general setting of [5]. Namely, if $\nu^{f_1},\nu^{f_2} : \langle x,t \mid t^n, W(x,t)\rangle \to \langle t \mid t^n\rangle$ $(n \geq 2)$ are retractions given by $\nu^{f_1}(t) = \nu^{f_2}(t) = t$ and $\nu^{f_1}(x) = t^{f_1}, \nu^{f_2}(x) = t^{f_2}$ then $K_1 = \ker(\nu^{f_1})$, $K_2 = \ker(\nu^{f_2})$ have cyclic presentations $\mathcal{G}_n(\rho^{f_1}(W(x,t)))$, $\mathcal{G}_n(\rho^{f_2}(W(x,t)))$, respectively (as defined in [5, p. 159]). Suppose that the shift automorphism has order n for either (and hence both) of these groups, and that K_1,K_2 are fundamental groups of closed, connected, orientable 3-manifolds M_1,M_2, respectively. Then M_1 is hyperbolic if and only if M_2 is hyperbolic. (Theorem 4.5 corresponds to the case $W(x,t) = x^l t x^k t x^{-l} t^{-2}$, as in (1), where $(k,l) \in \{(k,1),(1,l),(5,2),(2,5)\}$.) We record that, since K_1 is finite if and only if K_2 is finite, it is also the case that M_1 is spherical if and only if M_2 is spherical.

The significance of Theorem 4.5 is that the manifolds $M^{k/l}(n,0)$ are known to be hyperbolic in many cases, as we now describe. By [56, Corollary 2.1], if $n \geq 6$ is even then $M^{k/l}(n,0)$ is hyperbolic for all but finitely many pairs of coprime integers k, l, and it is conjectured [56, p. 658] that $n = 6$ and $k = l = 1$ is the only non-hyperbolic case (which is an affine Riemannian manifold by [22, Proposition 6]). Moreover, $M^{k/l}(n,0)$ is hyperbolic in each of the following cases: $k \geq 2, l = 1, n \geq 6$ and even [41, Theorem 3]; $k = 1, l \geq 2, n \geq 6$ and even [56, Corollary 3.5]; $k = l = 1, n \geq 8$ and even [22, Theorem C]. If $n = 4$ and $k = 1$ then $M^{k/l}(n,0) = M^{1/l}(4,0)$ is the lens space $L(4l^2+1, 2l)$ and $G^{1/l}(4,0) \cong \mathbb{Z}_{4l^2+1}$ by [56, Corollary 3.4]. For $k \geq 2$ the manifolds $M^{k/1}(4,0)$ are not hyperbolic and are described in [41, pp. 170–171]. In the next theorem we consider the manifolds $M^{k/l}(4,f)$ where $k \geq 2$.

Theorem 4.7 *Let $k \geq 2, l \geq 1$, where $\gcd(k,l) = 1$ and suppose $fk \equiv 0 \bmod 4$. Then $G^{k/l}(4,f)$ contains a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup and hence $M^{k/l}(4,f)$ is not hyperbolic.*

Proof We show that the subgroup A of $E = E^{k/l}(4)$ generated by x^k, tx^kt^{-1} is free abelian of rank 2. Then, since $\nu^f(x^k) = 1$ and $\nu^f(tx^kt^{-1}) = 1$, $A = \mathbb{Z} \oplus \mathbb{Z}$ is a subgroup of $\ker(\nu^f) = G^{k/l}(4,f)$, and hence $M^{k/l}(4,f)$ is not hyperbolic by [6, Theorem 3.3].

Let $G = \ker(\nu^0)$, so $G = F^{k/l}(4)$ and is generated by $x_i = t^ixt^{-i}$, and let $G_1 = \langle x_0, x_2 \mid x_0^k x_2^k \rangle$, $G_2 = \langle x_1, x_3 \mid x_1^k x_3^k \rangle$, $H = \langle a, b \mid ab = ba \rangle \cong \mathbb{Z}^2$. Let $\phi_1 : H \to G_1$, $\phi_2 : H \to G_2$, be given by $\phi_1(a) = x_0^k$, $\phi_1(b) = x_0^{-l}x_2^l$, $\phi_2(a) = x_3^{-l}x_1^l$, $\phi_2(b) = x_1^k$. We shall show that ϕ_1, ϕ_2 are injections. This implies that G can be expressed as an amalgamated free product $G = G_1 *_H G_2$, and $\phi_1(a) = x_0^k$, $\phi_1(b) = x_0^{-l}x_2^l$ generate a free abelian subgroup of G (compare [41, p. 171], which deals with the case $l = 1$). That is, x_0^k and x_1^k generate a free abelian subgroup of G and so x^k and tx^kt^{-1} generate a free abelian subgroup of E, as required.

