

New Design of Sparse Zero-Correlation-Zone Sequence Sets for Optimal Channel Estimation in (Generalized) Spatial Modulation Systems

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Abstract—Within the zero-correlation-zone (ZCZ), ZCZ sequence sets exhibit ideal correlation properties, which is highly advantageous for both wireless communications and radar sensing applications. Recently, to achieve optimal training for spatial modulation (SM), Pai *et al.* introduced the concept of sparse ZCZ (SZCZ) sequence sets, where each column contains only one non-zero element. In this paper, we first extend the SZCZ sequence set concept by permitting multiple non-zero elements per column, thereby accommodating training design requirements for generalized SM (GSM) systems. Then, we propose a direct construction of SZCZ sequence sets with parameter $(q^{n+k}, q^{m+n+k}, (q-1)q^{\pi(2)-1} + (q-2)q^{\pi(3)-1}, (q^n, q^{m+n}))$ based on restricted extended Boolean functions. Compared to existing works, the proposed SZCZ sequence sets exhibit a greater ZCZ width and can be applied to both SM and GSM training designs simultaneously. Simulation results show that compared with other training sequences, the channel estimation performance is significantly improved when the proposed SZCZ sequence set is used as the training sequences.

Index Terms—Sparse ZCZ sequence set, channel estimation, (generalized) spatial modulation, restricted extended Boolean function, frequency selective channel.

I. INTRODUCTION

A sequence set is termed a zero-correlation-zone (ZCZ) sequence set if its auto-correlation and cross-correlation functions are zero within a predefined region around the zero shift [1]. A ZCZ sequence set where this zone covers the entire sequence length is called a perfect sequence set. However, the existence of such perfect sets is precluded by the Welch bound [2]. Tang *et al.* established a theoretical upper bound for the ZCZ width, given by $Z \leq \frac{N}{M} - 1$, where M , N and Z denote the set size, sequence length, and ZCZ width, respectively [3]. For the specific case of binary sequence sets, it has been conjectured that $Z \leq \frac{N}{2M}$ [4]. A ZCZ sequence set that attains these bounds is considered optimal. Over the past two decades, numerous optimal ZCZ sequence sets have been constructed using various methods, including interleaving techniques, transformations of known sequences, and Boolean functions [5]–[12]. To date, ZCZ sequence sets

have found a wide range of applications. These include their use as spreading sequences in quasi-synchronous code-division multiple-access (QS-CDMA) systems [13], as pilot sequences for frequency synchronization and interference avoidance in multi-cell systems [14], [15], in waveform design for multiple-input multiple-output (MIMO) continuous-wave (CW) radar [16], [17], and as training sequences for optimal channel estimation in MIMO communication [18]–[20]. This paper focuses on the design of sparse ZCZ training sequence sets for spatial modulation (SM) and generalized SM (GSM) to achieve optimal channel estimation.

MIMO technology employs multiple antennas at both the transmitter and receiver to enhance data throughput and improve link reliability without requiring extra bandwidth or transmission power. It is now a cornerstone of modern wireless standards, including Wi-Fi, 4G LTE, and 5G NR [21]. However, these advantages come at the cost of stringent synchronization requirements, inter-antenna interference (IAI), and increased hardware complexity from additional radio-frequency (RF) chains. To achieve a balance, spatial modulation (SM) was proposed, which activates only one transmit antenna and its corresponding RF chain in any given time slot [22]–[24]. This characteristic inherently reduces IAI, lowers power consumption, and simplifies transceiver design. A longstanding challenge has been the lack of effective channel estimation schemes for SM over frequency-selective channels. Conventional MIMO training sequences [19], [20], [25], [26] are unsuitable as they do not account for the sparse signaling structure of SM. To address this, [27] and [28] proposed sequentially activating transmit antennas, assigning distinct cyclic shifts of a common pseudo-random sequence to each antenna. However, this approach may not support high-rate SM schemes due to its associated power efficiency loss and high pilot overhead [23]. In 2020, Liu *et al.* introduced the concept of cross Z-complementary pairs (CZCPs) for training sequence design in SM systems [29]. Several constructions of CZCPs with flexible parameters have been reported in [30]–[32]. Building on CZCPs, Huang *et al.* generalized the concept to cross Z-complementary sets (CZCSs) for training sequence design in SM systems [33]. Recently, they also proposed a construction of CZCSs with flexible length based on generalized Boolean functions [34]. Fundamentally, both CZCP and CZCS-based designs construct sparse zero-correlation-zone (SZCZ) sequence sets. However, their narrow ZCZ width restricts the number of tolerable multipath components.

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Channel estimation performance degrades sharply when the actual multipath count exceeds this width, a constraint inherent in their structure. Recently, Pai *et al.* proposed the concept of the sparse ZCZ (SZCZ) array (i.e., sequence set), whereby each column contains only one non-zero element, making it an excellent training sequence candidate for SM systems [35]. Moreover, two constructions of SZCZ arrays based on two-dimensional restricted generalized Boolean functions were proposed, and demonstrating the superiority of SZCZ arrays in channel estimation for SM systems. It is important to note that the SZCZ sequence set in [35] is specifically designed for SM systems and is not directly applicable to generalized SM (GSM) systems.

The spectral efficiency of SM is inherently limited, as it activates only a single RF chain in each time slot. To overcome this limitation, GSM was introduced, which activates multiple (but fewer than the total number of) RF chains per time slot [36], [37]. Consequently, a GSM symbol comprises two components: one selects the specific combination of active antennas, while the other specifies the constellation symbols transmitted from these active antennas. This architecture allows GSM to strike an excellent balance between spectral efficiency and RF hardware cost, thereby retaining the primary benefits of SM. Likewise, the design of training sequences for GSM is challenging. Due to the different sparse structure of the training matrix, the training sequence suitable for SM cannot be directly applied to GSM. In 2023, Zhou *et al.* proposed a new concept of symmetrical Z-complementary code set (S-ZCCS) for training sequence design in GSM systems [38]. The S-ZCCS is a subclass of the classical ZCCS [39], featuring a symmetric ZCZ. However, to eliminate inter-antenna interference, the S-ZCCS-based training framework requires the incorporation of zero padding in specific time slots, thereby increasing the pilot overhead. In the same year, Huang *et al.* developed GSM-compatible training sequence sets using CZCP mates. However, their design was severely constrained-supporting only two active TAs per time slot due to the limited availability of two sequences in CZCP [40]. Subsequently, in order to solve this problem, Huang *et al.* proposed another sequence family called enhanced CZCS (E-CZCS) for GSM training sequence design [41]. A careful observation reveals that the training sequence sets designed in [38], [40], [41] satisfy the following characteristics: 1) they are all ZCZ sequence sets; 2) they all have sparse structures and each column has more than one non-zero element. Inspired by this, this paper redefines the SZCZ sequence set to allow it to have flexible non-zero element entries in each column, so that it can be applicable to both SM systems and GSM systems.

Constructing a SZCZ sequence set is not easy, as it must simultaneously satisfy the requirements for a periodic ZCZ and flexible column sparsity. Essentially, the new concepts proposed by Zhou *et al.* and Huang *et al.* both aim to construct SZCZ sequence sets that meet GSM requirements. Observation reveals that restricting one Boolean function can only result in one non-zero element per column in the SZCZ sequence set. Taking inspiration from this, we restrict k variables in q^n different extended Boolean functions and ultimately obtained

a SZCZ sequence set with parameters of $(M = q^{n+k}, N = q^{m+n+k}, Z = (q-1)q^{\pi(2)-1} + (q-2)q^{\pi(3)-1}, (\hat{M} = q^n, \hat{N} = q^{m+n}))$, where $\hat{M}$ and $\hat{N}$ represent the number of non-zero elements in each column and row of SZCZ sequence sets, respectively. By adjusting the parameter n , the proposed construction is not only applicable to SM systems, but also to GSM systems. Simulation results show that the proposed SZCZ sequence set can achieve the minimum mean square error (MSE) of channel estimation in SM/GSM systems, which is better than other types of sequence sets.

The paper is structured as follows: Section II introduces the fundamental concepts and GSM system model. Section III first derives the design principles of the GSM training sequence set, then proposes the main construction of this paper and makes a detailed comparison with known results. Section IV provides some simulation result and analysis. Finally, Section V concludes with some remarks.

Notations: Throughout this paper, the following notations will be used: $\mathbb{Z}_\lambda = \{0, 1, 2, \dots, \lambda - 1\}$ represents the set of integers modulo a positive integer λ ; $\xi_\lambda = \exp(2\pi\sqrt{-1}/\lambda)$ is a primitive λ -th root of unity; $(\cdot)^*$, $(\cdot)^T$ and $(\cdot)^H$ represent the complex conjugation, the transpose and the Hermitian transpose, respectively; $(\cdot)_x$ represents the modulo x operation where x is a positive integer; $\text{Tr}(\cdot)$ represents the trace of the matrix; the lowercase bold letter $\mathbf{a}$ represents a sequence (or row vector) and the capital bold letter $\mathbf{A}$ represents a sequence set (or a matrix); $+$ and $-$ represent 1 and -1 , respectively.

II. PRELIMINARIES AND DEFINITIONS

A. Sequences

Given two complex-valued sequences $\mathbf{a} = [a_0, a_1, \dots, a_{N-1}]$ and $\mathbf{b} = [b_0, b_1, \dots, b_{N-1}]$ of length N , the periodic cross-correlation function (PCCF) of $\mathbf{a}$ and $\mathbf{b}$ at time-shift τ is defined as

$$R_{\mathbf{a}, \mathbf{b}}(\tau) = \begin{cases} \sum_{i=0}^{N-1} a_i b_{(i+\tau)_N}^*, & 0 \leq \tau < N; \\ \sum_{i=0}^{N-1} a_{(i-\tau)_N} b_i^*, & -N < \tau < 0; \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

Especially, when $\mathbf{a} = \mathbf{b}$, $R_{\mathbf{a}, \mathbf{b}}(\tau)$ is called the periodic auto-correlation function (PACF) of $\mathbf{a}$ and is abbreviated as $R_{\mathbf{a}}(\tau)$. Note that $R_{\mathbf{a}}(\tau) = R_{\mathbf{a}}^*(-\tau)$.

Definition 1. Let $\mathbf{S}$ be a sequence set of size M of length N , its matrix representation is

$$\mathbf{S} = \begin{bmatrix} \mathbf{s}_0 \\ \mathbf{s}_1 \\ \vdots \\ \mathbf{s}_{M-1} \end{bmatrix} = \begin{bmatrix} s_{0,0} & s_{0,1} & \cdots & s_{0,N-1} \\ s_{1,0} & s_{1,1} & \cdots & s_{1,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{M-1,0} & s_{M-1,1} & \cdots & s_{M-1,N-1} \end{bmatrix}, \quad (2)$$

where each row sequence has $\hat{N}$ non-zero elements and each column sequence has $\hat{M}$ non-zero elements. For a positive integer Z with $Z \leq N - 1$, the sequence set $\mathbf{S}$ is called an $(M, N, Z, (\hat{M}, \hat{N}))$ -SZCZ sequence set if the following conditions hold:

$$R_{\mathbf{S}_{i,j}}(\tau) = \begin{cases} \hat{N}, & i = j, \tau = 0; \\ 0, & i = j, 1 \leq |\tau| \leq Z; \\ 0, & i \neq j, 0 \leq |\tau| \leq Z. \end{cases} \quad (3)$$

Besides, the ZCZ ratio is defined as $\text{ZCZ}_{\text{ratio}} = MZ/N$, the larger $\text{ZCZ}_{\text{ratio}}$, the greater the width of ZCZ.

