

Golay-ZCZ Sequence Sets with Flexible Lengths Based on Matrices of Polynomials

Shibsankar Das*, Adrish Banerjee[†], Udaya Parampalli[‡] and Zilong Liu[§]

*Department of Mathematics, Patliputra University, Patna, India

[†]Department of Electrical Engineering, Indian Institute of Technology (IIT) Kanpur, Kanpur, India

[‡]School of Computing and Information Systems, University of Melbourne, Victoria, Australia

[§]School of Computer Science and Electronic Engineering, University of Essex, UK

Abstract—In this paper, we develop a generalized construction framework of zero-correlation zone (ZCZ) sequence sets based on para-unitary (PU) matrices. We show that any square PU matrix with unimodular sequences can generate ZCZ sequence sets by perfectly controlling its expanded product form, thereby improving upon existing results. The key idea is to impose specific conditions on the expanded product forms of PU matrices to enable accurate computation of periodic correlation functions. In addition, each constructed ZCZ sequence set is a PU column vector and hence it represents a Golay complementary set (GCS). In this work, we call them Golay-ZCZ sequence sets. The lengths, set sizes and ZCZ widths of the constructed Golay-ZCZ sequence sets are all flexible with the form of non-power-of-two. These Golay-ZCZ sequence sets can be directly applied to practical training-based channel estimation schemes in multiple-antenna systems.

I. INTRODUCTION

A Golay-ZCZ sequence set refers to a set of multiple sequences which represents both Golay complementary set (GCS) and zero-correlation zone (ZCZ) sequence set [1]. In [2], Tseng and Liu extended the idea of Golay complementary pairs to GCSs with two or more number of constituent sequences. GCSs have found applications in orthogonal frequency-division multiplexing (OFDM) schemes to reduce the peak-to-mean envelop power ratio (PMEPR) [3], and code-division multiple access (CDMA) to mitigate the interference [4]. Two distinct GCSs are said to be mutually orthogonal GCSs (MOGCSs) if the aperiodic cross-correlation sums between them are zero for any time-shift [2]. So far, several construction methods of GCSs and MOGCSs have been proposed with various lengths and set sizes [5]-[8]. In the literature, ZCZ sequence set was first introduced as spreading sequences to obtain interference-free performance in quasi-synchronous CDMA (QS-CDMA) communications [9]. ZCZ sequence sets have also been applied for channel estimation in multi-input and multi-output (MIMO) systems [10], radar [11], and pilot design [12]. In the literature, constructions of ZCZ sequence sets can be divided into the construction methods based on generalized Boolean functions [13], [14], perfect sequences [15], [16], perfect non-linear functions [17], mismatched filtering [18], and MOGCSs [19]-[22].

In [23], Vaidyanathan introduced the concept of para-unitary (PU) matrices in the study of filter-bank theory. A PU matrix refers to a matrix with polynomial entries in z^{-1} which

is unitary on the unit circle (i.e., $|z| = 1$). By using z -transforms of sequences, PU matrix becomes an effective tool to simplify the derivations of unimodular sequences with good aperiodic correlation properties. To construct Golay complementary pairs (and GCSs) based on PU matrices, a compact formulation written in z -domain is derived in [24]. Consequently, the applications of PU matrices have been extended to q -ary GCSs [25] and QAM GCSs [26]. To obtain q -ary sequences, [25] studied the utility of Butson-type Hadamard (BH) matrices. Later, it is shown in [27] that the use of equivalent BH matrices can increase the number of constructed complementary sequences from PU matrix approach. There has been tremendous research interest on the construction of MOGCSs based on PU matrices [7], [8], [27]-[29].

In contrast to the design of GCSs/MOGCSs, the application of PU matrices to the construction of ZCZ sequence sets is not straightforward. To obtain ZCZ sequence sets from PU matrices, all the constituent sequences in a PU column vector are to be unimodular, and the periodic auto-and cross-correlations have to be zero within the ZCZ width Z , conflicting with the feature of a PU column vector which has zero aperiodic auto-correlation sum property. This becomes a barrier to directly apply any PU matrix to obtain ZCZ sequence sets. Towards this, two natural questions arise:

- 1) What is the precise way to control the periodic auto-and cross-correlation properties of all the sequences from a PU column vector?
- 2) How can we directly construct Golay-ZCZ sequence sets from PU matrices without constructing CCC in advance?

