COMPLETE REDUCIBILITY IN BAD CHARACTERISTIC

ALASTAIR J. LITTERICK AND ADAM R. THOMAS

Abstract. Let G be a simple algebraic group of exceptional type over an algebraically closed field of characteristic $p > 0$. This paper continues a long-standing effort to classify the connected reductive subgroups of G. Having previously completed the classification when p is sufficiently large, we focus here on the case that p is bad for G. We classify the connected reductive subgroups of G which are not G-completely reducible, whose simple components have rank at least 3. For each such subgroup X, we determine the action of X on the adjoint module $L(G)$ and on a minimal non-trivial G-module, and the connected centraliser of X in G. As corollaries we obtain information on: subgroups which are maximal among connected reductive subgroups; products of commuting G-completely reducible subgroups; subgroups with trivial connected centraliser; and subgroups which act indecomposably on an adjoint or minimal module for G.

1. Introduction and results

This paper concerns the closed subgroups of reductive algebraic groups over algebraically closed fields. This study dates back to work of Dynkin in the 1950s, which classified reductive maximal connected subgroups of simple algebraic groups in characteristic zero [18], later extended to positive characteristic by Seitz [36, 37] and Liebeck and Seitz [25]. It is natural to try and extend this classification to *all* reductive subgroups. For instance, a coset space G/X for a reductive group G is affine if and only if the identity component X° is reductive [34]. Subgroups of G are also instrumental in studying the corresponding finite groups of Lie type [24, 13, 12, 14]. In characteristic zero, a closed subgroup X of G is reductive if and only if X is *G-completely reducible* (G-cr), meaning that whenever X is contained in a parabolic subgroup P of G, it lies in a Levi factor of P. Dynkin's results can then be used inductively to classify all reductive closed subgroups. In positive characteristic the study is significantly harder. With $P = QL$ the Levi decomposition of a parabolic subgroup of G, reductive subgroups of P may not have a P-conjugate in L. Instead, one must study actions of subgroups X_0 of L on the unipotent radical, and complements to Q in the semidirect product QX_0. Thus questions of *non-abelian cohomology* arise. See [29] for more motivation, history and background.

Now let G be simple of exceptional type in characteristic $p > 0$. Exceptional types are both interesting (since these groups exhibit the most complex behaviour) and achievable (due to the bounded rank). In [23] Liebeck and Seitz obtain a classification of all reductive subgroups of G which holds, for instance, if $p > 7$. The study in smaller characteristics, where cohomology and subgroup structure are far more complicated, was initiated by Stewart [40, 43] for types G_2 or F_4. Here, non-G-cr reductive subgroups exist only when $p \le 3$. Work of the present authors [30] solves the problem when p is *good for G*. Recall that for exceptional types, a prime p is called *bad* if $p = 2$ or 3, or $p = 5$ with G of type E_8; otherwise p is called *good*.

Our main result completes the classification of reductive subgroups of G whose simple factors each have rank at least 3, with no restriction on the characteristic.

Theorem 1. *Let G be a simple algebraic group of exceptional type over an algebraically closed field of characteristic $p > 0$. Let X be a semisimple subgroup of G, such that X is non-G-cr and each simple factor of X has rank at least 3. Then $p = 2$ or 3, and X is conjugate to exactly one subgroup listed in Tables 5–8 in Section 11, each of which is non-G-cr.*

For each such X, we give the connected centraliser $C_G(X)^\circ$ and the action of X on the Lie algebra of G and on a non-trivial module for G of least dimension.

Subgroups with factors of rank 1 or 2 are more numerous and more delicate to study, and are the subject of ongoing work of the authors.

The proof of Theorem 1 follows a similar blueprint to previous results [30, 43] (cf. also [29, §1.4.2]) but the arguments and final result are much more delicate. Besides more complicated cohomology sets, useful properties such as *separability* of subgroups (equality of the group-theoretic and infinitesimal centraliser dimensions) hold when p is good for G [19, Theorem 1.1] but often fail here. Also in good characteristic, many natural subgroups H are either *ascending hereditary*, meaning that connected H-cr subgroups of H are G-cr, or *descending hereditary*, meaning that connected G-cr subgroups of H are H-cr [29, Definition 2.47]. For instance, [2, Theorem 3.26] states that reductive subgroups containing a maximal torus of G are both ascending and descending hereditary. This fails in bad characteristic; an interesting example is given in [29, Example 2.40], where two classes of F_4-cr subgroups of type G_2 are non-B_4-cr, non-C_4-cr respectively. The study of such hereditary properties is the subject of ongoing work.

Remark 1.1. Table 4 shows that when $p = 2$, a group of type F_4 has two classes of subgroups of type B_3 which are non-F_4-cr, whereas Table 5 shows that a group of type E_6 has a unique class of such subgroups. Comparing actions on low-dimensional modules, we see that the two F_4 subgroup classes are not fused in E_6. Thus at least one class of subgroups are E_6-cr, so that F_4 is not descending hereditary in E_6. This contrasts with the case $p \neq 2$, where F_4 is both ascending and descending hereditary, by [35, Proposition 16.9] and [2, Corollary 3.21].

An interesting complication arising is the existence of non-G-cr subgroups which are maximal among connected reductive subgroups of G. For these, one cannot make use of prior results on known reductive subgroups of G, since all their maximal overgroups in G are parabolic. For brevity, we call these subgroups *MR*. The next result describes those occurring in Theorem 1.

Corollary 2. *With G and p as in Theorem 1, let X be an MR subgroup of G where each simple factor of X has rank at least* 3. *Then either X is a maximal connected subgroup of G or*

(i) *G has type E_6 and X is a Levi subgroup of type D_5T_1,*
(ii) *G has type E_7 and X is a Levi subgroup of type E_6T_1,*
(iii) *$(G, X, p) = (E_7, D_4, 2)$ and X is non-G-cr,*
(iv) *$(G, X, p) = (E_8, D_4, 2)$ and X is non-G-cr.*

Each subgroup in (i)–(iv) *is MR. Those in* (i), (ii) *and* (iv) *are unique up to conjugacy in G, and those in* (iii) *form infinitely many conjugacy classes.*

This corollary follows directly from Theorem 1 together with the classification of maximal connected subgroups [25]. The subgroups in (iii) were discovered in [21], and those in (iv) are new. Here and elsewhere, we abuse terminology and write *Levi subgroup of G* to mean a Levi factor of a parabolic subgroup of G.

For our next corollary, recall that a *subsystem subgroup* of G is a semisimple subgroup normalised by a maximal torus. These constitute a large class of subgroups of G corresponding to p-closed subsystems of the root system of G. Connected reductive subgroups not contained in a subsystem subgroup are both relatively rare and less straightforward to study. The following result complements its good-characteristic analogue [30, Corollary 6].

Corollary 3. *Let G and X be as in the hypothesis of Theorem 1. Then either X is contained in a proper subsystem subgroup of G, or $p = 2$ and one of the following holds:*

(i) *$(G, X) = (E_6, B_3)$ and X lies in a maximal subgroup F_4,*
(ii) *$(G, X) = (E_7, D_4)$ and X is MR,*
(iii) *$(G, X) = (E_8, D_4)$ and X is MR,*
(iv) *$(G, X) = (E_8, B_3)$ and X is contained in a subgroup from part (iii).*

Each subgroup in (i)–(iv) *lies in no proper subsystem subgroup of G. Those in* (i), (iii) *and* (iv) *are unique up to conjugacy in G, and those in* (ii) *form infinitely many conjugacy classes.*

Corollary 3 is proved by inspecting Tables 4–8: There, an explicit embedding of each subgroup X is given into a subsystem subgroup of G, when this is possible. The subgroups in (ii) and (iii) are MR so are in no proper reductive subgroup. By the Borel-de Siebenthal algorithm [6], when G has type E_6 each subsystem subgroup of G is contained in one of type A_1A_5, A_2^3 or D_5. Of these, only D_5 contains any subgroup of type B_3; but such a subsystem subgroup centralises a 1-dimensional torus of G, whereas the subgroup X in (i) does not. Similarly, Table 8 shows that the subgroup X in (iv) does not centralise a non-trivial torus in G, so could only be contained in a subsystem subgroup of maximal rank, with all simple factors of rank at least 3; these have types A_8, D_8 and A_4^2. The latter contains no subgroup B_3, and the former two act on $L(G)$ with indecomposable summands of dimension 80, 84 and 84; and 120 and 128, respectively [27, Lemma 11.2]. This is incompatible with the stated action of X, whose indecomposable summands have dimensions 30, 30, 62, 63 and 63.

Next, the main result of [3] states that the product of two commuting G-cr subgroups is again G-cr, as long as the characteristic p is good for G or $p > 3$. We conjecture that $p \neq 2$ is in fact sufficient [29, Remark 1.12]. Our next result is evidence in this direction.

Corollary 4. *Let G and X be as in the hypothesis of Theorem 1. If X is not simple then $p = 2$, G has type E_8 and X has type B_3^2. Each simple factor of X is G-cr.*

The first statement here follows immediately from the classification of non-G-cr subgroups in Theorem 1. The final statement uses the additional information contained in Tables 5–8. Specifically, the non-G-cr subgroups of type B_3 in Theorem 1 each have centraliser of rank at most 2, while the factors of the non-G-cr subgroup B_3^2 each centralise one another, hence are not among these non-G-cr subgroups.

For our next result, recall from [2, Lemma 3.17] that the centraliser of a G-cr subgroup is again G-cr, hence reductive. In particular, when G is semisimple a G-irreducible subgroup has trivial connected centraliser [28, Lemma 2.1]. When the characteristic is zero or sufficiently large, the converse holds: Since a reductive subgroup is G-cr, it is G-irreducible if and only if it is contained in no proper Levi subgroup of G, if and only if its connected centraliser is trivial. This fails in positive characteristic, and our next result describes the extent of this failure. This extends [30, Corollary 9]. The notation for the embeddings is given in Section 2.7.

Corollary 5. *Let G and X be as in the hypothesis of Theorem 1.*

Then $C_G(X)^\circ$ is reductive if and only if $C_G(X)^\circ = 1$, if and only if $p = 2$, G has type E_8 and X is either a subgroup $A_3 < D_8$ via $T(101)$, or $B_4 < D_8$ via 0001.

Although most of the non-G-cr subgroups in our classification do not have reductive centralisers, their centralisers are at least semidirect products. Indeed, when the centralisers are calculated, the reductive parts are found as $C_G(Y)$ for Y a G-cr overgroup of X.

Corollary 6. *Let G and X be as in the hypothesis of Theorem 1. Then $C_G(X)^\circ$ is a semidirect product of its unipotent radical and reductive part.*

Our final result highlights an interesting phenomenon regarding the actions of reductive subgroups of G on low-dimensional G-modules. Namely, one typically expects that a proper reductive subgroup will act with many indecomposable direct summands on a given irreducible G-module. However, we exhibit some subgroups which act indecomposably. This complements prior results such as [26] which classify subgroups acting irreducibly. In the following, we write $V_{\min}$ or V_n for a non-trivial G-module of least dimension, which is n.

Corollary 7. *Let G and X be as in the hypothesis of Theorem 1 and let $V = L(G)$ or $V_{\min}$. Then $V \downarrow X$ is indecomposable if and only if $p = 2$ and (G, X, V) is one of the following:*

(i) *$(G, X) = (F_4, B_3)$ with X in a short-root subgroup $\tilde{D}_4$ and $V = V_{26}$,*
(ii) *$(G, X) = (E_6, B_3)$ with $V = L(G)$,*
(iii) *$(G, X) = (E_7, D_4)$ with X an MR subgroup and $V = V_{56}$ or $V = L(G)$.*

Layout of the paper. After defining relevant notation in Section 2 and giving preliminary results in Section 3, Theorem 1 is proved in Sections 4–9. Each section corresponds to a Lie type of non-G-cr subgroups, split into subsections according to the strategy described in Section 3.1. Section 10 then determines the restrictions $L(G) \downarrow X$ and $V_{\min} \downarrow X$ for each X arising in Theorem 1, and the tables of these subgroups and associated data are given Section 11.

2. Notation

2.1. **Algebraic groups, roots, weights.** Throughout, all groups are linear algebraic groups over an algebraically closed field K of characteristic $p > 0$. We take the 'group variety' viewpoint [7], so that our algebraic groups are the K-points of Zariski-closed subgroups of general linear groups over K. All subgroups mentioned are closed and all group homomorphisms are variety morphisms. The terms 'simple' and 'semisimple' always refer to connected groups. Group actions are on the left, denoted by a dot, i.e. $(g, q) \mapsto g \cdot q$ for g in a group G and q in a G-set.

For a reductive algebraic group G, fix a maximal torus T of G and corresponding root system Φ. Let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be a base of simple roots and let $\{\lambda_1, \ldots, \lambda_l\}$ be the corresponding fundamental dominant weights, with the Bourbaki ordering [8, Ch. VI, Planches I-IX]. The notation $a_1 a_2 \ldots a_l$ will indicate a root $\sum a_i \alpha_i$ or a weight $\sum a_i \lambda_i$; context will prevent ambiguity.

The Weyl group $W = N_G(T)/T$ acts on $\mathbb{Z}\Phi \otimes \mathbb{R}$. For $\alpha \in \Phi$, let s_α denote the reflection in the hyperplane perpendicular to α, and let $U_\alpha = \{x_\alpha(t) : t \in K\}$ be the root subgroup corresponding to α. For $t \in K^*$, define

$$n_\alpha(t) = x_\alpha(t) x_{-\alpha}(-t^{-1}) x_\alpha(t) \in N_G(T),$$
$$h_\alpha(t) = n_\alpha(t) n_\alpha(-1) \in T,$$

so that $n_\alpha(t)$ maps to $s_\alpha \in W$, [9, §6.4]. Set $n_\alpha = n_\alpha(1)$ and for a simple root α_i let $s_i = s_{\alpha_i}$, $n_i = n_{\alpha_i}$ and $h_i(t) = h_{\alpha_i}(t)$.

The notation $\bar{X}$ denotes a subgroup of G generated by long root subgroups of G. If Φ has multiple root lengths then $\tilde{X}$ denotes a subgroup generated by short root subgroups of G.

2.2. **Modules.** For a reductive algebraic group G and a dominant weight λ of G, let $V_G(\lambda)$ denote the irreducible G-module of highest weight λ. When G is clear we write simply λ for the module; in particular 0 denotes the trivial irreducible G-module. The corresponding Weyl module for G is denoted $W(\lambda)$ or $W_G(\lambda)$, and the tilting module is denoted $T(\lambda)$ or $T_G(\lambda)$. We write $V^* = \mathrm{Hom}_K(V, K)$ for the dual of a G-module V. If $G = G_1 G_2 \ldots G_k$ is a commuting product of reductive algebraic groups then $(V_1, \ldots, V_k)$ denotes the G-module $V_1 \otimes \cdots \otimes V_k$, where V_i is a G_i-module for each i.

Let $F : G \to G$ be a Frobenius endomorphism which acts on root elements via $x_\alpha(t) \mapsto x_\alpha(t^p)$ for $\alpha \in \Phi$, $t \in K$. If V is a G-module afforded by a representation $\rho : G \to GL(V)$ then $V^{[r]}$ denotes the module afforded by $\rho^{[r]} := \rho \circ F^r$. Let $M_1, \ldots, M_k$ be G-modules and $m_1, \ldots, m_k$ be positive integers. Then $M_1^{m_1}/\ldots/M_k^{m_k}$ denotes a G-module having the same composition factors as $M_1^{m_1} \oplus \ldots \oplus M_k^{m_k}$. Furthermore, $V = M_1|\ldots|M_k$ denotes a G-module with socle series $V = V_1 > V_2 > \ldots > V_{k+1} = \{0\}$, so that $\mathrm{Soc}(V/V_{i+1}) = V_i/V_{i+1} \cong M_i$ for $1 \leq i \leq k$. We denote by $L(G)$ the adjoint G-module. For G of type F_4, E_6 and E_7 we respectively write $V_{26} = V_G(\lambda_4)$, $V_{27} = V_G(\lambda_1)$ and $V_{56} = V_G(\lambda_7)$ for these non-trivial modules of least dimension. Note that by Lemma 3.6, we only consider $G = F_4$ when $p = 2$ here, hence $V_G(\lambda_4)$ is irreducible of the stated dimension.

2.3. **Parabolic and Levi subgroups.** Let $J = \{\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}\} \subseteq \Pi$ and let $\Phi_J = \Phi \cap \mathbb{Z}J$. Then the standard parabolic subgroup of G corresponding to J is $P_J = \langle T, U_\alpha : \alpha \in \Phi_J \cup \Phi^+ \rangle$. This has Levi decomposition $P_J = Q_J L_J$ where $Q_J = R_u(P) = \langle U_\alpha : \alpha \in \Phi^+ \setminus \Phi_J \rangle$, and $L_J = \langle T, U_\alpha : \alpha \in \Phi_J \rangle$ is the standard Levi factor of P_J. If the derived subgroup L' has Lie type **X**, we call P_J and L_J an **X**-parabolic subgroup and **X**-Levi subgroup of G respectively. For instance when G has type E_6 the subgroup P_{2345} is a D_4-parabolic subgroup of G. Recall that 'Levi subgroup of G' means any Levi factor of a parabolic subgroup of G.

2.4. **Shape modules.** For a subset $J \subseteq \Pi$, the *level* of $\gamma = \sum_{\alpha \in \Pi} c_\alpha \alpha$ is the sum $\sum_{\alpha \in \Pi \setminus J} c_\alpha$, and the *shape* of γ (with respect to J) is $\sum_{\alpha \in \Pi \setminus J} c_\alpha \alpha$. Thus for a standard parabolic subgroup $P_J = Q_J L_J$ as above, Q_J is generated by root subgroups corresponding to roots of positive level. For each $i \geq 1$ we define

$$Q(i) = \langle U_\gamma : \gamma \text{ has level} \geq i \rangle ,$$

an L-stable normal subgroup of Q_J. The *i-th level of* Q is $Q(i)/Q(i+1)$, which is central in $Q/Q(i+1)$. By [1, Theorem 2 and Remark 1], each level is an L-module, and is a direct sum of *shape modules* V_S, where S runs over shapes of level i. Each V_S is either irreducible or indecomposable of length 2, the latter only if $(G,p) = (G_2, 2)$, $(G_2, 3)$ or $(F_4, 2)$. Moreover the high weight(s) of L on V_S can be determined combinatorially, cf. [23, Lemma 3.1].

2.5. **Centralisers.** When discussing subgroup centralisers, we write U_i for an i-dimensional unipotent group, and T_j for a j-dimensional torus. For instance, $C_G(X)^\circ = U_5 T_1$ means that $C_G(X)^\circ$ has a 5-dimensional unipotent radical, with quotient a 1-dimensional torus.

2.6. **Roots of G and its Levi subgroups.** If L is a standard Levi subgroup of G then roots of L are roots of G and we must be consistent in how we identify these. If L_0 is a simple factor of L then let $\Psi = \{\alpha'_1, \ldots, \alpha'_m\}$ be the simple roots of L_0, a subset of Π. Linearly order Π in such a way that $\alpha_i < \alpha_j$ whenever $i < j$. If L_0 has Lie type A_m, the embedding $L \to G$ is chosen such that α'_1 is the least simple root of G contained in Ψ. If L_0 has type E_6 or E_7 then $\alpha'_i = \alpha_i$ for all i. If L_0 has type D_4 then $(\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4) = (\alpha_2, \alpha_4, \alpha_3, \alpha_5)$. If L_0 has type D_5 then $(\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4, \alpha'_5) = (\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_2)$ or $(\alpha_6, \alpha_5, \alpha_4, \alpha_3, \alpha_2)$. If L_0 has type D_6 then $(\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4, \alpha'_5, \alpha'_6) = (\alpha_7, \alpha_6, \alpha_5, \alpha_4, \alpha_3, \alpha_2)$. Finally, if L has multiple components of the same type, then these components are ordered according to the position of their least simple root "α'_1" as an element of Π. For instance, if $G = E_7$ and $L = L_{12567}$, then L is a Levi subgroup of type $A_1 A_1 A_3$ and the first A_1 factor corresponds to α_1, and the second to α_2.

When G is simple of type E_7 or E_8, we will need to distinguish between certain isomorphic subsystem subgroups of G. For type E_7, there are two conjugacy classes of A_5-Levi subgroups, with representatives $A_5 = L_{24567}$ and $A'_5 = L_{34567}$. By [23, Table 8.2] we have $C_G(A_5)^\circ = A_2$ and $C_G(A'_5)^\circ = A_1 T_1$. Furthermore A'_5 is contained in a subgroup E_6 whereas A_5 is not. In E_8 there are two conjugacy classes of subgroups A_7, one given by the A_7-Levi subgroup $L_{1345678}$, which we denote A_7, and one denoted A'_7 contained in an E_7-Levi subgroup of G. Again by [*loc. cit.*] we have $C_G(A_7)^\circ = T_1$, and $C_G(A'_7)^\circ = A_1$.

2.7. **Embeddings into classical groups.** Finally, for a reductive algebraic group X, a rational X-module V corresponds to a representation from X into $H = \mathrm{SL}(V)$, $\mathrm{SO}(V)$ or $\mathrm{Sp}(V)$; for brevity, we write '$X \leq H$ via V' for this. When $V = V_X(\lambda)$ is irreducible we take this further and write '$X \leq H$ via λ.' We occasionally talk about chains '$X \leq Y \leq Z$ via λ', in which case the embedding refers to the first inclusion $X \to Y$; the inclusion $Y \to Z$ will be clear from context.

3. Preliminary Results

3.1. **A strategy for enumerating subgroups.** From now on, G denotes a simple algebraic group of exceptional Lie type over the algebraically closed field K. To prove Theorem 1 we follow the strategy developed in [43] and [30]. Each class of non-G-cr subgroups has a representative X in some standard parabolic subgroup P of G; take P to be minimal subject to containing X, and let X_0 be the image of X in the standard Levi factor L of P. By minimality, X_0 is

L-irreducible. If X is reductive then X is a complement (as abstract groups) to $Q := R_u(P)$ in the semidirect product QX_0. The cohomology set $H^1(X_0, Q)$ can be used to parametrise Q-conjugacy classes of such complements. The following now tells us that non-trivial elements of $H^1(X_0, Q)$ give rise to non-G-cr subgroups, and non-associated parabolic subgroups give rise to disjoint G-conjugacy classes of non-G-cr subgroups. Here, two parabolic subgroups are *associated* if their Levi factors are G-conjugate to one another.

Lemma 3.1 ([5, Theorem 5.8 and Proposition 5.14], [30, Lemma 3.26])**.** *Let X be a closed subgroup of G, let P be a parabolic subgroup of G containing X, and let X_0 be the image of X under projection to a Levi factor of P. Then X is G-conjugate to X_0 if and only if X is $R_u(P)$-conjugate to X_0.*

If P_1 and P_2 are minimal among parabolic subgroups of G containing X then P_1 and P_2 are associated, and the images of X in their respective Levi factors are G-conjugate.

The second part of this lemma allows us to break our enumeration of non-G-cr subgroups down, one subsection for each pair (L, X_0) with L representing a G-conjugacy class of Levi subgroups and X_0 an L-irreducible subgroup.

