

Universal Costas Matrices: Towards a General Framework for Costas Array Construction

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Abstract—Costas arrays are a special type of permutation matrices with ideal autocorrelation and low cross-correlation properties, making them valuable for radar, wireless communication, and integrated sensing and communication applications. This paper presents a novel unified framework for analyzing and discovering new Costas arrays. We introduce Universal Costas Matrices (UCMs) and Universal Costas Frequency Matrices (UCFMs) and investigate their structural characteristics. A framework integrating UCMs and UCFMs is proposed to pave the way for future artificial intelligence-assisted Costas array discovery. Leveraging the structural properties of UCMs and UCFMs, a reconstruction-based search method is developed to generate UCMs from UCFMs. Numerical results demonstrate that the proposed approach significantly accelerates the search process and enhances structural insight into Costas array generation.

I. INTRODUCTION

Costas arrays are permutation matrices in which the vectors connecting every pair of 1s are distinct. They were originally proposed to be used as frequency-hopping codes in radar/sonar systems [1]. They exhibit ideal autocorrelation properties, which are desirable for low probability of intercept radars to mitigate Doppler shift effects [2], [3]. These arrays are widely applied in modern radar waveform design research [4]–[6].

On the other hand, Costas arrays have been employed in multiple access spread spectrum wireless communication systems such as orthogonal frequency division multiplexing systems [7], [8]. Additionally, recent studies show that modified versions of Costas arrays can have low cross-correlation and optimal frequency hopping properties, as in orthogonal pattern division multiple access systems [3], [9], [10]. These features make Costas arrays attractive candidates for integrated sensing and communication (ISAC) applications [6], with their use in multi-user systems offering a potential solution to the sensing performance improvement challenge discussed in [11].

While the application areas of Costas arrays are broad, their generation techniques are limited. Conventionally, they can be generated with algebraic methods using Galois fields [12]–[14] and their structural properties are extensively studied [15]–[18]. Researchers also employed exhaustive search techniques for the enumeration of Costas arrays [19]–[23]. To the best of our knowledge, all Costas arrays have been enumerated only up to order 29 [21]. All known Costas arrays, whether obtained algebraically or through exhaustive search, are collected in a publicly available database [24]. Such a database could potentially be employed to enable artificial intelligence (AI)

based Costas array generation similar to those used in matrix completion problems [25], [26].

Despite these efforts, no general construction method for Costas arrays of any arbitrary order has been discovered to date. On the one hand, algebraic construction methods remain largely restricted to prime or prime-power orders. On the other hand, exhaustive search rapidly becomes infeasible due to its factorial growth in complexity. These ongoing challenges have motivated us to develop a unified analytical framework that enables the application of AI-assisted search and generative modeling approaches toward generalized Costas array generation. Moreover, such a unified representation could leverage existing datasets of known arrays to facilitate the prospective AI-based generation of new Costas arrays.

In this paper, a unified approach to analyze and search for new Costas arrays is proposed. We define Universal Costas Matrices (UCMs), where each row contains a distinct Costas array of order n in canonical form, and Universal Costas Frequency Matrices (UCFMs), which are $n \times n$ matrices that compactly represent the frequency of each element in the corresponding UCM. The contributions of the paper can be summarized as follows:

- UCMs and UCFMs are introduced as novel matrix representations for the analysis and search of Costas arrays.
- A theoretical analysis of the structural properties of UCMs and UCFMs is presented.
- An AI-enabled framework is proposed, where UCMs and UCFMs can be integrated into future generative AI-based search methods for Costas array discovery.
- A novel search method, i.e., the reconstruction of UCMs from UCFMs, which exploits the structural properties of UCMs and UCFMs, is proposed.

Furthermore, UCFM heatmaps are visualized to illustrate their structural symmetry, and the proposed reconstruction method is quantitatively evaluated. The results demonstrate that the proposed approach yields significant runtime improvement when compared with a benchmark method, while providing a foundation for future AI-based Costas array generation.

II. UNIVERSAL COSTAS MATRICES

In this section, universal Costas matrices (UCMs) and their structural properties are presented.

A. Preliminaries

Definition 1: A Costas array of order n is defined as an $n \times n$ permutation matrix that satisfies the condition of having

distinct vectors connecting every two 1's. The row indices of this permutation matrix give the Costas array.

