Quantitative Finance

ISSN: 1469-7688 (Print) 1469-7696 (Online) Journal homepage: www.tandfonline.com/journals/rquf20

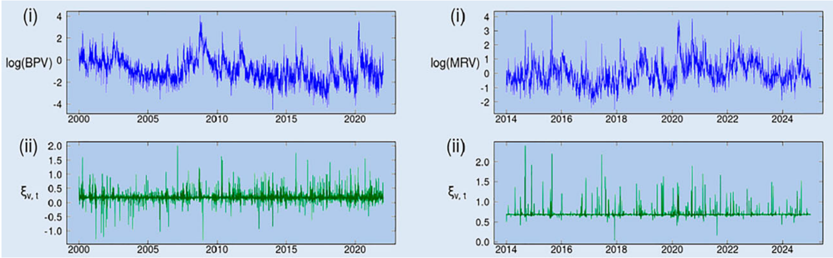
Modeling ex post variance jumps: implications for density and tail risk forecasting

John M. Maheu & Efthimios Nikolakopoulos

To cite this article: John M. Maheu & Efthimios Nikolakopoulos (01 Dec 2025): Modeling ex post variance jumps: implications for density and tail risk forecasting, Quantitative Finance, DOI: 10.1080/14697688.2025.2565290

To link to this article: https://doi.org/10.1080/14697688.2025.2565290

Quantitative Finance, 2025
https://doi.org/10.1080/14697688.2025.2565290



Modeling ex post variance jumps: implications for density and tail risk forecasting

JOHN M. MAHEU† and EFTHIMIOS NIKOLAKOPOULOS*‡

†DeGroote School of Business, McMaster University, 1280 Main Street West, Hamilton, ON L8S4M4, Canada
‡Essex Business School, University of Essex, Wivenhoe Park, Colchester CO4 3SQ, UK

(*Received 16 December 2024; accepted 11 September 2025*)

This paper focuses on modeling ex post variance jumps including several time-dependent arrival specifications to assess their importance to forecasts of daily returns and variance measures. The benchmark specification for variance measures includes two autoregressive components that capture the persistent and transitory elements. To this we add a jump process with either independent arrival rates, autoregressive conditional jump intensities, or a stochastic autoregressive jump arrival specification. Results from four major markets and four stocks show that ex post variance jumps are frequent and persistent. Modeling time-dependent variance jumps strongly improves ex post variance density forecasts for multiperiod forecast horizons and improves forecasts of the return density. There are economic benefits to modeling variance jumps as well. Models with time-dependent ex post variance jumps improve tail risk forecasting of value-at-risk and expected shortfall.

Keywords: Realized variance; Realized kernel; Bipower variation; Variance jumps; Tail risk; Density forecasts

1. Introduction

Financial asset prices present unexpected and large discontinuities defined as jumps, due to unexpected news flow in the market. In this paper, we focus on modeling jumps in ex post variance measures and investigate their importance to both forecasts of returns and variance measures. We develop bivariate models of daily returns and ex post variance in which we model contemporaneous and time-varying jumps in the ex post measures of realized variance (RV), realized kernel (RK), median realized variance (MRV) and bipower variation (BPV). Results show frequent and persistent ex post variance jumps. We test the ability of jumps in variance measures to improve forecasts of returns and variance densities over horizons of up to 60 days out-of-sample. Time-dependent variance jumps provide improvements in returns density forecasts including tail risk measures and strongly improve ex post variance density forecasts.

The development of realized variance estimators provides consistent and model-free ex post measures of the latent variance process. The importance of jumps in return volatility forecasts has been documented by Andersen *et al.* (2007, 2011). Their work and others, Chan and Maheu (2002), Maheu and McCurdy (2004), Lee (2012), Maheu *et al.* (2013), Maneesoonthorn *et al.* (2017), and Jeon *et al.* (2022), indicate significant time variation and clustering in jump arrivals in daily returns.†

While there is strong evidence for return jumps, variance jumps are found in Chen *et al.* (2024), Todorov and Tauchen (2011) and Eraker (2004). Caporin *et al.* (2015) use the autoregressive jump intensity (ARJ) framework of Chan and Maheu (2002) to estimate jumps in bipower realized range. In addition to jump clustering they find that ex post variance jumps can be explained by the credit default swap rates and the variance risk premium. Time variation in variance jumps is found in Maneesoonthorn *et al.* (2017) for S&P 500. Chan and Gray (2018) estimate volatility jumps and show that these coincide with news announcements.

How important are ex post variance jumps to forecasts of variance measures and returns? The answer is missing in the literature and is the purpose of this paper. Similar to Andersen *et al.* (2011) and Maheu and McCurdy (2011) we construct empirically realistic bivariate models to capture the dynamics of ex post variance measures and returns. Our main focus is

† Additional evidence for return jumps is Johannes *et al.* (1999), Due *et al.* (2000), Eraker (2004) and Lee and Mykland (2008).

Corresponding author. Email: e.nikolakopoulos@essex.ac.uk

on the importance of a jump process in ex post variance to the density forecasts of variance measures and returns.

We estimate a range of different jump models for the logarithm of daily realized variance measures (RM) from four markets and four individual firms. Our benchmark specification for log(RM) uses a two-component model of Maheu and McCurdy (2011) to capture the smooth autoregressive changes.† To avoid jump dynamics being a proxy for GARCH effects we include a GARCH specification for log(RM) innovations. To this model we add various jump specifications. The simplest is an independent arrival jump process that is common in the literature. We also consider a version of the ARJ model of Chan and Maheu (2002).

The more sophisticated new model for log(RM) has the two-component structure, with GARCH effects, but features the latent autoregressive process of Maheu and McCurdy (2008) to direct jump arrivals. This stochastic autoregressive jump model allows for jump clustering, like the ARJ model, but unlike that model, it has a random jump event conditional on last day's data. Jumps impact the conditional mean and volatility as well as higher order conditional moments of variance measures. We show how to estimate this model and infer jump times and jump sizes.

We test empirically four ex post variance measures: realized variance, a realized kernel estimator, median realized variance and bipower variation. Jumps in the two latter ones are undoubtedly variance jumps from integrated volatility, while for the former measures jumps could be from integrated volatility or from return jumps. To complete our models and account for return jumps all our specifications include a simple independent return jump process. Hence, only the ex post variance jump processes are different among the models tested. Including return jumps can also act as a correction to cases of noisy measures of variance.

We find large and persistent jumps in all realized measures for all four markets (SPX, DJI, FTSE, TSX) and four individual stocks (AAPL, AXP, IBM, NKE) considered. The ex post variance jumps are essentially time-dependent non-Gaussian dynamics that are not captured by standard conditional mean and variance frameworks. Predictive densities of the logarithmic realized measures indicate non-Gaussian distributions with fat-tails and evidence of asymmetry. The stochastic jump specification contributes to fat-tails in both the variance and return distribution.

We examine the predictive performance of the proposed models by multiperiod density forecasts over horizons of up to 60 days out-of-sample of both returns and log(RM). We find uniformly strong evidence for jumps in ex post variance measures. Modeling jumps in one form or another almost always leads to better multiperiod density forecasts of variance measures and returns. Of the jump specifications considered, those with time-varying arrival rates are preferred. Frequently the best density forecasts come from the the stochastic autoregressive jump specification. The stochastic autoregressive jump framework also stands out in forecasting the return density. The choice of the realized variance measure is also important to return density forecasts. Generally for the markets data, RV is preferred coupled with a time-varying arrival rate of jumps but there are important exceptions where RK and BP are preferred.‡ For the individual stocks, BPV or MRV are preferred coupled with a jump specification.

The importance of modeling ex post variance jumps is further reinforced by their economic benefits in value-at-risk (VaR) and expected shortfall (ES) of returns. There are strong improvements in jointly forecasting VaR and ES with jumps in the model of ex post variance. We find that models with time-varying arrival of jumps in ex post variance are the best performers in forecasting return tail risk.

The rest of the paper is organized as follows: Section 2 is a brief introduction to the ex post variance measures used, Sections 3, 4 and 5 present the specifications of the proposed models and their estimation steps. The forecasting process is detailed in Section 6. Section 7 presents the empirical application results and Section 8 concludes. A supplemental file contains details of posterior simulation steps and collects additional estimation results.

2. Ex post variance measures

The ex post variance of an asset's return is estimated with the nonparametric realized measures. Realized variance is the simplest measure. The important papers that provide theoretical foundation and applications of RV are from Andersen *et al.* (2001a), Andersen *et al.* (2001b), Barndorff-Nielsen and Shephard (2002a), Barndorff-Nielsen and Shephard (2002b) and Andersen *et al.* (2003). The baseline is a continuous-time stochastic volatility model. Including a jump component, as Press (1967), then the logarithmic asset price at time t, p_t, evolves under the following process

$$\mathrm{d}p_t = \mu_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}W_t + \xi_t \,\mathrm{d}q_t, \tag{1}$$

where μ_t and $\sigma_t(\sigma_t > 0)$ are the stochastic drift and diffusion processes, W_t is a standard Brownian motion, q_t is a Poisson process, uncorrelated with W_t, with $\mathrm{Prob}[\mathrm{d}q_t = 1] = \lambda$, $\mathrm{Prob}[\mathrm{d}q_t = 0] = 1 - \lambda$ and ξ_t is the jump size. Under (1) jumps are finite, rare and their frequency depends on λ. The daily continuously compounded logarithmic return at time t, r_t, is defined as

$$r_t \equiv p_t - p_{t-1} = \int_{t-1}^{t} \mu_\tau \,\mathrm{d}\tau + \int_{t-1}^{t} \sigma_\tau \,\mathrm{d}W_\tau + \sum_{t-1 \le \tau \le t |\, \mathrm{d}q=1} \xi_\tau \tag{2}$$

and the quadratic return variation is defined as the summation of integrated variance (IV) and the cumulative squared return

† This specification is similar to the Heterogeneous Autoregressive (HAR) model of Corsi (2009). Related extensions of this model are Andersen *et al.* (2007), Bollerslev *et al.* (2009), Andersen *et al.* (2011), Corsi (2009), Corsi and Renò (2012) and Caporin *et al.* (2015). Maheu and McCurdy (2011) find that their two-component model, which uses long- and short-run conditional variance components, forecasts RV better than a HAR model and provides improved returns density forecasts.

‡ Peiris *et al.* (2024) show that combining several volatility measures in a model can be beneficial for tail risk forecasting.

jump (RJ) term

$$\mathrm{QV}_t = \mathrm{IV}_t + \mathrm{RJ}_t = \int_{t-1}^{t} \sigma_\tau^2 \, \mathrm{d}\tau + \sum_{t-1 \le \tau \le t | \, \mathrm{d}q = 1} \xi_\tau^2 \quad (3)$$

For day t, given intraday returns $r_{t,i}$, $i = 1, \ldots, n$, Barndorff-Nielsen and Shephard (2002a) and Andersen *et al.* (2003) show that QV_t can be approximated by the realized variance estimator defined as

$$\mathrm{RV}_t = \sum_{i=1}^{n} r_{t,i}^2 \to \mathrm{QV}_t, \quad \text{as } n \to \infty. \quad (4)$$

RV_t is a consistent estimator of QV_t under no market microstructure noise. In practice, high-frequency returns contain market microstructure noise which makes RV_t biased and inconsistent, as Zhang *et al.* (2005), Hansen and Lunde (2006) and Bandi and Russell (2008) document.†

One way to minimize the effect of market frictions in RV is the use of kernel functions. In the general framework of Barndorff-Nielsen *et al.* (2008), the realized kernel estimation of QV_t is of the form

$$\mathrm{RK}_t = \mathrm{RV}_t + \sum_{\eta=1}^{H} k\left(\frac{\eta - 1}{H}\right) \{\gamma_\eta + \gamma_{-\eta}\}, \quad (5)$$

where $k(x)$, $x \in [0, 1]$ is a weight function and γ_η is the ηth realized autocovariance, $\gamma_\eta = \sum_{i=1}^{n} r_{t,i} r_{t,i-\eta}$. They provide detailed results on the choice of function $k(x)$ and the number of lags H.

RV_t and RK_t are not robust to return jumps. To estimate IV_t in (3), Barndorff-Nielsen and Shephard (2004) introduced the measure of bipower variation (BPV) which is robust to return jumps and is defined as

$$\mathrm{BPV}_t = \frac{\pi}{2} \sum_{i=2}^{n} |r_{t,i}||r_{t,i-1}| \to \mathrm{IV}_t, \quad \text{as } n \to \infty. \quad (6)$$

Ex post return jumps in (3) are typically estimated by subtracting BPV_t from RV_t.

The last measure we consider to approximate IV_t is the jump-robust Median RV (MRV) of Andersen *et al.* (2012), which is defined as

$$\mathrm{MRV}_t = \frac{\pi}{6 - 4\sqrt{3} + \pi} \left(\frac{n}{n-2}\right) \times \sum_{i=2}^{n-1} \mathrm{med}(|r_{t,i-1}|, |r_{t,i}|, |r_{t,i+1}|)^2. \quad (7)$$

2.1. Data

We consider two datasets. The first one is four markets consisting of daily open-to-close (log) returns, r_t, and their ex post variance measures of RV, two-scale RK and BPV from Oxford-Man Institute's Realized Library‡ (Heber *et al.* 2009). The four markets are: the Dow Jones Industrial Average Index (DJI), the Financial Times Stock Exchange 100 Index (FTSE), the S&P500 Index (SPX), and the Toronto Stock Exchange Index (TSX). RMs have been calculated from 5-minute returns using *subsampling*.

The second dataset contains individual stocks. It consists of daily open-to-close (log) returns and, their robust ex post variance measures of Parzen RK, MRV and BPV.§ RMs have been calculated using 5-minute returns obtained from Kibot.¶ We consider the following four stocks that are currently included in the Dow Jones 30: Apple Inc. (AAPL), American Express (AXP), International Business Machines (IBM), and Nike, Inc. (NKE).

Summary statistics are in supplemental file C, table 8. The returns have been converted to percentages while RV, RK, MRV and BPV data have been scaled by 100^2.

In the following we use RM_t as a generic term for either RV_t, RK_t, MRV_t or BPV_t and we denote the information sets as $r_{1:t} = \{r_1, \ldots, r_t\}$, $\mathrm{RM}_{1:t} = \{\mathrm{RM}_1, \ldots, \mathrm{RM}_t\}$ and $\mathcal{I}_{t-1} = \{r_{1:t-1}, \mathrm{RM}_{1:t-1}\}$.

3. Return equation

There are good reasons for moving beyond pure return based models and focusing on joint models of returns and RM. From both the forecasting side (Maheu and McCurdy 2011, Maneesoonthorn *et al.* 2017, Liu and Maheu 2018, Jin *et al.* 2019) and financial decision side (Fleming *et al.* 2003, Callot *et al.* 2017, Bollerslev *et al.* 2018, Liu and Maheu 2018) significant improvements can be obtained. Therefore we focus on bivariate discrete-time model specifications of returns and ex post variance. The integrated framework enables model comparison based on both the variance and returns distribution through multi-period forecasts. This allows us to see the importance of the ex post variance jump dynamics and how they impact both distributions.

We begin with the specification for daily returns which are assumed to follow,

$$r_t = \mu + J_{r,t}\xi_{r,t} + \sqrt{\mathrm{RM}_t}\epsilon_t, \quad \epsilon_t \sim \mathrm{N}(0, 1), \quad (8)$$

$$\xi_{r,t} \sim \mathrm{N}(\mu_r, \sigma_r^2), \quad J_{r,t} \in \{0, 1\}, \quad P(J_{r,t} = 1) = \lambda_r. \quad (9)$$

Equation (8) is a discrete-time version of (2). $J_{r,t}$ is the return jump indicator at time t which is a Bernoulli random variable with probability λ_r. The return jump size, $\xi_{r,t}$, follows a normal distribution with mean μ_r and variance σ_r^2. This return specification *is common to all of our models* and the main distinguishing feature of the bivariate models is how $\log(\mathrm{RM}_t)$ is modeled as well as what ex post measure is chosen for RM_t. We decompose

† See Zhang *et al.* (2005) and Aït-Sahalia and Mancini (2008) for the use of *subsampling* for realized measures robust to market frictions.