Remark 1. In [35], Pai *et al.* first introduced the concept of SZCZ for training design of SM. In their definition, the SZCZ parameter is denoted as (M, N, Z, S) , where $S = ((N - \hat{N})/N)$ indicates the sparsity level. Because the number of activated antennas in each time slot in GSM is flexible, unlike SM which only one activated antenna in each time slot, so we introduce the parameter $\hat{M}$ and redefine the SZCZ sequence set. Further, the parameter of the SZCZ sequence set we define are denoted by $(M, N, Z, (\hat{M}, \hat{N}))$. Here, S is not used because it can be calculated by N and $\hat{N}$. Especially, if $(\hat{M}, \hat{N}) = (M, N)$, the SZCZ sequence set $\mathbf{S}$ is reduced to a traditional dense ZCZ sequence set with parameter (M, N, Z) .

Remark 2. In Definition 1, if $M = \hat{M} = 1$ (i.e., sequence set $\mathbf{S}$ contains only one sequence denoted as $\mathbf{s}$) and $Z = N$, then $\mathbf{s}$ is called a perfect sequence.

Lemma 1. For any (M, N, Z) -ZCZ sequence set, the upper bound on ZCZ width [3] is given as

$$Z \leq \frac{N}{M} - 1. \quad (4)$$

However, for a binary (M, N, Z) -ZCZ sequence set, the upper bound on ZCZ width [4] is conjectured by

$$Z \leq \frac{N}{2M}. \quad (5)$$

Remark 3. According to Lemma 1, we have $\text{ZCZ}_{\text{ratio}} \leq 1$. In this paper, when $\text{ZCZ}_{\text{ratio}} = 1$, we refer to ZCZ sequence set as the optimal. Especially, if it is a binary ZCZ sequence set, when $\text{ZCZ}_{\text{ratio}} = 1/2$, we call it optimal.

B. Restricted Extended Boolean Functions

In this subsection, we will introduce the restricted extended Boolean functions [42], which are powerful tools for constructing SZCZ sequence sets. Before that, let us first review extended Boolean functions.

For two integers $q \geq 2$ and $\lambda \geq 2$ with $q|\lambda$, an extended Boolean function $f: \mathbb{Z}_q^m \rightarrow \mathbb{Z}_\lambda$ is comprised of m variables $x_1, x_2, \dots, x_m$ where $x_i \in \mathbb{Z}_q$ for $1 \leq i \leq m$. The sequence $\mathbf{f}$ associated to f is represented by $\mathbf{f} = (f_0, f_1, \dots, f_{q^m-1})$, where $f_i = f(i_1, i_2, \dots, i_m)$ and $(i_1, i_2, \dots, i_m)$ is the q -ary representation of integer $i = \sum_{j=1}^m i_j q^{j-1}$. Further, the λ -ary sequence $\mathbf{f}$ corresponds to the λ -phase complex-valued sequence is provided by $\xi_\lambda^{\mathbf{f}} = (\xi_\lambda^{f_0}, \xi_\lambda^{f_1}, \dots, \xi_\lambda^{f_{q^m-1}})$.

Let $J = \{j_1, j_2, \dots, j_k\}$ be a list of k indices with $1 \leq j_1 < j_2 < \dots < j_k \leq m$, $\mathbf{x}_J = (x_{j_1}, x_{j_2}, \dots, x_{j_m})$ and

$\mathbf{c} = (c_1, c_2, \dots, c_k) \in \mathbb{Z}_q^k$. A restricted extended Boolean function $f|_{\mathbf{x}_J=\mathbf{c}}$ obtained by restricting variables $\mathbf{x}_J$ in the f to the certain known $\mathbf{c}$. To simplify, consider $f|_{\mathbf{x}_J=\mathbf{c}}$ as the complex-valued sequence with component equal to $\xi_\lambda^{f_i}$ if $i_j = c_j$ for $1 \leq j \leq k$ and zero otherwise. Obviously, there are q^{m-k} non-zero elements in $f|_{\mathbf{x}_J=\mathbf{c}}$.

Example 1. Taking $q = \lambda = 3$, $m = 2$ and $f = 2x_1x_2 + x_1 + x_2 + 1$ as an example, then 3-phase sequence corresponding to f is given by

$$\xi_\lambda^{\mathbf{f}} = (\xi_3^1, \xi_3^2, \xi_3^0, \xi_3^2, \xi_3^2, \xi_3^2, \xi_3^1, \xi_3^2, \xi_3^1). \quad (6)$$

By setting $\mathbf{x}_J = (x_1)$, we have

$$\mathbf{F} = \begin{bmatrix} \mathbf{f}|_{\mathbf{x}_J=\mathbf{c}=(0)} \\ \mathbf{f}|_{\mathbf{x}_J=\mathbf{c}=(1)} \\ \mathbf{f}|_{\mathbf{x}_J=\mathbf{c}=(2)} \end{bmatrix} = \begin{bmatrix} \xi_3^1 & 0 & 0 & \xi_3^2 & 0 & 0 & \xi_3^1 & 0 & 0 \\ 0 & \xi_3^2 & 0 & 0 & \xi_3^2 & 0 & 0 & \xi_3^2 & 0 \\ 0 & 0 & \xi_3^0 & 0 & 0 & \xi_3^2 & 0 & 0 & \xi_3^1 \end{bmatrix}. \quad (7)$$

For matrix $\mathbf{F}$, we have $(\hat{M}, \hat{N}) = (1, 3)$.

C. Generalized Spatial Modulation

In this paper, we consider a broadband single-carrier GSM system with N_t transmit antennas (TAs), N_r receive antennas (RAs) and N_a radio frequency (RF) chains over frequency-selective channels as depicted in Fig. 1. During each time-slot t , N_a TAs out of N_t TAs are activated, and the active antennas send quadrature amplitude modulation (QAM) or phase-shift keying (PSK) modulation (the modulation order is M_{GSM}) constellation symbols accordingly. For simplicity, assume both N_t and M_{GSM} take powers of 2. In time slot t , $\lfloor \log_2 \binom{N_t}{N_a} \rfloor + N_a \log_2(M_{\text{GSM}})$ bits, denoted by $\mathbf{b}_t$, are transmitted over one GSM symbol $\mathbf{d}_t$. Specifically, $\lfloor \log_2 \binom{N_t}{N_a} \rfloor$ bits, denoted by $\mathbf{b}_t^{(1)}$, are used to activate N_a TAs, and $N_a \log_2(M_{\text{GSM}})$ bits, denoted by $\mathbf{b}_t^{(2)}$, are used to point to N_a constellation symbols.

TABLE I: An example of antenna activation patterns with $N_t = 4$ and $N_a = 2$.

Bits	Antenna activation patterns	Notes
00	$[1, 1, 0, 0]^T$	Active TAs: 1, 2
01	$[0, 1, 1, 0]^T$	Active TAs: 2, 3
10	$[0, 0, 1, 1]^T$	Active TAs: 3, 4
11	$[1, 0, 0, 1]^T$	Active TAs: 1, 4

Here is an example to explain the transmitter of GSM. Consider an GSM system with $N_t = 4$ TAs, $N_a = 2$, and using QPSK modulation, i.e., $M_{\text{GSM}} = 4$. Since $\binom{4}{2} = 6$, the length of $\mathbf{b}_t^{(1)} = \lfloor \log_2 \binom{4}{2} \rfloor = 2$, the mapping rules are shown in Table I. Assuming that an GSM block contains $T = 4$ GSM symbols and message bits is $[110110011101101111011]$, then $\mathbf{b}_1 = [11|0110]$, $\mathbf{b}_2 = [01|1101]$, $\mathbf{b}_3 = [10|1001]$ and $\mathbf{b}_4 = [11|1011]$. Taking $\mathbf{b}_1$ as an example, $\mathbf{b}_1^{(1)} = [11]$ and $\mathbf{b}_1^{(2)} = [0110]$. Referring to Table I, $\mathbf{b}_1^{(1)}$ corresponds to activating 1-st TA and 4-th TA, and $\mathbf{b}_1^{(2)}$ determines the QPSK symbols transmitted on the active antennas, i.e., "01"

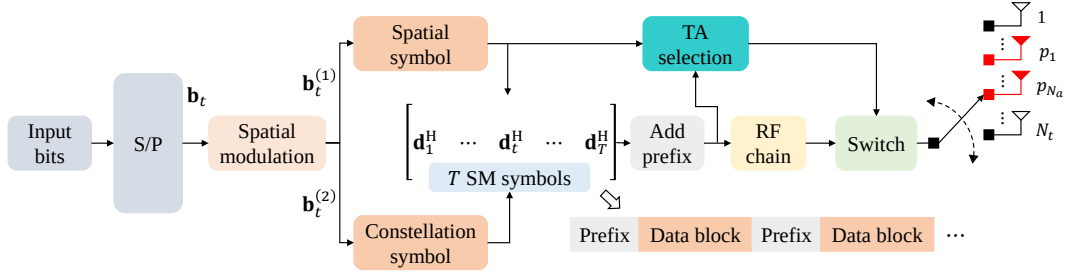


Fig. 1: A generic transceiver structure of SC-GSM systems.

$\rightarrow 1 \rightarrow \xi_4^1 \rightarrow 1\text{-st TA}$; “10” $\rightarrow 2 \rightarrow \xi_4^2 \rightarrow 4\text{-st TA}$. Therefore, the GSM block for the 4 GSM symbols can be formulated as follows:

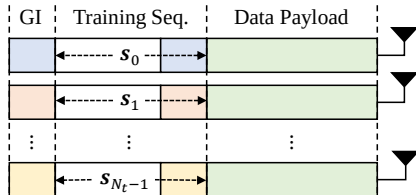
$$[\mathbf{d}_1^H | \mathbf{d}_2^H | \mathbf{d}_3^H | \mathbf{d}_4^H] = \begin{bmatrix} \xi_4^1 & 0 & 0 & \xi_4^2 \\ 0 & \xi_4^3 & 0 & 0 \\ 0 & \xi_4^4 & \xi_4^2 & 0 \\ \xi_4^2 & 0 & \xi_4^1 & \xi_4^3 \end{bmatrix}. \quad (8)$$

Remark 4. Since only N_a TAs are activated each time for a GSM symbol, the GSM symbol is sparse. In the SZCZ sequence set, the parameter $\hat{M}$ can be adjusted to adapt to this sparsity, and the SZCZ sequence set can be used as a training sequence set for the GSM system.

III. TRAINING DESIGN FOR GSM SYSTEMS

A. Design Principles

Wireless communications often use two channel estimation methods: training-based channel estimation and blind channel estimation. In this paper we consider the former, a generic training-based multiple-antenna transmission structure shown in Fig. 2. Before transmitting data payload, the training sequences $\mathbf{s}_0, \mathbf{s}_1, \dots, \mathbf{s}_{N_t-1}$ need to be sent in order for the receiver to estimate the channel state information (CSI). To resist inter symbol interference (ISI), it is necessary to add a guard interval (GI) before the training sequence, which can be a zero prefix (ZP), cyclic prefix (CP), or odd cyclic prefix (OP). It is crucial to note that, as demonstrated in [43], [44], the form of the GI fundamentally determines the correlation metric of training sequences: ZP corresponds to aperiodic correlation function, CP aligns with periodic correlation function, and OP induces odd periodic correlation function. Here we consider the CP scenario.