Definition 2: Let $C(n)$ be the number of all possible distinct Costas arrays of order n . A UCM of order n , denoted by U_n , is a $C(n) \times n$ matrix comprising all distinct Costas arrays of order n in each row. The rows of U_n are given canonically such that the first element of each row takes integer values from 1 to n , and the remaining elements in each row are arranged accordingly to preserve canonical ordering. The matrix is divided row-wise into n consecutive blocks, where the j^{th} block contains all Costas arrays whose first element is j . Thus, U_n can be shown as

$$U_n = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \cdots c_{r_1+1,1} \cdots c_{r_1+1,2} \cdots c_{r_1+1,n} \cdots \\ \vdots & \vdots & \ddots & \vdots \\ \cdots c_{r_2+1,1} \cdots c_{r_2+1,2} \cdots c_{r_2+1,n} \cdots \\ \vdots & \vdots & \ddots & \vdots \\ \cdots c_{r_{n-1}+1,1} \cdots c_{r_{n-1}+1,2} \cdots c_{r_{n-1}+1,n} \cdots \\ \vdots & \vdots & \ddots & \vdots \\ c_{C(n),1} & c_{C(n),2} & \cdots & c_{C(n),n} \end{bmatrix}, \quad (1)$$

where dashed lines separate canonical blocks, and r_j denotes the last row index of the j^{th} block corresponding to Costas arrays whose first element equals j . Formally, a UCM can be denoted as $U_n = \mathcal{U}(n)$, where $\mathcal{U}(\cdot)$ is the *UCM construction operator* that maps the set of all distinct Costas arrays of order n to a canonical matrix representation,

$$\mathcal{U}(n) : \{\text{All Costas arrays of order } n\} \rightarrow \mathbb{N}^{C(n) \times n}, \quad (2)$$

such that the rows of $\mathcal{U}(n)$ are ordered by their first element in ascending order, i.e., $U_n(r_1, 1) < U_n(r_2, 1), \forall r_1 < r_2$ to ensure a canonical block structure.

Example 1: U_4 can be shown in canonical order with 4 blocks separated by horizontal dashed lines as follows:

$$U_4 = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 1 & 3 & 4 & 2 \\ 1 & 4 & 2 & 3 \\ \cdots \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 1 & 4 \\ 2 & 4 & 3 & 1 \\ \cdots \\ 3 & 1 & 2 & 4 \\ 3 & 2 & 4 & 1 \\ 3 & 4 & 2 & 1 \\ \cdots \\ 4 & 1 & 3 & 2 \\ 4 & 2 & 1 & 3 \\ 4 & 3 & 1 & 2 \end{bmatrix}. \quad (3)$$

Definition 3: Based on the Definition 2, $S(n, k)$ is defined as the sum of the k^{th} column of U_n with elements $c_{m,k}$ where $k \in 1, 2, \dots, n$ as given by

$$S(n, k) = \sum_{m=1}^{C(n)} c_{m,k}. \quad (4)$$

In addition, it is useful to recall the elementary identity that for any permutation π of $\{1, \dots, n\}$ we have $\sum_{k=1}^n \pi(k) = \frac{n(n+1)}{2}$. Since a Costas array is a permutation, the sum of any

TABLE I: Number of Costas Arrays and Row/Column Sums in Universal Costas Matrices

Orders 3–16				Orders 17–29			
n	C(n)	D(n)	S(n)	n	C(n)	D(n)	S(n)
3	4	6	8	17	18276	153	164484
4	12	10	30	18	15096	171	143412
5	40	15	120	19	10240	190	102400
6	116	21	406	20	6464	210	67872
7	200	28	800	21	3536	231	38896
8	444	36	1998	22	2052	253	23598
9	760	45	3800	23	872	276	10464
10	2160	55	11880	24	200	300	2500
11	4368	66	26208	25	88	325	1144
12	7852	78	51038	26	56	351	756
13	12828	91	89796	27	204	378	2856
14	17252	105	129390	28	712	406	10324
15	19612	120	156896	29	164	435	2460
16	21104	136	179384	–	–	–	–

Costas array of order n or the sum of the elements in any row in U_n ($D(n)$) is given by

$$D(n) = \frac{n(n+1)}{2}. \quad (5)$$

B. Structural Properties and Theoretical Results

Theorem 1: In a universal Costas matrix U_n , all columns have equal sums, i.e.,

$$S(n, k) = S(n), \quad \forall k \in \{1, \dots, n\}. \quad (6)$$

Proof: Each Costas array comes in sets of four or eight since they belong to an equivalence class of polymorphs generated by the eight reflections and rotations [13]. Depending on diagonal symmetry, each polymorph set, i.e., orbit, has four or eight members obtained by 90° rotations or reflections of a Costas array permutation π and its inverse π^{-1} . For a fixed column k , four rotated polymorphs can be expressed as

$$\begin{aligned} \pi^{(1)}(k) &= \pi(k), & \pi^{(2)}(k) &= n+1-\pi(k), \\ \pi^{(3)}(k) &= \pi(n+1-k), & \pi^{(4)}(k) &= n+1-\pi(n+1-k), \end{aligned} \quad (7)$$