At this point we need two essential ingredients. Firstly, we need a description of the L-irreducible subgroups X_0 occurring in each L. This has been accomplished explicitly by the present authors in [31] and [45], which give the irreducible subgroups in each simple factor of each Levi subgroup L of G; a general L-irreducible semisimple subgroup is then a central product of these. Secondly, we must describe the sets $H^1(X_0, Q)$ for each such L-irreducible subgroup X_0, which parametrise the complements to Q in the semidirect product QX_0, up to Q-conjugacy.

3.2. **Cohomology and complements in semidirect products.** Recall that when X_0 and Q are algebraic groups with X_0 acting on Q via a morphism $X_0 \times Q \to Q$, complements to Q in the semidirect product QX_0 are in bijection with rational *1-cocycles*, variety morphisms $\phi : X_0 \to Q$ such that $\phi(xy) = \phi(x)(x \cdot \phi(y))$ for all $x, y \in X_0$. Here, a *complement* X to Q is by definition a closed subgroup of QX_0 satisfying (i) $QX = QX_0$, (ii) $Q \cap X = 1$, and (iii) $L(Q) \cap L(X) = \{0\}$, see [42, Definition 3.2.1] and [33, 4.3.1]. Two cocycles ϕ, ψ are *cohomologous* if there exists $q \in Q$ such that $\phi(x) = q^{-1}\psi(x)(x \cdot q)$ for all $x \in X_0$. This defines an equivalence relation on the set $Z^1(X_0, Q)$ of 1-cocycles, and the corresponding quotient is denoted $H^1(X_0, Q)$, and parametrises complements up to Q-conjugacy. The set $H^1(X_0, Q)$ has a distinguished element, which is the class of the map sending every element of X_0 to the identity of Q. In general $H^1(X_0, Q)$ is only a pointed set, but if Q is a KX_0-module then $H^1(X_0, Q)$ is naturally a K-vector space.

By [42, Lemma 3.6.1], a subgroup satisfying (i) and (ii) automatically satisfies (iii) when X_0 is connected reductive, Q is unipotent and $p \neq 2$. A subtlety occurs in characteristic 2, however. The issue is that closed reductive subgroups may arise as *abstract* complements in semidirect products, and thus the condition on Lie algebras must be relaxed if we are to account for all such subgroups. Fortunately, we have good control over this situation. When X_0 is defined over $\mathbb{F}_p \subset K$, we let $Q^{[1]}$ denote the algebraic X_0-group obtained by twisting the action of X_0 by a Frobenius morphism $X_0 \to X_0$ induced by the p-th power field automorphism $K \to K$. Then we get induced maps $Q \to Q^{[1]}$ and $H^1(X_0, Q) \to H^1(X_0, Q^{[1]})$, and the latter is always injective and is usually an isomorphism of pointed sets [42, Theorem 3.5.6]. The following is now a particular case of [42, Lemma 3.6.1].

Lemma 3.2. *Let X_0 be a simple algebraic group acting on a unipotent algebraic group Q via a morphism $X_0 \times Q \to Q$, and suppose that Q has a filtration by normal X_0-stable subgroups whose successive quotients are X_0-modules. Let X be an abstract complement to Q in the semidirect product QX_0. Then either:*

(i) *X is a complement to Q as algebraic groups, or*

(ii) $p = 2$, X_0 *has type* C_n, X *has type* B_n, *and some* X_0*-composition factor in the filtration of* Q *has high weight* λ_1.

In (ii), *if the groups and morphisms are all defined over the prime field then* X *corresponds to an element of* $H^1(X_0, Q^{[1]})$ *which is not in the image of* $H^1(X_0, Q) \to H^1(X_0, Q^{[1]})$. *Thus there is a bijection between* $H^1(X_0, Q^{[1]})$ *and the* Q*-conjugacy classes of closed, connected, reductive subgroups of* QX_0 *which are abstract complements to* Q.

Remark 3.3. For brevity, from now on whenever we say complement we mean abstract complement.

Thus to classify non-G-cr subgroups of G, we need to understand the sets $H^1(X_0, Q)$ for various groups X_0, and also $H^1(X_0, Q^{[1]})$ when X_0 has type C_n and $p = 2$. In many cases, this can be reduced to abelian cohomology calculations, using a long exact sequence. In particular:

Lemma 3.4 ([38, §I.5, Propositions 36 and 43])**.** *Let* Q *be an algebraic* X_0*-group and* R *an* X_0*-stable subgroup of* Q. *Then:*

(i) *There is an exact sequence of pointed sets*

$$0 \to R^{X_0} \to Q^{X_0} \to (Q/R)_0^X \to H^1(X_0, R) \to H^1(X_0, Q)$$

where $-^{X_0}$ *denotes the fixed-point subgroup under the action of* X_0.

(ii) *If moreover* R *is central in* Q, *then there is an exact sequence of pointed sets*

$$0 \to R^{X_0} \to Q^{X_0} \to (Q/R)^{X_0} \to H^1(X_0, R) \to H^1(X_0, Q) \to H^1(X_0, Q/R) \to H^2(X_0, R)$$

where $H^2(X_0, R)$ *is the second cohomology group defined for example in* [20, §II.4.2].

Note that these are only exact sequences of pointed sets, meaning that the image of each map is the pre-image of the distinguished element under the next. However, this is already sufficient to deduce much information about $H^1(X_0, Q)$. For instance, it also allows us to relate M-complete reducibility with G-complete reducibility, when $X \leq M \leq G$ is a chain of reductive subgroups:

Corollary 3.5. *Let* $M \leq G$ *be two reductive algebraic groups, and let* $P_M = Q_M L_M$ *be a parabolic subgroup of* M *containing a reductive subgroup* X, *such that* X *lies in no Levi factor of* P_M, *and write* X_0 *for the image of* X *in* L_M. *Furthermore, let* $P = QL$ *be a parabolic subgroup of* G *such that* $P_M = M \cap P$ *and* $Q_M = M \cap Q$.

If $(Q/Q_M)^{X_0} = \{0\}$ *then* X *is non-*G*-cr.*

Proof. The hypotheses imply that the map $(Q/Q_M)^{X_0} \to H^1(X_0, Q_M)$ is zero, and since X is non-M-cr, the latter set contains non-zero points. By exactness, these do not lie in the kernel of $H^1(X_0, Q_M) \to H^1(X_0, Q)$, and since X corresponds to such a point, X remains non-G-cr. □

3.3. **Approximating** $H^1(X_0, Q)$**.** We now recall a method, developed in [43] and applied extensively in [30], for approximating $H^1(X_0, Q)$. When Q is the unipotent radical of a parabolic subgroup P, recall that Q has a filtration by normal subgroups $Q(i)$ ($i = 1, \ldots, r$). We write $V_i = Q(i)/Q(i+1)$ for the corresponding levels, which are modules for a Levi subgroup L of P hence also for X_0 whenever $X_0 \leq L$. One can therefore form the direct sum

$$\mathbb{V} = \mathbb{V}_{X_0,Q} := \bigoplus_{i=1}^{r} H^1(X_0, V_i).$$

For $i = 1, \ldots, r$ one now defines partial maps $\rho_i : \mathbb{V} \to H^1(X_0, Q/Q(i+1))$ inductively [43, Definition 3.2.5]. Let $\mathbf{v} = ([\gamma_1], \ldots, [\gamma_r]) \in \mathbb{V}$. For $i = 1$ let $\rho_1(\mathbf{v}) = [\gamma_1]$. For $i > 1$, lift $\rho_{i-1}(\mathbf{v})$ to some element $[\Gamma_i] \in H^1(X_0, Q/Q(i+1))$ under the natural map $H^1(X_0, Q/Q(i+1)) \to H^1(X_0, Q/Q(i))$, *when this is possible*, and then set $\rho_i(\mathbf{v}) = [\Gamma_i][\gamma_i]$. As long as $[\Gamma_i]$ exists, this product is well-defined since $Q(i)/Q(i+1)$ is central in $Q/Q(i+1)$. If such a lift $[\Gamma_i]$ does not exist, declare ρ_i undefined at $\mathbf{v}$.

Each of these partial maps $\rho_i : \mathbb{V} \to H^1(X_0, Q/Q(i+1))$ turns out to be surjective (cf. [43, Proposition 3.2.6], [30, Lemma 3.10]), and we set $\rho = \rho_r$. As a particular corollary, if $H^1(X_0, V_i) = \{0\}$ for all levels V_i then $H^1(X_0, Q) = \{0\}$, and all complements to Q in QX are Q-conjugate to X. Similarly, in the setting of Lemma 3.2(ii) with X_0 of type C_n, if $H^1(X_0, V_i^{[1]}) = \{0\}$ for all i then no (abstract) complements of type B_n arise.

Finally, Q acts on itself by conjugation, and if $q \in Q^{X_0}$ then q serves to fuse together classes corresponding to different elements of $\mathbb{V}$. A full account is given in [30, pp. 5292–5294]. Complete details will be given when we make use of this action in our proofs; for the moment, we note only that if q maps to a point in $V_i^{X_0}$, then conjugation by q induces X_0-module homomorphisms $V_j \to V_{i+j}$ and linear maps $H^1(X_0, V_j) \to H^1(X_0, V_{i+j})$ for each j, and thereby a map $\mathbb{V} \to \mathbb{V}$. Typically, this will fuse together classes in such a way that a subgroup class represented by $(k_1, \ldots, k_m) \in \mathbb{V}$ will also be represented by an element with $k_i k_j = 0$ for particular indices i and j.

3.4. **From Q-conjugacy to G-conjugacy.** The theory so far describes non-G-cr semisimple subgroups in parabolic subgroups $P = QL$ *up to Q-conjugacy*; this is not the end of the story as elements of G can fuse classes together. We recall some details from [30, Section 3.5].

Firstly, the torus $Z(L)$ acts on Q and centralises each subgroup $X_0 \leq L'$, inducing an action on $H^1(X_0, Q)$. Fixing a basis of each space $H^1(X_0, V_i)$ where V_i is a level of Q, the map ρ allows us to parametrise complements to Q in QX_0 by certain m-tuples $(k_1, k_2, \ldots, k_m) \in \mathbb{V}$ for some m. The definition of ρ involves many arbitrary choices of lift, but subject to appropriate choices, the action of $Z(L)$ can be viewed as multiplying the coordinates k_i by appropriate scalars, fusing together all of the corresponding subgroup classes. The action of $Z(L)$ is fixed-point-free on the non-identity elements of Q since $C_G(Z(L)) = L$, and thus this action will be non-trivial in each coordinate of $\mathbb{V}$.

Also, elements of G can fuse classes of subgroups occurring in standard parabolic subgroups P_I and P_J, for not necessarily distinct subsets I and J of simple roots. By Lemma 3.1, the corresponding Levi subgroups L_I and L_J are G-conjugate in this case. In particular, we will be concerned with elements w of the Weyl group $W(G) = N_G(T)/T$, with pre-image $\dot{w} \in N_G(T)$, such that $\dot{w} \cdot L_I = L_J$. In this case, $\dot{w}$ sends each root subgroup in the unipotent radical of P_I to another root subgroup of G, which may or may not lie in P_J. Thus if we have a subgroup $X \leq P_I$, and if $X \leq UL_I$ for some subgroup $U \leq Q_I$ with $\dot{w} \cdot U \leq Q_J$, then X is G-conjugate to a subgroup of P_J. Again, when we use arguments of this form in Sections 4–9, we will give full details of the relevant Weyl group elements and how they fuse classes together.

3.5. **Lie types of non-G-cr subgroups.** We now describe those Lie types $\mathbf{X}$ of semisimple groups which admit non-zero cohomology on a relevant module.

Lemma 3.6. *Let G be a simple algebraic group of exceptional type in characteristic p, such that G has a non-G-cr semisimple subgroup of type $\mathbf{X}$ with simple components each of rank 3 or more. Then $(G, \mathbf{X}, p)$ appears in Table 1.*

TABLE 1. Types of semisimple non-G-cr subgroups

$\mathbf{X} \setminus G$	E_8	E_7	E_6	F_4
A_3, C_4, D_4	2	2		
B_3	2	2	2	2
C_3	3			
B_3^2, B_4	2			

Remark 3.7. In the course of proving Theorem 1 we will prove the converse to Lemma 3.6, that non-G-cr subgroups of each type do indeed exist. This was shown already in [41] for simple types, and exhibiting a non-G-cr subgroup of type B_3^2 (Section 5) completes the argument.

Proof. The simple types $\mathbf{X}$ which occur are given in [41, Theorem 1], so we need only consider types with more than one component. For such a type $\mathbf{X}$, suppose that G has a parabolic subgroup $P = QL$ with an L-irreducible subgroup X_0, also of type $\mathbf{X}$ (or of type C_n when $\mathbf{X} = B_n$), with $H^1(X_0, Q) \neq \{0\}$. From the exact sequence of cohomology (Lemma 3.4), we see that if $H^1(X_0, Q(i)/Q(i+1)) = \{0\}$ for each i, then $H^1(X_0, Q) = \{0\}$. Thus for a non-G-cr subgroup of type $\mathbf{X}$ to exist, we need $H^1(X_0, V) \neq \{0\}$ for some indecomposable X_0-module direct summand V of a level of Q.

In [31] and [45], an explicit list is given of the irreducible subgroups in each simple factor of a Levi subgroup of G. These are given up to conjugacy in the simple factor, except for factors of type D_7 in [31] when conjugacy under a graph automorphism is also allowed. Given this, in each Levi subgroup L of G, the L-irreducible semisimple subgroups are precisely the products of irreducible subgroups in each simple factor. In summary, the non-simple, semisimple L-irreducible subgroups X_0 arising in this way are as follows.

L	X_0
A_3^2	$X_0 = L'$
A_3A_4	$X_0 = L'$
D_6	A_3^2 via $(010, 0) + (0, 010)$
D_7	A_3D_4 via $(010, 0) + (0, \lambda_1)$
	B_3^2 via $(100, 0) + (0, 100)$ $(p \neq 2)$ or $0\|((100, 0) + (0, 100))\|0$ $(p = 2)$
	A_3B_3 via $(010, 0) + (0, 100) + 0$ $(p \neq 2)$ or $(010, 0) + (0, T(100))$ $(p = 2)$
	A_3B_3 via $(010, 0) + (0, 001)$

It remains to show that $H^1(X_0, V) = \{0\}$ for each such X_0 and each relevant V, except for X_0 of type B_3^2 with $p = 2$. By the Künneth formula for cohomology, if U_i is a module for a reductive group Y_i $(i = 1, 2)$ then $H^1(Y_1 \times Y_2, U_1 \otimes U_2) \cong \bigoplus_{j=0,1} H^j(Y_1, U_1) \otimes H^{1-j}(Y_2, U_2)$.

As an L-module, each level of Q is a direct sum of shape modules V_S as described in Section 2.4. The structure of such shape modules is known from prior work, e.g. [23, Lemma 3.1]. Explicitly, up to taking duals, for a simple factor of L of type A_n we obtain irreducible modules λ_1, λ_2, λ_3; for type D_n $(n \geq 4)$ we obtain λ_1, λ_{n-1} and λ_n; for type B_3 we obtain $W(100)$ and the irreducible module 001; for type C_3 we obtain $W(010)$; for type E_6 we obtain λ_1; and for type E_7 we obtain λ_7.

Once these shape modules V_S are known, determining the action of the L-irreducible subgroups X_0 on V_S is usually a matter of routine weight-space calculations. When X_0 is not simple, each simple factor of X_0 is contained in a proper Levi subgroup of L (hence of G) whose derived subgroup is simple of type A_3, A_4 or D_4. The non-zero shape modules occurring have dimension at most 10, and each L-module occurring in the filtration of Q then restricts to X_0 as a tensor product of modules for the factors of X_0. In particular, in each decomposition $U_1 \otimes U_2$ above, each factor U_i has dimension at most 10.

For most irreducible X_0-modules V of relevance to us, the first cohomology group $H^1(X_0, V)$ has already appeared in the literature. For example, [41, Lemma 4.6] collates many such results. Generally, such cohomology groups can be determined from the structure of the corresponding Weyl module and a dimension-shifting argument, see [20, Proposition 4.14] and the subsequent remark there. In turn, the structure of Weyl modules can be computed either using S. Doty's GAP routine [17] or by using the Weyl character formula together with the known weights of irreducible modules of low dimension [32]. As an elementary example, the Weyl module $W_{B_3}(100)$ has structure $100|0$, and dimension-shifting gives $H^1(B_3, 100) \cong H^0(B_3, 0) \cong K$.

In the end, for a simple factor Y of X_0, we find that the only irreducible Y-modules U with non-zero cohomology (and dimension at most 10) occur for factors of type B_3 and $U \cong V_{B_3}(100)$. Now, with the exception of $B_3^2 < D_7$ when $p = 2$, each such factor B_3 is contained in a proper Levi subgroup of type D_4, acting on the three 8-dimensional modules as a tilting module $T(100)$ or 001. Such a module has zero cohomology [15, Theorem 1.1], and thus if a non-G-cr subgroup X of type $\mathbf{X}$ exists, then $p = 2$ and $X_0 = B_3^2 < D_7$. □

Now that we have determined the possible types of non-G-cr semisimple subgroups X occurring, we next describe the relevant embeddings $X_0 \to L$ for Levi subgroups L of G with L-irreducible image; this follows directly from the main results of [31] and [45].

Lemma 3.8. *With $(G, \mathbf{X}, p)$ as in Table 1, let L be a Levi subgroup of G and $X_0 \leq L$ be an L-irreducible semisimple subgroup, where X_0 has type $\mathbf{X}$, or X_0 has type C_n when $\mathbf{X} = B_n$ ($n = 3$ or 4). Then either:*

(i) $p = 3$ and $(X_0, L') = (C_3, D_7)$ with $H \to L'$ via 010, or
(ii) $p = 2$ and the inclusion $X_0 \to L'$ is described in Table 2.

Table 2. Levis containing relevant irreducible subgroups X_0 when $p = 2$

L'	Embedding $X_0 \to L'$	L'	Embedding $X_0 \to L'$
B_3	A_3 via $010 + 0$	E_6	$C_4 < F_4$
C_3	A_3 via 010		$D_4 < C_4 < F_4$
D_4	B_3 via $T(100)$	A_7	C_4 via 1000
	B_3 via 001 (2 classes)		D_4 via 1000
A_5	C_3 via 100		B_3 via 001
	A_3 via 010	D_7	C_3 via $010 + 0$
D_5	B_4 via $T(1000)$		B_3^2 via $0\|((100, 0) + (0, 100))\|0$
D_6	A_3 via $010 + 010^{[r]}$ ($r \neq 0$; 2 classes)		B_3 via $0\|(100 + 100^{[r]})\|0$ ($r \neq 0$)
A_3^2	A_3 via $(100^{[r]}, 100^{[s]})$ or $(100^{[r]}, 001^{[s]})$ ($rs = 0$)		A_3 via 101

Finally, with Lemma 3.8 telling us which parabolic subgroups $P = QL$ to focus on, we consider the action of each relevant L-irreducible subgroup X_0 on the unipotent radical Q; in particular we need to know how X acts on each shape module V_S occurring in the levels of Q.

Proposition 3.9. *Let X_0 be a G-cr semisimple subgroup of G of type $\mathbf{X}$ as in Lemma 3.8, and let $P = QL$ be a parabolic subgroup of G, where $X_0 \leq L$ is L-irreducible. If $H^1(X_0, V^{[1]}) \neq \{0\}$ for some X_0-summand V of a shape module V_S occurring in a level of Q, then either:*

(i) $p = 3$ and $(X_0, L', V_S, V) = (C_3, D_7, V_{D_7}(\lambda_1), V_{C_3}(010))$ with $\dim H^1(X_0, V) = 1$, or
(ii) $p = 2$ and X_0, L', V_S and V appear in Table 3.

In (ii), if $(X_0, V) = (D_4, V_{D_4}(\lambda_2))$ then $\dim H^1(X_0, V) = 2$, otherwise $\dim H^1(X_0, V^{[1]}) = 1$. In the latter case we also have $\dim H^1(X_0, V) = 1$ unless X_0 has type C_n and $V_S \downarrow X_0$ has a high weight λ_1 (cf. Lemma 3.2).

Remark 3.10. In the course of proving Theorem 1 we will show, with two exceptions, that whenever X_0 has non-zero cohomology on a module V, this does in fact give rise to a non-G-cr subgroup of G. The exceptions occur in the A_3^2-parabolic subgroups of E_8, where X_0 has type A_3 and $V = V_X(210)$ occurs as a summand of a shape module, but $H^1(X_0, Q)$ nevertheless vanishes. This is caused by a shape module with non-zero second cohomology group, which by Lemma 3.4 obstructs the lifting of cocycles from a level of Q to Q itself.

Table 3. Shape modules with non-zero cohomology, $p = 2$

G	L'	$X_0 \le L$	$V_S \downarrow L'$	$V_S \downarrow X_0$
F_4	B_3	$X_0 = L'$	$100\|0$	$100\|0$
	C_3	$X_0 = L'$	$001\|100$	$001\|100$
E_6	A_5	C_3 via 100	λ_1, λ_5	100
E_7	A_5	C_3 via 100	λ_1, λ_5	100
			λ_3	$100\|001\|100$
		A_3 via 010	λ_2, λ_4	$101 + 0$
	E_6	$C_4 < F_4$	λ_1, λ_6	$0100 + 0$
		$D_4 < C_4 < F_4$	λ_1, λ_6	$0100 + 0$
E_8	A_5	C_3 via 100	λ_1, λ_5	100
			λ_3	$100\|001\|100$
		A_3 via 010	λ_2, λ_4	$101 + 0$
	A_3^2	A_3 via $(100, 100^{[1]})$	$(010, 100)$	210
		A_3 via $(100^{[1]}, 100)$	$(100, 010)$	210
	E_6	$C_4 < F_4$	λ_1, λ_6	$0100 + 0$
		$D_4 < C_4 < F_4$	λ_1, λ_6	$0100 + 0$
	A_7	C_4 via 1000	λ_1	1000
		D_4 via 1000	λ_2	$0\|0100\|0$
		B_3 via 001	λ_2	$0\|100\|010\|100\|0$
	D_7	B_3^2 via $0\|((100, 0) + (0, 100))\|0$	λ_1	$0\|((100, 0) + (0, 100))\|0$
		$B_3 < B_3^2$ via $0\|(100 + 100^{[r]})\|0$ $(r \neq 0)$	λ_1	$0\|(100 + 100^{[r]})\|0$
		A_3 via 101	λ_1	101

Proof of Proposition 3.9. As discussed in the proof of Lemma 3.6, each shape module V_S occurring is a Weyl module with a small range of possible high weights. In most cases, determining the action $V_S \downarrow X_0$ is a routine weight-space calculation, given the known embedding of X_0 into L'. For instance, when L' has type A_n, all modules lie in the tensor algebra of the natural module λ_1, and thus $V_S \downarrow X_0$ can be calculated by applying constructions to $\lambda_1 \downarrow X_0$ such as taking tensor products, summands etc. In other cases, the weights of X_0 on $V_{L'}(\lambda)$ can be determined by identifying a maximal torus of X_0 as a sub-torus of a maximal torus of L'. In yet further cases, the action follows from previous work of Seitz [36, Theorem 1]. For example, in the case of $C_3 < D_7$ via 010 when $p = 2$, the action of C_3 on the shape module $V_S = V_{D_7}(\lambda_6)$ is given by [36, Case IV$_8$, p. 283]. Similarly, for $A_3 < D_7$ via 101 when $p = 2$, the action of X_0 on $V_{D_7}(\lambda_6)$ is given by [36, Case S$_7$, p. 283].