We show that ϕ_1 is an injection, the argument for ϕ_2 being similar. Let $\alpha : G_1 \to \mathbb{Z}_k$ be given by $\alpha(x_0) = 1, \alpha(x_2) = 1 \in \mathbb{Z}_k$. Then $K = \ker(\alpha)$ is generated by $z = x_0^k$, $g_i = x_0^i x_2 x_0^{-(i+1)}$ $(0 \leq i \leq k-2)$, $g_{k-1} = x_0^{k-1}x_2$, and has a presentation

$$\langle z, g_0, \ldots, g_{k-1} \mid zg_0g_1 \ldots g_{k-2}g_{k-1}, zg_1g_2 \ldots g_{k-1}g_0, \ldots, zg_{k-1}g_0 \ldots g_{k-3}g_{k-2} \rangle.$$

Now let $\theta : K \to \mathbb{Z}^k$ be an epimorphism given by $\theta(g_0) = (1,0,\ldots,0)$, $\theta(g_1) = (0,1,\ldots,0), \ldots, \theta(g_{k-1}) = (0,0,\ldots,1)$, and $\theta(z) = (-1,-1,\ldots,-1)$. Let $l = qk + r$ where $q \geq 0$, $0 \leq r < k$ and note that $r \neq 0$, since $\gcd(k,l) = 1$. Then $x_0^{-l}x_2^l = (x_0^k)^{-q}x_0^{-r}(x_2^k)^q x_2^r = (x_0^k)^{-q}x_0^{-r}(x_0^{-k})^q x_2^r = z^{-2q-1}g_{k-r}g_{k-(r-1)} \cdots g_{k-1}$ and $x_2^k = z$. Therefore

$$\begin{aligned}\theta(x_0^{-l}x_2^l) &= (2q+1)(1,1,\ldots,1) + (0,\ldots,0,1,\ldots,1)\\ &= (2q+1,\ldots,2q+1,2q+2,\ldots,2q+2)\end{aligned}$$

and $\theta(x_2^k) = (-1,-1,\ldots,-1)$ so $\theta(x_0^{-l}x_2^l), \theta(x_2^k)$ generate a free abelian subgroup of rank 2 in $\mathbb{Z}^k$. Moreover, in G_1, $x_0^{-l}x_2^l$ and x_2^k commute and so generate a free abelian subgroup of G_1, and hence ϕ_1 is injective. ∎

Acknowledgements I thank Jim Howie for insightful comments on a draft of this article and Alessia Cattabriga for interesting discussions relating to this work, and for bringing pertinent results from [43] and [47] to my attention. I thank the referee for pointing out that [42, Theorem 8] provides a family of cyclic presentations that are 3-manifold spines where the Whitehead graph is of type (I.5), and for other valuable suggestions that improved the paper.