where the other four reflection polymorphs are obtained by replacing $\pi(k)$ with $\pi^{-1}(k)$. Each pair within the orbit is complementary:

$$\pi^{(i)}(k) + \pi^{(i+1)}(k) = n+1, \quad i \in \{1, 3, 5, 7\}. \quad (8)$$

Hence, the column sum over one set of polymorphs is

$$\sum_{t \in O(\pi)} \pi^{(t)}(k) = \frac{|O(\pi)|}{2}(n+1), \quad (9)$$

where $|O(\pi)| \in \{4, 8\}$ is the orbit size depending on the diagonal symmetry. This value depends only on n , not on k . Thus, stacking all polymorphs to form U_n yields

$$S(n, k) = S(n), \quad \forall k. \quad (10)$$

Corollary 1: The ratio of the sum of a column in a UCM of order n to the number of Costas arrays of order n is given by

$$\frac{S(n)}{C(n)} = \frac{n+1}{2}. \quad (11)$$

Proof: Summing all entries of U_n by rows gives $C(n)D(n)$, while summing by columns gives $nS(n)$. Equating and using

$D(n) = \frac{n(n+1)}{2}$ in (5) gives the result:

$$C(n) \frac{n(n+1)}{2} = nS(n) \quad (12)$$

$$\frac{S(n)}{C(n)} = \frac{n+1}{2}. \quad (13)$$

III. UNIVERSAL COSTAS FREQUENCY MATRICES

In this section, we present the universal Costas frequency matrices (UCFMs) and their symmetrical properties.

Definition 4: Let U_n be the UCM of order n . A UCFM of order n , denoted by F_n , is an $n \times n$ matrix that shows how frequently each number from 1 to n appears in each column of U_n . The $(i, k)^{th}$ element of F_n , i.e., $f_{i,k}$, counts the number of times the integer i occurs in the k^{th} column of U_n . Formally, the UCFM is defined as $F_n = \mathcal{F}(U_n)$ where $\mathcal{F}(\cdot)$ is the *UCFM transformation operator* that maps U_n to its corresponding frequency matrix F_n , defined elementwise as

$$[\mathcal{F}(U_n)]_{i,k} = f_{i,k} = \#\{r \mid U_n(r, k) = i\}, \quad (14)$$

where $\#\{\cdot\}$ denotes the cardinality of a set, i.e., the number of elements satisfying a given condition. For a better understanding, two samples of even and odd-numbered order UCFMs are given below.

Example 2: F_6 is given by

$$F_6 = \begin{bmatrix} 19 & 17 & 22 & 22 & 17 & 19 \\ 17 & 24 & 17 & 17 & 24 & 17 \\ 22 & 17 & 19 & 19 & 17 & 22 \\ 22 & 17 & 19 & 19 & 17 & 22 \\ 17 & 24 & 17 & 17 & 24 & 17 \\ 19 & 17 & 22 & 22 & 17 & 19 \end{bmatrix}, \quad (15)$$

where the first row shows the number of 1's in each column of U_6 , the second row shows the number of 2's in each column of U_6 , and so on. Similarly, F_5 is also shown below.

$$F_5 = \begin{bmatrix} 6 & 10 & 8 & 10 & 6 \\ 10 & 6 & 8 & 6 & 10 \\ 8 & 8 & 8 & 8 & 8 \\ 10 & 6 & 8 & 6 & 10 \\ 6 & 10 & 8 & 10 & 6 \end{bmatrix}, \quad (16)$$

where for instance, $f_{2,2} = 6$ shows the number of 2's in column 2 of U_5 . F_6 and F_5 shown in Example 2 also illustrate the two different symmetries as generalized below.

Theorem 2: Let F_n be the UCFM of order n . F_n shows horizontal, vertical, and diagonal reflection symmetries. For odd n , the symmetry axes pass through the middle row and column, while for even n , they lie between them.

Proof: As shown in Theorem 1, each Costas array (each row of U_n) belongs to an equivalence class of polymorphs (rotations and reflections) of size 4 or 8 depending on diagonal symmetry [21]. Let each array $\pi^{(r)}$ be represented by its 2-D binary form $B^{(r)} = [b_{i,k}^{(r)}]$ with $b_{i,k}^{(r)} = 1$ if $\pi^{(r)}(i) = k$, and 0 otherwise. F_n can be defined as the superposition of all such matrices as given by

$$F_n = \sum_r B^{(r)}, \quad f_{i,k} = \sum_r b_{i,k}^{(r)}. \quad (17)$$

For every $B^{(r)}$, its rotated and reflected polymorphs relocate the entries of 1 to positions related by index transformations of the form $(i, k) \mapsto (n+1-i, k)$, $(i, n+1-k)$, $(n+1-i, n+1-k)$, (k, i) , and their complementary reflections across the two diagonals. As a result, every value (i, k) in F_n has equal contributions with its symmetric counterparts:

$$f_{i,k} = f_{n+1-i,k} = f_{i,n+1-k} = f_{n+1-i,n+1-k} = f_{k,i} = f_{n+1-k,i} = f_{k,n+1-i} = f_{n+1-k,n+1-i}, \quad \forall i, k. \quad (18)$$