We now give details of two of the most involved cases, both with G of type E_8. When X_0 has type B_3, embedded in A_7 via 001 when $p = 2$, the three shape modules occurring have high weights λ_1, λ_3 and λ_6, hence restrict to X_0 as 001, $\wedge^3(001)$ and $\wedge^2(001)$ respectively (for the last, we use the isomorphism of the modules λ_6 and λ_2^* as well as the self-duality of 001). Now 001 is tilting for X_0 and therefore has trivial cohomology. The exterior cube $\wedge^3(001)$ is self-dual of dimension 56 and its weights are sums of triples of pairwise-distinct weights of 001; it follows that $\wedge^3(001)$ has a high weight 101. Inspecting [32], we find that $W_{X_0}(101)$ is irreducible of dimension 48 in characteristic 2; the highest remaining weight is 001. Since 101 and 001 are tilting in characteristic 2, it follows that $\wedge^3(001)$ splits as a direct sum $101 + 001$, and has trivial cohomology. Finally, similar calculations show that $\wedge^2(001)$ has composition factors $0^2/100^2/010$. In this case, the module turns out to be uniserial with socle series $0|100|010|100|0$. General theory tells us, for instance, that $\wedge^2(001)$ admits a unique non-zero X_0-homomorphism to K, up to scalars, since 001 supports a unique nondegenerate alternating bilinear form up to similarity. However the most straightforward way to determine the submodule structure is computationally, working with explicit matrices generating a sufficiently large finite subgroup such as $\mathrm{SO}_7(4) \leq X_0$, for which the module will still be uniserial.

Now that we know $\wedge^2(001) = 0|100|010|100|0$, we can work out its first cohomology group. The inclusion $100|0 \to \wedge^2(001)$ gives a long exact sequence of cohomology, part of which is

$$H^0(X_0, 0|100|010) \to H^1(X_0, 100|0) \to H^1(X_0, \wedge^2(001)) \to H^1(X_0, 0|100|010).$$

The first group here is zero, as is the right-hand group since $0|100|010$ is the dual of a Weyl module [25, Lemma 7.1.2(ii)] and such modules have zero first cohomology [16, Theorem 1]. Thus the middle groups are isomorphic, and $H^1(X_0, 100|0) = K$ since $T(100) = 0|100|0$, which has zero cohomology [*loc. cit.*].

For a final example, again with G of type E_8, consider X_0 of type B_3^2, embedded into L' of type D_7 via the module $V = 0|((100,0)+(0,100))|0$. Then $\dim H^1(X_0, V) = 1$: Consider the short exact sequence $\{0\} \to (100,0)|0 \to V \to 0|(0,100) \to \{0\}$ and the associated long exact sequence. We have $H^0(X_0, 0|(0,100)) = H^1(X_0, 0|(0,100)) = \{0\}$, the latter since it is the dual of a Weyl module. It follows that $H^1(X_0, (100,0)|0) \cong H^1(X_0, V) \cong K$, as claimed. Finally, the module $V_{\lambda_6}(\lambda_6) \downarrow X_0$ is a tensor product of spin modules for each factor, by [23, Proposition 2.7], in particular it is irreducible and has zero cohomology, so does not appear in Table 3. □

We single out a particular case of Proposition 3.9, both because it is a more involved calculation, and because we require additional details when classifying subgroups of type B_3 in Section 5.

Lemma 3.11. *Let $p = 2$, let X_0 be simple of type C_3 and let $M = \bigwedge^3 V_{X_0}(100)$. Then $H^1(X_0, M)$ vanishes, and $H^1(X_0, M^{[1]})$ is 1-dimensional. The inclusion $200 \to M^{[1]}$ induces an isomorphism of first cohomology groups, and the map $M^{[1]} \to 200$ induces the zero map on the first cohomology groups.*

Proof. The first statement holds since $M = 100|001|100$, which has no composition factors with a non-zero first cohomology group. Note, however, that $\dim H^1(X_0, 200) = 1$. Now, direct computation (e.g. using [17]) shows that the Weyl module $W(002) = 002|200|020|200|000|101|000$, a uniserial X_0-module. Thus $W(002)$ has no indecomposable quotient with composition factors $002/200/000$. Taking duals, we see that $H^1(X_0, 200|002)$ vanishes.

Next, the short exact sequence $\{0\} \to 200 \to M^{[1]} \to 200|002 \to \{0\}$ induces an exact sequence $\{0\} \to H^1(X_0, 200) \to H^1(X_0, M^{[1]}) \to H^1(200|002) = \{0\}$, which shows that $H^1(X_0, M^{[1]}) \cong H^1(X_0, 200) \cong K$. This also shows that the map $200 \to M^{[1]}$ induces an isomorphism on first cohomology groups.

Finally, the submodule $002|200$ of $M^{[1]}$ has one-dimensional first cohomology group, which we see from the short exact sequence $\{0\} \to 200 \to 002|200 \to 002 \to \{0\}$ inducing $\{0\} \to K \to H^1(X_0, 002|200) \to H^1(X_0, 002) = \{0\}$. Then the exactness of $\{0\} \to H^1(X_0, 002|200) \to H^1(X_0, M^{[1]}) \to H^1(X_0, 200)$ shows that the map $M^{[1]} \to 200$ induces the zero map on cohomology groups. □

3.6. **Exhibiting non-G-cr subgroups.** In many places, studying first cohomology will limit the number of classes of non-G-cr subgroups of a given type occurring. Here, we collect results which will be useful for showing that such subgroups do in fact exist. In places, we will be aware of non-M-cr subgroups when M is a proper reductive subgroup of G, and it will be helpful to know when such a subgroup is also non-G-cr; this is not automatic.

Lemma 3.12 ([2, Lemma 2.6 and Corollary 3.21])**.** *Let S be a linearly reductive subgroup of a connected reductive algebraic group G. Then S is G-cr, and if $H = C_G(S)^\circ$ then a subgroup of H is H-cr if and only if it is G-cr.*

In the particular case that S is a torus, $C_G(S)$ is a Levi subgroup of G and the result in this case was first proved in [39, Proposition 3.2].

The following is a minor generalisation of an example of M. Liebeck [2, Example 3.45].

Proposition 3.13. *Let X be a group with a faithful, irreducible module V of even dimension in characteristic $p = 2$, supporting a nondegenerate bilinear form.*

Then the orthogonal direct sum $W = V \perp V$ realises X as a non-H-cr subgroup of $H = \mathrm{Sp}(W)$, and of $\mathrm{SO}(W)$ if V is orthogonal.

Proof. We need only prove that X is non-H-cr. Write V', V'' for the left and right summands of W above, and fix an isometric X-module isomorphism $\phi : V' \to V''$. Then the proper, non-zero X-stable subspaces of W (other than V'' itself) all have the form V_λ, where

$$V_\lambda = \{v + \lambda\phi(v) \,:\, v \in V'\}.$$

Now, since ϕ is an isometry and V' is orthogonal to V'', we have $(v + \lambda\phi(v), w + \lambda\phi(w)) = (1 + \lambda^2)(v, w)$. Since the bilinear form is nondegenerate and $p = 2$, this expression is zero for all $v, w \in V$ if and only if $\lambda = 1$. Thus X preserves a unique nonzero totally isotropic subspace of W, hence X lies in a unique parabolic subgroup of $H = \mathrm{Sp}(W)$ and is therefore non-H-cr. Similarly, if V supports a nondegenerate quadratic form q then $q(v + \lambda\phi(v)) = (1 + \lambda^2)q(v)$ for all $v \in V'$, and again we find that V_1 is the unique nonzero X-stable totally singular subspace of W, so X is non-$\mathrm{SO}(W)$-cr. □

Remark 3.14. We will need this proposition in three situations: when $X = A_3 < H = D_6$ via $010 + 010$, when $X = B_3 < H = D_8$ via $001 + 001$ and when $X = D_4 < H = D_8$ via $1000 + 1000$. In each case this leads to two non-H-conjugate, non-G-cr subgroups of H, as can be seen for instance via their inequivalent actions on the half-spin modules for H; these subgroups will be conjugate under an outer automorphism of H. In general, one can prove using the orbit-stabiliser theorem, and the fact that X is unique up to conjugacy in the full isometry group of the form on $W = V \perp V$, that either one or two H-conjugacy classes of subgroups will occur in this way.

3.7. **Calculating centralisers.** We now discuss some details involved in calculating $C_G(X)^\circ$ for each X occurring in Theorem 1. Let P be minimal among parabolic subgroups of G containing X, with Levi decomposition $P = QL$, and let X_0 be its image in L. To begin, let V be an irreducible X-module occurring in the filtration of Q by X-stable normal subgroups. Since Q acts trivially on V, the action of X factors through X_0, giving a correspondence between the composition factors for X on the levels of Q, and those for X_0. Moreover when Q is abelian, or more generally when elements of X have the form $q_x x$ for some $x \in X$ and $q_x \in Z(Q)$, the entire action of X on Q factors through X_0, and in particular the fixed-point spaces Q^X and Q^{X_0} coincide. In this case, Q^{X_0} provides a connected unipotent subgroup of $C_G(X)$. Also when Q is abelian, the group Q^{X_0} itself is straightforward to determine from knowledge of Q as an X_0-module, using [1]. When Q is not abelian, however, two issues arise: The group $\dim Q^{X_0}$ need not be simply the number of trivial X_0-composition factors in each level, and also the elements q_x above need not commute with Q^{X_0}, so determining Q^X is more subtle.

To overcome the first issue, one can use the fact that X_0 is G-cr, which places useful constraints on the structure of $C_G(X_0)$; as discussed for instance in [31, §6.2]. In many cases of interest to us, Liebeck and Seitz [22, p. 333] have already determined $C_G(X_0)^\circ$, and we are able to use this to determine Q^{X_0} and hence Q^X.

For the issue regarding the relationship between Q^X and Q^{X_0}, this often comes down to showing that the elements q_x above commute with specific elements of Q^{X_0}, for instance by using the Chevalley commutator formula. Frequently, this reduces to the fact that some subgroup $Q(j)$ is abelian and contains both Q^{X_0} and all elements q_x. The most intricate calculations occur when G has type E_8 and X has type B_3, in which case we explicitly construct the relevant cocycles $x \to q_x$ in terms of root elements of G (Proposition 5.1), and thereby determine Q^X directly.

The following will also be useful in calculating $C_G(X)^\circ$ for each subgroup X in Theorem 1.

Lemma 3.15. *Let G be a reductive algebraic group of rank r with non-G-cr connected subgroup X, contained minimally in a parabolic subgroup $P = QL$ with L of semisimple rank s. Then $C_G(X)$ has rank at most $r - s - 1$.*

Proof. Suppose that $S \leq C_G(X)$ is a torus of rank $r - s$. To obtain a contradiction, it suffices to prove that X is contained in a parabolic subgroup of G whose Levi subgroups have semisimple rank less than s, since all minimal parabolic overgroups of X are associated (Lemma 3.1).

Let $M = C_G(S)$. Then M is a Levi subgroup of G containing X. Since X is non-G-cr, it follows from Lemma 3.12 that X is non-M-cr. In particular, $X < P_M = Q_M L_M$ where P_M is a proper parabolic subgroup of M and L_M is a Levi subgroup of M with semisimple rank less than s. It follows from [10, Propositions 2.6.6, 2.6.7], that there exists a parabolic subgroup $\hat{P}$ of G such that $P_M < \hat{P}$ and L_M is a Levi subgroup of $\hat{P}$. Since $X < P_M < \hat{P}$, we have reached a contradiction. □

Lemma 3.16. *Let X be a reductive subgroup of G. Then there exists a parabolic subgroup P with Levi decomposition $P = QL$, minimal subject to containing X, such that Q^X is a maximal connected unipotent subgroup of $C_G(X)$.*

Furthermore, let X_0 be the image of X under the projection $P \to L$, and let n be the total number of trivial X_0-composition factors across all the levels $Q(i)/Q(i+1)$, $i \geq 0$. Then $\dim Q^X \leq n$.

Proof. The existence of P follows directly from the Borel–Tits Theorem; if U is a maximal connected unipotent subgroup of $C_G(X)$ then U is the unipotent radical of UX and thus there exists a parabolic subgroup P of G which contains X, and contains U in its unipotent radical. Since $R_u(P_1) \geq R_u(P_2)$ whenever $P_1 \leq P_2$ are parabolic subgroups, we can take P to be minimal subject to containing X.

For the latter statement, note that the filtration $Q = Q(1) \geq Q(2) \geq \ldots$ consists of subgroups which are normal in P, in particular, the action of X on Q descends to an action of X on each level $Q(i)/Q(i+1)$. Moreover, Q acts trivially on this since $[Q(i), Q(j)] \subseteq Q(i+j)$, and thus the action of X factors through X_0. Thus we have a correspondence between X-composition factors and X_0-composition factors on each level. In particular, X and X_0 have the same number of trivial composition factors. Now the filtration of Q by X_0-stable normal subgroups $Q(i)$ gives rise to a filtration of Q^X by X-stable subgroups, and the corresponding quotients are trivial X-modules. The claim follows. □

Lemma 3.17. *Let P be a maximal parabolic subgroup of G, and suppose that P is minimal among parabolic subgroups containing a given non-G-cr subgroup X of G. Then:*

(i) *P is the unique proper parabolic subgroup of G containing X,*
(ii) *$N_G(X) \leq P$ and $C_G(X)^\circ \leq Q$.*

Proof. (i) By minimality, the image $\pi(X)$ is L-irreducible. Now let P_1 be any proper parabolic subgroup of G containing X; we will show that $P_1 = P$. Standard results on intersections of parabolic subgroups show that the Levi factor of P_1 is G-conjugate to L [30, Lemma 3.26]. We can then assume that P and L are standard, and by [10, §8] we can assume that $P_1 = n \cdot P_2$ for a standard parabolic subgroup P_2, where $n \in N_G(T)$ comes from a certain set of *distinguished double coset representatives* of the Weyl group. Now conjugation by n sends the standard Levi factor L_2 of P_2 to L. By maximality, L and L_2 correspond to removing a single node from the Dynkin diagram of G, hence n sends all the simple root subgroups in L_2 to positive roots; call the remaining simple root α. If $n \cdot \alpha$ is also positive then n preserves the set of positive roots of G, hence induces the identity element of $N_G(T)/T$, so $P_1 = P_2$. Then P and P_1 are both standard and both minimal with respect to containing X, hence $P = P_1$ as claimed. If instead $n \cdot \alpha$ is negative then n sends all roots of G with positive α-coefficient to negative roots. These are precisely the roots occurring in the unipotent radical of P_2, whereas the roots occurring in the unipotent radical of P are all positive; hence $P \cap P_1 = P \cap (n \cdot P_2) = L$. However, this intersection contains X, and by assumption X does not lie in L; this contradiction shows that this latter case does not occur, and thus $P_1 = P$.

(ii) By part (i) any element of G normalising X also normalises P, hence $N_G(X) \leq P$. Now by Lemma 3.15, $C_G(X)^\circ$ has rank 0 hence is unipotent, hence Lemma 3.16 gives us a (proper) parabolic subgroup P_0 containing X and having $C_G(X)^\circ$ as the fixed-point subgroup of its unipotent radical under X; by part (i) we have $P_0 = P$. □

4. Type A_3

4.1. G of type E_7.

4.1.1. *L' of type A_5.* There are three standard A_5-parabolic subgroups of G, namely P_{13456}, P_{34567} and P_{24567}. The first two are associated, with Levi subgroup labelled A_5' according to the convention discussed in Section 2.6, while P_{24567} forms its own association class, and has Levi subgroup labelled simply A_5. By Lemma 3.1, then, non-G-cr subgroups with irreducible image in the standard Levi subgroups L_{13456} and L_{34567} are not G-conjugate to those with irreducible image in L_{24567}.

First let $P = P_{13456}$ or P_{34567}, with Levi decomposition $P = QL$. Let X_0 be a simple subgroup of type A_3 contained in the derived subgroup L', with $X_0 \leq L'$ via 010 in the notation of Section 2.7. As described in Section 2.4, the levels of the unipotent radical Q are modules for L', whose high weights can be determined combinatorially. The L'-module structure of these levels, as well as their restrictions to X_0, are as follows.

	$Q/Q(2)$	$Q(2)/Q(3)$	$Q(3)/Q(4)$	$Q(4)$
L'_{13456}	$\lambda_3 + \lambda_5$	$\lambda_2 + 0$	λ_5	
$X_0 \leq L_{13456}$	$(010\|(200+002)\|010) + 010$	$101 + 0^2$	010	
L'_{34567}	$\lambda_1 + \lambda_2$	λ_3	λ_5	0
$X_0 \leq L_{34567}$	$010 + 101 + 0$	$010\|(200+002)\|010$	010	0

Here we have used $\lambda_2 \downarrow X_0 = \bigwedge^2(010) = 101+0$, and $\lambda_3 \downarrow X_0 = \bigwedge^3(010) = 010|(200+002)|010$. By Proposition 3.9, $H^1(X_0, 101) \cong K$ and all other summands in a level of Q have zero first cohomology group for X_0. Hence $\mathbb{V} \cong K$ (for each of the two choices of P).

The non-trivial torus $Z(L)$ centralises X_0 and acts on Q without fixed points, hence induces a non-trivial action on $\mathbb{V} = K$. As discussed in Section 3.3, it follows that there is at most one class of non-G-cr subgroups occurring in each parabolic. In Q_{13456} and Q_{34567} the modules of high weight λ_2 are respectively generated as an X_0-module by the images of $U_{0101111}$ in level 2 and U_{α_2} in level 1. Conjugation by the element $n_{1011111}n_{1010000} \in N_G(T)$ maps L_{13456} to L_{34567} and sends $U_{0101111}$ to U_{α_2}. We deduce that a non-G-cr subgroup in one of these parabolic subgroups is conjugate to a subgroup of the other. In particular, any non-G-cr subgroup arising here has a conjugate contained in P_{13456}. In this case, $Q(3)$ contains $U_{0101111}$ and so any non-G-cr subgroup arising here has a conjugate contained in $Q(3)X_0$. Note also that $[Q(3), Q(3)] = \{0\}$ by the Chevalley commutator formula, hence $Q(3)$ is abelian. Since $Q(3)/Q(4)$ has two trivial composition factors and since $H^1(X_0, 010) = \{0\}$, it follows that each complement to Q in QX_0 centralises a 2-dimensional unipotent subgroup, namely $Q(3)^{X_0}$.

We now exhibit an appropriate non-G-cr subgroup. By Remark 3.14, there are two D_6-classes of non-D_6-cr subgroups $A_3 \leq D_6$ via $010+010$; fix representatives Y and Z of these. It follows from Lemma 3.12 that Y and Z are non-G-cr, since D_6 is a Levi subgroup of G. Their actions on the 56-dimensional irreducible G-module are given in Table 6; pick Y and Z such that $V_{56} \downarrow Y = 010^4 + T(200) + T(002)$ and $V_{56} \downarrow Z = 010^4 + T(101)^2$. Then Y and Z are not conjugate in G since they are not $\mathrm{GL}(V_{56})$-conjugate, and Z is not contained in a conjugate of P_{13456} as its composition factors on V_{56} are distinct from those of X_0. By Proposition 3.9, each non-G-cr subgroup of type A_3 is contained in an A_5-parabolic subgroup. Thus Z lies in a conjugate of P_{24567} which we consider in a moment; since Y is not G-conjugate to Z, it follows that Y lies in a conjugate of P_{13456}.

We now calculate $C_G(Y)^\circ$. To begin, Lemma 3.15 shows that $C_G(Y)$ has rank at most 1. Since $Y \leq D_6$ we have $C_G(Y) \geq C_G(D_6) = \bar{A}_1$ [22, p.333 Table 2], thus $C_G(Y)$ has rank exactly 1. Now let U be a maximal connected unipotent subgroup of $C_G(Y)$. By Lemma 3.16 at least one of P_{13456} and P_{34567} contains a conjugate of U in its unipotent radical. In the table above, each Q has at most 2 trivial composition factors, hence also by Lemma 3.16 we have $\dim U \leq 2$. On the other hand, above we have exhibited a 2-dimensional unipotent subgroup of $Q = Q_{13456}$ centralised by all complements to Q in QX_0. Thus $\dim U = 2$. We have shown that $C_G(Y)^\circ$ is a connected algebraic group of rank 1 with a maximal connected unipotent subgroup of dimension 2. It follows that $C_G(Y)^\circ = U_1\bar{A}_1$ for some 1-dimensional connected unipotent group U_1.

We now consider the second association class of A_5-parabolic subgroups. Let $P = P_{24567} = QL$ and again let X_0 be a simple subgroup of type A_3, with $X_0 \leq L'$ via 010. The corresponding actions of L' and X_0 on Q are as follows

	$Q/Q(2)$	$Q(2)/Q(3)$	$Q(3)/Q(4)$	$Q(4)/Q(5)$	$Q(5)$
L'	$\lambda_2 + 0$	λ_2	λ_4	0	0
X_0	$101 + 0^2$	$101 + 0$	$101 + 0$	0	0

Thus $\mathbb{V} \cong K^3$. The X_0-modules 101 in levels 1, 2 and 3 of Q are respectively generated by the root groups U_{α_3}, $U_{1010000}$ and $U_{1122100}$. By [22, p.333, Table 3], $C_G(X_0)^\circ = G_2$ and thus Q^{X_0} is 6-dimensional. The trivial L-module in $Q/Q(2)$ is generated by $z_1(c) = x_{\alpha_1}(c)$, hence gives a subgroup of Q^{X_0} inducing an X_0-module isomorphism from the summand λ_2 of $Q/Q(2)$ to the summand λ_2 of $Q(2)/Q(3)$. Furthermore the trivial X_0-submodule of $Q(2)/Q(3)$ is generated by $z_2(c) = x_{1112100}(c)x_{1111110}(c)x_{1011111}(c)$, inducing a non-zero homomorphism $Q/Q(2) \to Q(3)/Q(4)$, and this is non-zero on the summand 101 since this trivial module lies outside of $Z(Q/Q(4)) = Q(3)/Q(4)$. Thus, as described in Section 3.3, when parametrising complements to Q in QX_0 by elements $(k_1, k_2, k_3) \in \mathbb{V}$, we may assume that at most one of k_1, k_2 and k_3 is non-zero. Moreover, the element $n_{\alpha_1} \in N_G(T)/T$ centralises L' and exchanges the roots of levels 1 and 2 whose roots give rise to the L'-modules of high weight λ_2. Finally, the following element of the Weyl group normalises L' and swaps the roots 1010000 and 1122100:

$$n_3n_4n_2n_5n_4n_3n_6n_5n_4n_2n_7n_6n_5n_4n_3.$$

By the discussion in Section 3.4, conjugation by these elements fuses together classes of complements corresponding to $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ of $\mathbb{V}$. Thus there is at most one non-G-cr subgroup in P_{24567}, up to G-conjugacy, corresponding to $(0,0,1) \in \mathbb{V}$. Above, we found a non-G-cr subgroup Z lying in this class of parabolic subgroups with irreducible image in L.