Fig. 2: Generic training-based GSM transmission structure with N_t TAs.

Assuming there are N_t TAs, N_r RAs and N_a RF chain in a synchronous GSM system and its training sequence set is given by

$$\mathbf{S} = \begin{bmatrix} s_{0,0} & s_{0,1} & \cdots & s_{0,N-1} \\ s_{1,0} & s_{1,1} & \cdots & s_{1,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{N_t-1,0} & s_{N_t-1,1} & \cdots & s_{N_t-1,N-1} \end{bmatrix}, \quad (9)$$

where $\mathbf{s}_m = (s_{m,0}, s_{m,1}, \dots, s_{m,N-1})$ is the training sequence transmitted by the $(m+1)$ -th TA, with an identical energy of $\sum_{l=0}^{N-1} |s_{m,l}|^2 = \hat{N}$ for $0 \leq m < N_t$. Since GSM only activates N_a TAs in each time slot, each column of the matrix $\mathbf{S}$ has only N_a non-zero elements. Consider a quasi-static frequency-selective channel with $(L+1)$ -multipath and channel matrix is denoted as

$$\mathbf{H} = [\mathbf{H}_0, \mathbf{H}_1, \dots, \mathbf{H}_L]_{N_r \times N_t(L+1)}, \quad (10)$$

where $\mathbf{H}_l \in \mathbb{C}^{N_r \times N_t}$ denotes the l -th multipath channel sub-matrix. Besides, let

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_0 \\ \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_L \end{bmatrix}_{N_t(L+1) \times N}, \quad (11)$$

where $\mathbf{X}_\tau$ represents the τ -th right cyclic shift of the matrix $\mathbf{S}$ in (9). Then, the discrete received signal can be represented as

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W}, \quad (12)$$

where $\mathbf{W} \in \mathbb{C}^{N_r \times N}$ a discrete uncorrelated white complex Gaussian noise matrix with zero-mean and variance σ_w^2 per dimension.

At the receiving end, we consider using the simple least square (LS) estimator, the estimated CSI matrix is

$$\hat{\mathbf{H}} = \mathbf{Y}\mathbf{X}^H(\mathbf{X}\mathbf{X}^H)^{-1}, \quad (13)$$

and the normalized MSE can be derived as [29]

$$\text{MSE} = \frac{\sigma_w^2}{N_t(L+1)} \text{Tr}((\mathbf{X}\mathbf{X}^H)^{-1}). \quad (14)$$

Clearly, the minimum MSE can be obtained if and only if $\mathbf{X}\mathbf{X}^H$ is a diagonal matrix, i.e.,

$$\mathbf{X}_i \mathbf{X}_j^H = \begin{cases} N \mathbf{I}_{N \times N}, & i = j; \\ \mathbf{0}_{N \times N}, & \text{otherwise,} \end{cases} \quad (15)$$

implying

$$R_{\mathbf{s}_i, \mathbf{s}_j}(\tau) = \begin{cases} \hat{N}, & i = j, \tau = 0; \\ 0, & i = j, 1 \leq |\tau| \leq L; \\ 0, & i \neq j, 0 \leq |\tau| \leq L. \end{cases} \quad (16)$$

Remark 5. For a GSM system, since only N_a TAs are activated in each time slot, there are two principles to the optimal training sequence set:

- **Principle 1:** Each column of the training sequence set $\mathbf{S}$ has only N_a non-zero elements.
- **Principle 2:** Training sequence set $\mathbf{S} = \{\mathbf{s}_n\}_{n=0}^{N_t-1}$ must meet the condition specified in (16).

Based on the two design principles in Remark 5, we can easily derive the following corollary.

Corollary 1. According to **Principle 1** and **Principle 2**, one can attain the minimum MSE if and only if GSM training sequence set is an $(M, N, Z, (\hat{M}, \hat{N}))$ -SZCZ sequence set with $M = N_t$, $Z \geq L$, and $\hat{M} = N_a$.

Remark 6. Taking $N_a = 1$, the GSM system degenerates into the SM system, the above analysis of training sequence sets also applies to SM systems, this has been confirmed in [35].

Remark 7. In (13), we obtain estimated channel $\hat{\mathbf{H}}$ through the LS estimator. Here we briefly analyze its complexity as follows: (1) Calculate $\mathbf{X}\mathbf{X}^H$: The complexity of matrix multiplication in this step is $\mathcal{O}((L+1)^2 N_t^2 N)$; (2) Calculate $(\mathbf{X}\mathbf{X}^H)^{-1}$: The complexity of inverting a square matrix is the cube of its order, so its complexity is $\mathcal{O}((L+1)^3 N_t^3)$; (3) Calculate $\mathbf{X}^H(\mathbf{X}\mathbf{X}^H)^{-1}$: The complexity of matrix multiplication in this step is $\mathcal{O}((L+1)^2 N_t^2 N)$; (4) Calculate $\mathbf{Y}\mathbf{X}^H(\mathbf{X}\mathbf{X}^H)^{-1}$: The complexity of matrix multiplication in this step is $\mathcal{O}((L+1) N_t N_r N)$. In summary, the complexity of obtaining the estimated channel $\hat{\mathbf{H}}$ is $\mathcal{O}((L+1)^3 N_t^3 + 2(L+1)^2 N_t^2 N + (L+1) N_t N_r N)$.

B. Proposed SZCZ Sequence Set for (G)SM System

In this subsection, we present a general method for constructing SZCZ sequence sets based on restricted extended Boolean functions. Our construction is different from that in [35] in that the parameter $\hat{M}$ can be flexibly adjusted. Therefore, by adjusting the parameters, the proposed SZCZ sequence set can be used for both SM and GSM systems.

Theorem 1. Given three integers $m \geq 0$, $k \geq 0$, $n \geq 0$ with $m+n+k \geq 1$. Let π be a permutation of set $\{1, 2, \dots, m+n+k\}$, $\mathbf{x}_I = [x_{\pi(m+1)}, x_{\pi(m+2)}, \dots, x_{\pi(m+n)}]$ and $\mathbf{x}_J = [x_{\pi(m+n+1)}, x_{\pi(m+n+2)}, \dots, x_{\pi(m+n+k)}]$. For any two integers $q \geq 2$ and $\lambda \geq 2$ with $q|\lambda$, define $f: \mathbb{Z}_q^{m+n+k} \rightarrow \mathbb{Z}_\lambda$ as

$$f = \frac{\lambda}{q} \sum_{l=1}^{m+n+k-1} x_{\pi(l)} x_{\pi(l+1)} + \sum_{l=1}^{m+n+k} c_l x_l + c_0, \quad (17)$$

where $c_l \in \mathbb{Z}_\lambda$ for $0 \leq l < m+n+k$ and $c_{m+n+k} \in \{0, \lambda/q\}$. If $\pi(1) = m+n+k$ when $n+k=1$, or $\pi(1) = m+n+k$

and $\{\pi(m+i) : 2 \leq i \leq n+k\} = \{m+i-1 : 2 \leq i \leq n+k\}$ when $n+k \geq 2$, then the set

$$\mathbf{F} = \left\{ \left(f + \frac{\lambda}{q} \sum_{\ell=1}^n d_\ell x_{\pi(m+\ell)} \right) |_{\mathbf{x}_J = \mathbf{g}} : 0 \leq d < q^n, \mathbf{g} \in \mathbb{Z}_q^k \right\} \quad (18)$$

forms an $(q^{n+k}, q^{m+n+k}, Z, (\hat{M}, \hat{N}))$ -SZCZ sequence set, where $(d_1, d_2, \dots, d_n)$ is the q -ary representation of d , $Z = (q-1)q^{\pi(2)-1} + (q-2)q^{\pi(3)-1}$, $\hat{M} = q^n$ and $\hat{N} = q^{m+n}$.

Proof. Refer to Appendix A. $\square$

Remark 8. By considering $\pi(2) = m$ and $\pi(3) = m-1$ in Theorem 1, we can obtain an $(q^{n+k}, q^{m+n+k}, (q-1)q^{m-1} + (q-2)q^{m-2}, (\hat{M}, \hat{N}))$ -SZCZ sequence set. When $q \rightarrow \infty$, we have

$$\begin{aligned} \text{ZCZ}_{\text{ratio}} &= \lim_{q \rightarrow \infty} \frac{q^{n+k} \times ((q-1)q^{m-1} + (q-2)q^{m-2})}{q^{m+n+k}} \\ &= 1, \end{aligned} \quad (19)$$

implying that $\mathbf{F}$ is asymptotically optimal. Especially, when $q = \lambda = 2$ (i.e., non-zero elements are defined on $\{1, -1\}$), we have $M = 2^{n+k}$, $N = 2^{m+n+k}$ and $Z = 2^{m-1}$, this means that $\mathbf{F}$ is an optimal SZCZ sequence set.

Remark 9. By taking $m = 0$ and $n = 2$ in Theorem 1, any sequence in the proposed SZCZ sequence set is a perfect sequence. Unlike [10, Remark 7], we can control the number of zero elements in a perfect sequence by adjusting the parameter k , while the work in [10, Remark 7] only applies to unimodular sequences.

Proof. Since $\frac{\lambda}{q}(d_1 x_{\pi(1)} + d_2 x_{\pi(2)})$ can be absorbed by the first-order term f , so we only need to consider that the generated sequence of $f|_{\mathbf{x}_J = \mathbf{g}}$ has a perfect periodic autocorrelation function. Let the sequence generated by function $f|_{\mathbf{x}_J = \mathbf{g}}$ be $\mathbf{f}_g$, where

$$f_{g,i} = \begin{cases} \xi_\lambda^{f_i}, & i_{\pi(n+\alpha)} = g_\alpha \text{ for } 1 \leq \alpha \leq k; \\ 0, & \text{otherwise.} \end{cases} \quad (20)$$

Let $N = q^{n+k}$ and $I_g = \{0 \leq i < N : i_{\pi(n+\alpha)} = g_\alpha, 1 \leq \alpha \leq k\}$. For $0 < \tau < N$, we need to prove that

$$R_{\mathbf{f}_g}(\tau) = \sum_{i=0}^{N-1} f_{g,i} f_{g,i+\tau}^* = \sum_{i,j \in I_g} \xi_\lambda^{f_i - f_j}, \quad (21)$$

where $j = (i + \tau)_N$.

Case 1: $i_{\pi(1)} = j_{\pi(1)}$. Since $i, j \in I_g$, then $i_{\pi(n+\alpha)} = j_{\pi(n+\alpha)} = g_\alpha$ for $1 \leq \alpha \leq k$. Furthermore, we have $i_{\pi(2)} \neq j_{\pi(2)}$; otherwise $i = j$. For $1 \leq y \leq q-1$, we let $i^{(y)}$ and $j^{(y)}$ be two integers that are distinct from i and j , respectively, in only one position $\pi(1)$, i.e., $i_{\pi(1)}^{(y)} = (i_{\pi(1)} + y)_q$ and $j_{\pi(1)}^{(y)} = (j_{\pi(1)} + y)_q$. Correspondingly, i and j are denoted as $i^{(0)}$ and $j^{(0)}$ respectively. By following the similar argument as provided in Case 1 of Appendix A, we have $\sum_{y=0}^{q-1} \xi_\lambda^{f_{j^{(y)}} - f_{i^{(y)}}} = 0$.