If a Costas array is symmetric along a diagonal (four polymorphs), some of these positions coincide, reducing the number of distinct equalities. Thus, the total contribution of all polymorphs remains balanced, and the element counts at symmetric positions in F_n become equal.

Corollary 2: The sum of all elements in a row or column of F_n gives the number of all Costas arrays of order n , i.e.,

$$C(n) = \begin{cases} \sum_{k=1}^n f_{k,m}, \forall m \\ \sum_{m=1}^n f_{k,m}, \forall k. \end{cases} \quad (19)$$

Proof: Given that F_n counts the frequency of each number in each column and it has $C(n)$ rows, the sum of each column in F_n is $C(n)$, since each number 1 to n appears only once in a Costas array of order n . Additionally, the reflection symmetry of F_n given in Theorem 2 provides that the sum of each column equals the sum of each row in F_n , which is also equal to $C(n)$.

Corollary 3: By using the elements of F_n , the sum of each column of U_n , i.e., $S(n)$, can be given by

$$S(n) = \sum_{k=1}^n k f_{k,m}, \quad \forall m \quad (20)$$

where $f_{k,m}$ is the element of F_n at the row k and column m .

Proof: Each $f_{k,m}$ counts how many times k appears in column m of U_n . Multiplying each count by k and summing over all rows gives the column sum $S(n)$, which is the same for all m (Theorem 1).

The theoretical foundations given in Sections II and III form the basis of the framework proposed in the next section.

IV. A GENERAL FRAMEWORK FOR COSTAS ARRAY DISCOVERY: RECONSTRUCTION OF UCMs FROM UCFMs

This section introduces a general framework for Costas array discovery that integrates UCMs, UCFMs, and AI-based generative models. Within this framework, a novel search method based on reconstructing UCMs from UCFMs is proposed and detailed below.

A. Motivation

As described in Section III, UCFMs provide a compact way to represent all Costas arrays of a given order, with 2-D $n \times n$ unique matrices. They exhibit highly structured and symmetrical patterns, that can be exploited to reconstruct the underlying Costas arrays. Furthermore, they can also be represented as images via heatmaps (illustrated in Section V).

With the advancement of AI techniques, subtle patterns in images can be distinguished. In particular, generative models such as generative adversarial networks [25], [27] have

demonstrated strong capabilities in synthesizing or extrapolating structured data distributions [26]. However, AI has not yet been applied to Costas array discovery due to the lack of a well-defined representation that connects Costas arrays with data-driven learning. Considering the compact representation of UCFMs, there is significant potential to leverage AI-based generative modeling to predict UCFMs of higher orders, thereby facilitating the discovery of new Costas arrays. To this end, we propose a general framework that integrates UCMs, UCFMs, and AI methods, as illustrated in Fig. 1. As shown in the upper branch of Fig. 1, all enumerated Costas arrays via exhaustive search can be converted into UCFMs (F_n), which are complete since all Costas arrays of order $n \leq 29$ are known. Additionally, algebraic methods can be used to generate incomplete UCFMs (F_n^*) using the large Costas array database available up to order $n \leq 1030$ [24], as shown in the lower branch of Fig. 1. These complete and incomplete UCFMs can thus serve as structured training data for potential generative AI models.

The predicted UCFMs can be obtained as the output of such models, as shown with $\hat{F}_n$ in Fig. 1. Then, these $\hat{F}_n$ s can be used to reconstruct the corresponding UCM ($\hat{U}_n$) via the inverse operation $\mathcal{F}^{-1}(\cdot)$, which represents the reconstruction of a UCM from UCFM. This reconstruction perspective provides a more holistic and computationally tractable approach to discovering new Costas arrays, as it leverages frequency constraints rather than direct combinatorial enumeration. We leave the prediction of UCFMs of arbitrary order n using generative AI models as future work, as it is non-trivial and beyond the scope of this paper. However, in this paper, we focus on the final block of Fig. 1, $\mathcal{F}^{-1}(\cdot)$, and present our detailed algorithm for the reconstruction process.