It remains to calculate $C_G(Z)^\circ$. To start, Z is conjugate to a subgroup corresponding to $(0,0,1) \in \mathbb{V}$, and we may therefore assume that Z is contained in $Q(3)X_0$, where $Q(3)$ is abelian. We claim that any complement to $Q(3)$ in $Q(3)X_0$ commutes with Q^{X_0}. To check this, we start by noting that $x_{1122100}(t)$ commutes with all elements of $Q^{X_0} \cap Q(3)$ since $Q(3)$ is abelian, so any complement to $Q(3)$ in $Q(3)X_0$ commutes with $Q^{X_0} \cap Q(3)$. Now Q^{X_0} is generated by $Q(3) \cap Q^{X_0}$ together with $z_1(c), z_2(c)$ given above and $z_3(c) = x_{0112100}(c)x_{0111110}(c)x_{0011111}(c)$. It is then a routine check using the Chevalley commutator formula that $x_{1122100}(t)$ also commutes with these three generators. Alternatively, one need not resort to explicit calculations since elements of Q^{X_0} induce X_0-module homomorphisms from the X_0-module generated by $x_{1122100}(t)$, which is of high weight 101 and in level 3, into levels 4 and above. But these homomorphisms are necessarily trivial because there are no vectors of non-zero weight in levels 4 or 5. This implies that elements of Q^{X_0} commute with $x_{1122100}(t)$.

At this point, we know that $C_G(Z)^\circ$ has rank 1 and contains a subgroup $\bar{A}_1$. By Lemma 3.16 we also know that $C_G(Z)$ has a maximal connected unipotent subgroup of dimension at most 6, and we have exhibited such a subgroup above, namely Q^{X_0}. It follows that $C_G(Z)^\circ = U_5\bar{A}_1$, where U_5 is a 5-dimensional connected unipotent subgroup.

4.2. G **of type** E_8**.**

4.2.1. *L' of type A_5.* The four standard A_5-parabolic subgroups of G are P_{13456}, P_{34567}, P_{45678} and P_{24567}. We start by considering $P = P_{13456} = QL$, with $X_0 = A_3 \leq L'$ via 010. The action of X_0 on the levels of Q is as follows:

$Q/Q(2)$	$Q(2)/Q(3)$	$Q(3)/Q(4)$	$Q(4)/Q(5)$
$(010\|(200+002)\|010)+010+0$	$101+010+0^2$	$101+010+0$	010^2

$Q(5)/Q(6)$	$Q(6)/Q(7)$	$Q(7)/Q(8)$	$Q(8)$
101	010	0	0

The summands of high weight 101 in levels $2, 3, 5$ are respectively generated by the images of the root groups $U_{01011110}$, $U_{01011111}$, and $U_{12232221}$. Furthermore, the elements $z_1(c) = x_{\alpha_8}(c)$, $z_2(c) = x_{01122210}(c)x_{11221110}(c)x_{11122110}(c)$ and $z_3(c) = x_{01122211}(c)x_{11221111}(c)x_{11122111}(c)$ commute with X_0. Then a standard Chevalley commutator formula calculation shows that for each $c \neq 0$, $z_1(c)$ induces an X_0-module isomorphism between the summands 101 in levels 2 and 3. Similarly, non-trivial elements $z_2(c)$ and $z_3(c)$ induce isomorphisms between the summands 101 in levels 3 and 5, and in levels 2 and 5, respectively. Parametrising complements to Q in QX_0 by $(k_1, k_2, k_3) \in \mathbb{V}$, this allows us to assume that $k_1k_2 = k_1k_3 = k_2k_3 = 0$. Moreover, n_{α_8} preserves the roots in L_{13456} and swaps the roots 01011110 and 01011111, the element $n_{12232111}n_{01122211}n_{10111100}n_2n_4n_3n_5n_4n_2n_8n_7$ preserves the roots in L_{13456} and sends $01011111 \mapsto 12232221 \mapsto 01011110$. Thus in this case, there is precisely one non-G-cr subgroup of type A_3 having irreducible image in the Levi factor.

If $P = QL$ is one of the three remaining standard A_5-parabolics then again, precisely three direct summands of some level of the unipotent radical are irreducible of high weight 101 for a subgroup $A_3 \leq L$ via 010. These are the only summands occurring with non-zero first cohomology group, and are respectively generated by the images of root subgroups corresponding to the following roots:

Parabolic	Roots
P_{34567}	α_2, 11111111, 12232111
P_{45678}	01110000, 11110000, 12232100
P_{24567}	α_3, 10100000, 11221000

Essentially identical calculations in these three parabolics show that, up to conjugacy, there is at most one non-G-cr subgroup of type A_3 in each, having irreducible image in the Levi factor. Finally, the following elements of the Weyl group of G send the roots of L_{13456} to those of the Levi factor in each other case, and send the root 01011110 to the given root β occurring in the corresponding unipotent radical:

Parabolic	Root β	Weyl group element
P_{34567}	α_2	$n_7n_6n_5n_4n_3n_1$
P_{45678}	01110000	$n_7n_6n_5n_4n_3n_1n_8n_7n_6n_5n_4n_3$
P_{24567}	10100000	$n_{112}n_{101}n_7n_6n_5n_4n_3$

We conclude that up to G-conjugacy there is exactly one non-G-cr subgroup of type A_3 with irreducible image in a Levi subgroup of type A_5.

Let $Y = A_3 \leq E_7$ represent one of the two non-E_7-cr subgroup classes (these are fused in G since they lie in a subgroup D_6, on which G induces a graph automorphism). Then Y is non-G-cr by Lemma 3.12 and contained in an A_5-parabolic subgroup of G.

It remains to calculate $C_G(Y)^\circ$. Since $C_G(E_7)^\circ = \bar{A}_1$ and $C_{E_7}(Y) = U_5\bar{A}_1$, it follows that $U_5\bar{A}_1^2 = C_{\bar{A}_1E_7}(Y) \leq C_G(Y)$. Checking the restriction of $L(G)$ to Y in Table 8, we see that $\dim C_G(Y) \leq 12$. We claim that Y is not separable in G, from which it follows that $U_5\bar{A}_1^2 = C_G(Y)$. To prove that Y is not separable in G we start by considering the restriction $L(G) \downarrow \bar{A}_1E_7 = L(\bar{A}_1E_7) + (1, V_{56})$. From this, and the restriction of $L(G) \downarrow Y$, we deduce that $\dim C_{L(\bar{A}_1E_7)}(Y) = 12$. But $C_{\bar{A}_1E_7}(Y)$ was already shown to be 11-dimensional in the above calculation. Therefore, Y is not a separable subgroup of $\bar{A}_1E_7$. Now, $(\bar{A}_1E_7, G)$ is a reductive pair and so by [4, Theorem 1.4], Y is not a separable subgroup of G, as required.

4.2.2. *L' of type A_3^2.* The two standard A_3^2-parabolic subgroups $P = QL$ of G are P_{134678} and P_{234678}. The filtration of the unipotent radical of P_{134678} has a module $(010, 100)$, generated as an X_0-module by the image of the root group $U_{01011000}$, and a module $(100, 010)$ generated by the image of the root group $U_{01122100}$. Thus if $A_3 \leq A_3^2$ via $(100^{[r]}, 100^{[s]})$, we find that $\mathbb{V}$ is 1-dimensional if $(r, s) = (1, 0)$ or $(0, 1)$, and $\mathbb{V} = \{0\}$ otherwise. We will show below that non-trivial cocycles $X_0 \to Q(2)/Q(3)$ and $X_0 \to Q(3)/(4)$ fail to lift to cocycles $X_0 \to Q$, so that $H^1(X_0, Q) = \{0\}$ and P_{134678} contains no non-G-cr subgroups of type A_3. (This occurs because of the presence of a module in the filtration with non-zero second cohomology group.)

In P_{234678}, the unipotent radical again gives rise to A_3^2-modules $(010, 100)$ and $(100, 010)$, respectively generated by the images of the root subgroups U_{α_5} and $U_{11122100}$. The Weyl group element $n_{\alpha_1}n_{\alpha_3}n_{\alpha_4}n_{\alpha_2}$ maps the standard Levi subgroup L_{134678} to L_{234678}, and respectively sends the root groups $U_{01011000}$ and $U_{11122100}$ to U_{α_5} and $U_{11122100}$. Thus any non-G-cr subgroup A_3 of P_{234678} is conjugate to a subgroup contained in P_{134678}.

We now show that $H^1(X_0, Q) = \{0\}$ when $Q = R_u(P_{134678})$. For this we use explicit calculations with root elements in Q; we give details for the case $X_0 \leq L'_{134678}$ via $(100, 100^{[1]})$, so that a module with non-zero first cohomology group appears only in $Q(2)/Q(3)$. We omit essentially identical calculations for other embeddings of X_0, where the Frobenius twist appears on the other factor, or where the dual module 001 is used in place of 100 in either factor.

Fix a maximal torus T_{X_0} of X_0, contained in the fixed maximal torus of G, and let β_1, β_2 and β_3 be a set of simple roots of X_0 with respect to T_{X_0}. If $V = V_{X_0}(210)$ is the irreducible X_0-module, then a non-trivial cocycle $X_0 \to V$ is determined by its restrictions to the three subgroups U_{β_1}, U_{β_2}, U_{β_3}, since these subgroups together generate the unipotent radical U_{X_0} of a Borel subgroup B_{X_0} of X_0, and we have isomorphisms $H^1(X_0, V) \cong H^1(B_{X_0}, V) \cong H^1(U_{X_0}, V)^{T_{X_0}}$ by [20, 4.7(c), 6.9(3)]. Moreover, consider an indecomposable extension $W = 0|V$, generated by a T_{X_0}-stable vector v of weight zero. Then the cocycles $X_0 \to V$, $x \mapsto \lambda(x \cdot v - v)$ give a complete set of representatives of the cohomology classes in $H^1(X_0, V)$. Thus each cocycle is cohomologous to some ϕ such that $\phi(x_{\beta_i}(1))$ is a sum of vectors of weight $c_i\beta_i$, $c_i \in \mathbb{N}$. Expressing the roots as elements of the weight lattice of X_0, an elementary calculation shows that each weight $\beta_1 = (2, -1, 0)$, $\beta_2 = (-1, 2, -1)$, $\beta_3 = (0, -1, 2)$ occurs in $V(210)$ with multiplicity one, and that $c_i\beta_i$ does not occur if $c_i > 1$. In particular, there are weight vectors $v_{(2,-1,0)}$, $v_{(-1,2,-1)}$, $v_{(0,-1,2)}$ such that ϕ is fully specified by $\phi(x_{\beta_1}(a)) = av_{(2,-1,0)}$, $\phi(x_{\beta_2}(b)) = bv_{(-1,2,-1)}$, $\phi(x_{\beta_3}(c)) = cv_{(0,-1,2)}$. Using the commutator relations for $x_{\beta_1}(a)$, $x_{\beta_2}(b)$ and $x_{\beta_3}(c)$, we calculate that

$$\phi(x_{\beta_1}(a)x_{\beta_2}(b)x_{\beta_3}(c)) = av_{(2,-1,0)} + bv_{(-1,2,-1)} + cv_{(0,-1,2)} + bc^2v_{(-1,0,3)} + ab^2v_{(0,3,-2)} + ab^2c^2v_{(0,1,2)}$$

for all $a, b, c \in K$, where each v_j is a non-zero vector of T_{X_0}-weight j.

We now realise T_{X_0} as a sub-torus of T. We realise A_3 as a subgroup of $A_3^2 = L'_{134678}$ via $(100, 100^{[1]})$ by defining $x_{\beta_1}(c) = x_1(c)x_6(c^2)$, $x_{\beta_2}(c) = x_3(c)x_7(c^2)$, $x_{\beta_3}(c) = x_4(c)x_8(c^2)$ for all $c \in K$. Then T_{X_0} is generated by elements $h_{\beta_i}(t) = n_{\beta_i}(t)n_{\beta_i}(-1)$, where we in turn define $n_{\beta_i}(t) = x_{\beta_i}(t)x_{-\beta_i}(-t^{-1})x_{\beta_i}(t)$, for $t \in K^*$.

Now, since T_{X_0} is a sub-torus of T, the T_{X_0}-weight spaces of $Q(2)/Q(3)$ are root subgroups of G. Direct calculation allows us to identify the six vectors v_j above as follows:

$$v_{(2,-1,0)} = x_{01011111}(\lambda_1)Q(3), \quad v_{(-1,2,-1)} = x_{01121110}(\lambda_2)Q(3), \quad v_{(0,-1,2)} = x_{11221100}(\lambda_3)Q(3),$$
$$v_{(-1,0,3)} = x_{01121111}(\lambda_4)Q(3), \quad v_{(0,3,-2)} = x_{11221110}(\lambda_5)Q(3), \quad v_{(0,1,2)} = x_{11221111}(\lambda_5)Q(3)$$

for some non-zero scalars $\lambda_1, \ldots, \lambda_6$. Finally, note that for a cocycle $\phi : X_0 \to Q$ and an arbitrary unipotent element $u \in X_0$, the element $\phi(u)u$ has order at most four, since it is a unipotent element in a complement to Q in QX_0. So we take the element $\phi(y(a,b,c))y(a,b,c)$ where $y(a,b,c) = x_{\beta_1}(a)x_{\beta_2}(b)x_{\beta_3}(c)$, and calculate:

$$y(a,b,c)^4 = x_{11221111}(ab^2c\lambda_1 + abc^2\lambda_2)x_{12232221}(ab\lambda_1\lambda_2).$$

In particular, this is only zero for all a, b and c if λ_1 and λ_2 are zero, which is the case if and only if ϕ is a coboundary.

4.2.3. *L' of type D_7.* Let $P = P_{2345678} = QL$ be the unique standard D_7-parabolic subgroup of G. The unipotent radical has two levels, with $Q/Q(2) \cong \lambda_6$ and $Q(2) = Z(Q) \cong \lambda_1$ as L'-modules.

Let $X_0 = A_3 \leq L'$ via 101. Then weight-space calculations show that $V_{L'}(\lambda_6) \downarrow X_0 = V_{L'}(\lambda_7) \downarrow X_0 = V_{X_0}(111)$ (cf. proof of [23, Proposition 2.12]; this is also stated in [36, p. 283, Case S7]), and these are the only modules of this high weight occurring in $L(G) \downarrow X_0$. In particular $\mathbb{V} = K$ and there is at most one non-G-cr subgroup of type A_3 minimally contained in P, up to G-conjugacy.

Now consider a subgroup Y of type A_3, embedded in a subsystem subgroup D_8 via the 16-dimensional module $T(101) = 0|101|0$. Since Y is reducible on the natural 16-dimensional module and Y preserves a nondegenerate quadratic form on 101 [36, p. 283, Case S7], it follows that Y lies in a D_7-parabolic subgroup of D_8, hence in a conjugate of P, and we can assume that the image of Y in L' is X_0. Note that Y is not D_8-conjugate to X_0 as they have incompatible actions on the natural 16-dimensional module, thus Y is non-D_8-cr.

In a D_7-parabolic of D_8 containing Y, the unipotent radical, call it Q_{D_8}, is abelian and isomorphic to $V_{D_7}(\lambda_1)$ as a module for L'. In G, we have $Q_{D_8} = Z(Q) = Q(2)$, and $Q_{D_8} \cong 101$ and $Q/Q_{D_8} \cong V_Y(111)$ as X_0-modules. Now the inclusion $Q_{D_8} \to Q$ induces a long exact sequence in cohomology, and since Q/Q_{D_8} has no fixed points under the action of X_0, Corollary 3.5 tells us that Y is non-G-cr.

Since P is a maximal parabolic subgroup of G and $Q^{X_0} = Q^Y = 1$, by Lemma 3.17 the connected centraliser of Y is trivial.

5. Type B_3 and B_3^2

5.1. **G of type F_4.** The classification of non-G-cr subgroups of type B_3 in G is given in [43, Lemma 4.4.3]. There are two classes of subgroups, which are respectively G-conjugates of $Y_1 < D_4 < B_4$ via $T(100)$ and $Y_2 < \tilde{D}_4 < C_4$ via $T(100)$. Moreover, Y_1 is contained in a B_3-parabolic subgroup and Y_2 is contained in a C_3-parabolic subgroup.

It remains to calculate $C_G(Y_1)^\circ$ and $C_G(Y_2)^\circ$. It suffices to calculate just one of these, as the exceptional graph morphism swaps Y_1 and Y_2 and thus their connected centralisers are isomorphic as abstract groups. We will calculate $C_G(Y_2)^\circ$. From the action of Y_2 on $L(G)$, given in Table 4, we see that $\dim C_G(Y_2) \leq 1$. On the other hand, Y_2 is contained in the parabolic subgroup $P_{234} = QL$, and Q has two levels, the second being $Z(Q)$ which is 1-dimensional and hence centralised by L'. Thus every complement to Q in QL' centralises this 1-dimensional unipotent subgroup and so $C_G(Y_2)^\circ = U_1$.

5.2. G of type E_6.

5.2.1. *L' of type A_5.* Let $P = P_{13456} = QL$ and $X_0 = C_3 \le L'$ via 100. The unipotent radical Q has two levels, with $Q(2) = Z(Q)$ a trivial 1-dimensional L'-module, and $Q/Q(2)$ isomorphic to the irreducible A_5-module λ_3. By Lemma 3.11, $H^1(X_0, Q/Q(2))$ vanishes and $H^1(X_0, (Q/Q(2))^{[1]})$ is 1-dimensional. Since $H^1(X_0, Q(2)) = \{0\}$ it follows from Lemma 3.4 that $H^1(X_0, Q)$ vanishes and $H^1(X_0, Q^{[1]})$ is at most 1-dimensional. Any non-G-cr complements to Q in QX_0 are of type B_3, by Lemma 3.2. Taking into account the action of the torus $Z(L)$ on $H^1(X_0, Q^{[1]})$ we see that up to G-conjugacy there is at most one non-G-cr subgroup B_3 in P with image X_0 under projection to L.

Let Y be a non-G-cr subgroup of type B_3 contained in a C_3-parabolic subgroup of F_4, as exhibited in [43, Theorem 1(A)]. Note that $Y < \tilde{D}_4 < F_4$, and this is how it is listed in Table 5. An embedding $F_4 \to E_6$ maps a C_3-parabolic subgroup of F_4 into P. Let R be the unipotent radical of the C_3-parabolic subgroup of F_4. Then R has a filtration $R/R(2) \cong V_{C_3}(\lambda_3)$, $R(2) \cong V_{C_3}(0)$ as C_3-modules. Thus R is an X_0-invariant normal subgroup of Q, and $(Q/R)^{[1]}$ is isomorphic to an X_0-module of shape 200|002. In particular this quotient has no fixed points, and by Corollary 3.5 this non-F_4-cr subgroup Y remains non-G-cr.

Since P is a maximal parabolic subgroup, Lemma 3.17 implies that $C_G(Y)^\circ = Q^Y$. Since $Q^{X_0} \le Z(Q)$ and is 1-dimensional it follows that $C_G(Y)^\circ = U_1$.

5.3. G of type E_7.

5.3.1. *L' of type A_5.* The three standard A_5-parabolic subgroups of G are P_{13456}, P_{34567} and P_{24567}. The unipotent radical of P_{24567} involves only modules of high weight 0, λ_2 and λ_4 for the Levi factor; in particular none of these modules have a non-zero first cohomology group for a subgroup C_3. As discussed in Section 4, the other two parabolic subgroups are associated, and the filtrations of their unipotent radicals each involve two modules of high weight λ_1 or λ_5, and one module of high weight λ_3, so $\mathbb{V} \cong K^3$ in each case.

To begin, let $P = P_{34567} = QL$, and let $X_0 = C_3 < L'_{34567}$ via 200. Then Q involves a single L'-module of each high weight λ_1, λ_3 and λ_5, respectively generated as an L'-module by the image of the root groups U_{α_1}, $U_{1111000}$ and $U_{1223210}$. The unipotent radical Q also involves an L'-module of high weight λ_2, generated by the image of the root group U_{α_2}. As an X_0-module, this is isomorphic to $\bigwedge^2(200) = 020 + 0$, and this trivial module induces a non-zero map of X_0-modules $Q/Q(2) \to Q(2)/Q(3)$. As discussed in the proof of Lemma 3.11, this induces an isomorphism of cohomology groups $H^1(X_0, 200) \to H^1(X_0, \bigwedge^3(200))$. Thus if we parametrise complements to Q in QX_0 by $(k_1, k_2, k_3) \in \mathbb{V}$, then we may assume $k_1 k_2 = 0$.

If $k_1 = 0$ then a Q-conjugate of the corresponding complement to Q in QX_0 is contained in the subgroup generated by X_0, $U_{1111000}$ and $U_{1223210}$. The Weyl group element $n_7n_6n_5n_4n_3n_1$ sends the roots in L_{34567} to roots in L_{13456}, and sends the roots 1111000 and 1223210 to roots in Q_{13456}. If instead $k_2 = 0$ then the Weyl group element:

$$n_7n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_2n_6n_5n_4n_3n_7n_6n_5n_4n_2$$

sends the roots in L_{34567} to roots in L_{13456}, and sends the roots α_1 and 1111000 to roots in Q_{13456}. Thus every non-G-cr complement to Q in QX_0 is G-conjugate to a subgroup of P_{13456}.

Now let $P = P_{13456} = QL$. Then Q has three levels and $X_0 = C_3 < L'$ acts on them as follows.

$Q/Q(2)$	$Q(2)/Q(3)$	$Q(3)$
(100|001|100) + 100	$010 + 0^2$	100

Thus $\mathbb{V} \cong K^3$, where the X_0-submodule of high weight 100 in 100|001|100 is generated by elements of the form $x_{0111000}(c)x_{0101100}(c)$ and the X_0-summands of high weight 100 are generated by the image of the root groups U_{α_7} and $U_{1223211}$.

The root group $U_{1223210}$ is a trivial L'-module, and induces a non-zero L'-module homomorphism between the two modules of high weight λ_5. Moreover, the corresponding Weyl group element $n_{1223210}$ swaps the roots α_7 and 1223211, and sends 0111000 and 0101100 to negative roots. Thus, parametrising complements to Q in QX_0 by triples $(k_1, k_2, k_3) \in \mathbb{V}$, we may assume that $k_2k_3 = 0$, and we may also swap k_2 and k_3, so long as $k_1 = 0$.

Taking into account the action of the 2-dimensional torus $Z(L)$, non-G-cr complements to Q in QX_0 correspond to one of the four triples $(1,0,0)$, $(0,0,1)$, $(1,1,0)$ and $(1,0,1)$. Thus there are at most four non-G-cr complements to Q in QX_0, up to G-conjugacy.