Case 2: $i_{\pi(1)} \neq j_{\pi(1)}$ and $i_{\pi(2)} = j_{\pi(2)}$. For $1 \leq y \leq q-1$, we let $i^{(y)}$ and $j^{(y)}$ be two integers that are distinct

$$\mathbf{F} = \begin{bmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{bmatrix} = \begin{bmatrix} + & + & + & - & + & + & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & + & + & - & - & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & + & + & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & + & - & + & + & + \\ + & - & + & + & + & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & + & - & + & + & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & + & + & - & + & + & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & + & + & - & - & - & + & - \end{bmatrix} \quad (25)$$

from i and j , respectively, in only one position $\pi(2)$, i.e., $i_{\pi(2)}^{(y)} = (i_{\pi(2)} + y)_q$ and $j_{\pi(2)}^{(y)} = (j_{\pi(2)} + y)_q$. By following the similar argument as provided in Case 3 of Appendix A, we have $\sum_{y=0}^{q-1} \xi_{\lambda}^{f_{j(y)} - f_{i(y)}} = 0$.

Case 3: $i_{\pi(1)} \neq j_{\pi(1)}$ and $i_{\pi(2)} \neq j_{\pi(2)}$. For $1 \leq y \leq q-1$, we let $i^{(y)}$ and $j^{(y)}$ be two integers that are distinct from i and j , respectively, in only one position $\pi(2)$, i.e., $i_{\pi(2)}^{(y)} = (i_{\pi(2)} + y)_q$ and $j_{\pi(2)}^{(y)} = (j_{\pi(2)} + y)_q$. Furthermore, we have

$$\begin{aligned} & f_{j(y)} - f_{i(y)} \\ &= \frac{\lambda}{q} (j_{\pi(1)} j_{\pi(2)}^{(y)} - i_{\pi(1)} i_{\pi(2)}^{(y)}) \\ &+ \frac{\lambda}{q} (j_{\pi(3)} j_{\pi(2)}^{(y)} - i_{\pi(3)} i_{\pi(2)}^{(y)}) \\ &+ c_{\pi(2)} (j_{\pi(2)}^{(y)} - i_{\pi(2)}^{(y)}) + c_{\pi(1)} (j_{\pi(1)} - i_{\pi(1)}) \\ &= \frac{\lambda}{q} \underbrace{(j_{\pi(1)} - i_{\pi(1)})}_z y + r, \end{aligned} \quad (22)$$

where

$$\begin{aligned} r &= \frac{\lambda}{q} i_{\pi(3)} [(j_{\pi(2)} + y)_q - (i_{\pi(2)} + y)_q] \\ &+ c_{\pi(2)} [(j_{\pi(2)} + y)_q - (i_{\pi(2)} + y)_q] \\ &+ \frac{\lambda}{q} (j_{\pi(1)} j_{\pi(2)} - i_{\pi(1)} i_{\pi(2)}) \\ &+ c_{\pi(1)} (j_{\pi(1)} - i_{\pi(1)}). \end{aligned} \quad (23)$$

Since $0 \leq i_{\pi(2)}, j_{\pi(2)}, y \leq q-1$ and $j_{\pi(1)} \neq i_{\pi(1)}$, so r is independent of y and $z \neq 0$. Then, we process

$$\sum_{y=0}^{q-1} \xi_{\lambda}^{f_{j(y)} - f_{i(y)}} = \xi_{\lambda}^r \sum_{y=0}^{q-1} \xi_{\lambda}^{zy} = 0. \quad (24)$$

Combining Case 1 to Case 3, we complete the proof. $\square$

Next, we give two examples for $q = 2$ and $q \neq 2$ to illustrate Theorem 1.

Example 2. Considering $q = \lambda = 2$, $m = 3$, $k = 1$, $n = 1$, $\pi = (5, 3, 2, 1, 4)$ and $c_l = 0$ for $0 \leq l \leq 5$. According to Theorem 1, $f = x_5 x_3 + x_3 x_2 + x_2 x_1 + x_1 x_4$, $\mathbf{x}_I = (x_1)$ and $\mathbf{x}_J = (x_4)$, then relevant sequence set $\mathbf{F}$ is an $(4, 32, 4, 2, 16)$ -SZCZ sequence set given by (25). The PACF of $\mathbf{f}_0$ and PCCF of $\mathbf{f}_0$ and $\mathbf{f}_1$ are shown in Fig. 3 for time-shifts $-20 \leq \tau \leq 20$. Obviously, each column of $\mathbf{F}$ has two non-zero elements and ZCZ width is 4.

Example 3. Let $q = 3$, $\lambda = 6$, $m = 2$, $k = 1$, $n = 1$, $\pi = (4, 2, 1, 3)$, $c_l = 0$ for $0 \leq l \leq 3$ and $c_4 = 2$. According to Theorem 1, $f = 2(x_4 x_2 + x_2 x_1 + x_1 x_3) + 2x_4$, $\mathbf{x}_I = (x_1)$ and $\mathbf{x}_J = (x_3)$, then $\mathbf{F}$ is an $(9, 81, 7, 3, 27)$ -SZCZ sequence set

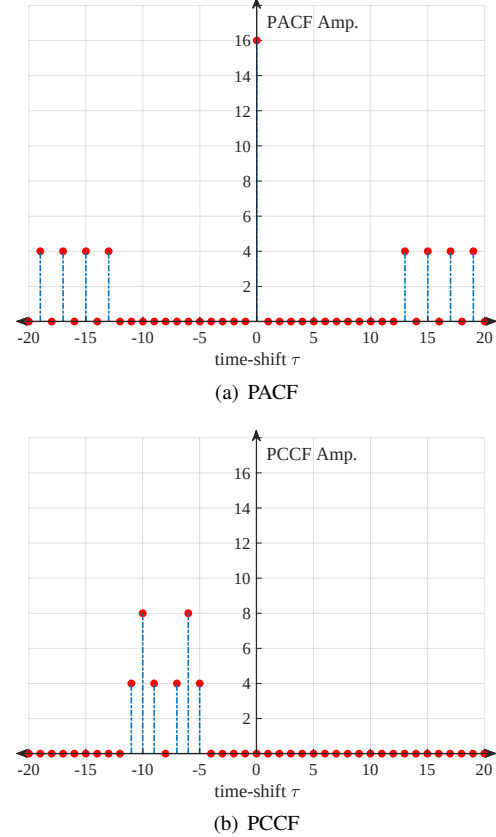


Fig. 3: PACF of $\mathbf{f}_0$ and PCCF of $\mathbf{f}_0$ and $\mathbf{f}_1$ in Example 2.

and the associated sequences of $f|_{\mathbf{x}_J=(0)}$ and $(f+2x_3)|_{\mathbf{x}_J=(2)}$ are shown in (26) and (27), respectively. The PACF of $\mathbf{f}_0$ and PCCF of $\mathbf{f}_0$ and $\mathbf{f}_5$ are shown in Fig. 4 for time-shifts $-20 \leq \tau \leq 20$. Obviously, the ZCZ width of $\mathbf{F}$ is $(3-1) \times 3^{2-1} + (3-2) \times 3^{2-2} = 7$.

Remark 10. In Theorem 1, we present a sequence set of length q^{m+n+k} and size q^{n+k} based on extended Boolean functions.. In contrast to iterative constructions that often require storing all generated sequences, our proposed method requires only the storage of key parameters, namely, the permutation π , the first-order coefficients $(c_1, c_2, \dots, c_{m+n+k})$, and the constant coefficient c_0 . A function-based construction then enables on-demand generation of specific sequence using these parameters and the corresponding logic circuits.

C. Comparison Between the Proposed Construction and Existing Results

In this subsection, we compare the proposed SZCZ sequence set with some known results to illustrate the innovativeness of the proposed construction.

TABLE II: Comparison of Different Method for Constructing SZCZ Sequence Set

Ref.	Largest ZCZ width	(Length, Size)	Based on	Max. ZCZ _{ratio}	Apply to
[29]	2^{m-1}	$(2^{n+m+1}, 2^n)$	$(2^m, 2^{m-1})$ -CZCP	$= 1/4$	SM
[33]	2^{m-n-k}	$(2^m, 2^n)$	$(2^k, 2^{m-n-k}, 2^{m-n-k})$ -CZCS	$= 1/4$	SM
[35]	2^{m-n-2}	$(2^m, 2^n)$	restricted generalized Boolean functions	$= 1/4$	SM
[35]	2^{m-n-1}	$(2^m, 2^n)$	restricted generalized Boolean functions	$= 1/2$	SM
[41]	$2^{m-k+v-1}$	$(2^{n-k+v+m}, 2^n)$	$(2^k, 2^v, 2^m, 2^{m-k+v-1})$ -E-CZCS	$= 1/2$	GSM
[38]	$2^{m-k+v-1} - 1$	$(2^{n-k+v+m} + 2L, 2^n)$	$(2^k, 2^v, 2^m, 2^{m-k+v-1} - 1)$ -S-ZCCS	$< 1/2$	GSM
Theorem 1	$(q-1)q^{\pi(2)-1} + (q-2)q^{\pi(3)-1}$	(q^{m+n+k}, q^{n+k})	restricted extended Boolean functions	$\geq 1/2$	SM/GSM

where L represents the number of multipath coefficients.

SZCZ sequence set obtained in [38] is less than $1/2$, while [41] is exactly $1/2$. In contrast, the proposed construction can exceed $1/2$.

5) *Comparison with [42] and [46]*: In our previous work [46], we initially proposed the concept of extended Boolean functions, which was subsequently refined in [42]. However, the focus of the current paper differs substantially from these earlier studies. While [46] investigated Z-complementary code sets (ZCCSs) and [42] examined sparse complete complementary codes (CCCs), the present work concentrates on SZCZ sequence sets. The sole commonality among these three studies lies in their shared mathematical framework.

For clarity, we have tabulated the pertinent parameters in Table II. Notably, the construction proposed in this paper exhibits a superior ZCZ ratio.

IV. SIMULATION RESULTS

This section presents a performance evaluation of the proposed SZCZ-based channel estimation scheme under quasi-static frequency-selective fading conditions with $(L+1)$ -multipath propagation, as characterized in (12).

A. Simulation of Channel Estimation for GSM

In the GSM simulation, we set the number of transmit antennas to be $N_t = 16$, the number of active antennas to be $N_a = 4$. At the same time, the training phase occupies 256 time slots, which means the length of the training sequence is $N = 256$, and the receiver has only one antenna. For comparison, we consider three training sequence sets in total.