B. Reconstruction Algorithm

The pseudocode of the reconstruction algorithm of a UCM from a complete UCFM, represented by the blue box in Fig. 1 with the operation $\mathcal{F}^{-1}(\cdot)$, is given in Algorithm 1. In this algorithm, we first determine the number of Costas arrays of the given order n from F_n using Corollary 2. Additionally, the block sizes and indices of U_n , where each block represents a canonical group of Costas arrays starting with a different element (see Definition 2 and Example 1), are determined using the first column of F_n . Then, a canonical exhaustive search is initiated using the benchmark Costas array search method proposed by Russo et al. [23] as implemented by $RUSO_SEARCH(n, a_0)$ in Algorithm 1. Russo's method employs symmetry reduction, bitmasking, forward checking (or look-ahead), and backtracking to accelerate the search. In

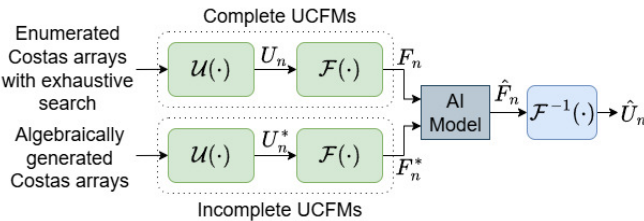


Fig. 1: A general framework to discover new Costas arrays using the UCMs/UCFMs and AI.

Algorithm 1 Reconstruction of the Universal Costas Matrix U_n from the Universal Costas Frequency Matrix F_n

Require: n : Order; F_n : Complete UCFM of size $n \times n$
Ensure: U_n : Canonical UCM of order n

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1:  $C(n) \leftarrow \sum_{v=0}^{n-1} F_n[v, 0] \triangleright C(n)$  from (19)
2: Initialize empty matrix  $U_n[C(n)][n]$ 
3: Initialize block sizes  $B[v] \leftarrow F_n[v, 0]$  for  $v = 0 \dots n-1$ 
4: Compute block start indices  $\text{start}[v]$  and counters  $\text{next}[v] \leftarrow \text{start}[v]$ , initialize  $\text{PlacedRows} \leftarrow \emptyset$ 
5: for  $a_0 = 0$  to  $n-1$  do
6:   if  $B[a_0] = 0$  then
7:     continue  $\triangleright$  Block already filled
8:   end if
9:    $G \leftarrow \text{RUSO\_SEARCH}(n, a_0) \triangleright$  Search Costas arrays of order  $n$  with first element  $a_0$  with Russo's method [23]
10:  for each canonical permutation  $p$  in  $G$  do
11:    if  $p \in \text{PlacedRows}$  then
12:      continue
13:    end if
14:     $\mathcal{O}(p) \leftarrow \text{POLYMORPHS}(p) \triangleright$  Generate polymorphs
15:    for each  $q$  in  $\mathcal{O}(p)$  do
16:      if  $q \notin \text{PlacedRows}$  and  $B[q[0]] > 0$  then
17:        Place  $q$  into block at position  $\text{next}[q[0]]$ 
18:         $\text{next}[q[0]] \leftarrow \text{next}[q[0]] + 1$ 
19:         $B[q[0]] \leftarrow B[q[0]] - 1$ 
20:         $\text{PlacedRows} \leftarrow \text{PlacedRows} \cup \{q\}$ 
21:      end if
22:    end for
23:    if  $B[a_0] = 0$  then
24:      break  $\triangleright$  Block fully filled; move to next block
25:    end if
26:  end for
27: end for
28: return  $U_n$ 

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particular, it encodes the difference triangles of the permutation using bitmasking, while the look-ahead process ensures that at least one valid future column position remains available without violating the uniqueness of the difference triangle. This allows pruning of infeasible branches early in the search.

Each Costas array generates up to 8 polymorphs, resulting in 3 or 7 additional arrays depending on the diagonal symmetry [13]. As applied in Russo's method, we generate polymorphs from each found Costas array. However, we treat them differently. Each generated polymorph is placed into the corresponding block as a row in U_n . When the corresponding block in U_n is fully filled, we stop the search for that block and move to the next block. When all the blocks are filled based on the block sizes in F_n , the reconstruction is completed. Next, the performance of this reconstruction method is demonstrated, and the theoretical results are validated.

V. RESULTS

In this section, numerical results to illustrate the theoretical findings as well as the performance results of the proposed reconstruction method elaborated in Section IV are presented.