‡ The Oxford-Man Realized Library was discontinued in 2022. MRV was available but we had concerns about the data quality and omitted it for the indices.

§ For individual stocks we used Parzen RK for our main results over plain RV as it had a lower bias.

¶ https://www.kibot.com/.

the conditional density as follows, $p(r_t, \log(\mathrm{RM}_t)|\mathcal{I}_{t-1}) = p(r_t|\log(\mathrm{RM}_t), \mathcal{I}_{t-1})p(\log(\mathrm{RM}_t)|\mathcal{I}_{t-1})$. We now turn to the specifications for $\log(\mathrm{RM}_t)$.

4. Benchmark models for ex post variance

4.1. Two-component model with GARCH (C-GARCH)

The benchmark model for RM has no jumps but captures the strong persistence through a two-component structure, allows for potential thick-tailed innovations and includes GARCH effects for the conditional variance (Corsi *et al.* 2008). The specification draws from the work of Andersen *et al.* (2007), Corsi (2009) and Maheu and McCurdy (2011) amongst others. The model C-GARCH-t is an extension of Maheu and McCurdy (2011) and along with (8)–(9), features two components to capture volatility's conditional mean persistence,

$$\log(\mathrm{RM}_t) = \omega + \sum_{i=1}^{2} \phi_i c_{i,t} + \rho u_{t-1} + e_t, \quad e_t \sim \mathrm{t}(0, \sigma_t^2, \nu), \tag{10}$$

$$\sigma_t^2 = \varpi + a e_{t-1}^2 + b\sigma_{t-1}^2, \tag{11}$$

$$c_{i,t} = (1-\alpha_i)\log(\mathrm{RM}_{t-1}) + \alpha_i c_{i,t-1}, \quad 0 < \alpha_i < 1, i = 1, 2, \alpha_1 > \alpha_2. \tag{12}$$

RM_t is the realized variance measure at time $t, t = 1, \ldots, T$. The conditional mean of $\log(\mathrm{RM}_t)$ is based on two components $c_{i,t}$ with different decay rates α_i with $\alpha_1 > \alpha_2$. Innovations to $\log(\mathrm{RM}_t)$ follow a Student-t distribution with scale parameter σ_t^2 that follows a GARCH(1,1) specification and degrees of freedom parameter ν. To ensure positivity of σ_t^2 we assume $\varpi > 0, a \geq 0, b \geq 0$.

The model is parsimonious and can produce mean-reverting forecasts for $\log(\mathrm{RM}_t)$ but still displays a slow autocorrelation function with α_1 close to one. Maheu and McCurdy (2011) show that multiperiod forecasts from this specification outperform other models, including versions of Corsi (2009) in the realized measurement equation, for returns density forecasting.

Parameter ρ captures the feedback due to return shocks, which is often called a leverage effect. The return shock is defined as,

$$u_t = (r_t - \mu)/\sqrt{\mathrm{RM}_t}. \tag{13}$$

The parameter ρ is expected to have a negative sign. This implies that a return drop will cause an increase in future variance and vice versa. This allows for a correlation between r_{t-1} and $\log(\mathrm{RM}_t)$ and is consistent with Yu (2005) and no arbitrage.

The benchmark C-GARCH-t model is fully defined with equations (8)–(12). We also test a second benchmark model referred to as C-GARCH in which the Student-t innovations in (10) are replaced by normal innovations.

The models C-GARCH and C-GARCH-t serve as the main benchmark specifications which feature no RM jump component. The return jump arrivals and sizes are unobserved and are estimated with the use of Markov chain Monte Carlo (MCMC) techniques. The model estimation algorithm is a hybrid of Gibbs and Metropolis-Hastings steps. See the supplemental file for details. All of the following models augment the benchmark specifications with RM jump dynamics of one form or another.

4.2. Autoregressive conditional jump intensity (C-GARCH-ARJ)

A natural benchmark is the autoregressive conditional jump intensity framework of Chan and Maheu (2002) which has been used by Caporin *et al.* (2015) to model variance jump dynamics. Compared to the stochastic autoregressive jump model introduced below the following C-GARCH-ARJ model assumes the conditional jump intensity is measurable with respect to the information set $\mathcal{I}_{t-1}$. The C-GARCH-ARJ model is defined with (8)–(9) and

$$\log(\mathrm{RM}_t) = \omega + \sum_{i=1}^{2} \phi_i c_{i,t} + \rho u_{t-1} + \sum_{k=1}^{n_t} \xi_{v,k} + v_t, \quad v_t \sim \mathrm{N}(0, \sigma_t^2), \tag{14}$$

$$\sigma_t^2 = \varpi + a e_{t-1}^2 + b\sigma_{t-1}^2, \quad e_t = \log(\mathrm{RM}_t) - \left(\omega + \sum_{i=1}^{2} \phi_i c_{i,t} + \rho u_{t-1}\right), \tag{15}$$

$$\xi_{v,k} \sim \mathrm{N}(\mu_v, \sigma_v^2), \; P(n_t = j|\mathcal{I}_{t-1}) = \frac{\exp(-\lambda_t)\lambda_t^j}{j!}, \quad j = 0, 1, 2, \ldots, \tag{16}$$

$$\lambda_t = \lambda_0 + \delta\lambda_{t-1} + \psi\zeta_{t-1}, \tag{17}$$

with $c_{i,t}$ defined in (12).

A key component of the model is the jump intensity error in (17), $\zeta_{t-1} \equiv \mathbb{E}[n_{t-1}|\mathcal{I}_{t-1}] - \lambda_{t-1}$. This is an ex post innovation since the term $\mathbb{E}[n_{t-1}|\mathcal{I}_{t-1}]$ is the inference on the average number of jumps at $t-1$ based on $t-1$ information while λ_{t-1} is the ex ante expected number of jumps using information up to $t-2$. By the definition of Poisson distribution, for time t, $\mathbb{E}[n_t|\mathcal{I}_t] = \sum_{j=0}^{\infty} jP(n_t = j|\mathcal{I}_t)$. To infer this probability, after observing RM_t, we use the following filter from the Bayes rule

$$P(n_t = j|\mathcal{I}_t) = \frac{\mathrm{N}(\log(\mathrm{RM}_t)|n_t = j, \mathcal{I}_{t-1})P(n_t = j|\mathcal{I}_{t-1})}{P(\log(\mathrm{RM}_t)|\mathcal{I}_{t-1})}, \quad j = 1, 2, \ldots \tag{18}$$

where $P(n_t = j|\mathcal{I}_{t-1})$ is from (16). The error term ζ_t is a martingale difference sequence, $\mathbb{E}[\zeta_t|\mathcal{I}_{t-1}] = 0$ so that the unconditional value of the jump intensity, given that $|\delta| < 1$, is $\mathbb{E}[\lambda_t] = \frac{\lambda_0}{1-\delta}$.

The C-GARCH-ARJ, similar to the C-GARCH-SARJ model that follows, can generate asymmetry in the distribution as well as thick tails in realized measures. See Chan and Maheu (2002) for conditional skewness and conditional kurtosis values.

For the models posterior estimation we use a hybrid of Metropolis-Hastings and Gibbs steps. See the supplemental file for details.

5. Stochastic autoregressive jumps (C-GARCH-SARJ)

In this section we introduce the main new model that combines elements of the last model with the stochastic autoregressive jump (SARJ) model of Maheu and McCurdy (2008) to jointly model returns and ex post variance measures. The RM jump intensity is governed by a latent and stochastic autoregressive process. This can capture jump clustering and nests non-persistent (iid) jumps. As above a second component of the conditional variance follows a GARCH(1,1).

The new model nests several of the previous specifications. It can capture asymmetry and thick tails in the ex post variance distribution as well as volatility and jump clustering. The C-GARCH-SARJ is specified with (8)–(9) and,

$$\log(\mathrm{RM}_t) = \omega + \sum_{i=1}^{2} \phi_i c_{i,t} + \rho u_{t-1} + J_{v,t}\xi_{v,t} + v_t,$$

$$v_t \sim \mathrm{N}(0, \sigma_t^2), \tag{19}$$

$$\sigma_t^2 = \varpi + a e_{t-1}^2 + b\sigma_{t-1}^2,$$

$$e_t = \log(\mathrm{RM}_t) - \left(\omega + \sum_{i=1}^{2} \phi_i c_{i,t} + \rho u_{t-1}\right), \tag{20}$$

$$\xi_{v,t} \sim \mathrm{N}(\mu_v, \sigma_v^2), \quad J_{v,t} \in \{0,1\},\ P(J_{v,t} = 1|z_t) = \lambda_t, \tag{21}$$

$$\lambda_t = \frac{\exp(z_t)}{1 + \exp(z_t)}, \tag{22}$$

$$z_t = \gamma_0 + \gamma_1 z_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathrm{N}(0,1). \tag{23}$$

$J_{v,t}$ is the jump indicator and $\xi_{v,t}$ is the RM jump size which follows an independent normal distribution. The true $J_{v,t}$, $\xi_{v,t}$ and λ_t are unobserved. They are estimated using observed data. If the data indicate a jump, then $J_{v,t}$ would be estimated close to 1. If there is no jump, $J_{v,t}$ would be approximately 0. If there is considerable uncertainty on the jump signal from the data then $\mathbb{E}(J_{v,t}|\mathcal{I}_T)$ would lie between 0 and 1.

λ_t is the time-varying RM jump intensity, independent of jumps in returns, and governed by the latent autoregressive process z_t. Conditional on z_t a jump occurrence follows a Bernoulli distribution with probability λ_t. The logistic function in (22) ensures that z_t is mapped in the interval $(0,1)$ for λ_t. z_t can be interpreted as the unobserved news flow into the market that impacts variance jumps. The parameter γ_1 captures the jump persistence and lies inside the unit circle, $|\gamma_1| < 1$. The innovations in (23) have a unit variance for identification. The error term e_t in (20) contains the contemporaneous jump component and allows it to affect future volatility through the GARCH factor.

This model has two main differences compared to the C-GARCH-ARJ. First, at most one jump can occur in the C-GARCH-SARJ model while the jump intensity in the C-GARCH-ARJ is governed by a Poisson distribution and can have more than one jump per day. The second and most important difference is in the time-varying jump intensity. The conditional jump intensity of the C-GARCH-ARJ has an ARMA functional form in (17) while the jump intensity in C-GARCH-SARJ model is directed by a latent stochastic AR process. Both can capture jump clustering but the SARJ allows for time-t shocks to affect λ_t while the ARJ does not.

Jumps in the C-GARCH-SARJ model will affect the whole distribution of ex post variance but some insight into the impact on the conditional moments are from

$$\mathbb{E}(\log(\mathrm{RM}_t)|\mathcal{I}_{t-1}, z_t) = \omega + \sum_{i=1}^{2} \phi_i c_{i,t} + \rho u_{t-1} + \lambda_t \mu_\xi, \tag{24}$$

$$\mathrm{Var}(\log(\mathrm{RM}_t)|\mathcal{I}_{t-1}, z_t) = \sigma_t^2 + \lambda_t \sigma_\xi^2, \tag{25}$$

which shows the variance of $\log(\mathrm{RM}_t)$ to be directed by the two components, GARCH and SARJ. In addition, jumps will cause conditional skewness and kurtosis.

The choice of RM as ex post measure of return variance will impact the nature of detected jumps. From (2) and its discrete-time version (8), when RV or RK are used as RM, ex post variance jumps are due to both price and variance. So one would expect little or no return jumps detected in (8), as they would be captured through the jump process in the RM equation. When MRV or BPV is used as RM, ex post variance jumps are solely due to volatility while the price jumps are likely to be captured by the jump process in (8).

We also consider several restricted versions of this model. The first, C-SARJ, sets $a = b = 0$, which omits the GARCH component while the second, C-IJ sets $a = b = 0$ and $\lambda_t = \lambda_v$ so that jumps are independent over time and have a constant jump probability.

5.1. Estimation

Since this model is new we present a brief overview of estimation. To estimate the model, we need to augment the parameters $\Theta = \{\theta, \mu, \mu_r, \sigma_r^2, \lambda_r, \mu_v, \sigma_v^2, \gamma\}$, where $\gamma = \{\gamma_0, \gamma_1\}$ and $\theta = \{\omega, \phi_1, \phi_2, \alpha_1, \alpha_2, \rho, \varpi, a, b\}$, with the latent vectors of the jump sizes $\xi_r = \{\xi_{r,1}, \ldots, \xi_{r,T}\}$, $\xi_v = \{\xi_{v,1}, \ldots, \xi_{v,T}\}$, the jump indicators $J_r = \{J_{r,1}, \ldots, J_{r,T}\}$, $J_v = \{J_{v,1}, \ldots, J_{v,T}\}$ and the jump intensities $z = \{z_1, \ldots, z_T\}$. Following Bayes rule, the model parameters have the following posterior

$$p(\Theta, \xi_r, J_r, \xi_v, J_v, z|\mathcal{I}_T) \propto p(\Theta) p(\xi_r, J_r, \xi_v, J_v, z|\Theta)$$
$$\times \prod_{t=1}^{T} \mathrm{N}(r_t|\mu + J_{r,t}\xi_{r,t}, \mathrm{RM}_t)$$
$$\times \mathrm{N}\left(\log(\mathrm{RM})_t \middle| \omega + \sum_{i=1}^{2} \phi_i c_{i,t} + \rho u_{t-1} \right.$$
$$\left. + J_{v,t}\xi_{v,t}, \sigma_t^2 \right). \tag{26}$$

By augmenting the parameter space with the auxiliary vectors ξ_r, J_r, ξ_v, J_v and z, we can obtain smoothed estimates of them through posterior simulation. Since there are no analytical results for the posterior in (26), we use MCMC methods to obtain a set of posterior draws from a series of conditional distributions. This allows a straightforward estimation with a set of conditional steps. By collecting a large sample (M) of posterior draws, $\{\Theta^{(m)}, \xi_r^{(m)}, J_r^{(m)}, \xi_v^{(m)}, J_v^{(m)}, z^{(m)}\}_{m=1}^{M}$, simulation consistent estimates of various posterior quantities can be computed.

The posterior draws of $\{\Theta, \xi_r, J_r, \xi_v, J_v, z\}$ are obtained through conditional distributions using the following steps:

(1) Sample (a) $p(\mu|\xi_r, J_r, \mathcal{I}_T)$, (b) $p(\mu_r|\sigma_r^2, \xi_r)$ and (c) $p(\sigma_r^2|\mu_r, \xi_r)$.
(2) Sample (a) $p(\xi_{r,t}|\mu_r, \sigma_r^2, J_{r,t}, \mathcal{I}_T)$ and (b) $p(J_{r,t}|\xi_{r,t}, \lambda_r, \mathcal{I}_T)$, $t = 1, \ldots, T$.
(3) Sample $p(\lambda_r|J_r)$.
(4) Sample $p(\theta|\Theta_{-\theta}, r_{1:T}, \log(\mathrm{RM}_{1:T}), z)$.
(5) Sample (a) $p(\mu_v|\sigma_v^2, \xi_v)$ and (b) $p(\sigma_v^2|\mu_v, \xi_v)$.
(6) Sample (a) $p(\xi_{v,t}|\mu_v, \sigma_v^2, J_{v,t}, \mathrm{RM}_t, \sigma_t^2, \theta)$ and (b) $p(J_{v,t}|\xi_{v,t}, z_t, \mathrm{RM}_t, \sigma_t^2, \theta)$, $t = 1, \ldots, T$.
(7) sample $p(z_t|z_{-t}, \gamma, J_{v,t})$, $t = 1, \ldots, T$.
(8) sample $p(\gamma|\Theta_{-\gamma}, z)$.

Details of the sampling steps are in supplemental file B.