- Firstly, we consider the training sequence set based on $(4, 2, 32, 7)$ -S-ZCCS proposed in [38]. Let $\mathcal{A} = \{\mathbf{A}^{(j)} : 0 \leq j \leq 3\}$ is an $(4, 2, 32, 7)$ -S-ZCCS, and $\mathbf{A}^{(j)} = \{\mathbf{a}_0^{(j)}, \mathbf{a}_1^{(j)}\}$. Furthermore, the S-ZCCS-based training sequence set is represented as follows:

$$\mathbf{A} = \left[\begin{array}{cccc|cccc|cccc} \mathbf{A}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}' & \mathbf{A}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}' \\ \mathbf{0} & \mathbf{A}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0}' & \mathbf{0} & \mathbf{A}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0}' \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_0 & \mathbf{0} & \mathbf{0}' & \mathbf{0} & \mathbf{0} & \mathbf{A}_1 & \mathbf{0} & \mathbf{0}' \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_0 & \mathbf{0}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_1 & \mathbf{0}' \end{array} \right], \quad (29)$$

where the sizes of $\mathbf{0}$ and $\mathbf{0}'$ are 4×32 and $4 \times L$, respectively, and

$$\mathbf{A}_i = \left[\begin{array}{c} \mathbf{a}_i^{(0)} \\ \mathbf{a}_i^{(1)} \\ \mathbf{a}_i^{(2)} \\ \mathbf{a}_i^{(3)} \end{array} \right]_{4 \times 32}. \quad (30)$$

In fact, the training sequence set generated based on $(4, 2, 32, 7)$ -S-ZCCS is a SZCZ sequence set with parameter $(16, 256 + 2L, 7, (4, 64))$.

- Secondly, we consider the training sequence set based on $(4, 2, 32, 8)$ -E-CZCS proposed in [41]. Let $\mathcal{B} = \{\mathbf{B}^{(j)} : 0 \leq j \leq 3\}$ is an $(4, 2, 32, 8)$ -E-CZCS, and $\mathbf{B}^{(j)} = \{\mathbf{b}_0^{(j)}, \mathbf{b}_1^{(j)}\}$. The E-CZCS-based training sequence set is represented as follows:

$$\mathbf{B} = \left[\begin{array}{cccc|cccc} \mathbf{B}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_1 \end{array} \right], \quad (31)$$

where the sizes of $\mathbf{0}$ is 4×32 , and

$$\mathbf{B}_i = \left[\begin{array}{c} \mathbf{b}_i^{(0)} \\ \mathbf{b}_i^{(1)} \\ \mathbf{b}_i^{(2)} \\ \mathbf{b}_i^{(3)} \end{array} \right]_{4 \times 32}. \quad (32)$$

It is not difficult to find that the training sequence set based on $(4, 2, 32, 8)$ -E-CZCS is an $(16, 256, 8, (4, 64))$ -SZCZ sequence set.

- Finally, we consider the proposed SZCZ sequence set in this paper. To design such a training sequence set, the parameters in Theorem 1 can be set to $q = \lambda = 4$, $n = 1$, $k = 1$ and $m = 2$, then we get an $(16, 256, 14, (4, 64))$ -SZCZ sequence set.

First, we conduct a symbol-level baseband simulation that excludes upsampling and pulse shaping. The MSE performances of the channel estimation based on different training sequence sets are shown in Figs. 6 and 7. Fig. 6 illustrates that the $(16, 256, 14, (4, 64))$ -SZCZ sequence set can achieve the minimum MSE, and compared to the training sequence sets based on $(4, 2, 32, 7)$ -S-ZCCS and $(4, 2, 32, 8)$ -E-CZCS, it has 7.23 dB and 2.71 dB gains, respectively when the number of multipaths is 14. The main source of the gain is that the proposed SZCZ sequence set has a larger ZCZ. In Fig. 7, we consider the MSE performance under different numbers of multipaths and $E_b/N_o = 15$ dB. Due to the ZCZ width of the SZCZ sequence set used being 14, it can achieve the minimum MSE when the number of multipath is less than 15. Since the ZCZ widths of the S-ZCCS-based and E-CZCS-based training sequence sets are 7 and 8, respectively, they cannot achieve optimal channel estimation when the number of multipaths exceeds 8 and 9. However, the SZCZ sequence set can still achieve optimal channel estimation until the multipath number increases to 15. At the same time, as the number of

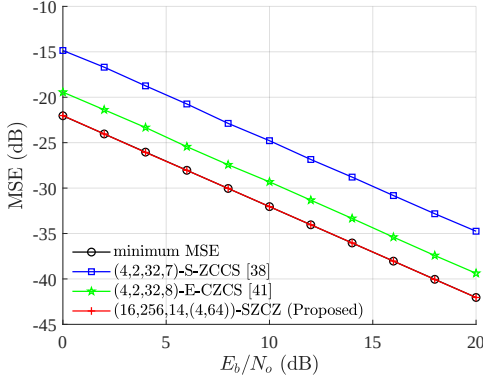


Fig. 6: No. of multipaths is 14 (GSM without upsampling and pulse shaping).

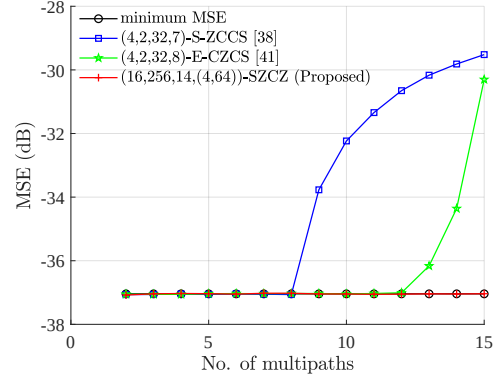


Fig. 7: E_b/N_o is 15 dB (GSM without upsampling and pulse shaping).

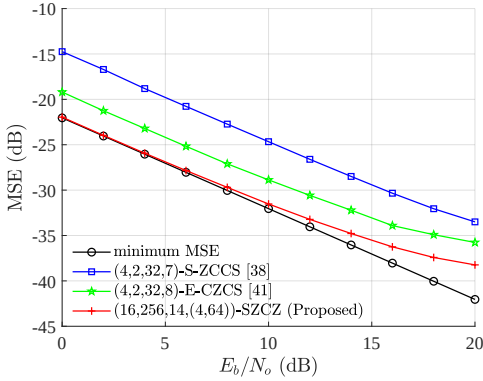


Fig. 8: No. of multipaths is 14 (GSM with upsampling and pulse shaping).

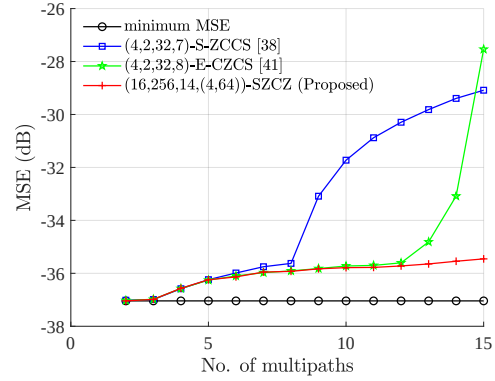


Fig. 9: E_b/N_o is 15 dB (GSM with upsampling and pulse shaping).

TABLE III: The MSE Losses Compared to The Minimum MSE (GSM)

No. of Multipaths	8	9	10	11	12	13	14	15
Training Seq.								
(16, 256, 14, (4, 64))-SZCZ	0 dB	0 dB	0 dB	0 dB	0 dB	0 dB	0 dB	0.01 dB
(4, 2, 32, 8)-E-CZCS-based	0 dB	0 dB	0 dB	0.01 dB	0.01 dB	0.90 dB	2.68 dB	6.77 dB
(4, 2, 32, 7)-S-ZCCS-based	0 dB	3.17 dB	4.75 dB	5.70 dB	6.36 dB	6.83 dB	7.18 dB	7.55 dB

multipaths gradually increases, the non-diagonal components of $\mathbf{X}\mathbf{X}^H$ (as show in (14)) gradually increase or become larger, thereby affecting the MSE of channel estimation, as described in (14). For a more intuitive understanding, the losses between different schemes and the minimum MSE are given in Table III.

Additionally, we also provide baseband simulations incorporating upsampling and pulse shaping, as shown in Figs. 8 and 9. A root-raised cosine FIR filter with a roll-off factor of 0.25 and an upsampling factor of 4 was employed in the simulation. It is clearly observed from both figures that all training sequence sets exhibit performance degradation. Although the proposed SZCZ sequence set does not achieve the minimum MSE at high E_b/N_o , it nevertheless exhibits the best performance among the three.

B. Simulation of Channel Estimation for SM

By adjusting n , Theorem 1 can also generate a training sequence set that satisfies the requirements of SM systems. In the SM simulation, we consider that the number of transmit antennas to be $N_t = 4$, the number of active antennas to be $N_a = 1$, the length of training sequences to be $N = 128$, and the receiver has only one antenna. For comparison, we consider for training sequence sets in total.

- Firstly, the (16,8)-CZCP-based training sequence set proposed in [29], which is generated as follows:

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{a}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{a}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{a}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{a}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{a}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{a}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{a}_1 \end{bmatrix}, \quad (33)$$

where

$$\begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \end{bmatrix} = \begin{bmatrix} + + + - + + - + + + - - - + - \\ + + + - + + - + - - - + + + - + \end{bmatrix} \quad (34)$$

is the (16, 8)-CZCP generated from [29, Construction 1], the size of $\mathbf{0}$ is 1×16 . Easy to verify, $\mathbf{A}$ is an (4, 128, 8, (1, 64))-SZCZ sequence set.

- Secondly, the (4, 8, 8)-CZCS-based training sequence set proposed in [33], which is generated as follows:

$$\mathbf{B} = \begin{bmatrix} \mathbf{b}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{b}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{b}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{b}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{b}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{b}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{b}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{b}_1 \\ \mathbf{b}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{b}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{b}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{b}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{b}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{b}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{b}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{b}_3 \end{bmatrix}, \quad (35)$$

where

$$\begin{bmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix} = \begin{bmatrix} + & + & + & - & + & + & - & + \\ + & + & + & - & - & - & + & - \\ - & - & - & + & - & - & + & - \\ + & + & + & - & - & - & + & - \end{bmatrix} \quad (36)$$

is an (4, 8, 8)-CZCS and the size of $\mathbf{0}$ is 1×8 . Actually, the sequence set $\mathbf{B}$ also is SZCZ sequence sets with a ZCZ width of 8.

- Finally, we consider the proposed SZCZ sequence set in this paper. To design such a training sequence set, the parameters in Theorem 1 can be set to $q = \lambda = 2$, $n = 0$, $m = 5$, and $k = 2$, then we get an (4, 128, 16, (1, 32))-SZCZ sequence set.

The MSE performances of the channel estimation based on different training sequence sets are shown in Figs. 10 and 11. Fig. 10 shows the variation of MSE with the number of multipaths when $E_b/N_0 = 15$ dB. From the Fig. 10, we observe that the MSE performance of the proposed (4, 128, 16, (1, 32))-SZCZ sequence set can consistently achieve the minimum MSE when the number of multipath paths is less than 17. On the contrary, the training sequence sets based on CZCPs and CZCSs can also maintain the minimum MSE when the multipath number is low, but their MSE performance drops sharply when the multipath number starts from 8 and 16 respectively. For a more intuitive understanding, we compare the MSE loss of different schemes with the minimum MSE in Table V. Specifically, we found that as the number of multipath paths increases, the reason for MSE loss in training sequence sets based on CZCP and CZCP is that $\mathbf{X}$ (as shown in (12)) is a row full rank matrix. Since there is an operation of inverting $\mathbf{X}\mathbf{X}^H$ in the LS estimator, if it is not row full rank, a large error will occur. In Table IV, we list the rank of matrix $\mathbf{X}$ under different training sequence sets. The comparison also illustrates the drawbacks of CZCP-based and CZCS-based training sequence sets, that is, the number of multipaths cannot exceed the length of the base sequence. Fig. 11 provides MSE performances with varying E_b/N_0 when the number of multipaths is $L + 1 = 17$. Obviously, since the number of multipaths exceeds the length of the initial CZCPs and CZCSs, the MSE performance of the training sequence set constructed based on the two is very poor. On the contrary, the proposed SZCZ sequence set can always achieve the minimum MSE.