In Fig. 3, the complete UCFMs for $5 \leq n \leq 29$ are illustrated as images via heatmaps in the first 5 rows of the figure. These

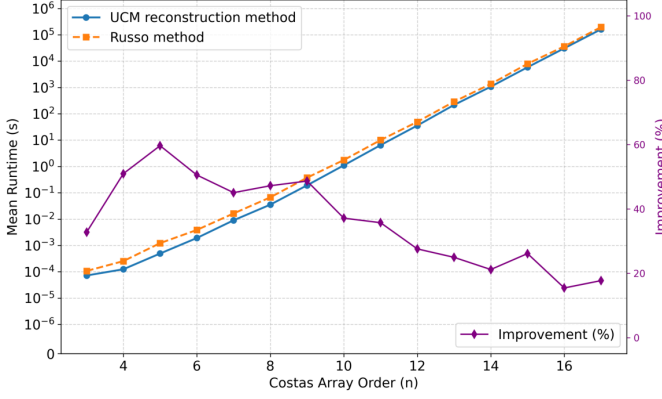


Fig. 2: Benchmark comparison results of the proposed UCM reconstruction method with the Russo method.

images are generated by applying (14) to the enumerated Costas arrays in [20], [21], [24]. Additionally, incomplete UCFMs are generated in row 6 of Fig. 3 using Beard’s database of algebraically generated Costas arrays [24]. In Fig. 3, the color scale is normalized based on the maximum UCFM value in each image, where brighter and darker regions indicate higher frequencies and lower frequencies, respectively. Rows 1 – 5 of this figure validate the horizontal, vertical, and diagonal reflection symmetries of UCFMs given in Theorem 2. Incomplete UCFMs still show the same symmetries since the database includes the polymorphs of the algebraically constructed Costas arrays.

In Fig. 2, the mean runtimes (left axis, log scale) of the proposed reconstruction method and Russo’s method in [23] are depicted for $4 \leq n \leq 17$. Each data point of this plot is the average of the 10 runs for $n < 13$ and 3 runs for $n \geq 13$ on a Dell PowerEdge T640 server, equipped with dual Intel Xeon Gold 2.1 GHz processors, and 128 GB RAM. The codes were implemented in Python 3.12, and no parallel processing was employed. Fig. 2 (right axis) shows the improvement in the mean runtime of the proposed method, calculated as $100 \times (1 - E\{t_{U/F}\}/E\{t_{\text{Russo}}\})$, where $E\{\cdot\}$ is the expectation operator, and $t_{U/F}$ and t_{Russo} are the runtimes of the proposed and Russo methods, respectively.

For small orders ($4 \leq n \leq 9$), the performance is significantly higher with respect to Russo’s method with a peak improvement of 59% at $n = 5$. After this peak, the performance gain decreases but stabilizes around 15% for $n \geq 13$. Even with only the exploitation of the first column of F_n and the block structure of U_n , there is a significant improvement in the performance. While parallel processing can enhance performance, a significant algorithmic improvement can be achieved by fully exploiting the residual frequency structure in F_n to reconstruct U_n .

VI. DISCUSSION AND CONCLUSION

This paper presents a holistic approach using UCMs and UCFMs to analyze and discover new Costas arrays. Their structural properties and theoretical foundations are examined, and a reconstruction-based search method is proposed to obtain UCMs from UCFMs. Numerical results demonstrate significant

improvements over the benchmark Russo method, highlighting the efficiency of the proposed approach. Furthermore, Costas arrays obtained through the proposed approach remain suitable for ISAC applications due to their ideal autocorrelation properties for sensing and low cross-correlation properties for multi-user communication.

Our observations and results suggest that any Costas array of order n may exist as a row within a UCM of the same order, rather than independently. This insight indicates that a unified, matrix-based perspective may be more effective than classifying Costas arrays as algebraic or sporadic. Thus, it facilitates a potential column-by-column search method to reconstruct a UCM.

The proposed framework also provides a foundation for AI-driven Costas array discovery. By integrating UCMs/UCFMs with generative AI models, the search for new arrays can be further accelerated. Although data scarcity is a challenge, data augmentation, pre-trained models used for combinatorial or matrix-completion problems, and generative modeling offer promising strategies for Costas array discovery.