6. Density forecasting

In Bayesian forecasting, the main focus is on the predictive density. We compare the various models through density forecasts as jumps will affect the whole distribution and measures based on point forecasts can miss the importance of modeling them. We conduct model comparison with the predictive likelihood, as suggested by Geweke (1994). In the following, we discuss the C-GARCH-SARJ model but the computations can easily be modified for the other specifications.

Given the posterior draws $\Psi = \{\Theta^{(m)}, z^{(m)}\}_{m=1}^{M}$, conditional on the information set $\mathcal{I}_t$, the predictive likelihood of $\log(\mathrm{RM}_t)$ can be approximated as

$$\begin{aligned}
&\hat{p}(\log(\mathrm{RM}_{t+1})|\mathcal{I}_t, \Psi) \\
&\quad = \int p(\log(\mathrm{RM}_{t+1})|\mathcal{I}_t, \Theta, z_t)\, p(\Theta, z_t|\mathcal{I}_t)\, \mathrm{d}\Theta\, \mathrm{d}z_t \\
&\quad \approx \frac{1}{M}\sum_{m=1}^{M} p\left(\log(\mathrm{RM}_{t+1})\middle|\Theta^{(m)}, z_{t+1}^{(m)}, z_t^{(m)}, \mathcal{I}_t\right)
\end{aligned} \tag{27}$$

where m is the mth posterior draw from (26), $m = 1, \ldots, M$, and z_{t+1} is a simulated value. The density of $\log(\mathrm{RM}_{t+1})$ is

$$\begin{aligned}
&p\left(\log(\mathrm{RM}_{t+1})\middle|\Theta^{(m)}, z_{t+1}^{(m)}, z_t^{(m)}, \mathcal{I}_t\right) \\
&\quad = \left(1 - \lambda_{t+1}^{(m)}\right) \mathrm{N}\left(\log(\mathrm{RM}_{t+1})\middle|\omega^{(m)} + \sum_{i=1}^{2}\phi_i^{(m)} c_{i,t+1}^{(m)} \right. \\
&\qquad \left. + \rho^{(m)} u_t, \sigma_{t+1}^{2(m)}\right) \\
&\qquad + \lambda_{t+1}^{(m)} \mathrm{N}\left(\log(\mathrm{RM}_{t+1})\middle|\omega^{(m)} + \sum_{i=1}^{2}\phi_i^{(m)} c_{i,t+1}^{(m)} + \rho^{(m)} u_t \right. \\
&\qquad \left. + \mu_v^{(m)}, \sigma_{t+1}^{2(m)} + \sigma_v^{2(m)}\right),
\end{aligned}$$

with

$$\begin{aligned}
&\sigma_{t+1}^{2(m)} = \varpi^{(m)} + a^{(m)} e_t^{2(m)} + b^{(m)} \sigma_t^{2(m)}, \\
&e_t^{2(m)} = \log(\mathrm{RM}_t) - \left(\omega^{(m)} + \sum_{i=1}^{2}\phi_i^{(m)} c_{i,t}^{(m)} + \rho^{(m)} u_{t-1}\right), \\
&\lambda_{t+1}^{(m)} = \frac{\exp\left(z_{t+1}^{(m)}\right)}{1 + \exp\left(z_{t+1}^{(m)}\right)}, \\
&z_{t+1}^{(m)} = \gamma_0^{(m)} + \gamma_1^{(m)} z_t^{(m)} + \varepsilon_{t+1}, \\
&\varepsilon_{t+1} \sim \mathrm{N}(0, 1), \\
&c_{i,t+1}^{(m)} = (1 - \alpha_i^{(m)}) \log(\mathrm{RM}_{t+1}) + \alpha_i^{(m)} c_{i,t}^{(m)}, \quad i = 1, 2, \quad \text{and} \\
&u_t = \left(r_t - \mu^{(m)}\right)/\sqrt{\mathrm{RM}_t}.
\end{aligned} \tag{28}$$

Similarly, the predictive likelihood of returns can be approximated as

$$\begin{aligned}
\hat{p}\left(r_{t+1}|\mathcal{I}_t, \Psi\right) &= \int p(r_{t+1}|\mathrm{RM}_{t+1}, \mathcal{I}_t, \Theta) \\
&\quad \times p\left(\log(\mathrm{RM}_{t+1}), \Theta, z_{t+1}|\mathcal{I}_t\right) \\
&\quad \mathrm{d}\log(\mathrm{RM}_{t+1})\, \mathrm{d}\Theta\, \mathrm{d}z_{t+1}, \\
&\approx \frac{1}{M}\sum_{m=1}^{M} p\left(r_{t+1}\middle|\Theta^{(m)}, z_{t+1}^{(m)}, z_t^{(m)}, \mathcal{I}_t\right),
\end{aligned} \tag{29}$$

with returns density being

$$\begin{aligned}
&p\left(r_{t+1}\middle|\Theta^{(m)}, z_{t+1}^{(m)}, z_t^{(m)}, \mathcal{I}_t\right) \\
&\quad = \lambda_r^{(m)} \mathrm{N}\left(r_{t+1}\middle|\mu^{(m)} + \mu_r^{(m)}, \widehat{\mathrm{RM}}_{t+1}^{(m)} + \sigma_r^{2(m)}\right) \\
&\qquad + (1 - \lambda_r^{(m)}) \mathrm{N}\left(r_{t+1}\middle|\mu^{(m)}, \widehat{\mathrm{RM}}_{t+1}^{(m)}\right),
\end{aligned} \tag{30}$$

where $\widehat{\mathrm{RM}}_{t+1}^{(m)} = \exp(x)$ and x is drawn as

$$\begin{aligned}
x \sim\ &\lambda_{t+1}^{(m)} \mathrm{N}\left(\omega^{(m)} + \sum_{i=1}^{2}\phi_i^{(m)} c_{i,t+1}^{(m)} + \rho^{(m)} u_t + \mu_v^{(m)}, \right. \\
&\left. \sigma_{t+1}^{2(m)} + \sigma_v^{2(m)}\right) \\
&+ \left(1 - \lambda_{t+1}^{(m)}\right) \mathrm{N}\left(\omega^{(m)} + \sum_{i=1}^{2}\phi_i^{(m)} c_{i,t+1}^{(m)} + \rho^{(m)} u_t, \sigma_{t+1}^{2(m)}\right).
\end{aligned} \tag{31}$$

Future returns can be simulated from its mixture distribution as

$$\begin{aligned}
\widehat{r}_{t+1}^{(m)} \sim\ &\lambda_r^{(m)} \mathrm{N}\left(\mu^{(m)} + \mu_r^{(m)}, \widehat{\mathrm{RM}}_{t+1}^{(m)} + \sigma_r^{2(m)}\right) \\
&+ (1 - \lambda_r^{(m)}) \mathrm{N}\left(\mu^{(m)}, \widehat{\mathrm{RM}}_{t+1}^{(m)}\right).
\end{aligned} \tag{32}$$

The above are used to step-by-step simulate returns and RM, and for likelihood evaluations for horizons longer than a period ahead.

The log-predictive likelihood for a block of data is the summation of individual log-predictive likelihoods of each estimation. The predictive likelihood of a model provides an *out-of-sample* forecast record of a model and naturally allows for model comparison. Two models can be compared using their log-predictive likelihood values with the larger value being favored and strongly favored if the improvement is more than five.

Here, for different forecast horizons $h, h = 1, \ldots, H$, the density forecasts are evaluated over an identical set τ of out-of-sample realized measures $\log(\mathrm{RM}_{T-\tau+1}), \ldots, \log(\mathrm{RM}_T)$ and returns $r_{T-\tau+1:T}$. The cumulative log-predictive likelihood for log(RM), for a forecast horizon h ($\mathrm{logPL}_{v,h}$), is calculated observation-by-observation as

$$\mathrm{logPL}_{v,h}(\log(\mathrm{RM}_{T-\tau+1:T})|\mathcal{I}_T, \Psi) = \sum_{t=T-\tau-h}^{T-h} \log\left(\hat{p}_h(\log(\mathrm{RM}_{t+h})|\mathcal{I}_t, \Psi)\right), \tag{33}$$

where the predictive likelihood on the right-hand side for $h = 1$ is given by (27). Similarly, the cumulative log-predictive likelihood for returns, for a forecast horizon h ($\mathrm{logPL}_{r,h}$), is calculated observation-by-observation as

$$\mathrm{logPL}_{r,h}(r_{T-\tau+1:T}|\mathcal{I}_T, \Psi) = \sum_{t=T-\tau-h}^{T-h} \log\left(\hat{p}_h(r_{t+h}|\mathcal{I}_t, \Psi)\right). \tag{34}$$

where the predictive likelihood on the right-hand side for $h = 1$ is given by (29).

7. Empirical application

7.1. Selection of priors

We assume that return jumps are infrequent with a jump intensity following a beta distribution, $\lambda_r \sim \mathrm{B}(0.01, 2)$. The other parameters in the return equation have priors: $\mu \sim N(0, 100)$, $\mu_r \sim N(0, 0.1)$ and $\sigma_r^2 \sim \mathrm{IG}(4/2, 1/2)$. For the independent ex post variance jumps we use an uninformative beta prior, $\lambda_v \sim \mathrm{B}(1, 1)$. The variance jump parameters have the following priors: $\mu_v \sim \mathrm{N}(0, 0.1)$, $\sigma_v^2 \sim \mathrm{IG}(4/2, 1/2)$, $\gamma \sim \mathrm{N}([0, 0.9]', I)\mathbf{1}_{\{|\gamma_1|<1\}}$, for $\gamma = [\gamma_0, \gamma_1]'$ with γ_1 being inside the unit circle. For each of the parameters in θ we use the uninformative independent prior N(0, 100) with various constraints as defined by the model or stationarity conditions imposed. The detailed list of priors is found in the supplemental file.

7.2. Posterior estimation results

Results from 50 000 MCMC posterior draws, after 20 000 burn-in, for SPX using log(BPV) and AAPL using log(MRV) are in table 1. These model estimates are broadly representative of estimates for the other indices and stocks, with additional results provided in supplemental file C tables 9–16.

As expected table 1 shows the component model has one large parameter value ($\alpha_1 = 0.93 - 0.96$) that captures the long-run movements of log-variance and a small value ($\alpha_2 = 0.31 - 0.49$) that directs the transitory part. Significant GARCH effects are present in both of the ex post measures.

Estimates from the C-SARJ, C-GARCH-SARJ and C-GARCH-ARJ all indicate a significant time-varying jump process in all variance measures. In these models, ex post variance jump arrival is highly persistent and the jump-size variance, σ_v^2, is relatively large. The jump-size mean, μ_v, is uniformly positive meaning that average jumps increase the variance.

The estimated ex post variance jumps (size, occurrence, intensity) are displayed in figures 1(a) for SPX and 1(b) for AAPL. These are variance shocks that are not captured by the conditional GARCH specification. Due to the flexible stochastic jump intensity z_t process, the models can capture non-persistent jumps as well as jump clusters. The AR(1) parameter γ_1, that captures the jump persistence in the C-GARCH-SARJ, is estimated as 0.724 in BPV and 0.766 in MRV. The frequent and persistent variance jumps are not only due to a volatile market. Although μ_v is positive in each series, many jump size declines are estimated in figure 1(a,b)(ii), indicating that jumps can provide a relief valve in dropping variance levels. Nevertheless, this positive relatively large μ_v will result in a conditionally skewed distribution for log(RM). This is a feature omitted in the pure GARCH models.

The inclusion of jumps has a noticeable impact on the conditional heteroskedasticity. As it is shown in figure 2, the GARCH in C-GARCH-SARJ framework mostly captures the persistence of RM's conditional variance while the transitory effects are captured by the stochastic jump process.

In the case of SPX, BPV is used to approximate the IV and there are frequent return jumps. In the case of AAPL, results indicate almost zero probability, ($\lambda_r \approx 0$) of return jumps. This is observed for all the individual stocks data and the measures we examine as we discuss in the following section. The question of which variance measure to use and the role of return jumps is an empirical question which we turn to later when we discuss the forecasts of returns which is common to all model specifications.

7.3. Source of jumps

In this section, we present empirical insights into the estimated ex post variance jumps. As discussed in Section 5, one would expect more frequent ex post variance jumps when RV or RK is used as RM, since these measures capture jumps arising from both volatility and price movements. Few to no jumps are expected in the return equation under these measures. When BPV or MRV is used, any detected jumps in the variance equation should, in theory, be attributed solely to volatility, while more frequent price jumps are expected in the return equation (8). Therefore, we anticipate more frequent ex post variance jumps under RV and RK than under BPV and MRV.

Table 1. Posterior estimation results for SPX-BPV and AAPL-MRV.

	SPX-BPV								AAPL-MRV							
	C-IJ		C-GARCH-ARJ		C-SARJ		C-GARCH-SARJ		C-IJ		C-GARCH-ARJ		C-SARJ		C-GARCH-SARJ	
	Mean	95% D.I.	Mean	95% D.I.	Mean	95% D.I.	Mean	95% D.I.	Mean	95% D.I.	Mean	95% D.I.	Mean	95% D.I.	Mean	95% D.I.
μ	0.092	[0.07, 0.12]	0.089	[0.06, 0.11]	0.087	[0.06, 0.11]	0.089	[0.06, 0.11]	0.094	[0.09, 0.10]	0.095	[0.09, 0.10]	0.094	[0.09, 0.10]	0.094	[0.09, 0.10]
μ_r	0.054	[− 0.07, 0.18]	0.059	[− 0.05, 0.18]	0.071	[− 0.04, 0.20]	0.062	[− 0.05, 0.18]	0.034	[− 0.58, 0.67]	− 0.128	[− 0.74, 0.50]	− 0.089	[− 0.62, 0.47]	− 0.012	[− 0.67, 0.62]
σ_r^2	0.266	[0.16, 0.43]	0.247	[0.14, 0.38]	0.246	[0.15, 0.37]	0.253	[0.16, 0.38]	0.498	[0.09, 2.00]	0.390	[0.083, 1.26]	0.453	[0.09, 1.55]	0.306	[0.09, 1.01]
λ_r	0.272	[0.18, 0.38]	0.287	[0.19, 0.42]	0.284	[0.19, 0.40]	0.280	[0.19, 0.40]	0.002	[0.00, 0.01]	0.002	[0.00, 0.02]	0.001	[0.00, 0.014]	0.001	[0.00, 0.02]
ω	− 0.076	[− 0.10, − 0.05]	− 0.063	[− 0.09, − 0.04]	− 0.062	[− 0.08, − 0.04]	− 0.069	[− 0.10, − 0.05]	− 0.082	[− 0.12, − 0.05]	− 0.091	[− 0.13, − 0.06]	− 0.095	[− 0.13, − 0.07]	− 0.082	[− 0.10, − 0.06]
ϕ_1	0.241	[0.18, 0.32]	0.205	[0.15, 0.27]	0.208	[0.16, 0.27]	0.224	[0.17, 0.30]	0.298	[0.22, 0.38]	0.322	[0.24, 0.40]	0.320	[0.26, 0.39]	0.329	[0.23, 0.43]
ϕ_2	0.729	[0.65, 0.79]	0.775	[0.71, 0.83]	0.771	[0.71, 0.82]	0.751	[0.68, 0.81]	0.630	[0.54, 0.70]	0.610	[0.53, 0.69]	0.609	[0.54, 0.67]	0.607	[0.50, 0.70]
α_1	0.929	[0.89, 0.96]	0.957	[0.93, 0.98]	0.956	[0.93, 0.98]	0.946	[0.92, 0.97]	0.930	[0.88, 0.96]	0.928	[0.89, 0.96]	0.944	[0.92, 0.97]	0.935	[0.88, 0.97]
α_2	0.436	[0.38, 0.49]	0.489	[0.43, 0.53]	0.485	[0.43, 0.53]	0.475	[0.42, 0.52]	0.330	[0.24, 0.41]	0.316	[0.22, 0.40]	0.339	[0.27, 0.41]	0.334	[0.21, 0.45]
ρ	− 0.104	[− 0.12, − 0.09]	− 0.101	[− 0.11, − 0.09]	− 0.101	[− 0.11, − 0.09]	− 0.100	[− 0.11, − 0.09]	− 0.034	[− 0.05, − 0.01]	− 0.037	[− 0.057, − 0.02]	− 0.033	[− 0.051, − 0.02]	− 0.035	[− 0.06, − 0.01]
σ^2	0.223	[0.20, 0.25]			0.171	[0.16, 0.19]			0.233	[0.21, 0.26]			0.219	[0.19, 0.24]		
ϖ			0.057	[0.02, 0.11]			0.004	[0.001, 0.01]			0.004	[0.001, 0.01]			0.003	[0.001, 0.01]
a			0.091	[0.06, 0.13]			0.032	[0.001, 0.06]			0.008	[0.003, 0.01]			0.006	[0.002, 0.01]
b			0.544	[0.19, 0.79]			0.944	[0.90, 0.99]			0.971	[0.94, 0.99]			0.982	[0.96, 0.99]
μ_v	0.223	[0.13, 0.34]	0.057	[0.01, 0.11]	0.097	[0.04, 0.15]	0.191	[0.10, 0.30]	0.764	[0.45, 1.17]	0.601	[0.36, 0.98]	0.614	[0.39, 0.90]	0.699	[0.47, 0.97]
σ_v^2	0.403	[0.31, 0.54]	0.206	[0.14, 0.33]	0.333	[0.29, 0.38]	0.365	[0.27, 0.60]	0.387	[0.19, 0.59]	0.317	[0.11, 0.51]	0.401	[0.26, 0.55]	0.397	[0.24, 0.58]
λ_v	0.212	[0.11, 0.32]							0.114	[0.05, 0.20]						
λ_0			0.007	[0.001, 0.02]							0.081	[0.02, 0.14]				
δ			0.990	[0.98, 1.00]							0.468	[0.12, 0.87]				
ψ			0.205	[0.04, 0.43]							0.101	[0.01, 0.26]				
γ_0					− 0.021	[− 0.06, 0.01]	− 0.527	[− 1.50, − 0.08]					− 0.422	[− 0.73, − 0.20]	− 0.655	[− 1.28, − 0.36]
γ_1					0.978	[0.97, 0.99]	0.724	[0.23, 0.95]					0.835	[0.72, 0.92]	0.766	[0.51, 0.89]