We now present four non-G-cr subgroups of type B_3, and we see that they are all non-conjugate by considering their actions on V_{56}, given in Table 6. Let $Y_1 < E_6$ be the non-E_6-cr subgroup from the previous section and let $Y_2 < A_6$ be embedded via $W(100)$. Then Y_1 and Y_2 are non-G-cr by Lemma 3.12. Subgroups Y_3 and Y_4 are embedded in the maximal rank subgroup A_7 via $T(100)$ and 001, respectively. To prove these are non-G-cr we note that they both act on V_{56} with two indecomposable summands of dimension 28 and hence are not contained in any Levi subgroup of G. The subgroup Y_3 is contained in a parabolic subgroup of A_7 and hence of G. The subgroup Y_4 is contained in a parabolic subgroup of G because $Y_4 < D_4 < C_4 < A_7$ and this C_4 subgroup is non-G-cr, as discussed in Section 8. Note that all Y_i subgroups must be contained in a parabolic subgroup of type A_5', since we have proved that a representative for every non-G-cr subgroup of type B_3 in G is contained P_{13456}.

It remains for us to calculate the connected centralisers of the four non-G-cr subgroups. Let us start by considering Y_1, which is contained in $F_4 < E_6$. We noted above that $C_{F_4}(Y_1)^\circ = U_1$. We also have that $C_G(F_4)^\circ = A_1$ (by [22, p.333, Table 3]) and so $U_1A_1 \leq C_G(Y_1)^\circ$. The socle series of the restriction of $L(G)$ to Y_1 given in Table 6 shows that $\dim C_{L(G)}(Y_1) \leq 4$. Thus $C_G(Y_1)^\circ = U_1A_1$.

Similarly, by considering the action of Y_2 on $L(G)$ we find that $\dim C_G(Y_2) \leq 3$. The non-G-cr subgroups corresponding to $(1,0,0)$ and $(0,0,1)$ both centralise tori and so Y_1 and Y_2 must be representatives of their conjugacy classes. In fact, one can check that Y_2 is conjugate to the subgroup corresponding to $(0,0,1)$ by calculating that $C_G(S)^\circ$ is of type A_6T_1 for the 1-dimensional torus $S \leq Z(L)$ centralising the root 1223211. Considering the structure of the unipotent radical of P_{13456} given above, we see that the non-G-cr subgroup corresponding to $(0,0,1)$, contained in $Q(2)X_0$, centralises a 2-dimensional unipotent subgroup of Q generated by the trivial X_0-modules in level 2 generated by $x_{1122111}(c)x_{11122111}(c)x_{0112221}(c)$ and $x_{1223210}(c)$. Therefore, $C_G(Y_2)^\circ = U_2T_1$.

Finally, the subgroups Y_3 and Y_4 are not contained in any Levi subgroups, as shown above, and therefore their connected centralisers are unipotent and they are each conjugate to precisely one of the subgroups corresponding to $(1,1,0)$ and $(1,0,1)$ (though we do not yet know which is which). Note that $C_{L(G)}(Y_i)$ contains $Z(L(G))$, a 1-dimensional subalgebra generated by a semisimple element, and hence $\dim C_G(Y_i) \leq \dim C_{L(G)}(Y_i) - 1$ for $i = 3,4$. In particular, using the restrictions in Table 6 we find that $\dim C_G(Y_3) \leq 2$ and $\dim C_G(Y_4) \leq 1$. Studying $P = P_{13456}$ as in the previous paragraph, we see that the subgroup corresponding to $(1,1,0)$ centralises a 1-dimensional subgroup of Q, whereas the subgroup corresponding to $(1,0,1)$ centralises a 2-dimensional unipotent subgroup of Q. It now follows that Y_3 is conjugate to the subgroup corresponding to $(1,0,1)$ with $C_G(Y_3) = U_2$ and similarly, Y_3 is conjugate to the subgroup corresponding to $(1,1,0)$ with $C_G(Y_4) = U_1$.

5.4. G of type E_8.

5.4.1. L' *of type* A_5. The four standard A_5-parabolic subgroups of G are P_{13456}, P_{34567}, P_{45678} and P_{24567}. We argue in this section as follows.

For $P = QL$ equal to each of the four standard parabolic subgroups, with $X_0 = C_3 < L'$ via 100, we find that P contains only finitely many complements to Q in QX_0, up to P-conjugacy. Moreover, each such subgroup is G-conjugate to a subgroup of P_{34567}. In fact, we can prove this latter claim without a complete analysis of the complements to Q in QX_0. We then proceed to consider P_{34567} in detail.

First let $P = P_{13456}$, and let $X_0 = C_3 < L'$ via 100. The X_0-modules with non-zero first cohomology group in the filtration of Q are generated by the images of the root groups U_α for $\alpha \in \{\alpha_7, \alpha_2, \alpha_7+\alpha_8, 12232110, 01122221, 12232111, 13354321\}$. Moreover there are four 1-dimensional subgroups of Q which centralise X_0 and induce non-zero homomorphisms between X_0-modules in levels of Q. We record these homomorphisms below. Writing (α,β) here means that a homomorphism is induced from the X_0-module generated by the image of U_α, to the X_0-module generated by the image of U_β:

Element	Induced homomorphism
$x_8(c)$	$(\alpha_7, \alpha_7+\alpha_8)$, $(12232110, 12232111)$
$x_{12232100}(c)$	$(\alpha_7, 12232110)$, $(\alpha_7+\alpha_8, 12232211)$, $(01122221, 13354321)$
$x_{01122210}(c)x_{11122110}(c)x_{11221110}(c)$	$(\alpha_7+\alpha_8, 01122221)$, $(12232111, 13354321)$
$x_{01122211}(c)x_{11122111}(c)x_{11221111}(c)$	$(\alpha_7, 01122221)$, $(12232110, 13354321)$

Working through all of the relations implied by these homomorphisms, as well as elements of $N_G(T)$ normalising L, it transpires that P_{13456} contains at most eight non-G-cr complements to Q in QX_0, up to G-conjugacy. However we do not need this; instead we show only that each such subgroup is conjugate to a subgroup of P_{34567}. For this, notice that the first homomorphism induced by $x_8(c)$ allows us to assume that $k_1k_2 = 0$, if we take an appropriate basis of the 7-dimensional space $\mathbb{V}$. Moreover, conjugation by the Weyl group element n_8 centralises X_0 and swaps α_7 and $\alpha_7+\alpha_8$, and also preserves the set of roots whose root groups in Q give rise to X_0-modules with non-zero first cohomology group. It follows that we can assume that $k_1 = 0$. So every complement to Q in QX_0 is Q-conjugate to a complement in Q_1X_0, where Q_1 is the smallest X_0-invariant subgroup of Q containing $Q(2)$ and $U_{\alpha_7+\alpha_8}$.

Now consider the element $n_7n_6n_5n_4n_3n_1$ of the Weyl group W. This sends the roots occurring in L_{13456} to the roots occurring in L_{34567}, and sends all of the roots giving modules with non-zero first cohomology group, other than α_7, to roots occurring in Q_{34567}. Thus this Weyl group element sends a Q-conjugate of each non-G-cr complement to Q in QX_0 to a subgroup of P_{34567}, as claimed.

Entirely similar calculations occur in the parabolic subgroups P_{45678} and P_{24567}. Up to conjugacy, we find that there are at most eight and three non-G-cr complements, respectively, to the unipotent radical in an appropriate semidirect product. For the case $P = P_{45678}$, these are all contained in the smallest X_0-invariant subgroup of P_{45678} generated by X_0 and $U_{23454321}$, $U_{11222100}$, $U_{11222110}$, $U_{10100000}$, $U_{12343210}$, U_{α_2} and $U_{22343210}$. The Weyl group element $n_3n_4n_5n_6n_7n_8$ sends the roots in L_{45678} to the roots in L_{34567} and sends each of these other root groups to root groups in Q_{34567}. Thus each non-G-cr complement in P_{45678} is G-conjugate to a subgroup of P_{34567}. Similarly each complement in P_{24567} is P_{24567}-conjugate to a complement contained in the smallest X_0-invariant subgroup generated by X_0 and $U_{22454321}$, $U_{11232111}$, $U_{11222211}$, $U_{10111111}$ and $U_{12343211}$. The following Weyl group element:

$$n_{00111111}n_2n_4n_3n_5n_4n_2n_6n_5n_4n_3n_7n_6n_5n_4n_2n_8$$

sends the roots in L_{24567} to the roots occurring in L_{34567}, and sends each of the root elements above to roots arising in the unipotent radical of P_{34567}; again we conclude that each non-G-cr occurring in P_{24567} is G-conjugate to a subgroup of P_{34567}.

So let $P = QL = P_{34567}$, and fix $X_0 = C_3 < L'$. The seven relevant summands of Q are generated by the images of the root groups U_{α_1}, U_{α_8}, $U_{01011111}$, $U_{11110000}$, $U_{12232100}$, $U_{22343211}$, $U_{23354321}$. All but the fourth of these give rise to an irreducible A_5-module with high weight λ_1 or λ_5, hence to a natural 6-dimensional X_0-module 100. The root group $U_{11110000}$ gives rise to an A_5-module λ_3, which restricts to X_0 as $100|001|100$. The submodule 100 is generated by the images of elements of the form $x_{11121000}(c)x_{11111100}(c)$, and as discussed in Lemma 3.11 the inclusion $100 \to 100|001|100$ induces an isomorphism in first cohomology (after applying a Frobenius twist).

Fix a basis of $\bigoplus_{i=1}^{7} H^1(X_0, Q(i)/Q(i+1)) = \mathbb{V} \cong K^7$, whose i-th member spans the first cohomology group of the i-th summand above, so that complements to Q in QX_0 are parametrised by 7-tuples $(k_1, k_2, \ldots, k_7)$.

The following elements generate Q^{X_0}, and the first six of them induce non-zero homomorphisms between X_0-modules in the levels of Q. Again, (α, β) means that a non-zero homomorphism is induced from the X_0-module generated by the image of U_α, to the X_0-module generated by the image of U_β:

Name	Elements	Induced homomorphisms	Implied conditions
$z_1(c)$	$x_{01011110}(c)x_{01111100}(c)x_{01121000}(c)$	$(\alpha_1, 11110000/12232100)$, $(\alpha_8, 01011111)$, $(22343211, 23354321)$	$k_1k_4 = 0$ or $k_1k_5 = 0$; if $k_1 = 0$ then $k_2k_3 = 0$; if $k_1 = k_2 = 0$ then $k_6k_7 = 0$
$z_2(c)$	$x_{10111111}(c)$	$(12232100, 22343211)$	$k_5k_6 = 0$
$z_3(c)$	$x_{11222211}(c)x_{11232111}(c)x_{11122221}(c)$	$(12232100, 23354321)$	$k_5k_7 = 0$
$z_4(c)$	$x_{22343210}(c)$	$(\alpha_8, 22343211)$, $(01011111, 23354321)$	$k_2k_6 = 0$; if $k_2 = 0$ then $k_3k_7 = 0$
$z_5(c)$	$x_{12343211}(c)x_{12243221}(c)x_{12233321}(c)$	$(\alpha_1, 22343211)$	$k_1k_6 = 0$
$z_6(c)$	$x_{13354321}(c)$	$(\alpha_1, 23354321)$	$k_1k_7 = 0$
$z_7(c)$	$x_{23465432}(c)$		

Moreover, there are a number of elements of the Weyl group of G which stabilise the set of simple roots in L_{34567} and conjugate together various complements to Q in QX_0. The element

$$w = n_2n_4n_3n_5n_4n_2n_6n_5n_4n_3n_7n_6n_5n_4n_2$$

normalises L' and swaps U_{α_1} with $U_{12232100}$; U_{α_8} with $U_{01011111}$; $U_{22343211}$ with $U_{23354321}$; and fixes $U_{11110000}$. This therefore swaps k_1 with k_5, and using the first induced homomorphism above, we can therefore take $k_1 = 0$. Hence using the same induced homomorphism, we can also assume $k_2k_3 = 0$.

If $k_2 \neq 0$ then $k_3 = k_6 = 0$. From the above induced homomorphisms we also have $k_5k_7 = 0$. If $k_5 = 0$ then applying the above Weyl group element lets us assume that $k_2 = 0$, which we deal with in a moment. So we can assume $k_5 \neq 0$, and we obtain the tuples $(0,1,0,1,1,0,0)$ and $(0,1,0,0,1,0,0)$, after considering the action of the three-dimensional torus $Z(L)$ on $H^1(X_0, Q^{[1]})$.

Now assume $k_2 = 0$. If $k_3 \neq 0$ then $k_7 = 0$ and $k_5k_6 = 0$. Moreover the element $n_{10111111}$ swaps k_5 and k_6, while stabilising k_3, k_4 and k_7 (it also sends α_1 and α_8 to negative roots, but this does not matter as $k_1 = k_2 = 0$). We can thus assume $k_5 = 0$. If $k_4 = k_6 = 0$ then we can apply the Weyl group element:

$$n_{22343211}n_{11122221}n_{00111110}n_{11110000}n_{11122100}n_5n_4n_6n_5n_8$$

which swaps k_3 and k_6, and therefore puts us in the $k_3 = 0$ case which we deal with in the next paragraph. Thus we assume that one of k_4 and k_6 is non-zero, and we obtain the tuples $(0,0,1,1,1,0,1,0)$, $(0,0,1,0,0,1,0)$ and $(0,0,1,1,0,0,0)$, after considering the action of $Z(L)$.

Now assume $k_1 = k_2 = k_3 = 0$. Then we have $k_6k_7 = k_5k_6 = k_5k_7 = 0$, so at most one of k_5, k_6 and k_7 is non-zero. Again, the element $n_{10111111}$ swaps k_5 and k_6, so we can assume that $k_5 = 0$. Then the Weyl group element w above lets us swap k_6 and k_7, so we can assume that $k_6 = 0$. We obtain the tuples $(0, 0, 0, 1, 0, 0, 1)$, $(0, 0, 0, 1, 0, 0, 0)$ and $(0, 0, 0, 0, 0, 0, 1)$, as well as $(0, 0, 0, 0, 0, 0, 0)$, corresponding to the G-cr class, after considering the action of $Z(L)$.

In summary, there are at most eight non-G-cr complements to Q in QX_0 up to conjugacy in G.

We now give eight non-G-cr subgroups of type B_3. Their actions on $L(G)$, given in Table 8, show that they represent different G-classes. Moreover, their composition factors on $L(G)$ show that they are all contained in a parabolic subgroup of type A_5. We let $Y_1 < E_6$, $Y_2 < A_6$ via $W(100)$, and $Y_3, Y_4 < A_7' < E_7$ via $T(100)$ and 001, respectively be the non-E_7-subgroups from Section 5.3.1, which are non-G-cr by Lemma 3.12. The subgroups $Y_5 < A_7$ via $T(100)$ and $Y_6 < D_7$ via $W(100) + W(100)^*$ are also non-G-cr by Lemma 3.12. Next we let $Y_7 < D_8$ via $T(100)^2$, which is a non-D_8-cr subgroup. From Table 8, we see that Y_7 acts on $L(G)$ with five indecomposable summands, two of which have dimension 28 and the other three have dimension 64. It now follows that Y_7 is not contained in any Levi subgroup of G and is hence non-G-cr. Finally, we let Y_8 be a non-G-cr subgroup of type B_3 from Lemma 10.9, contained in a non-G-cr subgroup of type D_4. The action of Y_8, given in Table 8, shows that it is not conjugate to the previous seven Y_i and not contained in any Levi subgroup of G (it has five direct summands on $L(G)$ of dimensions $30, 30, 62, 63, 63$).

It remains for us to calculate the connected centralisers of the subgroups Y_i. In this case, the following proposition gives us an explicit construction of the non-G-cr subgroups occurring here.

Proposition 5.1. *Let G be of type E_8, $p = 2$ and $P = P_{34567} = QL$. Let X_0 be a subgroup of type B_3 contained in P, with irreducible image X_0 of type C_3 in L. Then there exists $\mathbf{v} = (a_1, \ldots, a_7) \in \mathbb{V} \cong K^7$ such that X_0 is conjugate to $X_{\mathbf{v}} = \langle y_{\pm i}(t) : t \in K, 1 \leq i \leq 3\rangle$ for the following elements $y_{\pm i}(t)$:*

$$\begin{aligned}
y_{\pm 1}(t) &= x_{\pm 3}(t^2)x_{\pm 7}(t^2),\\
y_{\pm 2}(t) &= x_{\pm 4}(t^2)x_{\pm 6}(t^2),\\
y_3(t) &= x_5(t^2)x_{10111000}(a_1t)x_{00001111}(a_2t)x_{11222100}(a_3t)x_{11122110}(a_3t)x_{01122111}(a_4t)\\
&\quad x_{12233210}(a_5t)x_{22344321}(a_6t)x_{23465321}(a_7t),\\
y_{-3}(t) &= x_{-5}(t^2)x_{10110000}(a_1t)x_{00000111}(a_2t)x_{11221100}(a_3t)x_{11121110}(a_3t)x_{01121111}(a_4t)\\
&\quad x_{12232210}(a_5t)x_{22343321}((a_6 + a_1a_2a_5 + a_2a_3^2)t)x_{23464321}((a_7 + a_1a_4a_5 + a_3^2a_4)t).
\end{aligned}$$

Proof. By construction, the given elements lie in P, and when $a_1, \ldots, a_7$ are all zero these generate the L-irreducible subgroup of type C_3. It is also clear that the cocycles have image in the appropriate modules in the levels of Q. It remains, therefore, only to prove that each such subgroup is indeed a group of type B_3. This is now routine using [9, Theorem 12.1.1] (calculations assisted using Magma). □

Remark 5.2. To obtain the form of the implicit cocycles above, consider the restriction map $H^1(X_0, 200) \to H^1(Y_0, 200 \downarrow Y_0)$, where Y_0 is a subgroup of type A_1 of X_0 generated by $x_{\pm 5}(t)$. Then Y_0 is the derived subgroup of a Levi subgroup of X_0. Now $200 \downarrow Y_0 = 2 + 0^4$. It follows that the map $H^1(X_0, 200) \to H^1(Y_0, 200 \downarrow Y_0)$ is injective. Cocycles for subgroups of type A_1 have been described explicitly in [43, Lemma 3.6.2]. To lift this to a classification of cocycles $Y_0 \to Q$, one can use the relations defining groups of type A_1; see [43, Lemma 3.6.1], and [43, p. 45] for an explicit example.

Returning to the centraliser calculations, we first find the rank of $C_G(Y_i)$ by considering the smallest Levi subgroup in which Y_i is contained. Indeed, within our list of non-G-cr subgroups we have all non-L-cr subgroups of type B_3 for every Levi subgroup L, recalling that the first four subgroups represent the four E_7-classes of non-E_7-cr subgroups. Thus the rank is 2 for Y_1

and Y_2, since they are contained in E_6 and A_6, respectively. For $Y_3, \dots, Y_6$, the rank is 1 since they are contained in E_7, E_7, A_7, D_7, respectively. Finally, Y_7 and Y_8 are not contained in any Levi subgroup and hence the rank of its connected centraliser is 0.

Given the construction of all subgroups of type B_3 contained in P in Proposition 5.1 above, it is now routine to calculate $C_{Q^{X_0}}(X_{\mathbf{v}})$, since $Q^{X_0} = \langle z_i(c) : i = 1, \dots, 7\rangle$. In particular, notice that $a_1a_2a_5 + a_2a_3^2 = 0$ for $Z\mathbf{v} \in \mathbb{V}$ for each of our eight class representatives, and so the lift in the root group $U_{22343321}$ is trivial.

We now show that $C_G(Y_1) = U_1G_2$. We know that $C_{F_4}(Y_1) = U_1$ and $C_G(F_4) = G_2$. Since $\dim C_{L(G)}(Y_1) \leq 15$, from Table 8, we are done. Next we consider Y_2, and start by noting that $C_G(A_6) = \bar{A}_1T_1$. Furthermore, a routine calculation as above shows that the complements corresponding to $(0,0,0,1,0,0,0)$ and $(0,0,0,0,0,0,1)$ centralise Q^{X_0}. Therefore, $C_G(Y_2)^\circ$ contains a 7-dimensional unipotent subgroup. Since $\dim C_{L(G)}(Y_2) = 10$ it follows that $C_G(Y_2) = U_6A_1T_1$.

Next, we will find the connected centralisers of $Y_7 < D_8$ and Y_8. Recall that these centralisers are unipotent and so Y_7 and Y_8 must be conjugate to one of the subgroups corresponding to $(0,0,1,1,0,1,0)$ and $(0,1,0,1,1,0,0)$. We now check that the elements $z_2(c)$, $z_3(c)$, $z_5(c)$, $z_6(c)$, $z_7(c)$ centralise the complements corresponding to $(0,0,1,1,0,1,0)$ and $(0,1,0,1,1,0,0)$ and that $z_1(c)z_4(c)$ centralises the complement corresponding to $(0,0,1,1,0,1,0)$. We claim that $C_G(Y_7)^\circ = U_5$ and so Y_7 must be conjugate to the complement corresponding to $(0,1,0,1,1,0,0)$. To check the claim, we start by noting that $\dim C_G(Y_7) \leq 6$, using the relevant restriction given in Table 8. Since $Y_7 < D_8$, we have $\mathfrak{c} = C_{L(G)}(D_8) \subseteq C_{L(G)}(Y_7)$. Since the isogeny type of $D_8 < E_8$ is half-spin, it follows that $\mathfrak{c}$ is a 1-dimensional subalgebra generated by a semisimple element. Since $C_G(Y_7)$ is unipotent, it follows that $\mathrm{Lie}(C_G(Y_7))$ cannot contain $\mathfrak{c}$ and so $\mathrm{Lie}(C_G(Y_7))$, hence $C_G(Y_7)$, has dimension at most 5. It now follows that Y_8 is conjugate to the subgroup corresponding to $(0,0,1,1,0,1,0)$ and so $C_G(Y_8)^\circ$ has dimension at least 6. Checking the restriction in Table 8 shows that $\mathrm{Lie}(C_G(Y_8))$ is 6-dimensional and so $C_G(Y_8)^\circ = U_6$.

The remaining subgroups Y_3, Y_4, Y_5, Y_6 are each conjugate to exactly one of the four complements corresponding to $(0,0,0,1,0,0,1)$, $(0,0,1,1,0,0,0)$, $(0,0,1,0,0,1,0)$ and $(0,1,0,0,1,0,0)$; recall that their centralisers all have rank 1. A routine calculation shows that these complements centralise a 7, 6, 6 and 4-dimensional subgroup of Q^{X_0}, respectively. For the final two complements, only 5 and 3 generators of Q^{X_0} given above, respectively, are in the centraliser. However, in both cases the product of two of the excluded generators makes up the final generator of the centraliser of the complements.

Inspecting Table 8, we see that $\dim C_G(Y_i) = 10, 9, 8, 6$ for $i = 3, 4, 5, 6$, respectively. Next, we show that Y_3 and Y_4 are not separable in E_7T_1, i.e. $\dim C_{E_7T_1}(Y_i) < \dim C_{L(E_7T_1)}(Y_i)$. Since $L(E_7T_1)$ is a direct summand of $L(E_8)$, it then follows from [4, Theorem 1.5] that Y_3 and Y_4 are not separable in E_8. From the calculations in E_7 we find $\dim C_{E_7T_1}(Y_3) = 3$ and $\dim C_{E_7T_1}(Y_4) = 2$. We can easily check that the corresponding centraliser dimension in $L(E_7T_1)$ is one more in each case, using the restrictions $L(E_8) \downarrow E_7T_1 = L(E_7T_1) + (V_{56}, 1) + (V_{56}, -1) + (0, 1) + (0, -1)$ and $V_{56} \downarrow Y_3, Y_4$ in Table 6. Thus $\dim C_G(Y_3) \leq 9$ and $\dim C_G(Y_4) \leq 8$.