References

[1] N. Ankaralioglu and H. Aydin, *Some Dunwoody parameters and cyclic presentations*. Gen. Math. **16**(2008), no. 2, 85–93.
[2] H. Aydin, I. Gultekyn, and M. Mulazzani, *Torus knots and Dunwoody manifolds*. Sibirsk. Mat. Zh. **45**(2004), no. 1, 3–10.
[3] V. G. Bardakov and A. Y Vesnin, *A generalization of Fibonacci groups*. Algebra Logic **42**(2003), 131–160.
[4] J. Berge, *Heegaard*. https://github.com/3-manifolds/heegaard.
[5] W. A. Bogley, *On shift dynamics for cyclically presented groups*. J. Algebra **418**(2014), 154–173.
[6] P.J. Callahan and A. W. Reid, *Hyperbolic structures on knot complements*. Chaos, Solitons Fract. **9**(1998), no. 4, 705–738. Knot Theory and Its Applications: Expository Articles on Current Research.
[7] A. Cattabriga and M. Mulazzani, *Strongly-cyclic branched coverings of* $(1,1)$*-knots and cyclic presentations of groups*. Math. Proc. Cambridge Philos. Soc. **135**(2003), no. 1, 137–146.
[8] A. Cattabriga and M. Mulazzani, *All strongly-cyclic branched coverings of* $(1,1)$*-knots are Dunwoody manifolds*. J. London Math. Soc. (2) **70**(2004), no. 2, 512–528.
[9] A. Cattabriga and M. Mulazzani, *Representations of* $(1,1)$*-knots*. Fund. Math. **188**(2005), 45–57.
[10] A. Cattabriga, M. Mulazzani, and A. Vesnin, *Complexity, Heegaard diagrams and generalized Dunwoody manifolds*. J. Korean Math. Soc. **47**(2010), no. 3, 585–599.
[11] A. Cavicchioli, F. Hegenbarth, and A. C. Kim, *A geometric study of Sieradski groups*. Algebra Colloq. **5**(1998), no. 2, 203–217.
[12] A. Cavicchioli, D. Repovš, and F. Spaggiari, *Topological properties of cyclically presented groups*. J. Knot Theory Ramif. **12**(2003), no. 2, 243–268.
[13] A. Cavicchioli, F. Hegenbarth, and A. C. Kim, *On cyclic branched coverings of torus knots*. J. Geom. **64**(1999), nos. 1–2, 55–66.
[14] A. Cavicchioli, D. Repovš, and F. Spaggiari, *Families of group presentations related to topology*. J. Algebra **286**(2005), no. 1, 41–56.
[15] A. Cavicchioli, B. Ruini, and F. Spaggiari, *On a conjecture of M. J Dunwoody*. Algebra Colloq. **8**(2001), no. 2, 169–218.
[16] A. Cavicchioli and F. Spaggiari, *The classification of 3-manifolds with spines related to Fibonacci groups*. In: J. Aguadé *et al.* (eds.), Algebraic topology, Proc. Conf., S. Feliu de Guíxols/Spain, Lecture Notes in Mathematics, 1509, Springer Berlin, Heidelberg, 1990, pp. 50–78, 1992.
[17] I. Chinyere and G. Williams, *Fractional Fibonacci groups with an odd number of generators*. Topol. Appl. **312**(2022), Paper No. 108083.
[18] M. J. Dunwoody, *Cyclic presentations and* 3*-manifolds*. In: A. C. Kim and D. L. Johnson (eds.), Groups—Korea '94. Proceedings of the international conference, Pusan, Korea, August 18-25, 1994, Walter de Gruyter, Berlin, 1995.
[19] The GAP Group, *GAP—Groups, algorithms, and programming*, Version 4.13.1, 2024.
[20] L. Grasselli and M. Mulazzani, *Seifert manifolds and* $(1,1)$*-knots*. Sibirsk. Mat. Zh. **50**(2009), no. 1, 28–39.
[21] L. Grasselli and M. Mulazzani, *Genus one 1-bridge knots and Dunwoody manifolds*. Forum Math. **13**(2001), no. 3, 379–397.
[22] H. Helling, A. C. Kim, and J. L. Mennicke, *A geometric study of Fibonacci groups*. J. Lie Theory **8**(1998), no. 1, 1–23.
[23] H. Hilden, M. T. Lozano, and J. M. Montesinos-Amilibia, *On a remarkable polyhedron geometrizing the figure eight knot cone manifolds*. J. Math. Sci. **2**(1995), no. 3, 501–561.
[24] H. M. Hilden, M. T. Lozano, and J. M. Montesinos-Amilibia, *The arithmeticity of the figure eight knot orbifolds*. In: B. Apanasov *et al.* (eds.), Topology '90. Contributions from a research semester in low dimensional topology, held at Ohio State University Columbus, OH (USA), from Feb. through June 1990, Walter de Gruyter, Berlin, 1992, pp. 169–183.
[25] C. Hog-Angeloni and W. Metzler, *Geometric aspects of two-dimensional complexes*. In: C. Hog-Angeloni et al. (eds.), Two-dimensional homotopy and combinatorial group theory,