Notes: Results are from 50 000 MCMC posterior draws, after 20 000 burn-in sweeps. For each model are reported the parameter posterion mean and the 95% density interval. C-SARJ is a restricted version of the C-GARCH-SARJ with $a = b = 0$. C-IJ is a restricted version of C-SARJ with $\lambda_t = \lambda_v, \forall t$. The C-GARCH-ARJ and C-GARCH-SARJ are defined as

$$r_t = \mu + \xi_{r,t} J_{r,t} + \sqrt{\mathrm{RM}_t}\epsilon_t, \quad \epsilon_t \sim \mathrm{N}(0,1), \quad \xi_{r,t} \sim \mathrm{N}(\mu_r, \sigma_r^2), \quad J_{r,t} \in \{0,1\}, \quad P(J_{r,t}=1) = \lambda_r, \quad u_t \equiv (r_t - \mu)/\sqrt{\mathrm{RM}_t}.$$

C − GARCH − ARJ :

$$\log(\mathrm{RM}_t) = \omega + \sum_{i=1}^{2} \phi_i c_{i,t} + \rho u_{t-1} + \sum_{k=1}^{n_t} \xi_{v,k} + v_t, \quad v_t \sim \mathrm{N}(0, \sigma_t^2),$$

$$c_{i,t} = (1-\alpha_i)\log(\mathrm{RM}_{t-1}) + \alpha_i c_{i,t-1}, \quad 0 < \alpha_i < 1, \quad i = 1,2, \quad \alpha_1 > \alpha_2,$$

$$\sigma_t^2 = \varpi + a e_{t-1}^2 + b\sigma_{t-1}^2, \quad e_t = \log(\mathrm{RM}_t) - \left(\omega + \sum_{i=1}^{2} \phi_i c_{i,t} + \rho u_{t-1}\right)$$

$$\xi_{v,k} \sim \mathrm{N}(\mu_v, \sigma_v^2),$$

$$P(n_t = j | \mathcal{I}_{t-1}) = \frac{\exp(-\lambda_t)\lambda_t^j}{j!}, \quad j = 0,1,2,\ldots$$

$$\lambda_t = \lambda_0 + (\delta - \psi)\lambda_{t-1} + \psi \mathbb{E}[n_{t-1} | \mathcal{I}_{t-1}].$$

C − GARCH − SARJ :

$$\log(\mathrm{RM}_t) = \omega + \sum_{i=1}^{2} \phi_i c_{i,t} + \rho u_{t-1} + \xi_{v,t} J_{v,t} + v_t, \quad v_t \sim \mathrm{N}(0, \sigma_t^2),$$

$$c_{i,t} = (1-\alpha_i)\log(\mathrm{RM}_{t-1}) + \alpha_i c_{i,t-1}, \quad 0 < \alpha_i < 1, \quad i = 1,2, \quad \alpha_1 > \alpha_2,$$

$$\sigma_t^2 = \varpi + a e_{t-1}^2 + b\sigma_{t-1}^2, \quad e_t = \log(\mathrm{RM}_t) - \left(\omega + \sum_{i=1}^{2} \phi_i c_{i,t} + \rho u_{t-1}\right)$$

$$\xi_{v,t} \sim \mathrm{N}(\mu_v, \sigma_v^2), \quad J_{v,t} \in \{0,1\}, \quad P(J_{v,t} = 1 | z_t) = \lambda_t,$$

$$\lambda_t = \frac{\exp(z_t)}{1 + \exp(z_t)},$$

$$z_t = \gamma_0 + \gamma_1 z_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathrm{N}(0,1).$$

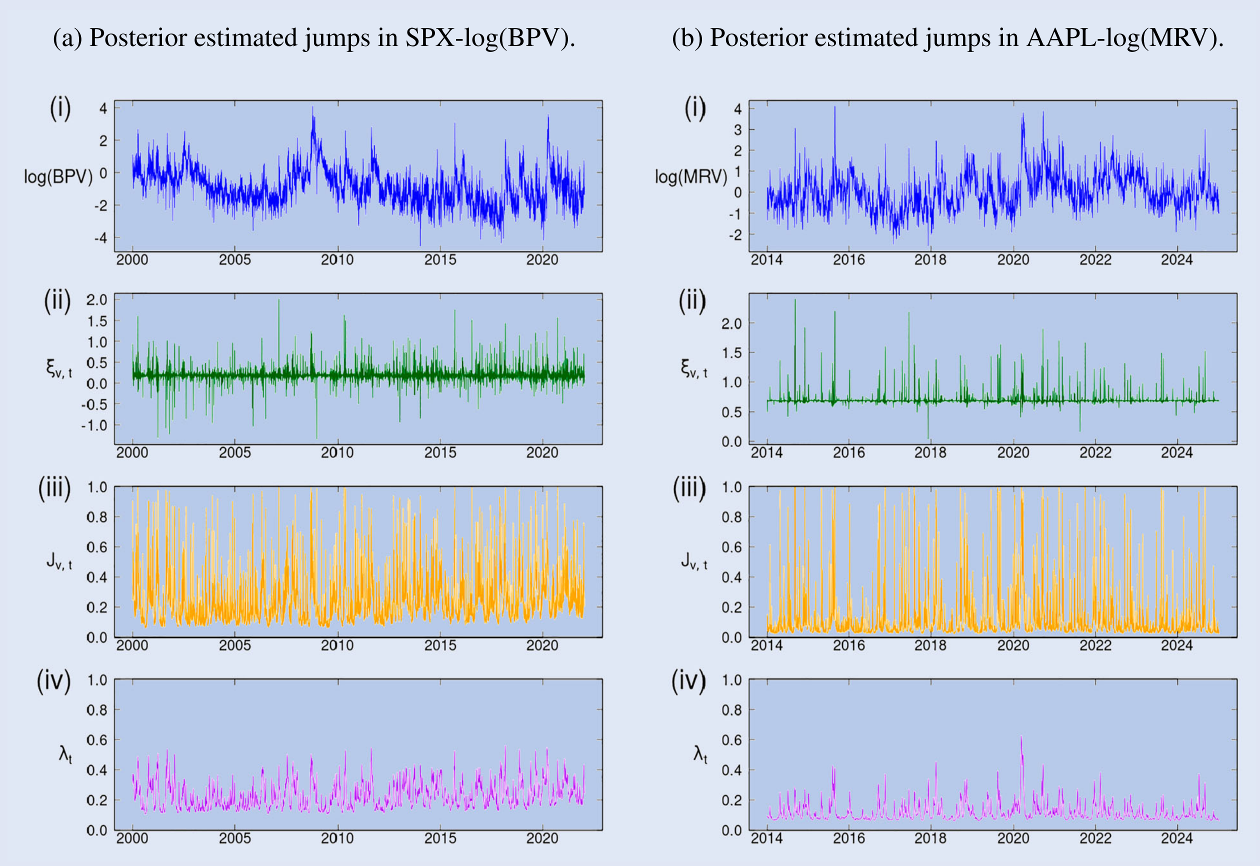


Figure 1. Posterior estimated ex post variance jumps from C-GARCH-SARJ model. From top to bottom: (i) log(RM), (ii) estimated jump size $\mathbb{E}(\xi_{v,t}|\mathcal{I}_T)$, (iii) estimated jump indication $\mathbb{E}(J_{v,t}|\mathcal{I}_T)$, and (iv) jump intensity $\mathbb{E}(\lambda_t|\mathcal{I}_T)$. (a) Posterior estimated jumps in SPX-log(BPV) and (b) Posterior estimated jumps in AAPL-log(MRV).

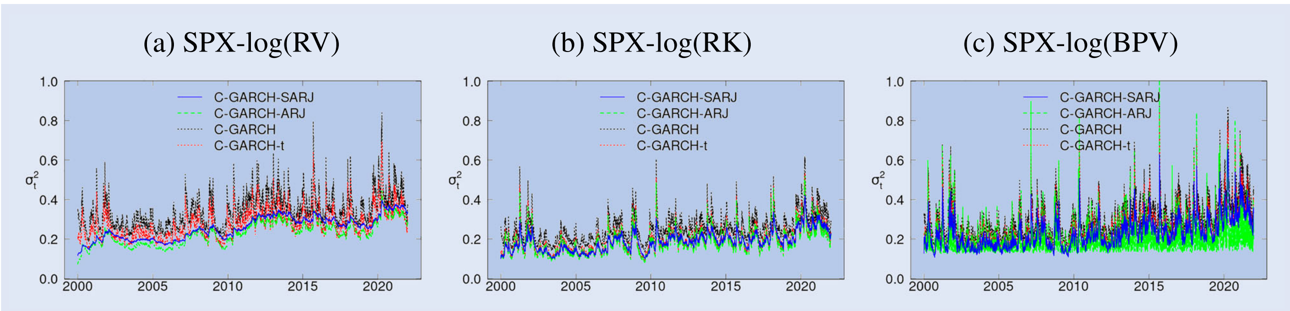


Figure 2. Estimated conditional heteroskedasticity, $\mathbb{E}(\sigma_t^2|\mathcal{I}_T)$, of (a) SPX-log(RV), (b) SPX-log(RK) and (c) SPX-log(BPV), from the GARCH(1,1) framework. (a) SPX-log(RV). (b) SPX-log(RK) and (c) SPX-log(BPV).

The implied unconditional jump intensity from the jump models is reported in table 2. When RV is used as RM, there are no return jumps in SPX, DJI and FTSE markets. For TSX the return jump probability is around 0.15, possibly as a correction to noise in RV data. Interestingly, as we already discussed for SPX, we see that RK gives frequent return jumps and BPV gives the most frequent ones in general, as expected.

In the case of BPV and MRV, what we see in figure 1(a,b) in (ii) are pure variance jumps and in (iii) are the variance jump arrivals. As discussed, in the case of RV and RK, the estimated ex post variance jumps may have a source in the variance or returns as seen in (3). This suggests that more jumps might be detected when RM = RV or RK compared to RM = BPV or MRV, which aligns with the trends observed in table 2, though notable exceptions exist. Of course jump frequency depends on the model specification or misspecification.

In the indices data, the C-IJ model gives less frequent jumps in BPV than in RV. In SPX, the time-dependent jump models estimate more frequent jumps in BPV than in RV and RK. Jumps in SPX-log(RV) occur approximately one every three days from the C-SARJ model with unconditional jump probability being 0.3541. For BPV the probability is 0.4246. These values are halved in the C-GARCH-SARJ model, where

Table 2. Unconditional jump intensity.

	RM	C-IJ Returns	C-IJ RM	C-GARCH-ARJ Returns	C-GARCH-ARJ RM	C-SARJ Returns	C-SARJ RM	C-GARCH-SARJ Returns	C-GARCH-SARJ RM
Panel A: Indices									
DJI	RV	5.2e − 5	0.1805	4.2e − 5	0.2967	6.2e − 5	0.3131	2.4e − 5	0.1221
	RK	0.0992	0.1084	0.0935	0.1557	0.0999	0.2044	0.0916	0.0877
	BPV	0.2745	0.0991	0.2708	0.5542	0.2678	0.2757	0.2696	0.1054
FTSE	RV	7.6e − 5	0.1885	9.8e − 5	0.2967	1.1e − 4	0.1835	1.2e − 4	0.1473
	RK	0.5681	0.1883	0.4672	0.3375	0.3497	0.1842	0.3763	0.1712
	BPV	0.5334	0.0917	0.5152	0.1223	0.5211	0.1141	0.5172	0.0972
SPX	RV	2.5e − 5	0.2302	7.8e − 5	0.3818	4.4e − 5	0.3541	3.9e − 5	0.1633
	RK	0.1559	0.1788	0.1634	0.1861	0.1594	0.3878	0.1611	0.1451
	BPV	0.2722	0.2116	0.2865	0.6490	0.2842	0.4246	0.2795	0.2244
TSX	RV	0.1525	0.2221	0.1537	0.4599	0.1567	0.2572	0.1484	0.1458
	RK	0.3918	0.2063	0.3957	0.3045	0.3934	0.2195	0.3944	0.2043
	BPV	0.4489	0.1486	0.4556	0.2146	0.4614	0.1340	0.4593	0.1240
Panel B: Stocks									
AAPL	RK	1.7e − 4	0.1872	2.7e − 4	0.2249	1.1e − 4	0.1887	5.9e − 5	0.1970
	MRV	0.0014	0.1144	0.0015	0.1525	0.0010	0.1579	0.0012	0.1170
	BPV	4.9e − 4	0.1373	0.0010	0.2486	3.5e − 4	0.2033	4.9e − 4	0.1617
AXP	RK	8.1e − 5	0.1547	1.5e − 4	0.1957	9.8e − 5	0.1641	9.6e − 5	0.1704
	MRV	9.2e − 4	0.0975	4.5e − 4	0.1079	1.4e − 4	0.1246	8.9e − 4	0.1036
	BPV	2.3e − 4	0.1123	6.9e − 4	0.2288	7.8e − 5	0.1419	2.0e − 4	0.1298
IBM	RK	4.6e − 5	0.3164	1.5e − 4	0.3450	4.9e − 5	0.4779	9.9e − 5	0.3518
	MRV	6.1e − 5	0.1044	3.0e − 4	0.1528	1.3e − 4	0.1356	0.0010	0.1334
	BPV	1.3e − 4	0.1040	1.1e − 4	0.1742	9.3e − 4	0.1397	1.8e − 4	0.1276
NKE	RK	3.6e − 4	0.0683	8.6e − 5	0.0086	2.5e − 4	0.0773	1.5e − 4	0.0953
	MRV	0.0096	0.1721	0.0046	0.3480	0.0049	0.1733	0.0014	0.2217
	BPV	0.0023	0.1772	3.9e − 4	0.2984	0.0015	0.2085	2.5e − 4	0.2021

Notes: The unconditional jump intensity $\mathbb{E}(\lambda_t)$ from the C-SARJ models is calculated by averaging $\{\lambda_t^{(m)}\}_{t=1}^T$ for all posterior draws m. For the C-GARCH-ARJ is calculated from the posterior means as: $\mathbb{E}(\lambda_t) = \frac{\lambda_0}{1-\delta}$.

the GARCH component captures some of the volatility but average λ_ts are still large with values 0.0877 − 0.2244.