These two fundamental research questions will be addressed in this work. We are interested in PU matrix approach due to its compact cascading structure. We first extend our earlier work [30] to obtain ZCZ sequence sets with flexible parameters. We draw out an important connection between ZCZ sequence sets and PU matrix. The key idea is to apply the intrinsic periodic auto-and cross-correlation properties of the row sequences of a specific unitary matrix C . We show that a larger class of ZCZ sequences with more flexible parameters can be obtained by employing equivalent forms of BH matrices and new block delay matrices. The resultant sequences form an asymptotically optimal ZCZ sequence set with respect to the

Tang-Fan-Matsufuji bound [9]. The constructed Golay-ZCZ sequence sets can be used to offer optimal channel estimation performance in MIMO systems based on the framework from [10].

The remainder of the paper is organized as follows. In Section II, we introduce background and notations on GCS, ZCZ, PMEPR and PU matrices. We present a generalized construction framework of Golay-ZCZ sequence sets in Section III. In Section IV, we briefly review and compare the parameters of the proposed Golay-ZCZ sequence sets with those of the existing methods. Finally, we summarize our work in Section V.

II. BACKGROUND AND NOTATIONS

A. Golay Complementary Sets (GCSs)

Let $\mathbf{x} = (x[0], x[1], \dots, x[L-1])$ and $\mathbf{y} = (y[0], y[1], \dots, y[L-1])$ be two length- L complex sequences with $x[l], y[l] \in \mathbb{C}$ for $l = 0, 1, \dots, L-1$. The aperiodic correlation function of \mathbf{x} and \mathbf{y} at the time-shift τ is denoted by $R_{\mathbf{x},\mathbf{y}}[\tau]$. It is called an aperiodic cross-correlation function (ACCF) if $\mathbf{x} \neq \mathbf{y}$; otherwise, it is called an aperiodic auto-correlation function (AACF). The AACF of \mathbf{x} is simply denoted as $R_{\mathbf{x}}[\tau]$. Also, the periodic correlation function of \mathbf{x} and \mathbf{y} at the time-shift τ is defined by

$$\hat{R}_{\mathbf{x},\mathbf{y}}[\tau] = R_{\mathbf{x},\mathbf{y}}[\tau] + R_{\mathbf{y},\mathbf{x}}^*[L-\tau]. \quad (1)$$

Then, $\hat{R}_{\mathbf{x},\mathbf{y}}[\tau]$ is said to be periodic cross-correlation function (PCCF) of \mathbf{x} and \mathbf{y} at the time-shift τ if $\mathbf{x} \neq \mathbf{y}$; otherwise, it is called a periodic auto-correlation function (PACF). The PACF of \mathbf{x} is simply denoted by $\hat{R}_{\mathbf{x}}[\tau]$. The z -transform of the AACF $R_{\mathbf{x}}[\tau]$ can be written as $R_{\mathbf{x}}(z)$. Let $\mathbf{x} = \{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{M-1}\}$ be a set of M sequences with identical length L .

Definition 1 (Golay Complementary Sets (GCSs)): The set \mathbf{x} is said to be an (M, L) -GCS [2] if

$$\sum_{m=0}^{M-1} R_{\mathbf{x}_m}(z) = \sum_{m=0}^{M-1} x_m(z) \cdot x_m^*(z^{-1}) = ML. \quad (2)$$