Similar to the previous subsection, Figs. 12 and 13 present channel estimation simulations incorporating upsampling and

pulse shaping. Compared to the case without upsampling and pulse shaping, all sequence sets exhibit degraded MSE performance. Nonetheless, the proposed sequence set maintains its leading performance.

V. CONCLUSION

In this paper, we have redefined the SZCZ sequence set as a special case of the original definition in [35]. In the new definition, we allow multiple non-zero elements per column to accommodate the number of activated antennas in GSM systems. This allows us to introduce a general construction of $(q^{n+k}, q^{m+n+k}, (q-1)q^{\pi(2)-1} + (q-2)q^{\pi(3)-1}, (q^n, q^{m+n}))$ -SZCZ sequence sets with large ZCZ width based on restricted extended Boolean functions, as shown in Theorem 1. Specifically, when $q = 2$, similar ZCZ sequence sets have been reported in [35, Theorem 2], but the structure in this paper is different, and when $q > 2$, the SZCZ sequence set proposed in this paper has a larger ZCZ width. Finally, we have applied the proposed SZCZ sequence set for optimal channel estimation over frequency-selective fading channels in GSM/SM systems. Simulation results show that compared to known training schemes, the proposed SZCZ sequence set can achieve the minimum MSE in GSM/SM systems.

APPENDIX A: PROOF OF THEOREM 1

Proof. In the first part, we consider the periodic auto-correlation of the sequence in $\mathbf{F}$. Let the sequence generated by function

$$f_g^{(d)} = \underbrace{\left(f + \frac{\lambda}{q} \sum_{\ell=1}^n d_\ell x_{\pi(m+\ell)} \right)}_{f^{(d)}}|_{\mathbf{x}_J=\mathbf{g}} \quad (37)$$

be $\mathbf{f}_g^{(d)}$, where $g = \sum_{l=1}^k g_l q^{l-1}$. Since $\frac{\lambda}{q} \sum_{\ell=1}^n d_\ell x_{\pi(m+\ell)}$ can be absorbed by the first-order term of f , so we only need to consider the auto-correlation function of $\mathbf{f}_g^{(0)}$. For simplicity, $\mathbf{f}_g^{(0)}$ is denoted as $\mathbf{f}_g$, where

$$f_{g,i} = \begin{cases} \xi_\lambda^{f_i}, & i_{\pi(m+n+\alpha)} = g_\alpha \text{ for } 1 \leq \alpha \leq k; \\ 0, & \text{otherwise.} \end{cases} \quad (38)$$

Let $v = m + n + k$, $N = q^v$, $Z = (q-1)q^{\pi(2)-1} + (q-1)q^{\pi(3)-1}$ and $I_g = \{0 \leq i < N : i_{\pi(m+n+\alpha)} = g_\alpha, 1 \leq \alpha \leq k\}$. Given $0 < \tau \leq Z$, we would like to show that

$$R_{\mathbf{f}_g}(\tau) = \sum_{i=0}^{N-1} f_{g,i} f_{g,j}^* = \sum_{i,j \in I_g} \xi_\lambda^{f_i - f_j} = 0, \quad (39)$$

where $j = (i + \tau)_N$. Let $(i_1, i_2, \dots, i_v)$ and $(j_1, j_2, \dots, j_v)$ be q -ary representations of i and j , respectively. Please note that $\pi(1) = m + n + k = v$. Now, we discuss the following cases.

Case 1: $i_{\pi(1)} = j_{\pi(1)}$. Suppose t is the smallest integer satisfying $i_{\pi(t)} \neq j_{\pi(t)}$. Since $i, j \in I_g$, so we have $i_{\pi(m+n+\alpha)} = j_{\pi(m+n+\alpha)} = g_\alpha$ for $1 \leq \alpha \leq k$, which means that $t \leq m + n$. For $1 \leq y \leq q-1$, let $i^{(y)}$ and $j^{(y)}$ be two integers which are distinct from i and j in only one position $\pi(t-1)$. Namely, $i_{\pi(t-1)}^{(y)} = (i_{\pi(t-1)} + y)_q$

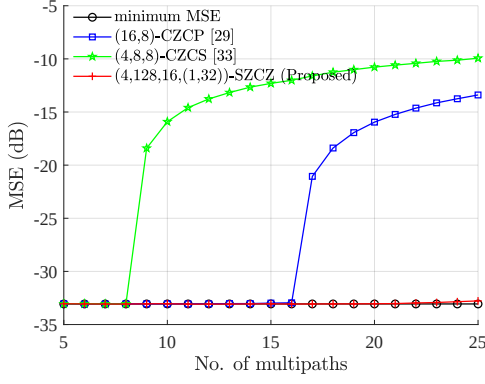


Fig. 10: E_b/N_o is 15 dB (SM without up-sampling and pulse shaping).

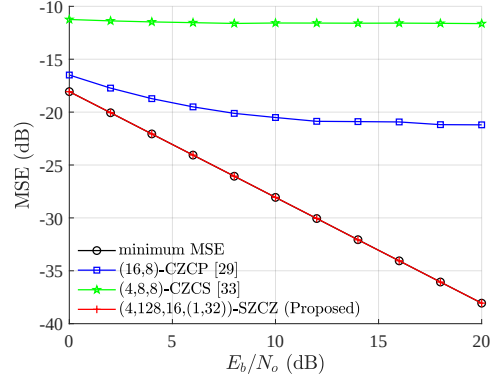


Fig. 11: No. of multipaths is 17 (SM without up-sampling and pulse shaping).

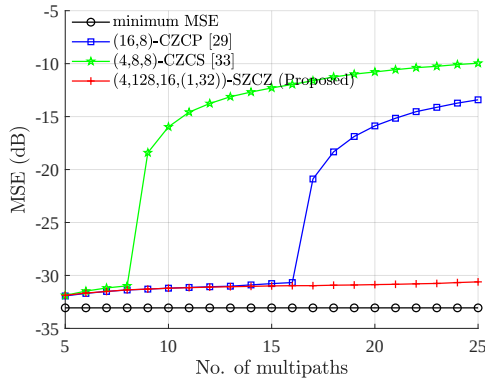


Fig. 12: E_b/N_o is 15 dB (SM with up-sampling and pulse shaping).

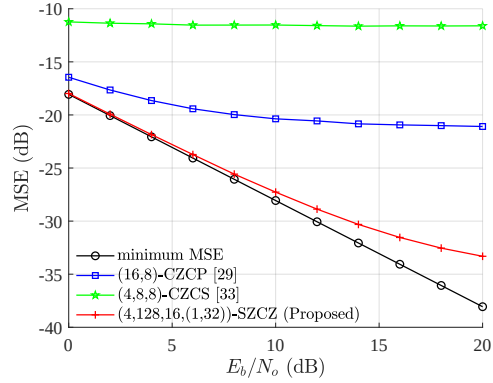


Fig. 13: No. of multipaths is 17 (SM with up-sampling and pulse shaping).

TABLE IV: Rank of $\mathbf{X}$ for different training sequence sets of length 128 under $N_t = 4$ TAs

Number of multipaths	6	8	10	12	14	16	18	20	22	24
Number of rows in $\mathbf{X}$	24	32	40	48	56	64	72	80	88	96
CZCP-Based	24	32	40	48	56	64	66	68	70	72
CZCS-Based	24	32	34	36	38	40	42	44	46	48
SZCZ sequence set	24	32	40	48	56	64	72	80	88	96

TABLE V: The MSE Losses Compared to The Minimum MSE (SM)

No. of Multipaths	6	8	10	12	14	16	18	20	22	24
Training Seq.										
(4, 128, 16, (1, 32))-SZCZ	0 dB	0 dB	0 dB	0 dB	0 dB	0 dB	0 dB	0.03 dB	0.07 dB	0.22 dB
(16, 8)-CZCP-based	0 dB	0 dB	0 dB	0.03 dB	0.04 dB	0.09 dB	14.76 dB	17.07 dB	18.45 dB	19.29 dB
(4, 8, 8)-CZCS-based	0 dB	0.02 dB	17.15 dB	19.290 dB	20.38 dB	21.06 dB	21.81 dB	22.27 dB	22.66 dB	22.94 dB

and $j_{\pi(t-1)}^{(y)} = (j_{\pi(t-1)} + y)_q$. Since the value of position $\{\pi(m+n+\alpha) : 1 \leq \alpha \leq k\}$ has not been changed, $i^{(y)}, j^{(y)} \in I_g$. Correspondingly, i and j are denoted as $i^{(0)}$ and $j^{(0)}$ respectively. Due to $j_{\pi(l)}^{(y)} = i_{\pi(l)}^{(y)}$ for $1 \leq l \leq t-1$, we have

$$f_{j^{(y)}} - f_{i^{(y)}} = \frac{\lambda}{q} \sum_{l=t-1}^{v-1} (j_{\pi(l)}^{(y)} j_{\pi(l+1)}^{(y)} - i_{\pi(l)}^{(y)} i_{\pi(l+1)}^{(y)}) + \sum_{l=t}^v c_{\pi(l)} (j_{\pi(l)}^{(y)} - i_{\pi(l)}^{(y)}). \quad (40)$$

Moreover, we know that $j_{\pi(l)}^{(y)} = j_{\pi(l)}$, $i_{\pi(l)}^{(y)} = i_{\pi(l)}$ for $l \neq$

$t-1$. Therefore, we have

$$f_{j^{(y)}} - f_{i^{(y)}} = \frac{\lambda}{q} (i_{\pi(t-1)} + y) \underbrace{(j_{\pi(t)} - i_{\pi(t)})}_z + \frac{\lambda}{q} \sum_{l=t}^{v-1} (j_{\pi(l)} j_{\pi(l+1)} - i_{\pi(l)} i_{\pi(l+1)}) + \sum_{l=t}^v c_{\pi(l)} (j_{\pi(l)} - i_{\pi(l)}) \pmod{\lambda}. \quad (41)$$

Since $j_{\pi(t)} \neq i_{\pi(t)}$ and $0 \leq j_{\pi(t)}, i_{\pi(t)} \leq q-1$, so $z =$

$j_{\pi(t)} - i_{\pi(t)} \not\equiv 0 \pmod{q}$. This means

$$\sum_{y=0}^{q-1} \xi_{\lambda}^{f_{j(y)} - f_{i(y)}} = \xi_{\lambda}^r \xi_q^{z i_{\pi(1)}} \sum_{y=0}^{q-1} \xi_q^{yz} = 0, \quad (42)$$

where r is the sum of the second and third terms of $f_{j(y)} - f_{i(y)}$.