In the individual stocks and for all the measures considered, the models give strong evidence of variance jumps and almost no evidence of return jumps. All the return jump intensities are below 1%. For AAPL, AXP and IBM, frequent variance jumps in MRV and BPV occur with probability ranging from 0.0975 to 0.2486. RK jumps on average arrive more frequently with probability in values 0.1641 − 0.3518, indicating the presence of price jumps within them. Interestingly, for the case of NKE, there are frequent variance jumps in MRV and BPV with occurrence probability higher than 0.17, while RK jumps are infrequent with probabilities of occurrence less than 0.1.

Figure 3 displays the posterior ex post variance jump probability from C-GARCH-SARJ. For DJI, FTSE and TSX the jumps in RV are clearly more frequent than the variance jumps in BPV but, this is not true for SPX. There are even periods that BPV has higher probability of jump occurrence than RV, for instance, in SPX from 2020 until the end of 2021.

Table 3 presents the regression of the estimated variance jump probability with RV onto the estimated variance jump probability with BPV for the C-GARCH-SARJ model. The variance jumps in BPV explain 53.4–70.3% of RV jumps. This suggests that the majority of jumps in RV are variance jumps and not return jumps.

It is difficult to fully disentangle where jumps are coming from. Nevertheless, we have found strong evidence of jumps in ex post variance measures and some evidence of jumps in returns. Next we consider which of these models and choice of RM are best for forecasts.

Table 3. Regression results of RV and BPV jumps probability.

	DJI	FTSE	SPX	TSX
const.	0.027	0.071	− 0.042	0.040
	(0.001)	(0.002)	(0.003)	(0.002)
β_J	0.906	0.784	0.915	0.853
	(0.008)	(0.010)	(0.010)	(0.010)
R^2	0.703	0.534	0.604	0.613

Notes: This table reports the regression results of the posterior probability of jumps in BPV on the posterior probability of jumps in RV, from the C-GARCH-SARJ model: $P(J_{v,t} = 1|\mathcal{I}_T, \text{RV}) = \text{const.} + \beta_J P(J_{v,t} = 1|\mathcal{I}_T, \text{BPV}) + e_t, e_t \sim \text{N}(0, \sigma_J^2)$. Standard errors are in parentheses.

7.4. Forecasting results

The accuracy of predictive mean point forecasts from different jump models for RM = RV, RK, MRV or BPV is reported in table 4 for all indices and stocks. Out-of-sample forecasts are computed for the last 500 daily observations of each data from December 2019 to the end of sample on December 31st, 2021, for the indices, and from January 5th, 2023 to December 31st, 2024, for the individual stocks. We perform from $h = 1$ step ahead forecasts up to $h = 60$ step ahead. In each case

Figure 3. Posterior estimated jump indication, $P(J_{v,t}=1|\mathcal{I}_T)$, from the C-GARCH-SARJ model for RV (top) and BPV (bottom) measures. (a) DJI. (b) FTSE. (c) SPX and (d) TSX.

the model is re-estimated at each point in the out-of-sample period to produce the forecasts for each of the h periods ahead. We report the daily ($h=1$), weekly ($h=5$), monthly ($h=20$) and 3-months ($h=60$) forecast horizons results.

Overall, there is not a great deal to distinguish between the models. In the case of indices, when RM = RK having no jumps is preferred, while RM = RV or RM = BPV having jumps in the specification helps to improve forecasts, mostly for the daily and weekly horizons. In the case of stocks, a variance jump model specification in most cases favors the daily ($h=1$) forecasts. For the longer horizons there does not seem to be a strong improvement of jumps. Although jumps do not have a large impact on predictive mean forecasts they have an important role for density forecasts which we consider next.

Multiperiod out-of-sample density forecasts results for log(RM) measures are in table 5. The table reports the cumulative log-predictive likelihood for all models for the four indices and the four stocks. Log-predictive Bayes factors (Kass and Raftery 1995) can be formed by subtracting two entries of a column in a subpanel with a common RM. A general result is that jumps improve log(RM) density forecasts. Overall among the models, a jump specification is preferred in 87 out of the 96 cases. The most sophisticated jump specification, C-GARCH-SARJ stands out, delivering the best density forecasts in 36 out of 96 cases. The second best is the C-GARCH-ARJ model with 28 out of 96 cases. In the 51 cases that either C-GARCH-SARJ or its restricted version C-SARJ are the best model, their log-Bayes factor with the second best model is mostly higher than 3, indicating strong preference for stochastic jump arrivals in ex post variance modeling.

Table 6 reports multiperiod out-of-sample return density forecasts for all the data. In these it is possible to make comparisons among all the three realized measures and the six models. In these joint models, returns are common but the choice of RM differs.

In the FTSE market, the C-GARCH-SARJ and C-SARJ models clearly stand out across all forecast horizons while for the DJI, SPX and TSX they are very competitive. For the AXP and NKE stocks, the C-GARCH-SARJ and C-SARJ models along with C-IJ deliver the best forecasts. For AAPL and IBM the C-GARCH benchmarks mostly stand out. The C-GARCH-SARJ model delivers the best return density forecasts in 10 out of 32 cases. The restricted version C-SARJ

Table 4. log(RM) RMSFE for different forecast horizons h.

Panel A: Indices	DJI				FTSE				SPX				TSX			
RM = RV	$h=1$	$h=5$	$h=20$	$h=60$	$h=1$	$h=5$	$h=20$	$h=60$	$h=1$	$h=5$	$h=20$	$h=60$	$h=1$	$h=5$	$h=20$	$h=60$
C-GARCH	0.6617	0.8909	**1.1685**	**1.3253**	0.7512	0.8781	1.0634	1.1800	0.6994	0.9679	**1.2630**	**1.4198**	0.6472	0.8784	1.2302	1.5925
C-GARCH-t	0.6631	0.8978	1.1863	1.3686	0.7441	0.8806	1.0748	1.2376	0.7016	0.9728	1.2798	1.4676	0.6528	0.8748	1.2281	1.6887
C-IJ	0.6631	0.8984	1.1855	1.3624	0.7446	0.8794	1.0662	1.2151	0.7004	0.9746	1.2856	1.4703	0.6305	0.8450	**1.1368**	1.2692
C-GARCH-ARJ	0.6613	0.9026	1.2140	1.4526	0.7522	**0.8760**	1.0839	1.2787	0.6985	0.9760	1.3118	1.5522	**0.6265**	0.8496	1.1468	1.2930
C-SARJ	0.6618	0.8920	1.1777	1.3585	**0.7428**	0.8796	1.0638	**1.2093**	0.7008	0.9724	1.2827	1.4715	0.6313	0.8454	1.1581	1.3157
C-GARCH-SARJ	**0.6580**	**0.8891**	1.1749	1.3431	0.7452	0.8796	**1.0620**	**1.2093**	**0.6973**	**0.9632**	1.2731	1.4353	0.6269	**0.8412**	1.1444	**1.2606**
RM = RK																
C-GARCH	0.5592	**0.8206**	**1.1326**	**1.2880**	**0.5868**	**0.7419**	**0.9651**	**1.1032**	**0.6231**	**0.9026**	**1.2225**	**1.3830**	**0.5411**	0.7815	1.0876	**1.2198**
C-GARCH-t	0.5619	0.8273	1.1588	1.3602	0.6054	0.7791	1.1004	1.5477	0.6282	0.9136	1.2511	1.4633	0.5479	0.7887	1.1003	1.2513
C-IJ	0.5610	0.8284	1.1598	1.3507	0.5912	0.7530	0.9853	1.1404	0.6267	0.9150	1.2595	1.4841	0.5506	0.7845	1.1143	1.3024
C-GARCH-ARJ	0.5588	0.8260	1.1655	1.3976	0.5902	0.7502	0.9908	1.1818	0.6244	0.9063	1.2644	1.5467	0.5523	**0.7759**	**1.0565**	1.3726
C-SARJ	**0.5584**	0.8222	1.1502	1.3425	0.5876	0.7480	0.9800	1.1391	0.6283	0.9159	1.2541	1.4689	0.5413	0.7837	1.0979	1.2662
C-GARCH-SARJ	0.5594	0.8257	1.1546	1.3435	0.5889	0.7439	0.9790	1.1246	0.6239	0.9096	1.2471	1.4389	0.5478	0.7840	1.0929	1.2627
RM = BPV																
C-GARCH	0.6496	0.8966	**1.1717**	**1.3092**	0.5995	0.7843	1.1776	1.8361	0.7274	0.9824	1.2672	**1.4135**	**0.6119**	0.8391	1.1422	**1.2575**
C-GARCH-t	0.6517	0.9002	1.1854	1.3387	0.6002	0.7794	1.1423	1.6782	0.7289	0.9861	1.2780	1.4349	0.6506	0.9604	1.6466	2.1273
C-IJ	0.6500	0.8995	1.1918	1.3621	0.5969	0.7528	0.9900	1.2007	0.7257	0.9931	1.2909	1.4795	0.6277	0.8570	1.1912	1.4039
C-GARCH-ARJ	0.6562	0.9026	1.2101	1.4243	0.5939	0.7560	1.0465	1.4060	0.7246	0.9919	1.3101	1.5401	0.6192	**0.8308**	**1.1184**	1.3706
C-SARJ	0.6510	0.9010	1.1993	1.3442	**0.5880**	**0.7411**	0.9777	**1.1008**	**0.7171**	**0.9796**	**1.2619**	1.4177	0.6150	0.8487	1.1590	1.3143
C-GARCH-SARJ	**0.6490**	**0.8954**	1.1874	1.3441	0.5891	0.7419	**0.9754**	1.1115	0.7283	0.9848	1.2804	1.4516	0.6341	0.8535	1.1591	1.3152

(*Continued*)

Table 4. Continued.

Panel B: Stocks	AAPL				AXP				IBM				NKE			
RM = RK	$h=1$	$h=5$	$h=20$	$h=60$	$h=1$	$h=5$	$h=20$	$h=60$	$h=1$	$h=5$	$h=20$	$h=60$	$h=1$	$h=5$	$h=20$	$h=60$
C-GARCH	0.6273	0.7048	0.7194	0.7608	0.6786	0.7230	**0.7210**	**0.7089**	0.7524	0.7917	0.7776	0.7768	0.6574	0.6639	0.6580	0.6481
C-GARCH-t	0.6268	**0.7015**	**0.7174**	**0.7521**	0.6805	0.7247	0.7222	0.7167	0.7535	0.7907	0.7824	0.7866	0.6550	**0.6635**	0.6589	0.6510
C-IJ	0.6275	0.7044	0.7232	0.7679	0.6784	0.7235	0.7243	0.7101	0.7509	0.7888	0.7780	**0.7760**	0.6573	0.6639	**0.6572**	**0.6445**
C-GARCH-ARJ	0.6303	0.7059	0.7284	0.7759	0.6805	**0.7154**	0.7235	0.7277	0.7547	0.7919	0.7813	0.7876	0.6617	0.6639	0.6621	0.6448
C-SARJ	0.6318	0.7053	0.7290	0.7887	0.6817	0.7254	0.7239	0.7136	0.7534	0.7908	**0.7763**	0.7794	0.6601	0.6657	0.6598	0.6511
C-GARCH-SARJ	**0.6252**	0.7076	0.7303	0.7907	**0.6764**	0.7241	0.7217	0.7183	**0.7508**	**0.7882**	0.7798	0.7796	**0.6548**	0.6636	0.6593	0.6474
RM = MRV																
C-GARCH	0.4944	0.5954	**0.6145**	**0.6501**	**0.4822**	**0.5694**	**0.5985**	0.5697	0.4950	**0.5695**	0.5894	0.5755	0.4765	0.5224	0.5430	0.5162
C-GARCH-t	0.4936	**0.5928**	0.6146	0.6540	0.4839	0.5737	0.6093	0.6035	0.4946	0.5745	0.6020	0.6086	**0.4750**	0.5235	0.5458	0.5279
C-IJ	**0.4929**	0.5964	0.6195	0.6656	0.4835	0.5721	0.6025	**0.5695**	**0.4941**	0.5734	0.5914	**0.5725**	0.4753	**0.5218**	**0.5415**	**0.5150**
C-GARCH-ARJ	0.4952	0.5981	0.6186	0.6794	0.4829	0.5697	0.6050	0.6081	0.4955	0.5888	0.6822	0.8718	0.4821	0.5241	0.5697	0.6065
C-SARJ	0.4954	0.6017	0.6336	0.6777	0.4844	0.5726	0.6001	0.5822	0.4969	0.5756	0.5943	0.5755	0.4815	0.5238	0.5424	0.5223
C-GARCH-SARJ	0.4943	0.6009	0.6336	0.6852	0.4841	0.5730	0.6011	0.5860	0.4962	0.5736	**0.5885**	0.5795	0.4768	0.5226	0.5435	0.5179
RM = BPV																
C-GARCH	**0.4903**	0.5931	0.6158	**0.6522**	0.4710	0.5619	0.5876	**0.5579**	0.4873	**0.5648**	0.5874	0.5706	0.4709	0.5140	0.5306	**0.5050**
C-GARCH-t	0.4905	**0.5909**	**0.6145**	0.6555	0.4717	0.5646	0.6003	0.6025	0.4879	0.5690	0.6012	0.6059	0.4717	0.5123	0.5363	0.5227
C-IJ	0.4905	0.5918	0.6187	0.6603	0.4708	0.5624	0.5923	0.5607	**0.4872**	0.5660	**0.5856**	**0.5701**	0.4693	0.5117	**0.5281**	0.5058
C-GARCH-ARJ	0.4928	0.5940	0.6221	0.7195	0.4732	**0.5600**	**0.5865**	0.6095	0.4896	0.6169	0.8254	1.2916	0.4772	0.5115	0.5561	0.6154
C-SARJ	0.4917	0.5946	0.6277	0.6916	0.4722	0.5657	0.5924	0.5775	0.4896	0.5706	0.5897	0.5774	0.4756	0.5115	0.5310	0.5093
C-GARCH-SARJ	0.4906	0.5978	0.6389	0.7065	**0.4701**	0.5622	0.5877	0.5690	0.4886	0.5688	0.5872	0.5806	**0.4688**	**0.5109**	0.5296	0.5111

Notes: The RMSFE are out-of-sample calculated for the last 500 days of each data set. Bold indicates the lowest value among the models.

Table 5. log(RM) cumulative log predictive likelihoods for different forecast horizons h.