B. ZCZ Sequence Sets

Let $\mathbf{x}(z) = [x_0(z), x_1(z), \dots, x_{M-1}(z)]^T$ be a set of M complex valued sequences of equal length L . The zero periodic auto-correlation zone Z_{ACZ} and the zero periodic cross-correlation zone Z_{CCZ} of this sequence set $\mathbf{x}(z)$ are defined as

$$Z_{ACZ} = \max \left\{ Z : \hat{R}_{\mathbf{x}_m}[\tau] = 0, \forall m, 0 < |\tau| \leq Z \right\}$$

$$Z_{CCZ} = \max \left\{ Z : \hat{R}_{\mathbf{x}_m, \mathbf{x}_{m'}}[\tau] = 0, \forall m \neq m', 0 \leq |\tau| \leq Z \right\}.$$

For the sequence set $\mathbf{x}(z)$, the interference free window (IFW) is defined by $Z = \min \{Z_{ACZ}, Z_{CCZ}\}$. Here, Z is called zero-correlation zone (ZCZ) width of PACFs and PCCFs. The set $\mathbf{x}(z)$ with IFW of Z is called a ZCZ sequence set of M sequences of equal length L and is denoted by (M, L, Z) -ZCZ sequence set. For an (M, L, Z) -ZCZ sequence set $\mathbf{x}(z)$, the upper bound, called the *Tang-Fan-Matsufuji bound*, on ZCZ width Z is given by [9] $Z \leq \frac{L}{M} - 1$. The performance

parameter η with respect to the *Tang-Fan-Matsufuji bound* can be defined by

$$\eta = \frac{M(Z+1)}{L}, \quad (3)$$

which can be used to evaluate the efficiency of the construction method. We observe that $0 < \eta \leq 1$. The sequence set $\mathbf{x}(z)$ is called an optimal (M, L, Z) -ZCZ sequence set if the *Tang-Fan-Matsufuji bound* is obtained with equality (i.e., $\eta = 1$).

Definition 2 (See [1]): A set $\mathbf{x}(z)$ of M sequences of equal length L is called an (M, L, Z) -Golay-ZCZ sequence set if it is both an (M, L) -GCS and (M, L, Z) -ZCZ sequence set.

A Golay-ZCZ sequence set $\mathbf{x}(z)$ is called an asymptotically optimal Golay-ZCZ sequence set if $\eta \rightarrow 1$ with respect to the *Tang-Fan-Matsufuji bound* when the length L increases.

C. Peak-to-Mean Envelope Power Ratio (PMEPR)

For a given q -PSK modulated sequence $(\hat{x}[0], \hat{x}[1], \dots, \hat{x}[L-1])$, the time-domain complex baseband OFDM signal can be written by

$$S_{\mathbf{x}}[t] = \sum_{l=0}^{L-1} \xi_q^{\hat{x}[l]} e^{2\pi\sqrt{-1}(f_c+l\Delta f)t}, \quad 0 \leq t \leq \frac{1}{\Delta f}, \quad (4)$$

where f_c denotes the carrier frequency, Δf refers to the subcarrier spacing, and L denotes the number of subcarriers in OFDM system. Here, we can write $\mathbf{x} = \left(\xi_q^{\hat{x}[l]} \right)_{l=0}^{L-1}$. The instantaneous envelope power ratio (IEPR) of the sequence \mathbf{x} is given by $P_{\mathbf{x}}[t] = \frac{|\sum_{l=0}^{L-1} \xi_q^{\hat{x}[l]} e^{2\pi\sqrt{-1}(f_c+l\Delta f)t}|^2}{L}$. Note that L is the peak amplitude of an OFDM signal with L subcarriers. Then, the peak-to-mean envelope power ratio (PMEPR) [3], [31] of the sequence \mathbf{x} can be defined by

$$\text{PMEPR}(\mathbf{x}) = \sup_{0 \leq t \leq \frac{1}{\Delta f}} P_{\mathbf{x}}[t]. \quad (5)$$

D. Para-unitary (PU) Matrices

A PU matrix is defined by a matrix of polynomials over z^{-1} which is unitary on the unit circle, i.e., $|z| = 1$ [23]. A matrix $\mathbf{X}(z)$ of polynomials with size $M \times K$ is defined as $\mathbf{X}(z) = [x_{l,m}(z)]$, where $x_{l,m}(z)$ is a polynomial representation over z^{-1} of the sequence $\mathbf{x}_{l,m}$ and $0 \leq l \leq M-1, 0 \leq m \leq K-1$. For a polynomial matrix $\mathbf{X}(z)$ over z^{-1} , we define the tilde operator as follows $\widetilde{\mathbf{X}}(z) = \mathbf{X}^H(z^{-1})$, where H is the Hermitian operation.