REFERENCES

- [1] J. P. Costas, “A study of a class of detection waveforms having nearly ideal range–doppler ambiguity properties,” *Proceedings of the IEEE*, vol. 72, no. 8, pp. 996–1009, 1984.
- [2] F. Güleç and E. Afacan, “Usage of costas arrays in low probability of intercept radars,” in *2015 23rd Signal Processing and Communications Applications Conference (SIU)*. IEEE, 2015, pp. 347–350.
- [3] J. Yao, R. Jiang, Z. Liu, and X. Wang, “Optimum frequency hop patterns constructed on a golomb costas array using two-dimension circular shift,” *IEEE Trans. Aerosp. Electron. Syst.*, 2024.
- [4] N. Touati, C. Tatkeu, A. Rivenq, C. Thierry, and Y. El Hillali, “Multi-user radar waveforms based on doubly coded costas signals,” *IET Radar, Sonar & Navigation*, vol. 11, no. 2, pp. 277–284, 2017.
- [5] B. Correll Jr, J. K. Beard, and C. N. Swanson, “Costas array waveforms for closely spaced target detection,” *IEEE Trans. Aerosp. Electron. Syst.*, vol. 56, no. 2, pp. 1045–1076, 2019.
- [6] D. Xing, C. Zhang, and Y. Zhang, “A phase-coded fmcw-based integrated sensing and communication system design for maritime search and rescue,” *Sensors*, vol. 25, no. 17, p. 5403, 2025.
- [7] S. V. Maric and E. L. Titlebaum, “A class of frequency hop codes with nearly ideal characteristics for use in multiple-access spread-spectrum communications and radar and sonar systems,” *IEEE Transactions on Communications*, vol. 40, no. 9, pp. 1442–1447, 1992.
- [8] S. V. Maric and O. Moreno, “Using costas arrays to construct frequency hop patterns for ofdm wireless systems,” in *2006 40th Annual Conference on Information Sciences and Systems*. IEEE, 2006, pp. 505–507.
- [9] J. Yao, R. Jiang, X. Wang, and Z. Liu, “Orthogonal pattern division multiple access utilizing optimal frequency hopping patterns constructed with welch costas arrays,” *IEEE Transactions on Green Communications and Networking*, 2024.
- [10] J. Yao, R. Jiang, and W. Heng, “Algebraic construction of optimal frequency hopping patterns based on welch costas arrays,” *IEEE Transactions on Vehicular Technology*, vol. 69, no. 2, pp. 1841–1854, 2019.
- [11] S. Lu *et al.*, “Integrated sensing and communications: Recent advances and ten open challenges,” *IEEE Internet of Things Journal*, vol. 11, no. 11, pp. 19 094–19 120, 2024.
- [12] S. W. Golomb and H. Taylor, “Constructions and properties of costas arrays,” *Proceedings of the IEEE*, vol. 72, no. 9, pp. 1143–1163, 1984.
- [13] K. Drakakis, “A review of costas arrays,” *Journal of Applied Mathematics*, vol. 2006, no. 1, p. 026385, 2006.
- [14] S. W. Golomb and G. Gong, “The status of costas arrays,” *IEEE Trans. Inf. Theory*, vol. 53, no. 11, pp. 4260–4265, 2007.
- [15] J. K. Beard, “Singular value decomposition of a matrix representation of the costas condition for costas array selection,” *IEEE Trans. Aerosp. Electron. Syst.*, vol. 57, no. 2, pp. 1139–1161, 2020.
- [16] J. Jedwab and L. Yen, “Costas cubes,” *IEEE Trans. Inf. Theory*, vol. 64, no. 4, pp. 3144–3149, 2017.