Case 2: $i_{\pi(1)} \neq j_{\pi(1)}$. In this case, we have $(j_{\pi(1)} - i_{\pi(1)})_q = 1$. Suppose not, namely $1 < (j_{\pi(1)} - i_{\pi(1)})_q < q$. If $j_{\pi(1)} > i_{\pi(1)}$, we have

$$\begin{aligned} \tau &= j - i \\ &= (j_{\pi(1)} - i_{\pi(1)})q^{\pi(1)-1} + \sum_{s=1}^{v-1} (j_s - i_s)q^{s-1} \\ &\geq 2q^{v-1} - (q-1) \sum_{s=1}^{v-1} q^{s-1} \\ &= q^{v-1} + 1. \end{aligned} \quad (43)$$

If $j_{\pi(1)} < i_{\pi(1)}$, we have $j_{\pi(1)} - i_{\pi(1)} < 0$, so $\tau = j - i + q^v$. Similarly, we can prove that $\tau \geq q^{v-1} + 1$. According to the permutation constraint in Theorem 1, we have $\pi(2) \leq m$ and $\pi(3) \leq m$. Due to $\tau \leq Z$ and $v = m + n + k$, this indicates that $\tau \geq q^{v-1} + 1$ and $Z = (q-1)q^{\pi(2)-1} + (q-2)q^{\pi(3)-1} \leq (q-1)q^{m-1} + (q-2)q^{m-2} = q^m - 2q^{m-2}$ are contradictory to each other. Therefore, we have $(j_{\pi(1)} - i_{\pi(1)})_q = 1$. Assuming $i_{\pi(2)} \neq j_{\pi(2)}$, let $i^{(y)}$ and $j^{(y)}$ be two integers which are distinct from i and j in only one position $\pi(1)$, i.e., $i_{\pi(1)}^{(y)} = (i_{\pi(1)} + y)_q$ and $j_{\pi(1)}^{(y)} = (j_{\pi(1)} + y)_q$, where $1 \leq y \leq q-1$. Similar to Case 1, we have $i^{(y)}, j^{(y)} \in I_g$. Therefore, we have

$$\begin{aligned} f_{j(y)} - f_{i(y)} &= \frac{\lambda}{q} \sum_{l=1}^{v-1} (j_{\pi(l)}^{(y)} j_{\pi(l+1)}^{(y)} - i_{\pi(l)}^{(y)} i_{\pi(l+1)}^{(y)}) + \sum_{l=1}^v c_{\pi(l)} (j_{\pi(l)}^{(y)} - i_{\pi(l)}^{(y)}) \\ &= \frac{\lambda}{q} (j_{\pi(1)}^{(y)} j_{\pi(2)}^{(y)} - i_{\pi(1)}^{(y)} i_{\pi(2)}^{(y)}) + \underbrace{c_{\pi(1)} (j_{\pi(1)}^{(y)} - i_{\pi(1)}^{(y)})}_{r_1} \\ &\quad + \underbrace{\sum_{l=2}^{v-1} (j_{\pi(l)} j_{\pi(l+1)} - i_{\pi(l)} i_{\pi(l+1)}) + \sum_{l=2}^v c_{\pi(l)} (j_{\pi(l)} - i_{\pi(l)})}_{r_2} \\ &= \frac{\lambda}{q} ((i_{\pi(1)} + y) \underbrace{(j_{\pi(2)} - i_{\pi(2)}) + j_{\pi(2)}}_z + r_1 + r_2). \end{aligned} \quad (44)$$

Since $c_{\pi(1)} \in \{0, \lambda/q\}$, so

$$(r_1)_q = \begin{cases} 0, & c_{\pi(1)} = 0; \\ \frac{\lambda}{q} (j_{\pi(1)} - i_{\pi(1)}), & c_{\pi(1)} = \frac{\lambda}{q}. \end{cases} \quad (45)$$

This indicates that r_1 is unrelated to y , so we can obtain

$$f_{j(y)} - f_{i(y)} = \frac{\lambda y}{q} z + \underbrace{\frac{\lambda}{q} (i_{\pi(1)} z + j_{\pi(2)}) + r_1 + r_2}_r, \quad (46)$$

where r is independent of y , hence

$$\sum_{y=0}^{q-1} \xi_{\lambda}^{f_{j(y)} - f_{i(y)}} = \xi_{\lambda}^r \sum_{y=0}^{q-1} \xi_q^{yz} = 0. \quad (47)$$

Case 3: $i_{\pi(1)} \neq j_{\pi(1)}$ and $i_{\pi(2)} = j_{\pi(2)}$. In this case, we can obtain $i_{\pi(3)} - j_{\pi(3)} \geq 2$. Suppose not, we have $1 - q \leq i_{\pi(3)} - j_{\pi(3)} \leq 1$. If $j_{\pi(1)} > i_{\pi(1)}$, we have

$$\begin{aligned} \tau &= j - i \\ &= (j_{\pi(1)} - i_{\pi(1)})q^{\pi(1)-1} + \sum_{\substack{s=1, s \neq \pi(2) \\ s \neq \pi(3)}}^{v-1} (j_s - i_s)q^{s-1} \\ &\quad + (j_{\pi(2)} - i_{\pi(2)})q^{\pi(2)-1} + (j_{\pi(3)} - i_{\pi(3)})q^{\pi(3)-1} \\ &\geq q^{v-1} - (q-1) \sum_{s=1}^{v-1} q^{s-1} + (q-1)q^{\pi(2)-1} \\ &\quad + (q-1)q^{\pi(3)-1} - q^{\pi(3)-1} \\ &= (q-1)q^{\pi(2)-1} + (q-2)q^{\pi(3)-1} + 1 \\ &= Z + 1. \end{aligned} \quad (48)$$

If $j_{\pi(1)} < i_{\pi(1)}$, so $\tau = j - i + q^v$. Similarly, we can prove that $\tau \geq Z + 1$. This contradicts $\tau \leq Z$, so we have $i_{\pi(3)} - j_{\pi(3)} \geq 2$. Let $i^{(y)}$ and $j^{(y)}$ be two integers which are distinct from i and j in only one position $\pi(2)$, i.e., $i_{\pi(2)}^{(y)} = (i_{\pi(2)} + y)_q$ and $j_{\pi(2)}^{(y)} = (j_{\pi(2)} + y)_q$, where $1 \leq y \leq q-1$, and let $(i^{(0)}, j^{(0)}) = (i, j)$. Therefore, we have

$$\begin{aligned} f_{j(y)} - f_{i(y)} &= \frac{\lambda}{q} (j_{\pi(1)} j_{\pi(2)}^{(y)} - i_{\pi(1)} i_{\pi(2)}^{(y)} \\ &\quad + j_{\pi(2)}^{(y)} j_{\pi(3)} - i_{\pi(2)}^{(y)} i_{\pi(3)}) \\ &\quad + c_{\pi(2)} (j_{\pi(2)}^{(y)} - i_{\pi(2)}^{(y)}) \\ &\quad + \sum_{l=3}^{v-1} (j_{\pi(l)} j_{\pi(l+1)} - i_{\pi(l)} i_{\pi(l+1)}) \\ &\quad + \sum_{l=1, l \neq 2}^v c_{\pi(l)} (j_{\pi(l)} - i_{\pi(l)}) \\ &= \frac{\lambda y}{q} (j_{\pi(1)} - i_{\pi(1)} \\ &\quad + j_{\pi(3)} - i_{\pi(3)}) + r \pmod{q}, \end{aligned} \quad (49)$$

where

$$\begin{aligned} r &= \frac{\lambda}{q} \sum_{l=3}^{v-1} (j_{\pi(l)} j_{\pi(l+1)} - i_{\pi(l)} i_{\pi(l+1)}) \\ &\quad + \sum_{l=1}^v c_{\pi(l)} (j_{\pi(l)} - i_{\pi(l)}) \\ &\quad + \frac{\lambda}{q} i_{\pi(2)} (j_{\pi(1)} - i_{\pi(1)} + j_{\pi(3)} - i_{\pi(3)}). \end{aligned} \quad (50)$$

Owing to $j_{\pi(3)} - i_{\pi(3)} \leq -2$ and $(j_{\pi(1)} - i_{\pi(1)})_q = 1$ implying that $z = j_{\pi(1)} - i_{\pi(1)} + j_{\pi(3)} - i_{\pi(3)} \neq 0$, we possess

$$\sum_{y=0}^{q-1} \xi_{\lambda}^{f_{j(y)} - f_{i(y)}} = \xi_{\lambda}^r \sum_{y=0}^{q-1} \xi_q^{yz} = 0. \quad (51)$$

Combining Case 1 to Case 3, the sequence $\mathbf{f}_g$ has a periodic zero auto-correlation zone with a ZCZ width of $(q-1)q^{\pi(2)-1} + (q-2)q^{\pi(3)-1}$.

In the second part, we would like to demonstrate that

$$R_{\mathbf{f}_g^{(d)}, \mathbf{f}_h^{(e)}}(\tau) = \sum_{i=0}^{L-1} f_{g,i}^{(d)} (f_{h,j}^{(e)})^* = \sum_{i \in I_g, j \in I_h} \xi_{\lambda}^{f_i^{(d)} - f_j^{(e)}} = 0 \quad (52)$$

for $|\tau| \leq Z$, where $d = e$ and $g = h$ do not hold at the same time, $j = (i + \tau)_N$. Below, we will discuss two cases.

Case 1: $g = h$ and $d \neq e$. If $\tau \neq 0$, similar to Case 1-3 in the first part, we can find $j^{(y)}, i^{(y)} \in I_g$ so that

$$\sum_{y=0}^{q-1} \xi_{\lambda}^{f_{j^{(y)}}^{(d)} - f_{i^{(y)}}^{(e)}} = 0. \quad (53)$$

If $\tau = 0$, since the linear coefficients of $f^{(d)}$ and $f^{(e)}$ are different, $\mathbf{f}_g^{(d)}$ and $\mathbf{f}_g^{(e)}$ are mutually orthogonal, so the conclusion is obvious.