London Mathematical Society Lecture Notes, 197, Cambridge University Press, Cambridge, 1993, pp. 1–50.
[26] J. Howie and G. Williams, *Fibonacci type presentations and 3-manifolds*. Topol. Appl. **215**(2017), 24–34.
[27] J. Howie and G. Williams, *Planar Whitehead graphs with cyclic symmetry arising from the study of Dunwoody manifolds*. Discrete Math. **343** 2020, no. 12, 112096.18.
[28] K.-W. Jeong, *On the cyclic branched coverings of the 2-bridge knot* $b(17,4)$. Algebra Colloq. **13**(2006), no. 1, 173–180.
[29] K.-W. Jeong and M.-O. Wang, *Notes on more Fibonacci groups*. Algebra Colloq. **15**(2008), no. 4, 699–706.
[30] D. L. Johnson and E. F. Robertson, *Finite groups of deficiency zero*. In: C. T. C. Wall (ed.), Homological group theory (Proc. Sympos Durham, 1977), volume 36 of London Mathematical Society Lecture Notes Series, Cambridge University Press, Cambridge, 1979, pp. 275–289.
[31] D. L. Johnson, J. W. Wamsley, and D. Wright, *The Fibonacci groups*. Proc. London Math. Soc. **29**(1974), no. 3, 577–592.
[32] D.L. Johnson, *Topics in the theory of group presentations*, volume 42 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 1980.
[33] G. Kim, Y. Kim, and A. Vesnin, *The knot* 5_2 *and cyclically presented groups*. J. Korean Math. Soc. **35**(1998), no. 4, 961–980.
[34] S. H. Kim and Y. Kim, *Torus knots and 3-manifolds*. J. Knot Theory Ramif. **13**(2004), no. 8, 1103–1119.
[35] S. H. Kim and Y. Kim, *On the 2-bridge knots of Dunwoody* $(1,1)$*-knots*. Bull. Korean Math. Soc. **48**(2011), no. 1, 197–211.
[36] S. H. Kim and Y. Kim, *On the polynomial of the Dunwoody* $(1,1)$*-knots*. Kyungpook Math. J. **52**(2012), no. 2, 223–243.
[37] S. H. Kim and Y. Kim, *The dual and mirror images of the Dunwoody 3-manifolds*. Int. J. Math. Math. Sci. (2013), 7 Art. ID 103209.
[38] S. H. Kim and Y. Kim, *On the generalized Dunwoody 3-manifolds*. Osaka J. Math. **50**(2013), no. 2, 457–476.
[39] T. Kozlovskaya, *Heegaard splittings of branched cyclic coverings of connected sums of lens spaces*. Bull. Korean Math. Soc. **54**(2017), no. 5, 1851–1857.
[40] C. Maclachlan, Generalisations of Fibonacci numbers, groups and manifolds. In: A.J. Duncan et al. (eds.), Combinatorial and geometric group theory. Proceedings of a workshop held at Heriot-Watt University, London Mathematical Society Lecture Notes Series, 204, Cambridge University Press, Edinburgh, 1993, pp. 233–238.1995.
[41] C. Maclachlan and A. W. Reid, *Generalised Fibonacci manifolds*. Transform. Groups **2**(1997), no. 2, 165–182.
[42] K. McDermott, *On the topology of the groups of type Z*. Topol. Appl. **307**(2022), 107767.
[43] J. Minkus, *The branched cyclic coverings of* 2 *bridge knots and links*. Mem. Amer. Math. Soc. **35**(1982), no. 255, iv+ 68.
[44] M. Mulazzani, $(1,1)$*-knots and Dunwoody manifolds*. In: D. Andrica and P. Blaga (eds.), Recent advances in geometry and topology, Cluj University Press, Cluj-Napoca, 2004, pp. 273–277.
[45] L. Neuwirth, *An algorithm for the construction of 3-manifolds from 2-complexes*. Math. Proc. Cambridge Philos. Soc. **64**(1968), 603–613.
[46] V. Noferini and G. Williams, *Smith forms of matrices in companion rings, with group theoretic and topological applications*. Linear Algebra Appl. **708**(2025), no. 1, 372–404.
[47] B. Ruini and F. Spaggiari, *On the structure of Takahashi manifolds*. Tsukuba J. Math. **22**(1998), no. 3, 723–739.
[48] B. Ruini, F. Spaggiari, and A. Vesnin, *On spines of Seifert fibered manifolds*. Aequationes Math. **65**(2003), nos. 1–2, 40–60.
[49] H. Seifert and W. Threlfall, *A textbook of topology*, volume 89 of Pure and Applied Mathematics, Academic Press Inc., New York, NY, 1980.[Harcourt Brace Jovanovich, Publishers], German edition. With a preface by Joan S. Birman, With "Topology of 3-dimensional fibered spaces" by Seifert, Translated from the German by Wolfgang Heil.
[50] A. J. Sieradski, *Combinatorial squashings, 3-manifolds, and the third homology of groups*. Invent. Math. **84**(1986), 121–139.
[51] F. Spaggiari and A. I. Telloni, *Isometry groups of some Dunwoody manifolds*. Algebra Colloq. **23**(2016), no. 1, 117–128.
[52] F. Spaggiari, *A geometric study of generalized Neuwirth groups*. Forum Math. **18**(2006), no. 5, 803–827.

[53] A. I. Telloni, *Combinatorics of a class of groups with cyclic presentation*. Discrete Math. **310**(2010), no. 22, 3072–3079.
[54] A. Umar, *Some remarks about Fibonacci groups and semigroups*. Commun. Algebra **25**(1997), no. 12, 3973–3977.
[55] A. Y. Vesnin and A. C Kim, *A topological study of the fractional Fibonacci groups*. Universität Bielefeld, Bielefeld, Preprint 96-049, 1996.
[56] A. Y Vesnin and A. C Kim, *Fractional Fibonacci groups and manifolds*. Sibirsk. Mat. Zh. **39**(1998), no. 4, 765–775.
[57] A. Y Vesnin and T. A. Kozlovskaya, *Brieskorn manifolds, generalized Sieradski groups and coverings of lens spaces*. Tr. Inst. Mat. Mekh. **23**(2017), no. 4, 85–97.

School of Mathematics, Statistics and Actuarial Science, University of Essex, Colchester, Essex CO4 3SQ, UK
e-mail: gerald.williams@essex.ac.uk