Panel A: Indices	DJI				FTSE				SPX				TSX			
RM = RV	$h = 1$	$h = 5$	$h = 20$	$h = 60$	$h = 1$	$h = 5$	$h = 20$	$h = 60$	$h = 1$	$h = 5$	$h = 20$	$h = 60$	$h = 1$	$h = 5$	$h = 20$	$h = 60$
C-GARCH	– 502.28	– 658.06	– 826.65	– 878.04	– 595.67	– 683.23	– 801.05	– 865.24	– 533.39	– 706.37	– 870.99	– 927.93	– 497.18	– 686.03	– 1061.73	– 1256.78
C-GARCH-t	– 506.60	– 658.21	– 795.33	– 875.54	– 564.01	– 657.35	– 750.39	– 849.69	– 534.72	– 706.31	– 850.20	– 924.64	– 478.17	– 626.27	– 828.10	– 1111.00
C-IJ	– 503.89	– 658.18	– 804.18	– 888.64	– 572.13	– 661.42	– 746.22	– 834.67	– 544.10	– 725.12	– 887.36	– 965.08	– 461.96	– 600.76	**– 766.59**	– 848.62
C-GARCH-ARJ	– 496.39	– 673.96	– 863.81	– 1029.97	– 561.07	– 652.86	– 787.21	– 941.30	**– 526.35**	– 723.67	– 938.97	– 1087.10	**– 449.47**	– 607.40	– 782.68	– 892.37
C-SARJ	– 506.43	– 657.83	– 793.18	– 862.84	– 561.29	– 652.51	– 734.15	– 819.94	– 535.71	– 710.43	– 867.52	– 924.07	– 456.78	**– 596.65**	– 772.14	– 859.38
C-GARCH-SARJ	**– 495.25**	**– 642.79**	**– 777.40**	**– 853.97**	**– 553.30**	**– 641.86**	**– 723.94**	**– 816.48**	– 529.16	**– 693.59**	**– 838.25**	**– 904.43**	– 461.54	– 596.80	– 772.64	**– 848.43**
RM = RK																
C-GARCH	– 419.38	– 620.27	– 821.38	– 907.84	– 449.81	– 570.59	– 729.44	– 803.58	– 479.62	– 677.30	– 865.09	– 945.08	– 391.76	– 588.96	– 861.66	– 936.74
C-GARCH-t	– 422.31	– 619.63	– 783.37	– 885.51	– 431.71	– 572.74	– 746.09	– 1072.91	– 483.50	– 677.72	– 839.14	– 936.23	– 387.74	– 565.66	**– 723.63**	**– 850.18**
C-IJ	– 425.13	– 623.42	– 795.92	– 887.54	– 425.44	– 559.52	– 693.80	– 814.95	– 500.61	– 707.77	– 905.14	– 1004.06	– 401.89	– 590.55	– 844.83	– 976.33
C-GARCH-ARJ	**– 415.06**	– 618.76	– 845.64	– 1038.23	**– 413.46**	– 559.14	– 722.88	– 909.02	**– 478.35**	– 677.32	– 931.84	– 1243.95	– 397.45	– 575.75	– 796.34	– 1059.86
C-SARJ	– 417.62	**– 609.38**	**– 769.44**	**– 849.26**	– 415.92	– 540.35	– 658.89	– 772.81	– 484.72	– 683.22	– 875.32	– 953.35	– 388.73	– 580.68	– 797.05	– 888.28
C-GARCH-SARJ	– 415.77	– 609.73	– 773.58	– 851.07	– 416.24	**– 538.68**	**– 658.45**	**– 768.99**	– 478.92	**– 671.02**	**– 838.40**	**– 916.41**	**– 386.89**	**– 556.15**	– 723.88	– 853.43
RM = BPV																
C-GARCH	**– 494.86**	– 665.52	– 825.75	– 903.14	– 463.13	– 605.61	– 931.82	– 1375.53	– 558.12	– 719.34	– 876.62	– 942.59	– 465.97	– 656.49	– 929.98	**– 958.34**
C-GARCH-t	– 501.46	– 668.70	– 813.47	– 890.64	– 440.07	– 582.99	– 781.34	– 1145.80	– 562.23	– 718.21	**– 866.95**	**– 942.04**	– 491.03	– 718.92	– 1075.20	– 1134.04
C-IJ	– 513.60	– 680.43	– 828.34	– 911.32	– 452.36	– 565.55	– 692.24	– 834.62	– 595.76	– 780.07	– 967.04	– 1032.61	– 477.89	– 647.08	– 859.27	– 991.15
C-GARCH-ARJ	– 495.51	– 662.02	– 854.18	– 1053.31	**– 431.67**	– 561.08	– 750.98	– 1033.23	**– 548.33**	**– 708.96**	– 897.75	– 1101.79	**– 462.82**	**– 629.29**	**– 841.21**	– 1033.19
C-SARJ	– 503.30	– 672.02	– 817.05	– 884.12	– 433.24	**– 540.71**	– 654.93	– 744.02	– 565.60	– 728.26	– 897.60	– 987.18	– 478.69	– 675.53	– 946.06	– 1006.34
C-GARCH-SARJ	– 497.38	**– 659.77**	**– 800.00**	**– 867.81**	– 432.67	– 544.60	**– 649.38**	**– 741.40**	– 561.79	– 717.83	– 872.56	– 942.39	– 483.84	– 676.15	– 956.83	– 1026.31

(*Continued*)

Table 5. Continued.

Panel B: Stocks	AAPL				AXP				IBM				NKE			
RM = RK	$h=1$	$h=5$	$h=20$	$h=60$	$h=1$	$h=5$	$h=20$	$h=60$	$h=1$	$h=5$	$h=20$	$h=60$	$h=1$	$h=5$	$h=20$	$h=60$
C-GARCH	– 475.30	– 534.70	– 553.77	– 583.45	– 517.78	– 550.59	– 562.13	– 578.80	– 568.15	– 593.29	– 595.71	– 606.16	– 497.66	– 506.05	– 518.43	– 528.13
C-GARCH-t	– 471.48	– 529.95	– **546.00**	– **578.20**	– 510.79	– 543.69	– 557.37	– 578.66	– 562.20	– 590.67	– 594.93	– 608.41	– **487.07**	– 499.85	– 513.01	– 523.73
C-IJ	– 472.00	– 533.04	– 554.55	– 591.06	– **507.95**	– 543.00	– 556.51	– 574.95	– **559.96**	– 588.63	– 590.12	– 601.02	– 489.07	– 504.05	– 515.77	– 525.77
C-GARCH-ARJ	– 472.66	– **529.94**	– 552.61	– 589.30	– 509.46	– **538.17**	– **549.14**	– **564.12**	– 560.56	– **587.26**	– **587.47**	– **597.58**	– 490.72	– 500.83	– 515.78	– 525.43
C-SARJ	– 476.51	– 535.15	– 558.35	– 604.31	– 509.92	– 545.84	– 559.16	– 579.78	– 564.87	– 590.32	– 591.37	– 602.40	– 492.26	– 506.10	– 517.17	– 526.95
C-GARCH-SARJ	– **471.47**	– 532.17	– 553.35	– 595.09	– 509.65	– 544.43	– 555.17	– 570.67	– 560.97	– 588.50	– 590.44	– 600.34	– 489.08	– **499.32**	– **512.56**	– **519.22**
RM = MRV																
C-GARCH	– 364.22	– 458.01	– 492.45	– 528.81	– 351.04	– 433.94	– 473.55	– 494.91	– 358.66	– 432.10	– 464.13	– 476.67	– 338.59	– 392.41	– 429.13	– 438.28
C-GARCH-t	– 357.48	– 449.30	– 480.43	– 523.73	– 335.75	– 427.40	– 473.40	– 504.25	– 345.53	– 428.12	– 464.66	– 487.28	– 330.47	– 385.47	– 420.19	– 431.01
C-IJ	– 356.28	– 454.11	– 483.85	– 525.28	– 328.99	– 419.22	– 466.72	– 490.96	– **337.37**	– 423.50	– 454.72	– 470.28	– **327.76**	– 381.45	– 415.87	– 421.95
C-GARCH-ARJ	– 352.26	– **445.72**	– **473.98**	– **520.58**	– 329.17	– **415.87**	– **458.71**	– 484.10	– 342.98	– 437.09	– 521.72	– 682.94	– 330.07	– **376.41**	– 422.69	– 459.78
C-SARJ	– **351.88**	– 452.34	– 485.76	– 528.23	– **327.65**	– 416.15	– 459.29	– **483.83**	– 339.84	– **421.92**	– 451.37	– **465.71**	– 330.84	– 380.79	– **411.51**	– **418.82**
C-GARCH-SARJ	– 354.38	– 449.11	– 487.06	– 532.10	– 331.34	– 419.44	– 463.40	– 490.83	– 338.05	– 422.10	– **450.50**	– 469.36	– 329.74	– 379.59	– 412.56	– 420.12
RM = BPV																
C-GARCH	– 358.95	– 453.73	– 488.77	– 527.08	– 338.57	– 427.22	– 465.05	– 486.37	– 351.76	– 427.07	– 460.95	– 473.69	– 332.46	– 384.13	– 421.63	– 434.02
C-GARCH-t	– 353.00	– 447.01	– 479.41	– 524.33	– 320.53	– 419.11	– 465.25	– 500.87	– 336.36	– 421.91	– 460.02	– 482.88	– 318.32	– 372.72	– 411.19	– 425.14
C-IJ	– 351.93	– 449.94	– 482.21	– **523.86**	– 315.18	– 411.46	– 459.35	– 485.95	– **326.96**	– **416.19**	– 447.98	– 463.71	– 314.26	– 366.66	– 403.02	– 412.01
C-GARCH-ARJ	– 348.88	– **445.00**	– **475.46**	– 569.26	– 315.13	– 407.45	– **443.44**	– 487.75	– 335.99	– 471.58	– 707.17	– 1290.04	– 314.87	– **361.63**	– 410.19	– 468.48
C-SARJ	– **348.57**	– 446.49	– 482.76	– 535.50	– 314.04	– 407.96	– 450.78	– 480.15	– 329.23	– 416.94	– **443.78**	– **460.89**	– 316.37	– 363.99	– 399.32	– 409.36
C-GARCH-SARJ	– 357.63	– 450.66	– 490.10	– 550.16	– **308.47**	– **404.06**	– 445.78	– **475.01**	– 330.22	– 417.00	– 444.16	– 462.74	– **308.85**	– 363.73	– **398.15**	– **404.27**

Notes: The predictive likelihood evaluations are for the last 500 days for each data set. Bold indicates the best value among the models.

Table 6. Returns cumulative log predictive likelihoods for different forecast horizons h.

Panel A: Indices	DJI				FTSE				SPX				TSX			
RM = RV	$h=1$	$h=5$	$h=20$	$h=60$	$h=1$	$h=5$	$h=20$	$h=60$	$h=1$	$h=5$	$h=20$	$h=60$	$h=1$	$h=5$	$h=20$	$h=60$
C-GARCH	– 620.88	– 665.57	– 729.08	– 758.44	– 749.91	– 767.70	– 805.25	– 838.17	– 592.46	– 654.51	– 717.39	– 744.98	– 471.83	– 513.59	– 702.50	– 712.49
C-GARCH-t	– 621.25	– 667.47	– 731.03	– 769.75	– 745.38	– 766.30	– 802.67	– 860.79	– 591.98	– 656.92	– 721.32	– 755.44	– 471.15	– 507.48	– 654.08	– 742.75
C-IJ	– 620.26	– 667.56	– 729.99	– 765.83	– 744.17	– 763.39	– 797.65	– 849.36	– 590.57	– 658.44	– 724.70	– 761.79	– 469.37	– 505.57	– 598.07	– 605.51
C-GARCH-ARJ	– 620.49	– 671.94	– 762.22	– 845.80	– 745.71	– 765.03	– 832.84	– 933.74	– 591.95	– 663.86	– 764.95	– 828.64	– 467.34	– 511.12	– 662.83	– 660.97
C-SARJ	– 621.05	– **664.32**	– 722.57	– 756.40	– **742.96**	– 761.12	– **789.64**	– 838.23	– 591.23	– 655.38	– 713.93	– 745.05	– 468.04	– 503.65	– 588.80	– 618.08
C-GARCH-SARJ	– 619.27	– 665.11	– **721.71**	– 756.32	– 744.29	– **759.75**	– 791.28	– **835.30**	– 592.00	– 651.63	– 712.95	– **736.33**	– **466.22**	– **501.66**	– 597.50	– 591.08
RM = RK																
C-GARCH	– 613.56	– 665.86	– 736.17	– 770.99	– 774.52	– 789.11	– 835.93	– 874.25	– 590.14	– 656.61	– 728.38	– 756.02	– 475.27	– 520.17	– 659.14	– 630.85
C-GARCH-t	– 615.38	– 668.85	– 741.22	– 789.44	– 780.12	– 805.01	– 882.75	– 1020.54	– 590.48	– 657.35	– 732.48	– 774.58	– 474.99	– 518.82	– 600.16	– 625.05
C-IJ	– 614.25	– 667.31	– 738.69	– 777.96	– 763.92	– 780.11	– 827.16	– 873.38	– 590.91	– 664.27	– 745.90	– 787.84	– 474.01	– 515.38	– 628.15	– 629.35
C-GARCH-ARJ	– 613.36	– 668.38	– 764.16	– 837.87	– 772.21	– 800.88	– 924.60	– 1019.01	– 590.10	– 663.23	– 776.79	– 893.90	– 477.44	– 509.89	– 561.84	– 562.60
C-SARJ	– **612.78**	– 665.67	– 726.62	– 763.87	– 762.93	– 773.35	– 811.08	– 855.71	– **589.01**	– 657.81	– 722.29	– 754.35	– 471.10	– 513.10	– 625.04	– 623.12
C-GARCH-SARJ	– 614.07	– 664.69	– 731.10	– 765.04	– 762.52	– 775.99	– 819.40	– 845.18	– 589.33	– 656.74	– 723.85	– 751.50	– 471.36	– 510.46	– 590.99	– 622.74
RM = BPV																
C-GARCH	– 619.34	– 668.10	– 731.18	– 763.38	– 766.44	– 792.35	– 909.34	– 1103.93	– 592.95	– 649.98	– 714.12	– 738.36	– 479.51	– 517.79	– 647.18	– 633.38
C-GARCH-t	– 620.15	– 669.13	– 735.48	– 769.42	– 767.81	– 794.55	– 887.03	– 1045.12	– 592.39	– 649.65	– 718.04	– 742.18	– 478.73	– 519.63	– 816.51	– 992.44
C-IJ	– 619.31	– 670.36	– 732.00	– 770.80	– 757.62	– 772.84	– 818.69	– 880.53	– 589.67	– 658.55	– 726.36	– 779.64	– 480.63	– 521.93	– 653.82	– 651.55
C-GARCH-ARJ	– 618.04	– 666.68	– 754.73	– 854.58	– 761.91	– 788.37	– 891.85	– 1034.99	– 591.53	– **648.62**	– 729.33	– 811.69	– 477.74	– 504.50	– **557.15**	– **552.06**
C-SARJ	– 619.84	– 666.19	– 723.54	– **749.94**	– 750.18	– 765.20	– 804.69	– 850.13	– 590.27	– 648.89	– 714.63	– 751.13	– 477.08	– 519.51	– 654.20	– 632.07
C-GARCH-SARJ	– 619.97	– 667.04	– 727.95	– 760.20	– 753.93	– 764.21	– 803.60	– 849.82	– 592.19	– 648.99	– **712.87**	– 740.66	– 480.28	– 516.88	– 647.43	– 631.44

(*Continued*)

Table 6. Continued.