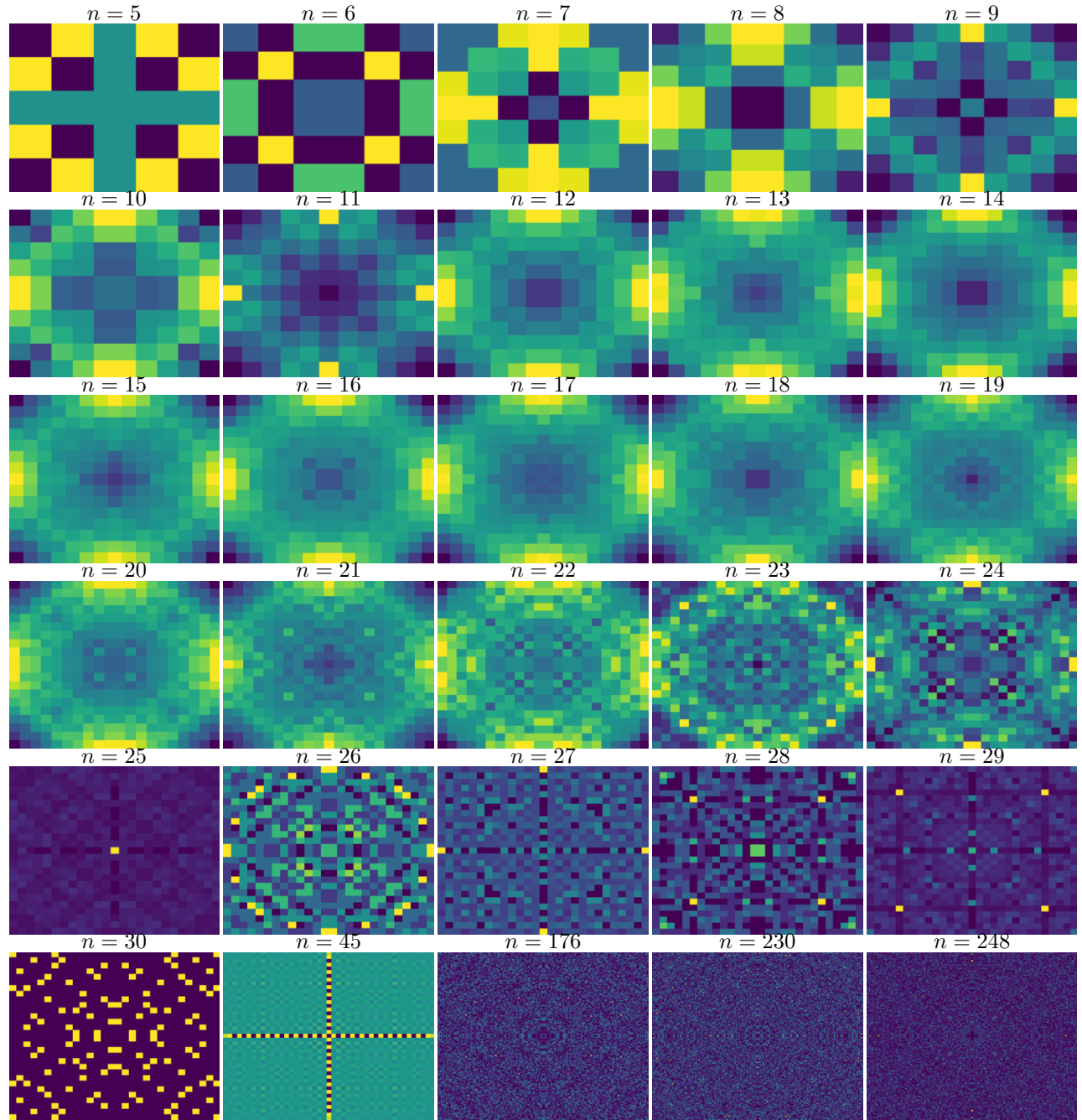


Fig. 3: Complete UCFMs (F_n) for $5 \leq n \leq 29$ (rows 1-5) and incomplete UCFMs for $n = 30, 45, 176, 230, 248$ (only row 6).

- [17] L. Warnke, B. Correll, and C. N. Swanson, "The density of costas arrays decays exponentially," *IEEE Trans. Inf. Theory*, vol. 69, no. 1, pp. 575–581, 2022.
- [18] J. Torres and I. Rubio, "Multidimensional costas arrays and their periodicity," *IEEE Trans. Inf. Theory*, vol. 69, no. 8, pp. 5032–5040, 2023.
- [19] K. Drakakis *et al.*, "Results of the enumeration of costas arrays of order 27," *IEEE Trans. Inf. Theory*, vol. 54, no. 10, pp. 4684–4687, 2008.
- [20] K. Drakakis, F. Iorio, and S. Rickard, "The enumeration of costas arrays of order 28 and its consequences," *Adv. Math. Commun.*, vol. 5, no. 1, pp. 69–86, 2011.
- [21] K. Drakakis, F. Iorio, S. Rickard, and J. Walsh, "Results of the enumeration of costas arrays of order 29," *Adv. Math. Commun.*, vol. 5, no. 3, pp. 547–553, 2011.
- [22] J. K. Beard, J. C. Russo, K. G. Erickson, M. C. Monteleone, and M. T. Wright, "Costas array generation and search methodology," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 43, no. 2, pp. 522–538, 2007.
- [23] J. C. Russo, K. G. Erickson, and J. K. Beard, "Costas array search technique that maximizes backtrack and symmetry exploitation," in *2010 44th Annual Conf. on Inf. Sci. and Sys. (CISS)*. IEEE, 2010, pp. 1–8.
- [24] J. Beard, "Costas arrays and enumeration to order 1030," *IEEE Dataport*, 2017. [Online]. Available: <https://dx.doi.org/10.21227/H21P42>
- [25] J. Yoon, J. Jordon, and M. Schaar, "Gain: Missing data imputation using generative adversarial nets," in *International conference on machine learning*. PMLR, 2018, pp. 5689–5698.
- [26] S. Wang *et al.*, "Generative adversarial matrix completion network based on multi-source data fusion for mirna–disease associations prediction," *Briefings in Bioinformatics*, vol. 24, no. 5, p. bbad270, 2023.
- [27] I. J. Goodfellow *et al.*, "Generative adversarial nets," *Advances in neural information processing systems*, vol. 27, 2014.