Definition 3 (See [23]): An $M \times K$ matrix $\mathbf{X}(z)$ of polynomials is said to be a PU matrix if

$$\widetilde{\mathbf{X}}(z) \cdot \mathbf{X}(z) = c \cdot \mathbf{I}_K, \quad (6)$$

where c is a real positive constant.

For a PU matrix, the degree is defined as the minimum number of delay elements needed to implement it. The connection between a PU matrix and complete complementary codes (CCC) is given by the following lemma.

Lemma 1 (See [28]): An $M \times M$ polynomial matrix $\mathbf{X}(z)$ represents a polyphase (M, M, L) -CCC if and only if it is an $M \times M$ unimodular PU matrix of degree $L-1$.

III. GENERALIZED CONSTRUCTION OF GOLAY-ZCZ SEQUENCE SETS WITH FLEXIBLE LENGTHS

To achieve more sequence candidates, Ma *et al.* discussed a new class of delay matrices by defining $(mN)!$ different delay permutations instead of $N!$ in [7]. We will show that we can utilize $(mN)!$ different delay permutations instead of $N!$ in the PU expanded form [28] by properly defining a new class of block delay matrices.

Let $M = p^{m+1}$ for some positive integers p and m . For the ease of presentation, we consider the case when $M = 3^{m+1}$ with $p = 3$ for some positive integer m . For $n = 0, 1, \dots, N-1$ with $N \in \mathbb{N}$, we define the n -th block delay matrix of size $3^{m+1} \times 3^{m+1}$ as follows

$$\mathcal{D}_n(z) = \mathbf{I}_3 \otimes \mathbf{D}_3^{3^\pi(mn)}(z) \otimes \mathbf{D}_3^{3^\pi(mn+1)}(z) \otimes \dots \otimes \mathbf{D}_3^{3^\pi(mn+m-1)}(z), \quad (7)$$

where $\mathbf{D}_3(z) = \text{diag}([1 \ z^{-1} \ z^{-2}])$ and π is a permutation of integers $\{0, 1, \dots, (mN-1)\}$. Note that $\mathbf{D}_3^a(z) = \text{diag}([1 \ z^{-a} \ z^{-2a}])$ for some positive integer a .

Let $M = 3^{m+1}$ for some positive integer m . Let $\mathbf{U}_{3^{m+1}}^{(n+1)}$ and $\mathbf{U}_{3^{m+1}}^{(0)}$ be *BH* matrices (formal definition of *BH* matrix can be found in [6]) of size $3^m \times 3^m$ and $3^{m+1} \times 3^{m+1}$, respectively, for $n = 0, 1, 2, \dots, N-1$. We take the n -th block delay matrix $\mathcal{D}_n(z)$ as defined in (7). We now construct the n -th block *BH* matrix of size $3^{m+1} \times 3^{m+1}$ as follows $\mathbf{U}_n = \mathbf{I}_3 \otimes \mathbf{U}_{3^{m+1}}^{(n+1)}$, where $n \in \{0, 1, 2, \dots, N-1\}$ for any positive integer $N \in \mathbb{N}$. Then, a new recursive PU generator is defined by

$$\mathcal{G}_{n+1}(z) = \mathbf{U}_n \mathcal{D}_n(z) \mathbf{P}_n \mathcal{G}_n(z) \mathbf{Q}_n, \quad (8)$$

where \mathbf{P}_n and \mathbf{Q}_n are two arbitrary $3^{m+1} \times 3^{m+1}$ permutation matrices with the initial matrix $\mathcal{G}_0(z) = \mathbf{U}_{3^{m+1}}^{(0)}$. According to (8), the expanded product form is given by

$$\mathcal{G}_N(z) = \prod_{n=N-1}^0 \left(\mathbf{U}_n \mathcal{D}_n(z) \mathbf{P}_n \right) \mathbf{U}_{3^{m+1}}^{(0)} \mathbf{Q}_0. \quad (9)$$