Panel B: Stocks	AAPL				AXP				IBM				NKE			
RM = RK	$h=1$	$h=5$	$h=20$	$h=60$	$h=1$	$h=5$	$h=20$	$h=60$	$h=1$	$h=5$	$h=20$	$h=60$	$h=1$	$h=5$	$h=20$	$h=60$
C-GARCH	– 694.04	– 700.42	– 705.71	– 710.09	– 771.04	– 771.68	– 770.46	– 772.21	– 636.08	– 639.35	– 638.38	– 640.33	– 782.29	– 777.89	– 779.97	– 779.09
C-GARCH-t	– 693.63	– **700.11**	– **705.13**	– 707.33	– 771.34	– 772.50	– 771.81	– 773.36	– 636.44	– 639.21	– 638.53	– 640.81	– 780.91	– 777.27	– 779.57	– 778.61
C-IJ	– 694.83	– 701.24	– 707.60	– 711.86	– 769.60	– 771.54	– 771.09	– 771.22	– 635.94	– 639.79	– 638.57	– 640.31	– 781.33	– 777.80	– 779.25	– 778.28
C-GARCH-ARJ	– 695.74	– 700.28	– 705.79	– 711.79	– 770.94	– 772.11	– 771.82	– 772.37	– 638.02	– 640.00	– 637.89	– 638.44	– 781.61	– 778.02	– 779.78	– 778.94
C-SARJ	– 695.61	– 701.58	– 710.31	– 714.36	– 770.11	– 771.73	– 770.30	– 772.22	– 637.20	– 640.46	– 638.61	– 641.16	– 781.92	– 777.32	– 778.90	– 779.37
C-GARCH-SARJ	– 693.51	– 701.45	– 706.45	– 714.25	– 769.76	– 772.08	– 770.81	– 771.90	– 635.70	– 639.63	– 639.54	– 640.76	– 782.04	– 777.31	– 779.72	– 778.27
RM = MRV																
C-GARCH	– 693.00	– 701.51	– 707.48	– 708.98	– 768.57	– 772.41	– 771.17	– 770.55	– 635.40	– 635.98	– 634.87	– 636.49	– 783.80	– 775.91	– 777.23	– 776.01
C-GARCH-t	– 692.75	– 700.55	– 706.01	– 705.65	– 768.55	– 773.49	– 772.78	– 773.58	– 635.17	– 636.25	– **634.84**	– 636.54	– 782.41	– 776.23	– 778.26	– 776.54
C-IJ	– 693.82	– 701.94	– 708.91	– 709.74	– **766.58**	– 771.50	– 771.35	– 770.01	– 634.71	– **635.44**	– 636.16	– 637.95	– **779.83**	– 775.85	– 776.85	– 774.67
C-GARCH-ARJ	– 694.15	– 701.48	– 707.84	– 710.66	– 769.06	– 774.28	– 772.96	– 775.50	– 635.23	– 638.07	– 642.68	– 670.50	– 782.32	– 775.92	– 783.17	– 785.19
C-SARJ	– 694.78	– 703.36	– 710.45	– 713.73	– 767.18	– 771.13	– 769.90	– 772.48	– 635.41	– 636.03	– 636.66	– 638.12	– 781.94	– 774.70	– 777.42	– 774.74
C-GARCH-SARJ	– 693.28	– 701.87	– 710.08	– 714.95	– 767.30	– 773.13	– 771.30	– 770.64	– 634.95	– 636.00	– 635.99	– 638.48	– 781.36	– 774.64	– 778.98	– 774.51
RM = BPV																
C-GARCH	– **692.28**	– 701.52	– 707.90	– 709.91	– 769.93	– 772.30	– 770.92	– 770.45	– 634.01	– 636.17	– 635.29	– **636.25**	– 783.71	– 776.01	– 777.55	– 775.74
C-GARCH-t	– 692.39	– 700.71	– 705.54	– **705.24**	– 769.69	– 773.50	– 773.51	– 774.63	– 634.00	– 636.20	– 634.97	– 636.26	– 782.93	– 775.72	– 778.08	– 777.77
C-IJ	– 693.29	– 701.59	– 708.58	– 710.71	– 767.32	– 771.64	– 770.73	– **769.88**	– **633.30**	– 636.71	– 635.76	– 637.90	– 780.07	– 774.80	– 777.37	– **774.07**
C-GARCH-ARJ	– 692.73	– 701.43	– 707.72	– 713.86	– 769.39	– 773.83	– 771.60	– 773.32	– 634.99	– 642.50	– 666.03	– 775.57	– 784.63	– 776.86	– 783.73	– 785.61
C-SARJ	– 693.99	– 703.13	– 710.73	– 715.16	– 768.96	– 771.95	– **769.09**	– 770.95	– 634.10	– 637.88	– 637.10	– 639.92	– 780.59	– 774.22	– 777.39	– 776.19
C-GARCH-SARJ	– 692.67	– 702.96	– 711.50	– 715.96	– 767.13	– **771.12**	– 770.33	– 770.41	– 634.30	– 636.58	– 635.98	– 638.44	– 780.81	– **773.21**	– **776.58**	– 775.16

Notes: The predictive likelihood evaluations are for the last 500 days for each data set. Bold indicates the best value for each forecast horizon among all models and RMs.

is the second best with 7 out of 32 cases. The benchmarks C-GARCH-ARJ and C-IJ models are the best in 3 and 6 out of 32 cases, respectively. In the indices cases that either C-GARCH-SARJ or C-SARJ are the best model, their log-Bayes factor with the second best model is higher than 1 and mostly higher than 5. For the individual stocks, all the models seem to perform very close and the best ones emerge marginally. Overall, with 26 out of 32 cases where an ex post variance jump specification is the best, highlights the importance of modeling the variance jump dynamics for predicting the returns distribution.

Regarding the choice of ex post variance measure for forecasting return density, the jump-robust MRV and BPV provide the best forecasts in 19 out of 32 cases while RV and RK are the best in 13 out of 32 cases. For the indices, RV is usually preferred, while the jump-robust BPV is preferred for the individual stocks.

To understand the model differences we turn to plots of log-predictive distributions. Figure 4 displays the one-step ahead out-of-sample log-predictive densities from the full sample. For the three realized measures, there are similar trends. The C-GARCH-t gives a very fat-tailed shape, perhaps too fat tailed based on density forecast comparisons above while the C-GARCH-ARJ produces thinner tails for both returns and RM. The C-GARCH-SARJ shows clear asymmetry by putting more probability mass to the right-hand side of the distribution of log(RV), log(RK) and log(BPV). These findings support previous literature results that empirically RV is not log-normally distributed.

7.5. *Tail risk forecasting*

This section considers the importance of ex post variance jumps to return tail forecasts. The models are compared based on the tail risk measures of value-at-risk (VaR) and expected shortfall (ES).

VaR is a quantile measure based on a confidence level ϵ. Given an investment time interval the VaR is the least potential loss that could happen to an investment ϵ% of the time. For a one period investment, $\mathrm{VaR}^{\epsilon}_{t+1}$ is defined as $P[r_{t+1} \leq \mathrm{VaR}^{\epsilon}_{t+1} | \mathcal{I}_t] = \epsilon$. VaR does not consider the magnitude of the potential loss. This is better accessed by ES which is the measure of expected loss conditional on exceeding the VaR value. For a one-period horizon, ES is defined as $\mathrm{ES}^{\epsilon}_{t+1} = \mathbb{E}[r_{t+1} | r_{t+1} \leq \mathrm{VaR}^{\epsilon}_{t+1}, \mathcal{I}_t]$.

To estimate these values we take a large number of simulated draws from each model's predictive density $\{r^{(m)}_{t+h}\}^{M}_{m=1}$, from (32) for $h = 1$,† and compute the empirical quantile for the associated $\mathrm{VaR}^{\epsilon}_{t+1}$. The ES is estimated from these simulations as

$$\mathrm{ES}^{\epsilon}_{t+1} = \sum_{i=1}^{M} \hat{r}^{(m)}_{t+1} \mathbf{1}_{\left\{\hat{r}^{(m)}_{t+1} \leq \mathrm{VaR}^{\epsilon}_{t+1}\right\}} \Big/ \sum_{i=1}^{M} \mathbf{1}_{\left\{\hat{r}^{(m)}_{t+1} \leq \mathrm{VaR}^{\epsilon}_{t+1}\right\}}.$$

† We only consider the daily tail risk forecast since for longer horizons it would be better to consider the cumulative return distribution from close-to-close daily returns. This would not allow us to use the MRV and BPV measures.

To compare models we use the joint loss function of Patton *et al.* (2019) defined as

$$\mathcal{L}(r_{t+1}, \mathrm{VaR}^{\epsilon}_{t+1}, \mathrm{ES}^{\epsilon}_{t+1}, \epsilon) = -\frac{\left(\mathrm{VaR}^{\epsilon}_{t+1} - r_{t+1}\right)}{\epsilon \mathrm{ES}^{\epsilon}_{t+1}} \mathbf{1}_{\{r_{t+1} \leq \mathrm{VaR}^{\epsilon}_{t+1}\}} + \frac{\mathrm{VaR}^{\epsilon}_{t+1}}{\mathrm{ES}^{\epsilon}_{t+1}} + \log\left(-\mathrm{ES}^{\epsilon}_{t+1}\right) - 1. \tag{35}$$

Smaller average $\mathcal{L}(r_{t+1}, \mathrm{VaR}^{\epsilon}_{t+1}, \mathrm{ES}^{\epsilon}_{t+1}, \epsilon)$ values over the out-of-sample data indicate more accurate tail measures.

Table 7 reports the models out-of-sample performance in jointly forecasting VaR and ES at the 1% and 5% levels, based on the average loss function from (35), across both equity indices and individual stocks. Although our analysis is from a Bayesian perspective, a referee suggested the Model Confidence Set ranking (MCS) of Hansen *et al.* (2011) for the best models at the 95% confidence level. The results are included in the table.

For equity indices (Panel A), the SARJ models tend to outperform alternatives in both levels across most indices and realized measures. The C-SARJ model, for instance, frequently achieves the lowest loss and is included in the MCS for DJI, FTSE, and TSX, especially when using RK and BPV as realized measures. The C-GARCH-ARJ model also shows strong performance, particularly for DJI and TSX.

For individual stocks (Panel B), the SARJ models again perform competitively. In particular, C-SARJ and C-GARCH-SARJ achieve low losses and strong MCS inclusion for stocks like AXP and IBM. The results suggest that incorporating stochastic jump dynamics significantly improves tail risk forecasting for both indices and individual assets. The benchmarks C-GARCH(-t) are strong performers for AAPL and in a handful of occasions for the rest of series.

Like the density forecasts in table 6 there is no dominant RM for tail risk forecasting. This is consistent with Jin *et al.* (2021) who show that the method of realized covariance construction has important implications for forecast accuracy and financial decisions. This suggests the need for either pretesting for the optimal RM for an asset or the use of more sophisticated approaches that average or pool different models over different RM specifications.

Figure 5 provides more information about the impact of ex post variance jumps in tail risk forecasting. It displays the different in period-by-period values of (35) for the C-GARCH-SARJ (jump) model minus the C-GARCH (no jump) model. Positive values favor the C-GARCH-SARJ model. Throughout the out-of-sample forecasting period the models perform similarly. However, the jump model records large gains over the no jumps model in a handful of occasions where the daily stock return plummets.

In summary, models with sophisticated ex post variance jump processes improve return tail risk measure forecasting for the markets and most of the stocks we consider.

Table 7. Tail risk forecasting results.

		DJI				FTSE				SPX				TSX			
		1%		5%		1%		5%		1%		5%		1%		5%	
Panel A: Indices	RM	Loss	MCS	Loss	MCS	Loss	MCS	Loss	MCS	Loss	MCS	Loss	MCS	Loss	MCS	Loss	MCS
C-GARCH	RV	1.202	17	0.806	11	2.050	5	1.215	10	1.342	6	0.784		1.281		0.627	
C-GARCH-t		1.143	11	0.822	14	2.073	6	1.198	3	1.320	8	0.784	7	1.251		0.640	
C-IJ		1.135	10	0.824	15	2.086	9	1.218	9	1.425	14	0.833		1.009	5	0.580	13
C-GARCH-ARJ		1.136	16	0.802	10	2.003	4	1.178	2	1.434		0.777	5	0.995	2	0.551	3
C-SARJ		1.180	13	0.807	6	**1.904**	**1**	**1.161**	**1**	1.416	12	**0.726**	**1**	1.015	4	0.578	12
C-GARCH-SARJ		1.073	5	0.783	3	2.030	3	1.192	5	1.260	2	0.768		1.014	3	0.562	5
C-GARCH	RK	1.064	4	0.795	7	2.678		1.396		1.261	4	0.789	6	1.075	8	0.561	7
C-GARCH-t		1.110	7	0.802	9	2.747		1.459		1.359	7	0.828		1.127	10	0.576	9
C-IJ		1.108	9	0.808	12	2.362	10	1.327	8	1.416	11	0.848		1.183	11	0.579	10
C-GARCH-ARJ		1.039	2	0.800	8	2.699		1.363		1.431	13	0.855		1.139	12	0.595	15
C-SARJ		**1.005**	**1**	0.772	2	2.367		1.356		1.371	10	0.822		1.045	7	**0.533**	**1**
C-GARCH-SARJ		1.071	6	0.792	5	2.370	7	1.289	7	1.336	9	0.812		1.052	6	0.557	6
C-GARCH	BPV	1.149	14	0.815	16	2.650		1.362		1.278	5	0.744	4	1.200		0.571	8
C-GARCH-t		1.131	12	0.836		2.546		1.363		**1.212**	**1**	0.741	3	1.375		0.644	
C-IJ		1.124	8	0.825	13	2.238	8	1.262	6	1.490		0.813		1.181	13	0.581	11
C-GARCH-ARJ		1.250		**0.771**	**1**	2.260	11	1.301		1.413	15	0.728	2	**0.970**	**1**	0.555	2
C-SARJ		1.079	3	0.782	4	1.995	2	1.208	4	1.668		0.868		1.134	9	0.558	4
C-GARCH-SARJ		1.174	15	0.815	17	2.198	12	1.266	11	1.255	3	0.775		1.203	14	0.577	14

(*Continued*)

Table 7. Continued.

		AAPL				AXP				IBM				NKE			
		1%		5%		1%		5%		1%		5%		1%		5%	
Panel B: Stocks	RM	Loss	MCS	Loss	MCS	Loss	MCS	Loss	MCS	Loss	MCS	Loss	MCS	Loss	MCS	Loss	MCS
C-GARCH	RK	1.062	8	0.748	9	1.576	7	1.061	11	1.118	8	0.703	6	1.175	10	0.884	4
C-GARCH-t		1.063	6	0.744	5	1.554	5	1.057	8	1.110	4	0.722	15	1.195	14	**0.872**	**1**
C-IJ		1.049	5	0.741	3	1.514	2	1.040	2	1.104	2	0.694	2	1.182	13	0.890	13
C-GARCH-ARJ		1.110	12	0.780	17	1.570	3	1.062	10	1.114	7	0.733		1.182	11	0.892	9
C-SARJ		1.074	13	0.745	4	**1.504**	**1**	**1.034**	**1**	1.137	15	0.695	3	1.216	16	0.892	14
C-GARCH-SARJ		1.059	9	0.745	7	1.621	13	1.059	6	**1.086**	**1**	0.709	9	1.167	12	0.883	3
C-GARCH	MRV	1.087	11	0.742	12	1.635	10	1.062	4	1.138	16	0.710	12	1.140	3	0.901	12
C-GARCH-t		1.055	7	0.744	13	1.685	16	1.082	17	1.139	14	0.715	16	1.161	8	0.899	10
C-IJ		1.055	16	0.742	11	1.683	15	1.082	18	1.113	9	0.704	10	1.163	9	0.883	2
C-GARCH-ARJ		1.081	15	0.752	14	1.715	18	1.064	3	1.111	6	**0.689**	**1**	1.195	15	0.914	17
C-SARJ		1.119		0.760	16	1.597	4	1.070	5	1.143	13	0.697	7	1.143	7	0.892	7
C-GARCH-SARJ		1.045	4	0.740	6	1.697	14	1.074	14	1.120	12	0.698	8	1.145	2	0.905	16
C-GARCH	BPV	**1.019**	**1**	**0.733**	**1**	1.601	9	1.070	15	1.141	11	0.712	14	1.158	6	0.889	6
C-GARCH-t		1.053	3	0.735	2	1.674	12	1.086	16	1.185	17	0.711	13	1.196		0.904	15
C-IJ		1.065	10	0.739	10	1.577	8	1.075	12	1.109	5	0.697	5	1.142	4	0.891	11
C-GARCH-ARJ		1.097		0.756	15	1.711	17	1.067	7	1.226		0.724	17	1.225		0.941	
C-SARJ		1.066	14	0.775		1.589	6	1.077	13	1.107	3	0.696	4	**1.114**	**1**	0.889	8
C-GARCH-SARJ		1.027	2	0.740	8	1.598	11	1.065	9	1.116	10	0.705	11	1.138	5	0.886	5

Notes: The table reports the average out-of-sample losses, in the columns Loss, calculated as: $\frac{1}{\tau}\sum_{t=T-\tau-1}^{T-1}\mathcal{L}(r_{t+1}, \text{VaR}_{t+1}^{\epsilon}, \text{ES}_{t+1}^{\epsilon}, \epsilon)$, for $\epsilon = 1\%$ and 5%. The MCS columns report the rank of models in the 95% model set. Models without a rank are not included in the set. Bold indicates the top ranked model, among all the models and realized measures. The results are for the last 500 days of each data set.