Case 2: $g \neq h$. If $\tau = 0$, since $f = \sum_g f|_{x_j=g}$, the indices of the non-zero elements in $\mathbf{f}_g^{(d)}$ and $\mathbf{f}_h^{(e)}$ do not intersect, that is, $R_{\mathbf{f}_g^{(d)}, \mathbf{f}_h^{(e)}}(\tau) = 0$. For $\tau \neq 0$, if $i_{\pi(1)} \neq j_{\pi(1)}$, the proof process is similar to Case 2 and Case 3 in the first part, which we will omit here. If $i_{\pi(1)} = j_{\pi(1)}$, suppose t is the smallest integer satisfying $i_{\pi(t)} \neq j_{\pi(t)}$. Since $\{\pi(m+i) : 2 \leq i \leq n+k\} = \{m+i-1 : 2 \leq i \leq n+k\}$ when $n+k \geq 2$, then $\pi(2) \leq m$, this means that the binary representation of τ is $(\tau_1, \tau_2, \dots, \tau_m, 0, 0, \dots, 0)$ and there is at least one $1 \leq w \leq m$ that makes $\tau_{\pi(w)} \neq 0$. Furthermore, we have $t \leq m+1$. By following a similar argument as given by Case 1 in the first part, we can also find $j^{(y)} \in I_h$ and $i^{(y)} \in I_g$ satisfying

$$\sum_{y=0}^{q-1} \xi_{\lambda}^{f_{j^{(y)}}^{(d)} - f_{i^{(y)}}^{(e)}} = 0, \quad (54)$$

which complete the proof. $\square$

REFERENCES

- [1] P. Z. Fan, N. Suehiro, N. Kuroyanagi, and X. M. Deng, "Class of binary sequences with zero correlation zone," *Electronics Letters*, vol. 35, no. 10, pp. 777–779, 1999.
- [2] L. Welch, "Lower bounds on the maximum cross correlation of signals (corresp.)," *IEEE Transactions on Information Theory*, vol. 20, no. 3, pp. 397–399, 1974.
- [3] X. H. Tang, P. Z. Fan, and S. Matsufuji, "Lower bounds on correlation of spreading sequence set with low or zero correlation zone," *Electronics Letters*, vol. 36, no. 6, pp. 551–552, 2000.
- [4] S. Matsufuji, N. Suehiro, N. Kuroyanagi, and P. Fan, "Spreading sequence sets for approximately synchronized cdma system with no co-channel interference and high data capacity," in *Proc. 2nd Int. Symp. Wireless Personal Multimedia Communications*, 1999, pp. 333–339.
- [5] H. Torii, M. Nakamura, and N. Suehiro, "A new class of zero-correlation zone sequences," *IEEE Transactions on Information Theory*, vol. 50, no. 3, pp. 559–565, 2004.
- [6] Z. Zhou, X. Tang, and G. Gong, "A new class of sequences with zero or low correlation zone based on interleaving technique," *IEEE Transactions on Information Theory*, vol. 54, no. 9, pp. 4267–4273, 2008.
- [7] X. Deng and P. Fan, "Spreading sequence sets with zero correlation zone," *Electronics Letters*, vol. 36, no. 11, p. 1, 2000.
- [8] X. Tang and W. H. Mow, "A new systematic construction of zero correlation zone sequences based on interleaved perfect sequences," *IEEE Transactions on Information Theory*, vol. 54, no. 12, pp. 5729–5734, 2008.
- [9] C.-Y. Chen and S.-W. Wu, "Golay complementary sequence sets with large zero correlation zones," *IEEE Transactions on Communications*, vol. 66, no. 11, pp. 5197–5204, 2018.
- [10] C.-Y. Pai, Y.-J. Lin, and C.-Y. Chen, "Optimal and almost-optimal Golay-ZCZ sequence sets with bounded PAPRs," *IEEE Transactions on Communications*, vol. 71, no. 2, pp. 728–740, 2023.
- [11] D. Zhang and T. Hellesteth, "Sequences with good correlations based on circular florentine arrays," *IEEE Transactions on Information Theory*, vol. 68, no. 5, pp. 3381–3388, 2022.
- [12] Z. Liu, Y. L. Guan, and U. Parampalli, "A new construction of zero correlation zone sequences from generalized reed-muller codes," in *2014 IEEE Information Theory Workshop (ITW 2014)*. IEEE, 2014, pp. 591–595.
- [13] M. Addad, A. Djebbari, and I. Dayoub, "Performance of ZCZ codes in QS-DS-CDMA communication systems," *Signal Processing*, vol. 164, pp. 146–150, 2019.
- [14] W. Zhang, F. Zeng, X. Long, and M. Xie, "Improved mutually orthogonal ZCZ polyphase sequence sets and their applications in OFDM frequency synchronization," in *2010 6th International Conference on Wireless Communications Networking and Mobile Computing*. IEEE, 2010, pp. 1–5.
- [15] R. Zhang, X. Cheng, M. Ma, and B. Jiao, "Interference-avoidance pilot design using ZCZ sequences for multi-cell MIMO-OFDM systems," in *2012 IEEE Global Communications Conference (GLOBECOM)*. IEEE, 2012, pp. 5056–5061.
- [16] H. Haderer, R. Feger, C. Pfeffer, and A. Stelzer, "Millimeter-wave phase-coded CW MIMO radar using zero-and low-correlation-zone sequence sets," *IEEE Transactions on Microwave Theory and Techniques*, vol. 64, no. 12, pp. 4312–4323, 2017.
- [17] L. Xu and Q. Liang, "Zero correlation zone sequence pair sets for MIMO radar," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 48, no. 3, pp. 2100–2113, 2012.
- [18] S.-A. Yang and J. Wu, "Optimal binary training sequence design for multiple-antenna systems over dispersive fading channels," *IEEE Transactions on Vehicular Technology*, vol. 51, no. 5, pp. 1271–1276, 2002.
- [19] W. Yuan, P. Wang, and P. Fan, "Performance of multi-path MIMO channel estimation based on ZCZ training sequences," in *2005 IEEE International Symposium on Microwave, Antenna, Propagation and EMC Technologies for Wireless Communications*, vol. 2. IEEE, 2005, pp. 1542–1545.
- [20] S. Hu, Z. Liu, Y. L. Guan, C. Jin, Y. Huang, and J.-M. Wu, "Training sequence design for efficient channel estimation in MIMO-FBMC systems," *IEEE Access*, vol. 5, pp. 4747–4758, 2017.
- [21] K. Zheng, L. Zhao, J. Mei, B. Shao, W. Xiang, and L. Hanzo, "Survey of large-scale MIMO systems," *IEEE Communications Surveys & Tutorials*, vol. 17, no. 3, pp. 1738–1760, 2015.
- [22] R. Y. Mesleh, H. Haas, S. Sinanovic, C. W. Ahn, and S. Yun, "Spatial modulation," *IEEE Transactions on Vehicular Technology*, vol. 57, no. 4, pp. 2228–2241, 2008.
- [23] P. Yang, Y. Xiao, Y. L. Guan, K. Hari, A. Chockalingam, S. Sugiura, H. Haas, M. Di Renzo, C. Masouros, Z. Liu *et al.*, "Single-carrier SM-MIMO: A promising design for broadband large-scale antenna systems," *IEEE Communications Surveys & Tutorials*, vol. 18, no. 3, pp. 1687–1716, 2016.
- [24] M. Wen, B. Zheng, K. J. Kim, M. Di Renzo, T. A. Tsiftsis, K.-C. Chen, and N. Al-Dhahir, "A survey on spatial modulation in emerging wireless systems: Research progresses and applications," *IEEE Journal on Selected Areas in Communications*, vol. 37, no. 9, pp. 1949–1972, 2019.
- [25] C. Fragouli, N. Al-Dhahir, and W. Turin, "Training-based channel estimation for multiple-antenna broadband transmissions," *IEEE Transactions on Wireless Communications*, vol. 2, no. 2, pp. 384–391, 2003.
- [26] P. Fan and W. H. Mow, "On optimal training sequence design for multiple-antenna systems over dispersive fading channels and its extensions," *IEEE Transactions on Vehicular Technology*, vol. 53, no. 5, pp. 1623–1626, 2004.
- [27] S. Wang, Y. Li, M. Zhao, and J. Wang, "Energy-efficient and low-complexity uplink transceiver for massive spatial modulation MIMO," *IEEE Transactions on Vehicular Technology*, vol. 64, no. 10, pp. 4617–4632, 2015.
- [28] S. Wang, Y. Li, and J. Wang, "Multiuser detection in massive spatial modulation MIMO with low-resolution ADCs," *IEEE Transactions on Wireless Communications*, vol. 14, no. 4, pp. 2156–2168, 2015.
- [29] Z. Liu, P. Yang, Y. L. Guan, and P. Xiao, "Cross Z-complementary pairs for optimal training in spatial modulation over frequency selective channels," *IEEE Transactions on Signal Processing*, vol. 68, pp. 1529–1543, 2020.

- [30] A. R. Adhikary, Z. Zhou, Y. Yang, and P. Fan, "Constructions of cross Z-complementary pairs with new lengths," *IEEE Transactions on Signal Processing*, vol. 68, pp. 4700–4712, 2020.
- [31] H. Zhang, C. Fan, Y. Yang, and S. Mesnager, "New binary cross Z-complementary pairs with large CZC ratio," *IEEE Transactions on Information Theory*, vol. 69, no. 2, pp. 1328–1336, 2022.
- [32] Z.-M. Huang, C.-Y. Pai, and C.-Y. Chen, "Binary cross Z-complementary pairs with flexible lengths from Boolean functions," *IEEE Communications Letters*, vol. 25, no. 4, pp. 1057–1061, 2020.
- [33] —, "Cross Z-complementary sets for training design in spatial modulation," *IEEE Transactions on Communications*, vol. 70, no. 8, pp. 5030–5045, 2022.
- [34] Z.-M. Huang, C. Hu, Z. Liu, and C.-Y. Chen, "Cross Z-complementary sets with flexible lengths for optimal training design in spatial modulation," *IEEE Transactions on Communications*, 2024.
- [35] C.-Y. Pai, Z. Liu, and C.-Y. Chen, "Sparse zero correlation zone arrays for training design in spatial modulation systems," *arXiv preprint arXiv:2411.13878*, 2024.
- [36] A. Younis, N. Serafimovski, R. Mesleh, and H. Haas, "Generalised spatial modulation," in *2010 conference record of the forty fourth Asilomar conference on signals, systems and computers*. IEEE, 2010, pp. 1498–1502.
- [37] J. Wang, S. Jia, and J. Song, "Generalised spatial modulation system with multiple active transmit antennas and low complexity detection scheme," *IEEE Transactions on Wireless Communications*, vol. 11, no. 4, pp. 1605–1615, 2012.
- [38] Y. Zhou, Z. Zhou, Z. Liu, Y. Yang, P. Yang, and P. Fan, "Symmetrical Z-complementary code sets for optimal training in generalized spatial modulation," *Signal Processing*, vol. 208, p. 108990, 2023.
- [39] P. Fan, W. Yuan, and Y. Tu, "Z-complementary binary sequences," *IEEE Signal Processing Letters*, vol. 14, no. 8, pp. 509–512, 2007.
- [40] Z.-M. Huang, C. Hu, Z. Liu, and C.-Y. Chen, "On the mates of cross Z-complementary pairs for training sequence design in generalized spatial modulation," *IEEE Access*, vol. 11, pp. 145 231–145 237, 2023.
- [41] Z.-M. Huang, C.-Y. Pai, Z. Liu, and C.-Y. Chen, "Enhanced cross Z-complementary set and its application in generalized spatial modulation," *IEEE Open Journal of the Communications Society*, vol. 5, pp. 4674–4690, 2024.
- [42] B. Shen, Y. Yang, Z. Zhou, and S. Mesnager, "Constructions of spectrally null constrained complete complementary codes via the graph of extended boolean functions," *IEEE Transactions on Information Theory*, vol. 69, no. 9, pp. 6028–6039, 2023.
- [43] P. Spasojevic and C. N. Georgiades, "Complementary sequences for ISI channel estimation," *IEEE Transactions on Information Theory*, vol. 47, no. 3, pp. 1145–1152, 2001.
- [44] S. Wang and A. Abdi, "MIMO ISI channel estimation using uncorrelated Golay complementary sets of polyphase sequences," *IEEE Transactions on Vehicular Technology*, vol. 56, no. 5, pp. 3024–3039, 2007.
- [45] K. G. Paterson, "Generalized Reed-Muller codes and power control in OFDM modulation," *IEEE Transactions on Information Theory*, vol. 46, no. 1, pp. 104–120, 2000.
- [46] B. Shen, H. Meng, Y. Yang, and Z. Zhou, "New constructions of Z-complementary code sets and mutually orthogonal complementary sequence sets," *Designs, Codes and Cryptography*, vol. 91, no. 2, pp. 353–371, 2023.