Theorem 1: Let $\mathcal{G}_N(z)$ be a generating matrix with the expanded product form (9). Then, each entry of $\mathcal{G}_N(z)$ represents a unimodular sequence of length 3^{mN} .

Proof: Let $M = 3^{m+1}$. Based on (9), the j -th column vector of $\mathcal{G}_N(z)$ can be expressed as $\mathbf{x}_{N,j}(z) = \mathcal{G}_N(z) \cdot \mathbf{e}_j^T$, where \mathbf{e}_j denotes the j -th row vector of the identity matrix \mathbf{I}_M with $j = 0, 1, \dots, M-1$. The j -th column vector is given by

$$\mathbf{x}_{n,j}(z) = \mathbf{U}_n \cdot \mathcal{D}_n(z) \cdot \mathbf{P}_n \cdot \mathbf{x}_{n-1,j}(z), \quad (10)$$

where $\mathbf{x}_{0,j}(z) = \mathbf{U}_{3^{m+1}}^{(0)} \mathbf{Q}_0 \cdot \mathbf{e}_j^T$. Let $\mathbf{x}_{n,j}(z) = [x_{n,j}^{(0)}(z), x_{n,j}^{(1)}(z), \dots, x_{n,j}^{(M-1)}(z)]^T$ be constructed by (10) with $\mathbf{U}_{3^m}^{(n)} = [u_{pq}^{(n)}]_{3^m \times 3^m}$ such that $|u_{pq}^{(n)}| = 1$ at the n -th iteration. Then, the sequence elements constructed by (10) can be written as

$$x_{n,j}^{(p+l3^m)}(z) = \sum_{q=0}^{3^m-1} u_{pq}^{(n)} \cdot x_{n-1,j}^{(p_n(q+l3^m))}(z) \cdot z^{-f_{mn}(q)}, \quad (11)$$

where $p = 0, 1, \dots, 3^m - 1, l = 0, 1, 2$ and p_n is the permutation corresponding to \mathbf{P}_n , and $f_{mn}(x)$ is

$$f_{mn}(x) = \sum_{j=0}^{m-1} x_j 3^{\pi(mn+j)}, \quad (12)$$

where x_j denotes the j -th bit of the ternary representation of x given by $x = \sum_{j=0}^{m-1} x_j 3^j$ for $x = 0, 1, \dots, 3^m - 1$. In the time-domain, we have

$$x_{n,j}^{(p+l3^m)}[k] = u_{pk(n-1)}^{(n)} \cdot x_{n-1,j}^{(p_n(k(n-1)+l3^m))}[k - f_{mn}(q)], \quad (13)$$

where $k = \sum_{i=0}^{n-1} k_i \cdot (3^m)^i$ with the i -th digit k_i of k in the 3^m -ary representation. ■

By employing the expanded PU product form, we construct the j -th sequence set $\mathbf{x}_j(z)$ consisting of p^{m+1} length- $p^{m(N+2)+2}$ sequences for $j = 0, 1, \dots, p^{m+1} - 1$.