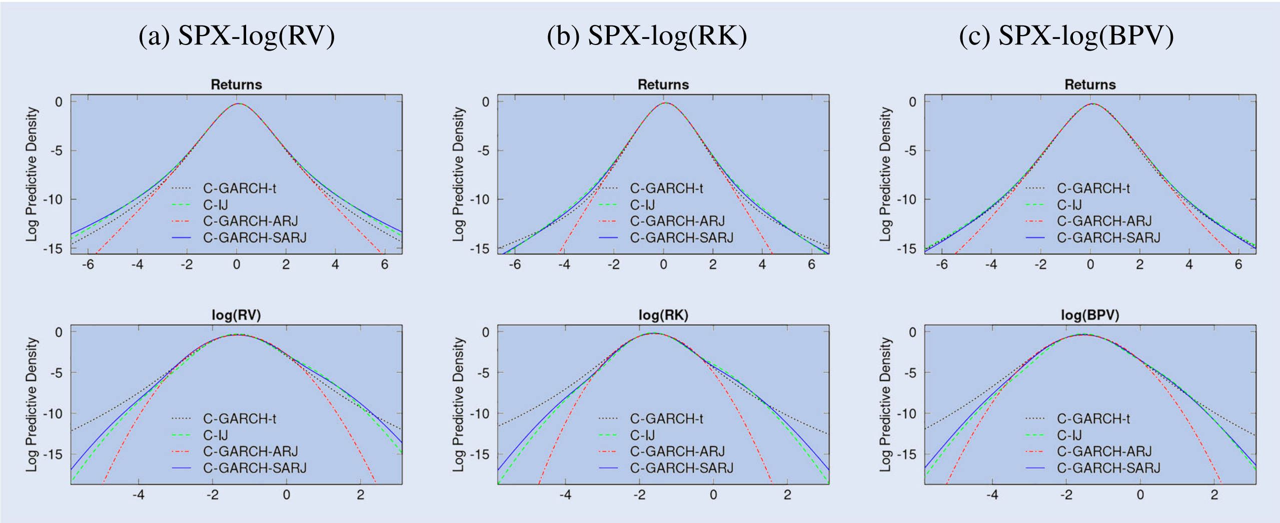


Figure 4. Daily out-of-sample log-predictive densities for SPX returns (top row) and log-variance (bottom row) from models with log(RV) in (a), log(RK) in (b) and log(BPV) in (c). The densities are calculated by evaluating grids with the full sample posterior draws. (a) SPX-log(RV). (b) SPX-log(RK) and (c) SPX-log(BPV).

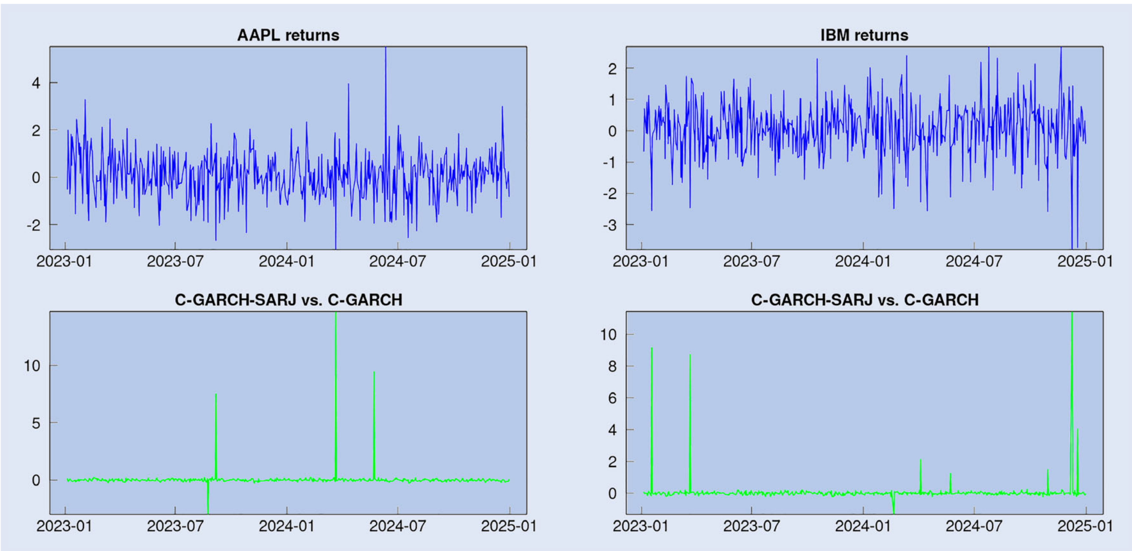


Figure 5. Tail risk forecasting comparison between C-GARCH-SARJ (jumps model) against C-GARCH (no jumps model), with the measure MRV. Top: realized returns. Bottom: $\mathcal{L}_{\mathrm{C-GARCH}}(r_{t+1}, \mathrm{VaR}^{1\%}_{t+1}, \mathrm{ES}^{1\%}_{t+1}, 1\%) - \mathcal{L}_{\mathrm{C-GARCH-SARJ}}(r_{t+1}, \mathrm{VaR}^{1\%}_{t+1}, \mathrm{ES}^{1\%}_{t+1}, 1\%)$. Values greater than zero are gains of C-GARCH-SARJ against C-GARCH.

8. Concluding remarks

This paper considers the importance of jump dynamics in ex post variance measures through an empirically realistic discrete-time joint model framework of returns and their ex post variance measure. To this we add a jump process in the variance equation. We consider a number of popular jump specifications as well as time-dependent versions. These latter versions allow jump arrivals to cluster. We also include a jump process in the return equation to capture price jumps.

Applied to data from four markets and four stocks we find evidence of time dependent ex post variance jumps. Depending on the realized measure, these are either pure variance jumps or a mix with return jumps. In all individual stocks, we find no evidence of return jumps, for the three realized measures we examine. Jumps strongly improve density forecasts of realized variance measures and returns. However, they have minimal impact on point forecasts. We find that models with more sophisticated time-varying jump arrivals are preferred.

The choice of the realized variance measure is also important to return density forecasts. Generally for the indices, RV is preferred coupled with a time-varying arrival rate of jumps but there are important exceptions where RK and BPV are preferred. For the individual stocks, BPV or MRV are preferred coupled with a jump specification.

We also consider the impact of ex post variance jumps in returns tail risk forecasting. Our proposed models with sophisticated ex post variance jump processes improve the jointly forecasts of value-at-risk and expected shortfall.

Our work on ex post variance jumps could be expanded into a multivariate setting. Modeling correlated jumps as well as spillover effects from one jump process to another would be a good way to capture tail risk dynamics amongst several assets. We leave this for future work.

Acknowledgments

We are grateful for helpful comments from two anonymous referees, an associate editor as well as feedback from Chenxing Li, Ilias Chronopoulos, Lazaros Symeonidis, and the participants of Essex Business School Finance Group Workshop. The authors declare no conflict of interest. Maheu thanks SSHRC of Canada for financial support.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Supplemental data

Supplemental data for this article can be accessed online at http://dx.doi.org/10.1080/14697688.2025.2565290.

References

Aït-Sahalia, Y. and Mancini, L., Out of sample forecasts of quadratic variation. *J. Econom.*, 2008, **147**(1), 17–33.
Andersen, T.G., Bollerslev, T., Diebold, F.X. and Ebens, H., The distribution of realized stock return volatility. *J. Financ. Econ.*, 2001a, **61**(1), 43–76.
Andersen, T.G., Bollerslev, T., Diebold, F.X. and Labys, P., The distribution of realized exchange rate volatility. *J. Am. Stat. Assoc.*, 2001b, **96**(453), 42–55.
Andersen, T.G., Bollerslev, T., Diebold, F.X. and Labys, P., Modeling and forecasting realized volatility. *Econometrica*, 2003, **71**(2), 579–625.
Andersen, T.G., Bollerslev, T. and Diebold, F.X., Roughing it up: Including jump components in the measurement, modeling, and forecasting of return volatility. *Rev. Econ. Stat.*, 2007, **89**(4), 701–720.
Andersen, T.G., Bollerslev, T. and Huang, X., A reduced form framework for modeling volatility of speculative prices based on realized variation measures. *J. Econom.*, 2011, **160**(1), 176–189.
Andersen, T.G., Dobrev, D. and Schaumburg, E., Jump-robust volatility estimation using nearest neighbor truncation. *J. Econom.*, 2012, **169**(1), 75–93.
Bandi, F.M. and Russell, J.R., Microstructure noise, realized variance, and optimal sampling. *Rev. Econ. Stud.*, 2008, **75**(2), 339–369.
Barndorff-Nielsen, O.E. and Shephard, N., Econometric analysis of realized volatility and its use in estimating stochastic volatility models. *J. R. Stat. Soc.: Ser. B (Stat. Methodol.)*, 2002a, **64**(2), 253–280.
Barndorff-Nielsen, O.E. and Shephard, N., Estimating quadratic variation using realized variance. *J. Appl. Econom.*, 2002b, **17**(5), 457–477.
Barndorff-Nielsen, O.E. and Shephard, N., Power and bipower variation with stochastic volatility and jumps. *J. Financ. Econom.*, 2004, **2**(1), 1–37.
Barndorff-Nielsen, O.E., Hansen, P.R., Lunde, A. and Shephard, N., Designing realized kernels to measure the ex post variation of equity prices in the presence of noise. *Econometrica*, 2008, **76**(6), 1481–1536.
Bollerslev, T., Kretschmer, U., Pigorsch, C. and Tauchen, G., A discrete-time model for daily S&P500 returns and realized variations: Jumps and leverage effects. *J. Econom.*, 2009, **150**(2), 151–166.
Bollerslev, T., Patton, A.J. and Quaedvlieg, R., Modeling and forecasting (un)reliable realized covariances for more reliable financial decisions. *J. Econom.*, 2018, **207**(1), 71–91.
Callot, L.A.F., Kock, A.B. and Medeiros, M.C., Modeling and forecasting large realized covariance matrices and portfolio choice. *J. Appl. Econom.*, 2017, **32**(1), 140–158.
Caporin, M., Rossi, E. and Magistris, P.S. d., Volatility jumps and their economic determinants. *J. Financ. Econom.*, 2015, **14**(1), 29–80.
Chan, K.F. and Gray, P., Volatility jumps and macroeconomic news announcements. *J. Fut. Mark.*, 2018, **38**(8), 881–897.
Chan, W.H. and Maheu, J.M., Conditional jump dynamics in stock market returns. *J. Bus. Econ. Stat.*, 2002, **20**(3), 377–389.
Chen, J., Clements, M.P. and Urquhart, A., Modeling price and variance jump clustering using the marked hawkes process. *J. Financ. Econom.*, 2024, **22**(3), 743–772.
Corsi, F., A simple approximate long-memory model of realized volatility. *J. Financ. Econom.*, 2009, **7**(2), 174–196.
Corsi, F. and Renò, R., Discrete-time volatility forecasting with persistent leverage effect and the link with continuous-time volatility modeling. *J. Bus. Econ. Stat.*, 2012, **30**(3), 368–380.
Corsi, F., Mittnik, S., Pigorsch, C. and Pigorsch, U., The volatility of realized volatility. *Econom. Rev.*, 2008, **27**(1-3), 46–78.
Due, D., Pan, J. and Singleton, K., Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 2000, **68**(6), 1343–1376.
Eraker, B., Do stock prices and volatility jump? Reconciling evidence from spot and option prices. *J. Finance.*, 2004, **59**(3), 1367–1403.
Fleming, J., Kirby, C. and Ostdiek, B., The economic value of volatility timing using "realized" volatility. *J. Financ. Econ.*, 2003, **67**(3), 473–509.
Geweke, J, Bayesian comparison of econometric models, Technical Report, Citeseer, 1994.
Hansen, P.R. and Lunde, A., Realized variance and market microstructure noise. *J. Bus. Econ. Stat.*, 2006, **24**(2), 127–161.
Hansen, P.R., Lunde, A. and Nason, J.M., The model confidence set. *Econometrica*, 2011, **79**(2), 453–497.
Heber, G., Lunde, A., Shephard, N. and Sheppard, K, Oxford-Man Institute's Realized Library, version 0.1, 2009.
Jeon, Y., McCurdy, T.H. and Zhao, X., News as sources of jumps in stock returns: Evidence from 21 million news articles for 9000 companies. *J. Financ. Econ.*, 2022, **145**(2), 1–17.
Jin, X., Maheu, J.M. and Yang, Q., Bayesian parametric and semiparametric factor models for large realized covariance matrices. *J. Appl. Econom.*, 2019, **34**(5), 641–660.
Jin, X., Liu, J. and Yang, Q., Does the choice of realized covariance measures empirically matter? A Bayesian density prediction approach. *Econometrics*, 2021, **9**(4), 45.
Johannes, M., Kumar, R. and Polson, N.G., State dependent jump models: How do US equity indices jump. Woking Paper, University of Chicago, 1999.
Kass, R.E. and Raftery, A.E., Bayes factors. *J. Am. Stat. Assoc.*, 1995, **90**(430), 773–795.
Lee, S.S., Jumps and information flow in financial markets. *Rev. Financ. Stud.*, 2012, **25**(2), 439–479.
Lee, S.S. and Mykland, P.A., Jumps in financial markets: A new nonparametric test and jump dynamics. *Rev. Financ. Stud.*, 2008, **21**(6), 2535–2563.
Liu, J. and Maheu, J.M., Improving markov switching models using realized variance. *J. Appl. Econom.*, 2018, **33**(3), 297–318.
Maheu, J.M. and McCurdy, T.H., News arrival, jump dynamics, and volatility components for individual stock returns. *J. Finance.*, 2004, **59**(2), 755–793.
Maheu, J.M. and McCurdy, T.H, Modeling foreign exchange rates with jumps. In *Forecasting in the Presence of Structural Breaks and Model Uncertainty*, 2008 (Emerald Group Publishing Limited: Leeds).

Maheu, J.M. and McCurdy, T.H., Do high-frequency measures of volatility improve forecasts of return distributions? *J. Econom.*, 2011, **160**(1), 69–76.

Maheu, J.M., McCurdy, T.H. and Zhao, X., Do jumps contribute to the dynamics of the equity premium?*J. Financ. Econ.*, 2013, **110**(2), 457–477.

Maneesoonthorn, W., Forbes, C.S. and Martin, G.M., Inference on self-exciting jumps in prices and volatility using high-frequency measures. *J. Appl. Econom.*, 2017, **32**(3), 504–532.

Patton, A.J., Ziegel, J.F. and Chen, R., Dynamic semiparametric models for expected shortfall (and value-at-risk). *J. Econom.*, 2019, **211**(2), 388–413.

Peiris, R., Wang, C., Gerlach, R. and Tran, M.-N., Semi-parametric financial risk forecasting incorporating multiple realized measures. *Quant. Finance*, 2024, **24**(12), 1823–1837.

Press, S.J., A compound events model for security prices. *J. Bus.*, 1967, **40**(3), 317–335.

Todorov, V. and Tauchen, G., Volatility jumps. *J. Bus. Econ. Stat.*, 2011, **29**(3), 356–371.

Yu, J., On leverage in a stochastic volatility model. *J. Econom.*, 2005, **127**(2), 165–178.

Zhang, L., Mykland, P.A. and Aït-Sahalia, Y., A tale of two time scales: Determining integrated volatility with noisy high-frequency data. *J. Am. Stat. Assoc.*, 2005, **100**(472), 1394–1411.