We claim that $C_G(Y_i)$ contains a subgroup A_1 in all four remaining cases. For Y_3, Y_4 this is clear, since they are contained in E_7, and $C_G(E_7)^\circ = \bar{A}_1$. The subgroup Y_5 acts as $T(100)$ on $V_{A_7}(\lambda_1)$, and since $T(100)$ is self-dual it follows that $Y_5 < C_4 < A_7$. By [22, p.333 Table 3, Lemma 4.9], $C_G(C_4) = A_1$ and hence $A_1 \leq C_G(Y_5)$. For Y_6, we note that $Y_6 < B_3^2 < B_6 < D_7$. Then, [22, p.333 Table 3] shows that $C_G(B_6) = A_1$ and hence $A_1 \leq C_G(Y_6)$.

We now have enough information to conclude that $C_G(Y_3) = U_6\bar{A}_1$, $C_G(Y_4) = U_5\bar{A}_1$, $C_G(Y_5) = U_5A_1$ and $C_G(Y_6) = U_3A_1$.

5.4.2. *L' of type A_7.* Let $P = P_{1345678} = QL$ and $X_0 = B_3 < L' = A_7$ be embedded via 001. Then the unipotent radical Q has three levels, with X_0 actions

$Q/Q(2)$	$Q(2)/Q(3)$	$Q(3)$
$101+001$	$0\|100\|010\|100\|0$	001

By Proposition 3.9, $H^1(X_0, Q(2)/Q(3)) = K$, so $\dim H^1(X_0, Q) \leq 1$ and, considering the action of $Z(L)$, we find at most one non-G-cr complement to Q in QX_0, up to conjugacy in G.

Let Y be a subgroup of $B_3 < D_8$, embedded via 001+001. By Proposition 3.13 and Remark 3.14 there are two D_8-classes of such subgroups and both are non-D_8-cr. Now D_8 has two classes of Levi subgroup A_7, one of which is a Levi subgroup of G, and one of which is the subgroup A_7'. Considering the images of the non-D_8-cr subgroups in these Levi factors, then, one class arises from a G-cr subgroup contained in a subgroup E_7. So even if such subgroups are non-G-cr, they will arise from minimal parabolic subgroups with a Levi factor not conjugate to L. The other class, however, has irreducible image in A_7, which remains a Levi subgroup of G, so we only need to prove that these non-D_8-cr subgroups remain non-G-cr. Since $(Q/Q(2))^{X_0} = \{0\}$ and $Q(2)$ is the full unipotent radical of an A_7-parabolic of D_8, this follows from Corollary 3.5.

By Lemma 3.17 we have $C_G(Y)^\circ = C_Q(Y)^\circ$. In an A_7-parabolic subgroup of D_8, the unipotent radical is the 28-dimensional module $V_{A_7}(\lambda_2)$, and this is therefore a subgroup of Q isomorphic to $Q(2)/Q(3)$. Since this has a 1-dimensional space of fixed points under Y, we conclude that $C_G(Y)^\circ = U_1$.

5.4.3. *L' of type D_7.* In this case, L' has irreducible subgroups of types B_3 and B_3^2 that we need to consider; we start with B_3^2. Let $P = P_{2345678} = QL$ and $X_0 = B_3^2 < L' = D_7$ via $0|((100,0)+(0,100))|0$. Then the unipotent radical Q has two levels, with $Q/Q(2) \cong \lambda_6$ and $Q(2) \cong \lambda_1$ as L'-modules, and therefore $Q/Q(2) \downarrow X_0 \cong (001,001)$ and $Q(2) \downarrow X_0 = 0|((100,0)+(0,100))|0$.

From Proposition 3.9 we have $H^1(X_0, Q(2)) \cong K$ and $H^1(X_0, Q/Q(2)) = \{0\}$, and taking into account the action of $Z(L)$, we deduce that there is at most one non-G-cr complement to Q in QX_0, up to conjugacy in G.

Consider the subgroup $Y = B_3^2 < D_8$ via the 16-dimensional B_3^2-module $(0|(100,0)|0) + (0|(0,100)|0)$. Each 8-dimensional factor has a nonsingular vector fixed by B_3^2. Thus the 2-dimensional subspace spanned by these contains a singular vector fixed by B_3, and thus this subgroup B_3^2 is contained in a D_7-parabolic subgroup of D_8. Since this is the unique non-zero totally singular subspace fixed by B_3^2, we deduce that this subgroup is non-D_8-cr. Since $Q/Q(2)$ is an irreducible Y-module, Corollary 3.5 applies, and Y is also non-G-cr. A routine application of Lemma 3.17 shows that $C_G(Y)^\circ = U_1$.

Now, if $X_0 = B_3 < L'$ via $100 \otimes 100^{[r]}$ $(r > 0)$, identical arguments to the above show that P contains at most one non-G-cr complement to Q in QX_0, up to conjugacy. A representative is given by taking a subgroup $Z < D_8$ via $T(100) + T(100)^{[r]}$. Again, Corollary 3.5 tells us that this non-D_8-cr subgroup is non-G-cr, and Lemma 3.17 tells us that $C_G(Z)^\circ = U_1$. Note that, for each $r > 0$, the class representative Z is a diagonal subgroup of Y.

6. Type B_4

Let $P = P_{1345678} = QL$ and X_0 be of type C_4 embedded in L' via 1000. Then the action of X_0 on the levels of Q are as follows:

$Q/Q(2)$	$Q(2)/Q(3)$	$Q(3)$
$1000+0100$	$\bigwedge^2(1000)$	1000

Since 0010 and $\bigwedge^2(1000) = T(0100) = 0|0100|0$ are tilting, their first cohomology groups vanish, hence $\mathbb{V} \cong K^2$. The trivial submodule in level 2 lifts to a subgroup of Q^{X_0}, since $C_G(X_0)^\circ = A_1$ by [22, p.333 Table 3]. It follows easily that this induces a non-zero X_0-module homomorphism

$Q/Q(2) \to Q(3)$. We deduce that complements are parametrised by $(k_1, k_2) \in \mathbb{V}$ with $k_1k_2 = 0$. After considering the action of $Z(L)$, there are therefore at most two non-G-cr subgroups of type B_4 in such a parabolic subgroup, up to conjugacy.

Let Y_1 be the subgroup of type B_4 embedded in A_8 via $V_{A_8}(\lambda_1) \downarrow Y_1 = 1000|0$. Since $p \neq 3$ and A_8 is the centraliser of an element of order 3, [2, Corollary 3.1] implies that Y_1 is non-G-cr. There are two D_8-conjugacy classes of subgroups of type B_4 acting as $V_{B_4}(\lambda_4)$ on the natural module for D_8. By [44, Lemma 7.4], one of these subgroups, denoted $B_4(\ddagger)$ in *ibid*, is non-G-cr and non-conjugate to Y_1, let this subgroup be Y_2. Therefore Y_1 and Y_2 are representatives of two non-conjugate non-G-cr subgroup classes contained in P.

Since P is a maximal parabolic subgroup it follows from Lemma 3.17 that $C_G(Y_1)^\circ$ and $C_G(Y_2)^\circ$ are equal to Q^{Y_1} and Q^{Y_2}, respectively. In Table 8 we see that $\dim C_{L(G)}(Y_i) = 1$ for $i = 1, 2$. Since $Y_2 < D_8$ and $C_{L(G)}(D_8)$ is a 1-dimensional subalgebra generated by a semisimple vector, it follows that $C_G(Y_2)^\circ$ is trivial. Now, the complement to Q in P corresponding to $(0, 1) \in \mathbb{V}$ centralises the lift of the trivial submodule in level 2, since $Q(2)$ is abelian. It follows that Y_1 is conjugate to this non-G-cr subgroup and $C_G(Y_1)^\circ = U_1$.

7. Type C_3

Recall that in Proposition 3.9, this is the only case where $p = 3$. Let $P = P_{2345678} = QL$ be the unique standard D_7-parabolic subgroup of G and let $X_0 = C_3 < L'$ via 010. Then $Q/Q(2) \downarrow X_0 = 110 + 001$ and $Q(2) \downarrow X_0 = 010 + 0$. By Proposition 3.9, we have $\mathbb{V} \cong K$ and hence by considering the action of $Z(L)$ we have at most one G-conjugacy class of non-G-cr complements to Q in QX_0.

Let Y be a subgroup of type C_3 embedded in D_8 via $V_{D_8}(\lambda_1) = T(010)+0$. Then Y is non-D_8-cr and hence non-E_8-cr by Lemma 3.12, since D_8 is the centraliser of an involution in E_8 when $p \neq 2$. It follows that Y must be contained in P since it is the only parabolic that possibly contains non-G-cr subgroups of type C_3 when $p = 3$.

Since P is maximal, using Lemma 3.17 we have $C_G(Y)^\circ = C_Q(X_0) = Q(2)^{X_0} = U_1$.

8. Type C_4

The case $G = E_7$ is covered by [21, Lemma 2.7], yielding one conjugacy class of non-G-cr subgroups, with representative $Y = C_4 < A_7$ embedded via 1000. The connected centraliser of Y is U_1, as given in [22, p.333 Table 3]. So we may assume that $G = E_8$.

Let $P = P_{123456} = QL$ be the unique standard E_6-parabolic subgroup of G and let X_0 of type C_4 be a representative of the unique class of E_6-irreducible subgroups. Then X_0 acts on the levels of Q as follows:

$Q/Q(2)$	$Q(2)/Q(3)$	$Q(3)/Q(4)$	$Q(4)/Q(5)$	$Q(5)$
$0100 + 0^2$	$0100 + 0$	$0100 + 0$	0	0

By Proposition 3.9, we have $\mathbb{V} \cong K^3$. The images of U_{α_7}, $U_{\alpha_7+\alpha_8}$, and $U_{01122221}$ generate the X_0-module 0100 in levels 1, 2 and 3, respectively. Moreover, three trivial summands in levels 1 and 2 are respectively generated by the image of the elements $x_{\alpha_8}(c)$, $x_{11221110}(c)\ x_{11122110}(c)\ x_{01122210}(c)$ and $x_{11221111}(c)\ x_{11122111}(c)\ x_{01122211}(c)$. It then follows that we may assume that $k_1k_2 = k_1k_3 = k_2k_3 = 0$. The element $n_8 \in N_G(T)$ centralises X_0 whilst swapping U_{α_7} and $U_{\alpha_7+\alpha_8}$, and the element

$$n_7n_6n_5n_4n_3n_2n_4n_5n_6n_7n_1n_3n_4n_5n_6n_2n_4n_5n_3n_4n_1n_3n_2n_4n_5n_6n_7$$

centralises X_0 whilst swapping $U_{\alpha_7+\alpha_8}$ and $U_{01122221}$. We therefore conclude that there is at most one G-conjugacy class of non-G-cr complements to Q in QX_0.

Let Y be the non-G-cr subgroup of type C_4 contained in E_7. Then Y is non-G-cr by Lemma 3.12 and hence a conjugate of Y is contained in P. The connected centraliser of Y is calculated in [22, Lemma 4.9] and is U_5A_1. Note that $Y < A_1C_4 < D_8 < G$ and so $C_G(Y)^\circ$ is again a semidirect product of its unipotent radical and its reductive part.

9. Type D_4

In this section we complete our analysis of non-G-cr subgroups by considering those of type D_4, so that according to Lemma 3.6 we have $p = 2$ and $G = E_7$ or E_8.

9.1. **G of type E_7.** In this case, according to [21, Theorem 1], there are infinitely many classes of subgroups of type D_4 contained in an E_6-parabolic subgroup of G, each having irreducible image in the Levi factor. Exactly one such class consists of G-completely reducible subgroups. Let $P = QL$ be an E_6-parabolic subgroup. Then Q is abelian, and is an irreducible L'-module of high weight λ_1. Thus the non-G-cr subgroups of type D_4 occurring are complements to Q in QX_0, where X_0 is generated by the short root subgroups of $F_4 < L'$. We have $Q \downarrow X_0 = 0100{+}0$, hence $H^1(X_0, Q)$ is 2-dimensional. Fixing a basis so that each $(a, b) \in K^2$ gives a complement to Q in QX_0, the action of the 1-dimensional torus $Z(L)$ shows that complements corresponding to $(a\mu, b\mu)$, as μ varies over K^*, are all G-conjugate. Next note that L' contains a subgroup $D_4.S_3$. This outer automorphism group S_3 acts on Q and thus on the collection of complements to Q in QX_0. Concretely, this can be realised as an element of order 3 sending a complement corresponding to (a, b) to a complement corresponding to $(b, a + b)$, and an element of order 2 swapping complements corresponding to (a, b) and (b, a). Thus all G-classes of non-G-cr complements correspond to a pair $(1, a)$, as a varies over K.

Next, we consider potential overgroups of these non-G-cr subgroups. First, we note that by Lemma 3.15, the connected centraliser of every such subgroup is unipotent. Suppose that M is a reductive maximal connected subgroup of G containing a non-G-cr subgroup Z of type D_4. Then each simple factor of M must have rank at least 4, otherwise Z would project trivially to some simple factor M_0 and give $M_0 < C_G(Z)$. Thus M must be of type A_7 by [26, Theorem 1]. Let $Y = D_4 < A_7 < G$ via 1000. Since $N_G(A_7)/A_7$ has order 2, Y centralises an involution in G and thus lies in a parabolic subgroup of G. From Table 8, we see that Y acts with two 28-dimensional indecomposable summands on V_{56}. It follows that Y is not contained in any Levi subgroup of G and is therefore non-G-cr.

Thus we have shown that precisely one of the non-G-cr subgroups is contained in a proper reductive overgroup in G, and the remaining infinitely many classes are MR.

It remains to calculate the connected centralisers. As Q is abelian, $Q^{X_0} = Q^Z$ for any non-G-cr subgroup Z contained in P. It follows from Lemma 3.17(ii) that $C_G(Z)^\circ = U_1$.

9.2. **G of type E_8.**

9.2.1. *L' of type A_7.* Let $P = QL$ with L' of type A_7 and $X_0 < L'$ embedded via 1000. Then Q has three levels, with X_0-actions

$Q/Q(2)$	$Q(2)/Q(3)$	$Q(3)$
$1000 + 0011$	$0\vert 0100\vert 0$	1000

Proposition 3.9 implies that $\mathbb{V} \cong K$ and thus there exists at most one class of non-G-cr complement to Q in QX_0. By Proposition 3.13 and Remark 3.14, there are two D_8-classes of non-D_8-cr subgroups of type D_4 which act as $1000 + 1000$ on $V_{D_8}(\lambda_1)$. Embedding $D_8 < G$, since the subgroups lie in a proper parabolic subgroup of D_8, they lie in a proper parabolic subgroup of G. We now prove that these subgroups are both non-G-cr. We start by noting that the action of each subgroup on $L(G)$, given in Table 8, has three indecomposable summands of dimension

64 and no indecomposable summand of dimension less than 28. For a contradiction, suppose that Y is a subgroup in either class and that L is minimal among proper Levi subgroups of G containing Y. Since L' has rank at most 7 and Y has rank 4, it follows that L' is simple, and in fact L' has type A_7, D_4, D_5, D_6, D_7, E_6 or E_7. Knowledge of high weights on $L(G)$ [23, Table 10.1] shows that only D_7 can potentially have a subgroup acting with three indecomposable summands of dimension 64. But in this case, $L(G) \downarrow D_7 = \lambda_1^2 + T(\lambda_2) + \lambda_6 + \lambda_7$ and any subgroup of D_7 acts on $L(G)$ with an indecomposable summand of dimension at most 14. This contradiction proves that both classes of non-D_8-cr subgroups of type D_4 are in no proper Levi subgroup of G, hence are non-G-cr.

The two class representatives have different composition factors on $L(G)$. By Proposition 3.9, only A_7-parabolics and E_6-parabolics can contain non-G-cr subgroups of type D_4, and it follows that each class of parabolics gives rise to precisely one of these non-G-cr subgroup classes. The fact that the subgroup $X_0 < A_7$ has a composition factor of high weight 0011 shows that the first subgroup in Table 8 is contained in P.

Finally, we calculate $C_G(Y)^\circ$ for Y in this class of non-G-cr subgroups. Since P is maximal, we can apply Lemma 3.17. There is a C_4 overgroup of X_0 in A_7 and $C_G(C_4)^\circ = A_1$ by [22, p.333 Table 3]). It follows that the trivial submodule in $Q(2)/Q(3)$ lifts to a subgroup of Q and so Q^{X_0} is 1-dimensional. Moreover, $Y < Q(2)X_0$ and $Q^{X_0} \leq Q(2)$ which is abelian, so $Q^Y = Q^{X_0}$. Hence $C_G(Y) = U_1$.

9.2.2. *L' of type E_6.* Let $P = QL$ be the standard E_6-parabolic of G and let X_0 be the E_6-irreducible subgroup of type D_4 generated by short root subgroups in an F_4 subgroup in E_6. Then Q has five levels, with actions of L' and X_0 as follows.

	$Q/Q(2)$	$Q(2)/Q(3)$	$Q(3)/Q(4)$	$Q(4)/Q(5)$	$Q(5)$
L'	$\lambda_1 + 0$	λ_1	λ_6	0	0
X_0	$0100 + 0^2$	$0100 + 0$	$0100 + 0$	0	0

The trivial summand in $Q/Q(2)$, generated by the image of the root group U_{α_8}, induces a non-zero E_6-module homomorphism $Q/Q(2) \to Q(2)/Q(3)$. In the action of X_0 on λ_1 and λ_6, the modules of high weight 0100 in the three levels are respectively generated by the images of the root groups U_{α_7}, $U_{\alpha_7+\alpha_8}$ and $U_{01122221}$. Note that X_0 is contained in a subgroup $Y_0 = C_4 < E_6$, and as described in Section 8, the additional trivial modules arising in $Q/Q(2)$ and $Q(2)/Q(3)$ respectively induce non-zero Y_0-module homomorphisms $\psi : Q(2)/Q(3) \to Q(3)/Q(4)$ and $\psi' : Q/Q(2) \to Q(3)/Q(4)$.

Now $H^1(X_0, 0100) \cong K^2$, hence $\mathbb{V} \cong K^6$ and we denote elements of $\mathbb{V}$ by (ϕ_1, ϕ_2, ϕ_3), where $\phi_i \in H^1(X_0, Q(i)/Q(i+1))$. We identify elements in the different copies of $H^1(X_0, 0100)$ such that $x_{\alpha_8}(c)$ induces the map

$$(\phi_1, \phi_2, \phi_3) \mapsto (\phi_1, \phi_2 + c\phi_1, \phi_3)$$

and the additional trivial modules from $Q/Q(2)$ and $Q(2)/Q(3)$ respectively induce maps

$$(\phi_1, \phi_2, \phi_3) \mapsto (\phi_1, \phi_2, \phi_3 + c\phi_2), \qquad (\phi_1, \phi_2, \phi_3) \mapsto (\phi_1, \phi_2, \phi_3 + c\phi_1).$$

Again, since $X_0 < Y_0$ it follows that X_0 is centralised by $n_8 \in N_G(T)$, which swaps the root groups U_{α_7} and $U_{\alpha_7+\alpha_8}$, and centralises $U_{01122221}$. Moreover, the element

$$w = n_7 n_6 n_5 n_4 n_3 n_2 n_4 n_5 n_6 n_7 n_1 n_3 n_4 n_5 n_6 n_2 n_4 n_5 n_3 n_4 n_1 n_3 n_2 n_4 n_5 n_6 n_7$$

also centralises X_0, and swaps $U_{\alpha_7+\alpha_8}$ and $U_{01122221}$ whilst mapping U_{α_7} to a negative root subgroup. Similarly, the element

$$w' = n_7 n_6 n_5 n_4 n_2 n_3 n_1 n_4 n_3 n_5 n_4 n_2 n_6 n_5 n_4 n_3 n_1 n_7 n_6 n_5 n_4 n_2 n_3 n_4 n_5 n_6 n_7$$

centralises X_0, swaps U_{α_7} with $U_{01122221}$ and maps $U_{\alpha_7+\alpha_8}$ to a negative root subgroup.

We now use the above elements to parametrise complements to Q in QX_0, including X_0 itself, by elements of $\mathbb{P}^1 \cup \{3 \text{ points}\}$. To do this, we divide into a number of cases.

In the case that $(\phi_1, \phi_2) = (0, 0)$, conjugating by w' sends the corresponding complement to one corresponding to $(\phi_3, 0, 0) \in \mathbb{V}$. Such a subgroup is contained in a Levi subgroup of G of type E_7, and is non-E_7-cr. We will shortly show that such subgroups are G-conjugate if and only they are E_7-conjugate, hence there are infinitely many such subgroups, as described in Section 9.1.

Now suppose that $(\phi_1, \phi_2) \neq (0, 0)$ but that ϕ_1 and ϕ_2 are multiples of one another under the above identification of the copies of $H^1(X_0, 0100)$. Then, conjugating by n_8 if necessary, we can assume that $\phi_1 \neq 0$. Conjugating by an appropriate element $x_{\alpha_8}(c)$ then allows us to assume that $\phi_2 = 0$. This complement therefore corresponds to an element of the form $(\phi_1, 0, \phi_3) \in \mathbb{V}$ where $\phi_1 \neq 0$. If ϕ_3 is a multiple of ϕ_1 under the identification above, then conjugating by an appropriate X_0-fixed point of $Q(2)/Q(3)$ shows that the complement corresponds to $(\phi_1, 0, 0)$, which again lies in an E_7 Levi subgroup of G as above. Thus we may assume that $\{\phi_1, \phi_3\}$ form a basis of $H^1(X_0, 0100)$. Now, conjugation by the element w' and the X_0-fixed points of $Q(2)/Q(3)$ and $Z(L)$, fuse together all classes corresponding to any such basis of $H^1(X_0, 0100)$ (this is essentially the fact that $\mathrm{GL}_2(K)$ is transitive on ordered bases of its natural module). Thus this case gives rise to at most one new class of subgroups. In Lemma 10.8 we will show that the indecomposable direct summands of such a group on $L(G)$ are incompatible with $L(G) \downarrow E_7$, so that these subgroups are not contained in a subgroup of type E_7.

Finally, suppose that $\{\phi_1, \phi_2\}$ form a basis of $H^1(X_0, 0100)$. Conjugating by the X_0-fixed points which give rise to the maps ψ and ψ' above, we may assume that $\phi_3 = 0$. Conjugation by the element n_8, together with U_{α_8} and $Z(L)$, again fuses together all classes corresponding to triples $(\phi_1, \phi_2, 0)$ where $\{\phi_1, \phi_2\}$ is a basis of $H^1(X_0, 0100)$. Thus in this case we find at most one more non-G-cr subgroup, up to conjugacy. Again, in Lemma 10.8 we derive the actions of these subgroups on $L(G)$, and see that these are incompatible with the subgroup being contained in E_7, or being conjugate to any of the non-G-cr subgroups found above. Thus these do indeed give rise to a distinct class of non-G-cr subgroups.