Theorem 2: Let $M = p^{m+1}$ for some positive integers m and p . Then, the generalized PU-based construction of unimodular Golay-ZCZ sequence sets is given by

$$\mathbf{x}_j(z) = \prod_{n=N-1}^0 \left\{ \mathbf{U}_n \mathcal{D}_n(z) \mathbf{P}_n \right\} \mathbf{U}_{p^{m+1}}^{(0)} \mathbf{Q}_0 \cdot \mathbf{D}_{p^{m+1}} \left(z^{p^{mN}} \right) \cdot \mathbf{F}_{p^{m+1}} \cdot \mathbf{D}_{p^{m+1}} \left(z^{p^{m(N+1)+1}} \right) \cdot \mathbf{f}_j^T, \quad (14)$$

where $\mathbf{D}_{p^{m+1}}(z) = \text{diag} \left(1, z^{-1}, \dots, z^{-(p^{m+1}-1)} \right)$ and \mathbf{f}_j is the j -th row vector of a $p^{m+1} \times p^{m+1}$ DFT matrix $\mathbf{F}_{p^{m+1}}$. The proposed Golay-ZCZ sequence sets have the set size $M = p^{m+1}$, length $L = p^{m(N+2)+2}$, and ZCZ $Z = (p^{m+1}-1)p^{mN}$.

Proof: We omit the proof due to space constraint. ■

Example 1: Let $M = 3^{2+1} = 27$ and $N = 2$ with $m = 2$. We have $mN = 4$. The sequence length is given by $3^{mN} = 3^4 = 81$. For $n \in \{0, 1\}$, the n -th block delay matrix of size 27×27 is given by

$$\mathcal{D}_n(z) = \mathbf{I}_3 \otimes \mathbf{D}_3 \left(z^{3^\pi(2n)} \right) \otimes \mathbf{D}_3 \left(z^{3^\pi(2n+1)} \right), \quad (15)$$

where $\pi = [2, 0, 3, 1]$ is a permutation of the integers $\{0, 1, 2, 3\}$ and $\mathbf{D}_3(z) = \text{diag}([1 \ z^{-1} \ z^{-2}])$. In the delay matrix $\mathbf{D}_3^{3^\pi(2n)}(z) \otimes \mathbf{D}_3^{3^\pi(2n+1)}(z)$, the delay elements are given by $z^0, z^{-3^\pi(2n+1)}, z^{-2 \cdot 3^\pi(2n+1)}, z^{-3^\pi(2n)}, z^{-(3^\pi(2n)+3^\pi(2n+1))}, z^{-(3^\pi(2n)+2 \cdot 3^\pi(2n+1))}, z^{-2 \cdot 3^\pi(2n)}, z^{-(2 \cdot 3^\pi(2n)+3^\pi(2n+1))}, z^{-(2 \cdot 3^\pi(2n)+2 \cdot 3^\pi(2n+1))}$ for the permutation π and $n = 0, 1$. In (15), we can use $4! = 24$ permutations to obtain different block delay matrices. Let $\mathbf{U}_{3^{m+1}}^{(n+1)} = \mathbf{F}_9, \mathbf{U}_{3^{m+1}}^{(0)} = \mathbf{F}_{27}$ and $\mathbf{P}_n = \mathbf{Q}_0 = \mathbf{I}_{27}$ for $n = 0, 1$. Based on the recursive construction method (8), an 27×27 unimodular PU matrix of length $3^{mN} = 3^4 = 81$ is given by

$$\mathcal{G}_2(z) = \mathbf{U}_1 \mathcal{D}_1(z) \mathbf{P}_1 \mathbf{U}_0 \mathcal{D}_0(z) \mathbf{P}_0 \mathbf{U}_{3^{m+1}}^{(0)} \mathbf{Q}_0 \quad (16)$$

Let \mathbf{f}_0 be the 1-st row vector of *BH* matrix \mathbf{F}_{27} with $j = 0$. By applying the expanded product (14), we construct a sequence set given by