In summary, supposing that all potential complements above actually exist, we have shown that non-G-cr subgroups correspond to one of: $(\phi, 0, 0)$ with ϕ a non-zero point of $H^1(X_0, 0100)$; or to $(\phi, 0, \psi)$ or to $(\phi, \psi, 0)$ for a fixed basis $\{\phi, \psi\}$ of $H^1(X_0, 0100) \cong K^2$. In the former case, these are parametrised by elements $(a, b) \in K^2$ modulo the action of $Z(L)$ and a subgroup S_3, as in 9.1 above. In each of the latter two cases we get at most one extra non-G-cr subgroup.

Let us first consider the infinitely many non-G-cr subgroups corresponding to elements $(\phi, 0, 0) \in \mathbb{V}$, each contained in a Levi subgroup M of type E_7. We will show that two such subgroups are G-conjugate if and only if they are M'-conjugate. Let Y be such a non-G-cr subgroup. To start, note that M' centralises a connected simple subgroup $\bar{A}_1$ (generated by root subgroups corresponding to $\pm\alpha_0$), hence so does Y. Moreover, note that conjugation by w' sends Y to a subgroup corresponding to $(0, 0, \phi) \in \mathbb{V}$; since this is contained in $Q(3)X_0$, it follows that Y commutes with the 6-dimensional unipotent subgroup Q^{X_0} (the induced Y-module homomorphisms $Q(3) \to Q$ induced by elements of Q^{X_0} are all zero as their image lies in $Q(4)$, and neither $Q(4)/Q(5)$ nor $Q(5)$ have a non-trivial composition factor). By Lemma 3.16, Q^{X_0} is a maximal unipotent subgroup of $C_G(Y)$, for all subgroups Y corresponding to an element $(0, 0, \phi) \in \mathbb{V}$. And Lemma 3.15 shows that $C_G(Y)$ is rank 1. Since Q^{X_0} meets the subgroup $\bar{A}_1$ in a 1-dimensional connected unipotent group, we conclude that $C_G(Y)^\circ = U_5\bar{A}_1$.

Thus if Y and Z are two non-G-cr complements to Q in QX_0, each contained in M' (a fixed copy of E_7), we have $C_G(Y)^\circ = C_G(Z)^\circ = UH$, where U is a 5-dimensional unipotent subgroup and H is simple of type A_1. Now suppose $g \cdot Y = Z$ for some $g \in G$. Then g normalises $C_G(Y)^\circ$ and so H and $g \cdot H$ are simple subgroups of UH of type A_1. A quick calculation shows that UH is G-conjugate to the subgroup generated by Q^{X_0} and $U_{-\alpha_8}$, with the subgroup A_1 generated by the root subgroups $U_{\pm\alpha_8}$. Using the commutator relations in G, it follows that U has a filtration by H-modules of high weight 0 and 1. Each of these modules has zero first cohomology group, hence $H^1(H,U)$ vanishes and $g \cdot H$ is U-conjugate to H (note that although $H^1(H,1^{[1]})$ is 1-dimensional, the resulting non-trivial complements cannot be G-conjugate to H since they are not algebraically isomorphic to H; the projection $UH \to H$ has a scheme-theoretic kernel on restriction to H). Therefore $(ug) \cdot H = H$ for some $u \in U$. Since $U \leq C_G(Y) = C_G(Z)$, we have $(ug) \cdot Y = u \cdot Z = Z$, and $ug \in N_G(H) = HE_7$. Finally, this means we may take some $h \in H$ such that $(hug) \cdot Y = h \cdot Z = Z$ with $hug \in E_7$; hence Y and Z are conjugate under an element of E_7.

We now claim there are indeed two distinct classes of non-G-cr subgroups of type D_4 contained in P but not contained in E_7; we exhibited the first of these in the previous case of $P_{1345678}$. Indeed, there is a non-D_8-cr subgroup Y_1 acting as $1000 + 1000$ on $V_{D_8}(\lambda_1)$ which is contained in P. Now let Y_2 be the subgroup generated by the set of eight Chevalley generators $y_{\pm\beta_i}(t)$ for $i = 1,\dots,4$ in Proposition 10.5 with $a_1 = a_4 = 1$ and $a_i = 0$ for $i = 2,3,5,6$. This is a subgroup of type D_4 contained in $P_{1345678}$ by construction. Furthermore, in Lemma 10.8, we show that Y_2 acts with an indecomposable summand of dimension 188. No maximal reductive subgroup of G or proper Levi subgroup of G has an indecomposable summand of this dimension or higher, hence Y_2 is non-G-cr and maximal among proper connected reductive subgroups of G.

Finally we calculate $C_G(Y_1)^\circ$ and $C_G(Y_2)^\circ$. These are unipotent groups because neither Y_1 nor Y_2 is contained in a Levi subgroup of G. Inspecting Table 8, we find that $\dim C_G(Y_1) \leq 5$. Since $Y_1 < D_8$, it follows (as for the subgroup Y_7 in Section 5.4.1) that $\dim C_G(Y_1) \leq 4$. Again inspecting Table 8, we find that $\dim C_G(Y_2) \leq 5$. We know from the above analysis that these two extra subgroups come from the cases $(\phi_1, 0, \phi_3) \in \mathbb{V}$, which are conjugate to $(0, \phi_1, \phi_3) \in \mathbb{V}$, and $(\phi_1, \phi_2, 0) \in \mathbb{V}$, where ϕ_1, ϕ_3 and ϕ_1, ϕ_2 form a basis of K^2, respectively (though we do not yet know which corresponds to Y_1 or Y_2). It is routine to check that a subgroup corresponding to $(0, \phi_1, \phi_3)$ centralises a 5-dimensional subgroup of Q^{X_0}, and such a subgroup is therefore conjugate to Y_2 and $C_G(Y_2)^\circ = U_5$. Another routine check shows that a subgroup corresponding to $(\phi_1, \phi_2, 0)$ centralises a 4-dimensional subgroup of Q^{X_0}; this is therefore a conjugate of Y_1 and $C_G(Y_1)^\circ = U_4$.

10. Actions of non-G-cr subgroups

In this section we determine the restrictions $L(G) \downarrow X$ and $V_{\min} \downarrow X$ for the non-G-cr subgroups X arising in Theorem 1. When X is not of type D_4, it has a proper reductive overgroup in G and we use this to deduce the required restrictions. More details about this method can be found in [30, Section 10]. The case when X has type D_4 and $p = 2$ is by far the most difficult to analyse; the remainder of this section is dedicated to giving a full treatment of this scenario.

So now let $p = 2$ and suppose firstly that $G = E_7$, with X contained in $P = QL$ with L' of type E_6, such that the image X_0 of X in L lies in a subgroup of type F_4 and is generated by short root subgroups of this. Recall from Section 9 that Q is abelian, with $Q \downarrow X_0 = 0100 + 0$, and the G-conjugacy class of X corresponds to $\phi \in K^2$ where we can take $\phi = (1, a)$ with $a \neq 0$. Furthermore, precisely one of these infinitely many classes consists of subgroups lying in a proper reductive overgroup of type A_7, and the rest are MR (maximal among proper connected reductive subgroups of G).

Lemma 10.1. *In the above set-up, let $Y < X$ be a subgroup of type B_3. Then Y is non-G-cr, and is conjugate to the subgroup $B_3 < A_7$ via* 001 *given in Table 6.*

Proof. Let X_0 be the (G-cr) image of X in the Levi subgroup L, and let $Z_0 < Y_0 < X_0$ where Z_0 is simple of type A_3 and Y_0 is the image of Y (simple of type B_3). We will show that Z_0 can be chosen such that the restriction map $H^1(X_0, 0100) \to H^1(Z_0, 0100)$ sends the non-zero cohomology class giving rise to X, to a non-zero class. It then follows that the corresponding complement, Z, is non-G-cr, and also restriction of this class to $H^1(Y_0, 0100)$ is non-zero, so that Y is also non-G-cr.

In a simply connected group, the derived subgroup of a Levi factor is again simply connected. Therefore, the Lie algebra of an A_3-Levi subgroup in a simply connected group of type D_4 has socle series $101|0$ as an A_3-module. Since $L(A_3)$ is a submodule of $L(D_4) = 0100|0^2$, and since $H^1(D_4, 0100) \cong K^2$ and $H^1(A_3, 101) \cong K$ it follows that the restriction map $H^1(D_4, 0100) \to H^1(L', 0100)$ has a 1-dimensional kernel, for each choice of A_3-Levi subgroup L. Moreover the group of graph automorphisms S_3 acts on the centre of $L(D_4)$ and on $H^1(D_4, 0100)$, giving each the structure of an irreducible 2-dimensional S_3-module. In particular this action preserves no 1-space, and it follows that for each non-trivial cohomology class in $H^1(D_4, 0100)$, there is a simple subgroup A_3 whose class is not in the kernel of the restriction map.

As above, let $Z_0 < Y_0 < X_0$ where Y_0 has type B_3 and Z_0 has type A_3. Choosing Z_0 so that the cohomology class giving rise to X does not lie in the kernel of $H^1(X_0, Q) \to H^1(Z_0, Q)$, the class remains non-trivial for both Y_0 and Z_0. Therefore the corresponding complements $Z < Y < X$ are respectively not Q-conjugate to Z_0, Y_0 and X_0, hence are non-G-cr. Considering the composition factors of Z and Y on $L(G)$ and on $V_{\min}$, we identify Z as the subgroup in Table 6 with $V_{\min} \downarrow Z = 010^4 + T(101)^2$. Then $\dim(V_{\min}^Z) = 2$. Looking at the first three subgroups B_3 in Table 6, the corresponding subgroups A_3 will have a four-dimensional fixed-point space on $V_{\min}$, and therefore Y is a conjugate of the final subgroup B_3, embedded in the subsystem subgroup of type A_7 of G via 001, as claimed. □

Proposition 10.2. *In the above set-up:*

(i) *If $X < A_7$ then $V_{\min} \downarrow X = (0|\lambda_2|0)^2$.*
(ii) *If X is MR then $V_{\min} \downarrow X = 0^2|\lambda_2^2|0^2$ is indecomposable.*

Proof. The restriction for the subgroup $X < A_7$ follows readily since A_7 acts on $V_{\min}$ with two direct summands: the alternating squares of the natural 8-dimensional A_7-module and its dual.

So suppose that X is MR and take a subgroup $Y < X$ of type B_3; by Lemma 10.1 this is a non-G-cr subgroup contained in a subsystem subgroup H of type A_7, via 001. Inspecting Table 6, the module $V_{\min} \downarrow Y$ is a direct sum of two indecomposable 28-dimensional modules, both of which are H-submodules. Since H is maximal among proper connected subgroups of G, it follows that H is the identity component of the full stabiliser in G of this direct-sum decomposition of $V_{\min}$. Since X is not contained in such a subsystem subgroup of G, we see that X cannot stabilise such a direct-sum decomposition and so $V_{\min} \downarrow X$ is indecomposable as claimed. To obtain the socle series in this case, note that $V_{\min} \downarrow X$ is self-dual with composition factors $0^4/\lambda_2^2$. There are no uniserial X-modules of shape $\lambda_2|0^i|\lambda_2$ for $i \geq 0$, since these would be generated by a highest weight vector and would hence be an image of the Weyl module $\lambda_2|0^2$, which is absurd. Since $V_{\min} \downarrow X$ is indecomposable, it follows that it cannot have any irreducible quotient λ_2; since $\mathrm{Ext}^1(\lambda_2|0^2, 0^2|\lambda_2) = \{0\}$ by [20, Proposition 4.13], the stated socle series is the only remaining possibility. □

Proposition 10.3. *In the above set-up:*

(i) *If $X < A_7$ then*

$$L(G) \downarrow X = (\lambda_2|(2\lambda_1 + 0^2)|\lambda_2) + (\lambda_2|(2\lambda_3 + 2\lambda_4 + 0)|(\lambda_2 + 0)) + 0.$$

(ii) *If X is MR then $L(G) \downarrow X$ is indecomposable with socle series*

$$\lambda_2^2|(2\lambda_1 + 2\lambda_3 + 2\lambda_4 + 0^3)|(0^2 + \lambda_2^2).$$

Proof. In Proposition 10.2 we showed that the subgroups X in (i) lie in a subgroup A_7, which acts on $L(G)$ as a direct sum $V_{A_7}(\lambda_1 + \lambda_7) + V_{A_7}(\lambda_4) + 0$. Take $Y < X$ of type B_3, recalling from Lemma 10.1 that this is a conjugate of the final subgroup in Table 6). The indecomposable summands of X on $L(G)$ have the dimensions of the stated modules. The module structure itself follows at once from inspecting $L(G) \downarrow Y$ and comparing this with the action of Y on each X-composition factor.

For (ii), we again take $Y < X$ of type B_3 and consider its action on $L(G)$ in Table 6. This has three indecomposable summands, denoted $M_1 = Z(L(A_7)) = Z(L(G))$, M_{62} and M_{70}, where the subscript denotes the dimension. Now, Y lies in a subsystem subgroup H of type A_7 in G, and the stabilisers of M_{62}, M_{70} and $M_{62} + M_{70}$ all contain H. Since H is maximal in G, it in fact equals each of these stabilisers. However, we proved in Section 9 that X is not contained in any subgroup of type A_7, so X does not stabilise M_{62}, M_{70} or $M_{62} + M_{70}$. This shows that $L(G) \downarrow X$ is indecomposable. Similarly, since X does not stabilise either of the indecomposable Y-direct summands of the 132-dimensional A_7-module $L(G)/M_1 \cong M_{62} + M_{70}$, we see that X is indecomposable on $L(G)/Z(L(G))$. Finally, consider the G-module $T(\lambda_2)$. This has a 133-dimensional indecomposable submodule $L(G)$, hence X is either indecomposable on $T(\lambda_2)$ or X preserves a 1-dimensional complement to $L(G)$. However, note that as an H-module, $T(\lambda_2)$ has indecomposable summands of dimension 1, 1, 63 and 70, and the subgroup Y preserves only these direct summands and no more. Thus each X-module complement to $L(G)$ in $T(\lambda_2)$ is an H-submodule, and again H is the full stabiliser of such a submodule. Since X is not contained in a subgroup of type A_7, we again conclude that $T(\lambda_2)$ is an indecomposable X-module.

To determine the socle series, we first work with the G-module $T_{E_7}(\lambda_2)$, which is self-dual, indecomposable and contains $L(G)$ as a submodule. To begin, since $T_{E_7}(\lambda_2)$ has exactly one X-composition factor of each high weight $2\lambda_1$, $2\lambda_3$ and $2\lambda_4$, none of these can appear as a submodule or quotient, since the self-duality of $T_{E_7}(\lambda_2)$ would imply the existence of a direct summand with one of these weights.

Next, $L(G)$ has a 1-dimensional centre since $p = 2$ and G is simply connected, and $L(Q) \downarrow X = 0 + \lambda_2$, which does not meet $Z(L(G))$ and thus furnishes a second trivial submodule and a submodule λ_2; these latter submodules consist entirely of nilpotent elements of $L(G)$. The subgroup $Y < X$ does not have a 3-dimensional trivial submodule in its action on $L(G)$, and we conclude that the fixed-point space of X on $L(G)$ is 2-dimensional. Next, note that for a rational cocycle ϕ, the maps $x \mapsto \phi(x)x$ and $\phi(x)x \mapsto x$ are mutually inverse isomorphisms of algebraic groups between X and its image in the E_6-Levi subgroup of G. This image in the Levi subgroup is of adjoint type, since its action on $V(\lambda_7)$ is a direct sum $0^2 + \lambda_2^2$. Thus X is adjoint, so $L(G)$ has a submodule $L(X) = 0^2|\lambda_2$, and moreover this submodule λ_2 does not consist entirely of nilpotent elements, hence is not equal to the submodule λ_2 furnished by $L(Q)$. Now, note that since $T_{E_7}(\lambda_2)$ has exactly four X-composition factors of high weight λ_2, we cannot have three of these in the socle, since this would imply the existence of a proper direct summand. Thus $L(G) \downarrow X$ has socle $0^2 + \lambda_2^2$. It follows that $T(\lambda_2)$ also has socle $0^2 + \lambda_2$, otherwise it would have socle $0^3 + \lambda^3$, which would imply that $T(\lambda_2) = L(G) + 0$, whereas we know it is in fact indecomposable for X.

Using the fact that $T_{E_7}(\lambda_2)$ admits a quotient λ_2^2, together with the fact that $2\lambda_1$, $2\lambda_3$ and $2\lambda_4$ have trivial first cohomology, we see that all of the composition factors $2\lambda_1$, $2\lambda_3$ and $2\lambda_4$ must appear as submodules once we factor out the socle of $T_{E_7}(\lambda_2)$, as well as all of the remaining trivial composition factors. Hence the same is also true for $L(G)$. Finally, by self-duality of $T_{E_7}(\lambda_2)$, it follows that the remaining two composition factors λ_2^2 do not appear in this second socle layer of $T(\lambda_2)$, hence they do not appear in the second socle layer of $L(G)$. The given socle series for $L(G) \downarrow X$ now follows, as well as

$$T_{E_7}(\lambda_2) \downarrow X = \lambda_2^2|(2\lambda_1 + 2\lambda_3 + 2\lambda_4 + 0^4)|(0^2 + \lambda_2^2).$$

□

Remark 10.4. Note that combining $T_{E_7}(\lambda_2) \downarrow X$ above with Proposition 10.2(ii), we immediately deduce the module structure of $L(E_8) \downarrow X$.

Finally, we need to determine the action of X of type D_4 in $G = E_8$ when X is contained in no reductive overgroup. To do this, we require an explicit construction of such a subgroup in an

E_6-parabolic P of G. For completeness, we construct of a conjugate of every non-G-cr subgroup of type D_4 in P.

Proposition 10.5. *Let G be of type E_8, $p = 2$ and let $P = QL$ be the standard E_6-parabolic subgroup of G. Let X be a subgroup of type D_4 contained in P, with irreducible image X_0 in L. Then, identifying $\mathbb{V} = (H^1(X_0, \lambda_2))^3 = K^6$, there exists $\mathbf{v} = (\phi_1, \phi_2, \phi_3) = (a_1, \ldots, a_6) \in \mathbb{V}$ such that X is conjugate to $X_{\mathbf{v}} = \langle y_{\pm i}(t) : t \in K,\ 1 \leq i \leq 4\rangle$ for the following elements $y_{\pm i}(t)$.*

$$\begin{aligned}
y_{\pm 1}(t) &= x_{\pm 3}(t)x_{\pm 5}(t),\\
y_{\pm 2}(t) &= x_{\pm 1}(t)x_{\pm 6}(t),\\
y_{3}(t) &= x_{0011000}(t)x_{0001100}(t)x_{11232110}(a_1t)x_{11232111}(a_3t)x_{12354321}(a_5t),\\
y_{-3}(t) &= x_{-0011000}(t)x_{-0001100}(t)x_{11111110}(a_1t)x_{11111111}(a_3t)x_{12233321}(a_5t),\\
y_{4}(t) &= x_{0111000}(t)x_{0101100}(t)x_{12232110}(a_2t)x_{12232111}(a_4t)x_{13354321}(a_6t),\\
y_{-4}(t) &= x_{-0111000}(t)x_{-0101100}(t)x_{10111110}(a_2t)x_{10111111}(a_4t)x_{11233321}(a_6t).
\end{aligned}$$

Proof. By construction, the given elements lie in P, and when $a_1, \ldots, a_6$ are all zero these generate the L-irreducible subgroup X_0, as shown in [21, p. 444]. It is also clear that the cocycles have image in the appropriate modules in the levels of Q. It remains, therefore, only to prove that each such subgroup is indeed a group of type D_4. This is now routine using [9, Theorem 12.1.1]; our calculations were assisted using Magma. □

Remark 10.6. The form of the cocycles in the above construction was derived using the restriction $H^1(X_0, 0100) \to H^1(Y_0, 0100 \downarrow Y_0)$, where Y_0 is the subgroup of type A_1^2 of X_0 generated by $x_{\pm 0011000}(t)x_{\pm 0001100}(t)$ and $x_{\pm 0111000}(t)x_{\pm 0101100}(t)$. Then Y_0 is the derived subgroup of a Levi subgroup of X_0. Since $1000 \downarrow Y_0 = (1,1) + 0^4$, it follows that $0100 \downarrow Y_0 = (2,0) + (0,2) + (1,1)^4 + 0^6$, a completely reducible module. As mentioned earlier, in a simply connected semisimple algebraic group, the derived subgroup of any Levi subgroup is again simply connected. The Lie algebra of a simply connected simple group of type D_4 has shape $0100|0^2$, and the Lie algebra of a simply connected semisimple group of type A_1A_1 has shape $((2,0)|0) + ((0,2)|0)$. It follows that the X_0-module $0100|0^2$ restricts to Y_0 as $(2,0)|0 + (0,2)|0 + (1,1)^4 + 0^6$, and in particular the map $H^1(X_0, 0100) \to H^1(Y_0, 0100 \downarrow Y_0)$ is injective. Finally, we use the explicit description of cocycles for subgroups of type A_1 in [43, Lemma 3.6.2].

With this construction in hand, the following result allows us to deduce the action of the algebraic group using a well-chosen finite subgroup.

Lemma 10.7 ([24, Proposition 1.4])**.** *Let X be a connected simple algebraic group and let Y be a finite subgroup of X. Suppose V is a finite-dimensional X-module satisfying the following conditions:*

(i) *Every X-composition factor is irreducible for Y;*
(ii) *for each pair of X-composition factors M, N of V, the restriction map $\mathrm{Ext}^1_X(M,N) \to \mathrm{Ext}^1_Y(M,N)$ is injective;*
(iii) *for each pair of X-composition factors M, N of V, if $M \downarrow Y \cong N \downarrow Y$ then $M \cong N$ as X-modules.*

Then X and Y fix exactly the same subspaces of V.

Lemma 10.8. *Let $G = E_8$, $p = 2$ and X be an MR non-G-cr subgroup of type D_4. Then $L(G) \downarrow X$ has the following socle series:*

$$T(\lambda_2)^2 + (0|\lambda_2^3|(0^4 + 2\lambda_1 + 2\lambda_3 + 2\lambda_4)|(\lambda_2^2 + 0^2)|(\lambda_2 + 0)).$$

Proof. We apply Lemma 10.7 to X and its finite subgroup $Y := X(4) \cong \mathrm{SO}_8^+(4)$, acting on $L(G)$. To do so, we must check the three conditions hold. Conditions (i) and (iii) are immediate from inspection of the X-composition factors of $L(G)$. Condition (ii) follows from [11, Theorem 7.4], noting that $\langle \lambda, \alpha_j \rangle \leq 3$ for $j \in \{1, \ldots, 4\}$ for all highest weights λ occurring in $L(G) \downarrow X$.

Therefore, X and Y stabilise exactly the same subspaces of $L(G)$. In particular, the claimed socle series for $L(G) \downarrow X$ follows from the socle series for $L(G) \downarrow Y$. The latter socle series can be calculated in Magma, using the generators in Proposition 10.5 with $a_1 = a_4 = 1$ and $a_i = 0$ for $i = 2, 3, 5, 6$. □

Lemma 10.9. *Let G be of type E_8, $p = 2$ and X be an MR non-G-cr subgroup of type D_4. If $Y < X$ is of type B_3 then Y is non-G-cr and acts with five direct summands on $L(G)$ of dimensions 30, 30, 62, 63, 63.*

Proof. A group of type D_4 contains three conjugacy classes of maximal subgroups of type B_3. From the construction of X in Proposition 10.5, we can write down generators for the three classes. Using Magma we find they each contain a finite subgroup $\mathrm{SO}_7(4)$ which acts with five direct summands on $L(G)$ of dimensions $30, 30, 62, 63, 63$. Another routine application of Lemma 10.7 yields that these are actually the direct summands of each maximal subgroup of type B_3. That these are non-G-cr now follows since no Levi subgroup can contain a subgroup acting with summands of the given dimensions. □

Remark 10.10. It follows from the classification of non-G-cr subgroups of type B_3 in $G = E_8$ when $p = 2$ that all subgroups of type B_3 in an MR subgroup of type D_4 are conjugate to one another. We do not state this in the previous lemma, because the lemma itself used in the proof of the classification.

11. Tables of embeddings for Theorem 1

We now give the tables listing non-G-cr subgroups X of simple algebraic groups G as in Theorem 1, which have been enumerated in Sections 4–9. Within each table, each given X indicates a unique G-conjugacy class of subgroups, except where explicitly stated otherwise.

The first column in each table gives the Lie type of X. For each type, we use a horizontal line starting from Column 2 to distinguish between non-G-cr subgroups minimally contained in different association classes of parabolic subgroups. For convenience, the order is the same as the order in which we have considered the parabolic subgroups in the proof.

The second column either gives a proper reductive overgroup M of X or 'MR', indicating that X is maximal among connected reductive subgroups of G, cf. Corollaries 2 and 3. When M is classical we use the notation of Section 2.7 to indicate its embedding. When M is exceptional, it turns out that X is always non-M-cr and we give the same overgroup of X as in the table for M (when X is MR in M, we just list M).

The next columns give the action of X on a non-trivial G-module of least dimension, and on the adjoint module $L(G)$ (when these modules are different); the structure of the Weyl modules and tilting modules in these columns is given in Appendix A.

The final column gives the structure of $C_G(X)^\circ$ using the notation in Section 2.5. The symbol (†) indicates that X is separable in G, i.e. $L_{C_G(X)}(X) = L(C_G(X))$. We recall that G is simply connected, so in the case $G = E_7$, where G is not a separable subgroup of itself, the inseparability of many subgroups X is an artefact of this choice.

Table 4: Non-G-cr subgroups of $G = F_4$, $p = 2$

X	Embedding of X	$V_{26} \downarrow X$	$L(G) \downarrow X$	$C_G(X)^\circ$
B_3	D_4 via $T(100)$	$T(100) + 001^2 + 0^2$	$(100\|010\|100\|0^2) + T(100) + 001^2$	U_1
	$\tilde{D}_4$ via $T(100)$	$100\|010\|100$	$((002 + 0^2)\|(200 + 100)\|(010 + 002 + 0)\|100) + 0$	U_1 (†)

Table 5: Non-G-cr subgroups of $G = E_6$, $p = 2$

X	Embedding of X	$V_{27} \downarrow X$	$L(G) \downarrow X$	$C_G(X)^\circ$
B_3	$\tilde{D}_4 < F_4$	$(100\|010\|100) + 0$	$100\|(010 + 002 + 0^2)\|(200 + 100^2)\|(010 + 002 + 0)\|(100 + 0)$	U_1 (†)

Table 6: Non-G-cr subgroups of $G = E_7$, $p = 2$

X	Embedding of X		Module actions	$C_G(X)^\circ$
A_3	D_6 via $010 + 010$	$V_{56} \downarrow X =$	$010^4 + T(200) + T(002)$,	$U_5\bar{A}_1$
		$L(G) \downarrow X =$	$W(101) + W(101)^* + T(101)^4 + T(020) + 0^3$	
	D_6 via $010 + 010$	$V_{56} \downarrow X =$	$010^4 + T(101)^2$	$U_1\bar{A}_1$
		$L(G) \downarrow X =$	$W(101) + W(101)^* + T(200)^2 + T(002)^2 + T(020) + 0^3$	
B_3	E_6	$V_{56} \downarrow X =$	$(100\|010\|100)^2 + 0^4$	U_1A_1 (†)
		$L(G) \downarrow X =$	$(100\|(010 + 001 + 0^2)\|(200 + 100^2)\|(010 + 001 + 0)\|(100 + 0)) + (100\|010\|100)^2 + 0^3$	
	A_6 via $W(100)$	$V_{56} \downarrow X =$	$W(100) + W(100)^* + ((0 + 010)\|100) + (100\|(0 + 010))$	U_2T_1 (†)
		$L(G) \downarrow X =$	$(100\|(010 + 0)\|200\|(010 + 0)\|100) + W(100) + W(100)^* + (100\|002\|100\|(010 + 0))+$ $(010\|100\|(002 + 0)\|100) + 0$	
	A_7 via $T(100)$	$V_{56} \downarrow X =$	$(100\|(010 + 0)\|(100 + 0))^2$	U_2
		$L(G) \downarrow X =$	$(100\|(010 + 0^2)\|(200 + 100)\|(100 + 010 + 0)\|(100 + 0))+$ $(100\|(010 + 002)\|100^2\|(010 + 002 + 0)\|(100 + 0)) + 0$	
	A_7 via 001	$V_{56} \downarrow X =$	$(0\|100\|010\|100\|0)^2$	U_1
		$L(G) \downarrow X =$	$(100\|010\|100\|(002 + 0)\|100\|(010 + 0)\|100)+$ $(100\|010\|100\|(002 + 0^2)\|(200 + 100)\|(010 + 0)\|(100 + 0)) + 0$	

C_4	A_7 via 1000	$V_{56} \downarrow X =$	$T(0100)^2$	U_1
		$L(G) \downarrow X =$	$(0100\|0\|2000\|0\|0100) + (0100\|(0001+0)\|(0100+0)) + 0$	
D_4	MR (∞ classes)	$V_{56} \downarrow X =$	$0^2\|0100^2\|0^2$	U_1
		$L(G) \downarrow X =$	$0100^2\|(2000+0020+0002+0^3)\|(0^2+0100^2)$	
	A_7 via 1000	$V_{56} \downarrow X =$	$(0\|0100\|0)^2$	U_1
		$L(G) \downarrow X =$	$(0100\|(2000+0^2)\|0100) + (0100\|(0020+0002+0)\|(0100+0)) + 0$	

Table 7: Non-G-cr subgroups of $G = E_8$, $p = 3$

X	Embedding of X	$L(G) \downarrow X$	$C_G(X)^\circ$
C_3	D_8 via $T(010) + 000$	$200 + T(010) + T(101) + W(110) + W(110)^*$	U_1 (†)

Table 8: Non-G-cr subgroups of $G = E_8$, $p = 2$

X	Embedding of X	$L(G) \downarrow X$	$C_G(X)^\circ$
A_3	D_6 via $010 + 010$	$T(200)^2 + T(101)^4 + W(101) + W(101)^* + T(020) + 010^8 + T(002)^2 + 0^6$	$U_5\bar{A}_1^2$ (†)
	D_8 via $T(101)$	$(101\|(020+0^2)\|(210+012+101)\|(020+0)\|(101^2+0)) + 111^2$	1
B_3	E_6	$(100\|(010+001+0^2)\|(200+100^2)\|(010+001+0)\|(100+0)) + (100\|010\|100)^6 + 0^{14}$	U_1G_2 (†)
	A_6 via $W(100)$	$(100\|(010+0)\|200\|(010+0)\|100) + W(100)^3 + (W(100)^*)^3 + ((0+010)\|100)^2+$ $(100\|(0+010))^2 + (100\|002\|100\|(010+0)) + (010\|100\|(002+0)\|100) + 0^4$	$U_6\bar{A}_1T_1$ (†)
	A_7 via $T(100)$	$T(200) + T(100)^4 + (100\|(010+0)\|(100+0))^2 + (010\|002\|100\|010)^2$	U_5A_1 (†)
	$A_7' < E_7$ via $T(100)$	$(100\|(010+0^2)\|(200+100)\|(100+010+0)\|(100+0)) + (100\|(010+0)\|(100+0))^4+$ $(100\|(010+002)\|100^2\|(010+002+0)\|(100+0)) + 0^4$	$U_6\bar{A}_1$
	$A_7' < E_7$ via 001	$(100\|010\|100\|(002+0)\|100\|(010+0)\|100) + (0\|100\|010\|100\|0)^4+$ $(100\|010\|100\|(002+0^2)\|(200+100)\|(010+0)\|(100+0)) + 0^4$	$U_5\bar{A}_1$
	D_7 via $W(100) + W(100)^*$	$W(100)^2 + (W(100)^*)^2 + (100\|(010+0)\|200\|(010+0)\|100) + ((0+010)\|100\|0)+$ $(0\|100\|(0+010)) + T(002)^2$	U_3A_1 (†)
	D_8 via $T(100)^2$	$(100\|010\|100\|0^2) + (0\|100\|(010+0)\|100) + T(200) + T(002)^2$	U_5
	non-G-cr MR D_4	$T(010)^2 + (100\|(010+0^2)\|(200+100)\|(100+010+0)\|(100+0))+$ $(100\|010\|100\|(002+0)\|100\|(010+0)\|100\|0) + (0\|100\|010\|100\|(002+0)\|100\|(010+0)\|100)$	U_6 (†)

	D_8 via 001^2	$(0\vert 100\vert 010\vert 100\vert 0)^2 + 101^2 + 001^4 + T(002)$	U_1 (†)
	D_8 via $T(100) + T(100)^{[r]}$ $(r \neq 0)$	$(0\vert(100 + 100^{[r]})\vert(010 + 010^{[r]} + 0)\vert(100 + 100^{[r]})\vert 0^2)+$ $(0\vert(100 + 100^{[r]})\vert(100 \otimes 100^{[r]} + 0^2)\vert(100 + 100^{[r]})\vert 0) + (001 \otimes 001^{[r]})^2$	U_1
B_3^2	D_8 via $(0\vert(100, 0)\vert 0) + (0\vert(0, 100)\vert 0)$	$(0\vert((100, 0) + (0, 100))\vert((010, 0) + (0, 010) + 0)\vert((100, 0) + (0, 100))\vert 0^2)+$ $(0\vert((100, 0) + (0, 100))\vert((100, 100) + 0^2)\vert((100, 0) + (0, 100))\vert 0) + (001, 001)^2$	U_1 (†)
B_4	A_8 via $W(1000)$	$(1000\vert 0\vert 0100\vert 0\vert 2000\vert 0\vert 0100\vert 0\vert 1000) + (1000\vert(0010 + 0)\vert 0100\vert 0)+$ $(0\vert 0100\vert 0\vert(1000 + 0010))$	U_1 (†)
	D_8 via 0001	$(1000\vert 0\vert 0100\vert(0010 + 0)\vert 0100\vert 0\vert(1000 + 0))+$ $(1000\vert 0\vert 0100\vert 0\vert 2000\vert(0010 + 0)\vert 0100\vert 0\vert 1000)$	1
C_4	$A_7 < E_7$	$(0100\vert 0\vert 2000\vert 0\vert 0100) + T(0100)^4 + (0100\vert(0001 + 0)\vert(0100 + 0)) + 0^4$	$U_5\bar{A}_1$
D_4	D_8 via $1000 + 1000$	$(0^2\vert 0100^2\vert 0^2) + T(2000) + T(0011)^2$	U_1
	E_7 (∞ classes)	$(0100^2\vert(2000 + 0020 + 0002 + 0^4)\vert(0^2 + 0100^2)) + (0^2\vert 0100^2\vert 0^2)^2 + 0^2$	$U_5\bar{A}_1$ (†)
	$A_7 < E_7$	$(0100\vert(2000 + 0^2)\vert 0100) + (0\vert 0100\vert 0)^4 + (0100\vert(0020 + 0002 + 0)\vert(0100 + 0)) + 0^4$	$U_5\bar{A}_1$
	D_8 via $1000 + 1000$	$(0100\vert 0^2) + (0^2\vert 0100) + T(2000) + T(0020) + T(0002)$	U_4
	MR	$T(0100)^2 + (0\vert 0100^3\vert(0^4 + 2000 + 0020 + 0002)\vert(0100^2 + 0^2)\vert(0100 + 0))$	U_5 (†)

Appendix A. Ancillary data

Here we provide the socle series for the Weyl and tilting modules occurring in Tables 4–8. These mostly follow from the Weyl character formula and knowledge of the weights of low-dimensional irreducible modules, as given for instance in [32]. Explicit calculations are also facilitated using S. Doty's software package [17]. For tilting modules, attention is focused on indecomposable modules $T(\lambda)$, since direct sums and summands of tilting modules are tilting; moreover tensor products of tilting modules are tilting. Finally, the uniqueness of $T(\lambda)$ for each dominant weight λ implies that $T(\lambda) \cong T(-w_0\lambda)^*$ where w_0 is the longest element of the Weyl group; for us this is often sufficient information to determine the submodule structure of $T(\lambda)$.

Two of the most complicated tilting modules occurring in this paper arise for groups of type B_3 in characteristic 2, specifically $T(200)$ and $T(002)$. The former is $T(100) \otimes T(100)$ and the latter is $001 \otimes 001$. Using a sufficiently large finite subgroup, we can use Lemma 10.7 to perform explicit computations, e.g. in Magma, to obtain the precise module structure. By [11, Corollary 7.5], a subgroup $\Omega_7(4)$ is 'sufficiently large' for our purposes.

X	p	λ	Socle Series of $W(\lambda)$
A_3	2	101	$101\|0$
B_3	2	100	$100\|0$
C_3	3	110	$110\|001$
B_4	2	1000	$1000\|0$

X	p	λ	Socle Series of $T(\lambda)$
A_3	2	101	$0\|101\|0$
		020	$101\|(020+0)\|(101+0)$
		200	$010\|200\|010$
		002	$010\|002\|010$
B_3	2	100	$0\|100\|0$
		010	$0\|100\|(010+0)\|100\|0^2$
		200	$100\|(010+0^2)\|(200+100+0)\|(100+010+0)\|(100+0^2)$
		002	$0\|100\|101\|100\|(002+0)\|100\|(101+0)\|100\|0$
C_3	3	010	$0\|010\|0$
		101	$010\|(101+0)\|010$
C_4	2	0100	$0\|0100\|0$
D_4	2	$2\lambda_i$, $i=1,3,4$	$0\|0100\|0\|(2\lambda_i+0)\|0100\|0$
		0100	$0^2\|0100\|0^2$
		0011	$1000\|0011\|1000$

Acknowledgements

The authors thank the London Mathematical Society for support through a Scheme 4 grant, and the MFO for a Research in Pairs visit. They also thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the programme *Groups, Representations and Applications: New perspectives*, when much of the work on this paper was undertaken. This was supported by: EPSRC grant number EP/R014604/1.

The first author acknowledges support from the Alexander von Humboldt Foundation, Germany, as well as a Scheme 9 *Research Reboot* grant from the London Mathematical Society.

The second author is supported by EPSRC grant EP/W000466/1.

References

1. H. Azad, M. Barry, and G. Seitz, *On the structure of parabolic subgroups*, Comm. Algebra **18** (1990), no. 2, 551–562. MR 1047327 (91d:20048)
2. Michael Bate, Benjamin Martin, and Gerhard Röhrle, *A geometric approach to complete reducibility*, Invent. Math. **161** (2005), no. 1, 177–218. MR 2178661 (2007k:20101)
3. ______, *Complete reducibility and commuting subgroups*, J. Reine Angew. Math. **621** (2008), 213–235. MR 2431255 (2009h:20051)
4. Michael Bate, Benjamin Martin, Gerhard Röhrle, and Rudolf Tange, *Complete reducibility and separability*, Trans. Amer. Math. Soc. **362** (2010), no. 8, 4283–4311. MR 2608407 (2011i:20068)
5. ______, *Closed orbits and uniform S-instability in geometric invariant theory*, Trans. Amer. Math. Soc. **365** (2013), no. 7, 3643–3673. MR 3042598
6. A. Borel and J. De Siebenthal, *Les sous-groupes fermés de rang maximum des groupes de Lie clos*, Comment. Math. Helv. **23** (1949), 200–221. MR 0032659 (11,326d)
7. Armand Borel, *Linear algebraic groups*, second ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991. MR 1102012 (92d:20001)
8. N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968. MR 0240238 (39 #1590)
9. Roger W. Carter, *Simple groups of Lie type*, John Wiley & Sons, London-New York-Sydney, 1972, Pure and Applied Mathematics, Vol. 28. MR 0407163 (53 #10946)
10. ______, *Finite groups of Lie type*, Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York, 1985, Conjugacy classes and complex characters, A Wiley-Interscience Publication. MR 794307 (87d:20060)
11. E. Cline, B. Parshall, L. Scott, and W. van der Kallen, *Rational and generic cohomology*, Invent. Math. **39** (1977), no. 2, 143–163. MR 0439856 (55 #12737)
12. David A. Craven, *The maximal subgroups of the exceptional groups $F_4(q)$, $E_6(q)$ and ${}^2E_6(q)$ and related almost simple groups*, 2021.
13. ______, *Maximal* PSL_2 *subgroups of exceptional groups of Lie type*, Mem. Amer. Math. Soc. **276** (2022), no. 1355, v+155. MR 4386346
14. ______, *On the maximal subgroups of $E_7(q)$ and related almost simple groups*, 2022.
15. Stephen Donkin, *A filtration for rational modules*, Mathematische Zeitschrift **177** (1981), no. 1, 1–8. MR 611465
16. ______, *Rational representations of algebraic groups*, Lecture Notes in Mathematics, vol. 1140, Springer-Verlag, Berlin, 1985, Tensor products and filtration. MR 804233 (87b:20054)
17. Stephen Doty, *Weyl modules*, Gap Package, ver. 1.1.
18. E. B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Mat. Sbornik N.S. **30(72)** (1952), 349–462 (3 plates). MR 0047629 (13,904c)
19. Sebastian Herpel, *On the smoothness of centralizers in reductive groups*, Trans. Amer. Math. Soc. **365** (2013), no. 7, 3753–3774. MR 3042602
20. Jens Carsten Jantzen, *Representations of algebraic groups*, second ed., Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, Providence, RI, 2003. MR 2015057 (2004h:20061)
21. Martin W. Liebeck, Jan Saxl, and Donna M. Testerman, *Simple subgroups of large rank in groups of Lie type*, Proc. London Math. Soc. (3) **72** (1996), no. 2, 425–457. MR 1367085 (96k:20087)
22. Martin W. Liebeck and Gary M. Seitz, *Subgroups generated by root elements in groups of Lie type*, Ann. of Math. (2) **139** (1994), no. 2, 293–361. MR 1274094 (95d:20078)
23. ______, *Reductive subgroups of exceptional algebraic groups*, Mem. Amer. Math. Soc. **121** (1996), no. 580, vi+111. MR 1329942 (96i:20059)
24. ______, *On the subgroup structure of exceptional groups of Lie type*, Trans. Amer. Math. Soc. **350** (1998), no. 9, 3409–3482. MR 1458329 (99j:20055)
25. ______, *The maximal subgroups of positive dimension in exceptional algebraic groups*, Mem. Amer. Math. Soc. **169** (2004), no. 802, vi+227. MR 2044850 (2005b:20082)
26. ______, *Subgroups of exceptional algebraic groups which are irreducible on an adjoint or minimal module*, J. Group Theory **7** (2004), no. 3, 347–372. MR 2063402
27. ______, *Unipotent and nilpotent classes in simple algebraic groups and Lie algebras*, Mathematical Surveys and Monographs, vol. 180, American Mathematical Society, Providence, RI, 2012. MR 2883501
28. Martin W. Liebeck and Donna M. Testerman, *Irreducible subgroups of algebraic groups*, Q. J. Math. **55** (2004), no. 1, 47–55. MR 2043006 (2005b:20087)
29. Alastair J. Litterick, David I. Stewart, and Adam R. Thomas, *Complete reducibility and subgroups of exceptional algebraic groups*, London Mathematical Society Lecture Note Series, p. 191–245, Cambridge University Press, 2024.
30. Alastair J. Litterick and Adam R. Thomas, *Complete reducibility in good characteristic*, Transactions of the American Mathematical Society **370** (2018), no. 8, 5279–5340.

31. ______, *Reducible subgroups of exceptional algebraic groups*, Journal of Pure and Applied Algebra (2018).
32. Frank Lübeck, *Small degree representations of finite Chevalley groups in defining characteristic*, LMS J. Comput. Math. **4** (2001), 135–169 (electronic). MR 1901354 (2003e:20013)
33. George J. McNinch, *Levi decompositions of a linear algebraic group*, Transform. Groups **15** (2010), no. 4, 937–964. MR 2753264 (2012b:20110)
34. R. W. Richardson, *Affine coset spaces of reductive algebraic groups*, Bull. London Math. Soc. **9** (1977), no. 1, 38–41. MR 437549
35. ______, *Conjugacy classes of n-tuples in Lie algebras and algebraic groups*, Duke Math. J. **57** (1988), no. 1, 1–35. MR 952224
36. Gary M. Seitz, *The maximal subgroups of classical algebraic groups*, Mem. Amer. Math. Soc. **67** (1987), no. 365, iv+286. MR 888704 (88g:20092)
37. ______, *Maximal subgroups of exceptional algebraic groups*, Mem. Amer. Math. Soc. **90** (1991), no. 441, iv+197. MR 1048074 (91g:20038)
38. Jean-Pierre Serre, *Galois cohomology*, english ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002, Translated from the French by Patrick Ion and revised by the author. MR 1867431 (2002i:12004)
39. ______, *Complète réductibilité*, Astérisque (2005), no. 299, Exp. No. 932, viii, 195–217, Séminaire Bourbaki. Vol. 2003/2004. MR 2167207 (2006d:20084)
40. David I. Stewart, *The reductive subgroups of G_2*, J. Group Theory **13** (2010), no. 1, 117–130. MR 2604850 (2011c:20099)
41. ______, *Non-G-completely reducible subgroups of the exceptional algebraic groups*, International Mathematics Research Notices (2013).
42. ______, *On unipotent algebraic G-groups and 1-cohomology*, Trans. Amer. Math. Soc. **365** (2013), no. 12, 6343–6365. MR 3105754
43. ______, *The reductive subgroups of F_4*, Mem. Amer. Math. Soc. **223** (2013), no. 1049, vi+88. MR 3075783
44. Adam R. Thomas, *Simple irreducible subgroups of exceptional algebraic groups*, J. Algebra **423** (2015), 190–238.
45. ______, *The irreducible subgroups of exceptional algebraic groups*, Mem. Amer. Math. Soc. **268** (2020), no. 1307, v+191. MR 4218306