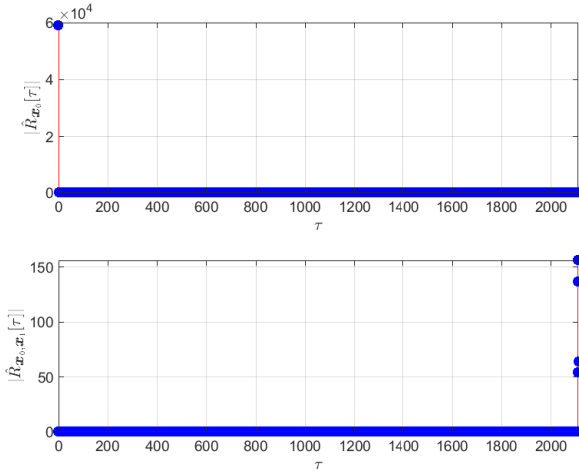


Fig. 1: The PACF of \mathbf{x}_1 and PCCF between \mathbf{x}_1 and \mathbf{x}_2

$$\mathbf{x}_0(z) = \left(\mathbf{I}_3 \otimes \left(\mathbf{F}_9 \left(\mathbf{D}_3^{3^3}(z) \otimes \mathbf{D}_3^3(z) \right) \mathbf{F}_9 \cdot \left(\mathbf{D}_3^{3^2}(z) \otimes \mathbf{D}_3^1(z) \right) \right) \right) \mathbf{F}_{27} \cdot \mathbf{D}_{27} \left(z^{3^4} \right) \cdot \mathbf{F}_{27} \cdot \mathbf{D}_{27} \left(z^{3^7} \right) \cdot \mathbf{f}_0^T, \quad (17)$$

where $\mathbf{D}_{27}(z) = \text{diag}(1, z^{-1}, \dots, z^{-26})$. Based on (17), the m -th sequence $x_0^{(m)}(z)$ of length 3^{10} from the set $\mathbf{x}_0(z)$ is

$$x_0^{(m)}(z) = \sum_{n=0}^{26} \sum_{i=0}^{26} \sum_{l=0}^{80} f_{n,i} f_{i,0} g_{m,n}[l] z^{-((3^3 i + n)81 + l)}, \quad (18)$$

where $0 \leq m \leq 26$. One can observe that $\mathbf{x}_0(z)$ is a PU column vector of size 27 and length 3^{10} , and hence it is a $(27, 3^{10})$ -GCS. In Fig. 1, we plot the PACFs of $x_0^{(0)}(z)$ and PCCFs of $x_0^{(0)}(z)$ and $x_0^{(1)}(z)$ for the time-shifts $\tau = 0, 1, \dots, (26 \cdot 3^4 + 3)$. We observe that the ZCZ width is $26 \cdot 3^4$. The performance parameter η for $(27, 3^{10}, 26 \cdot 3^4)$ -Golay-ZCZ sequence set is given by $\eta = \frac{27(26 \cdot 3^4 + 1)}{3^{10}} = 0.963$. In Fig. 2, we plot the IEPRs only for the first six sequences of the set $\mathbf{x}_0(z)$. Through numerical analysis, it is verified that the maximum value of the IEPRs for all the sequences in $\mathbf{x}_0(z)$ is 5.29. Thus, the PMEPR of $\mathbf{x}_0(z)$ is 5.29, which is much less than the theoretical upper bound of 27.

IV. COMPARATIVE STUDY WITH EXISTING WORKS

We illustrate a detailed comparison of our proposed PU design method with the existing constructions in Table I. From Table I, we can see that our proposed construction method can generate Golay-ZCZ sequence sets with more available sequence lengths for a given set size. The advantages of our proposed construction method based on PU matrices are twofold: Firstly, it offers more choices in Golay-ZCZ parameter values compared to the existing works [1], [13]-

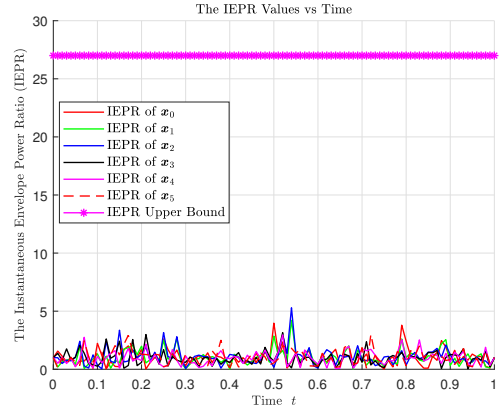


Fig. 2: The IEPRs of $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$, and \mathbf{x}_5

[17], [32], [33]. Secondly, it provides a compact formulation¹ in matrix forms of Golay-ZCZ sequence sets with performance parameter $\eta \geq 1/2$. For example, a $(9, 243, 24)$ -Golay-ZCZ sequence set cannot be generated by the existing works [1], [13], [14], [15], [16], [17], [32], [33]. On the other hand, our proposed construction method based on PU matrices has an advantage over [19], [20], [21], and [22] given the fact that we can directly construct Golay-ZCZ sequence sets without generating MOGCSs (and CCCs) in beforehand.

Remark 1: Based on [27], an efficient implementation of *Theorem 2* can be obtained with only adders.

V. CONCLUSIONS

In this paper, we have derived a connection between Golay-ZCZ sequence sets and PU matrix theory. We have shown that the periodic auto-and cross-correlation properties of the constructed sequences are completely governed by the last two unitary matrices in the expanded product form of a PU matrix. The proposed Golay-ZCZ sequence sets of length $p^{m(N+2)+2}$ have PMEPR at most p^{m+1} . The constructed Golay-ZCZ sequence set can be used in the training design for MIMO system. In addition, the proposed Golay-ZCZ sequence sets can directly be utilized in a QS-CDMA system to interference-free multi-user transmission.

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¹By compact formulation, we mean a construction framework that uses only matrix multiplications.

Table I: Summary of Existing Construction Methods on ZCZ Sequence Sets

Methods	Refs.	Set Size	Length	ZCZ Width	Constraints	Based on
Boolean Functions	[13]	2^{k+1}	2^{n+k+2}	2^n	$n, k \geq 2$	Reed-Muller Codes
	[32]	2^k	2^m	$2^{m-k} - 2^{m-k-z}$	$z \leq k, m \geq k+z$	Boolean Functions
	[14]	d^ν	d^m	$d^{\pi(2)-1}(d-1) + d^{\pi(3)-1}(d-2)$	$d, q \geq 2, dq, \nu \leq m-1, \pi$ is permutation	Extended Boolean Functions
	[1]	2^k	2^m	$2^{\pi(2)-1}$	$m \geq 2, k \leq m-1, \pi$ is permutation	Generalized Boolean Functions
Perfect Sequences	[33]	l_1	ll_1^{π}	$(l-2)l_1^{\pi-1}$	$l = l_0l_1, 1 \leq l_0, l_1 \leq l$	Length- l Perfect Sequence
	[15]	$2M$	$2N$	L	$L = \min\{\min_{e \neq f \in E}(2d_0, 2d_1), \min_{e, f \in E}(2d_2+1, 2d_3+1)\}$	E is the set of shift sequences
	[16]	M	N	$N/M-1$	$N = \prod_{m=0}^{N-1} M_m, 0 \leq n \leq M$	Length- M Perfect Sequence
	[17]	p	p^2	p	p is prime	Perfect non-linear functions
MOGCS or CCC	[19]	PN	QNL	N_ZL	$Q \geq 2, P = Q/(N_Z+1), N_Z \leq N-1, L = \text{Seed Seq. Length}$	(N, N, L) -CCC
	[20]	MP	$mMNQ$	nN	$(m, n) = (2, 1), (3, 2), (4, p-1)$ with $p \leq 4, M, Q \geq 2$	$M \times MN$ MOGCS
	[21]	M	$2MN$	N	CCC exists for length N	(M, M, N) -CCC
	[22]	M	M^2N	$(M-1)N$	CCC exists for length N	(M, M, N) -CCC
PU Matrices	[30]	M	M^2P^N	$(M-1)P^N$	$M \geq 2, P \geq 1, P M, N \in \mathbb{N}$	$M \times M$ PU Matrix
	<i>Theorem 2</i>	p^{m+1}	$p^{m(N+2)+2}$	$(p^{m+1}-1)p^{mN}$	$p \geq 2, m \geq 1, N \in \mathbb{N}$	$p^{m+1} \times p^{m+1}$ PU Matrix

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