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Comparison measures of fuzzy inference rules based on generalizations of jaccard indices applied to inference method selection

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Abstract
A fuzzy rule-based system with the Generalized Modus Ponens (GMP) requires choosing the inference mechanism and its operators. These parameters affect the output of the system. In this paper, we aim to select the best possible output with respect to an input from a set of inference methods for solving the GMP through a decision-making system. To develop this system, we propose a fuzzy rule comparison measure that returns the similarity degree between two fuzzy *if-then* rules, in which the influence of both antecedent and consequent fuzzy sets is considered. To construct such comparison measures, we develop some generalizations of the Jaccard index based on grouping and overlap indices. Finally, we illustrate our decision-making system in a gastric cancer survival length prediction problem by evaluating various inference configurations across distinct clinical patient profiles and applying the proposed method to select the best one. Our results show that the generalized Jaccard index performs better than the classical Jaccard index at identifying similarities between fuzzy rules. Moreover, our selection approach effectively identifies the best inference method, in this case study, a specific configuration within the interpolation-based methods.

Keywords Fuzzy logic · Fuzzy rule comparison measure · Fuzzy reasoning

Mathematics Subject Classification 03E72 · 03B52 · 68T37

Abbreviations

Abbreviation	Full form
AI	Artificial Intelligence
FIS	Fuzzy inference system
CRI	Compositional rule of inference
CRP	C-reactive protein value
MCFR	Measure for comparing fuzzy rule
XAI	Explainable Artificial Intelligence

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GMP	Generalized Modus Ponens
TSK	Takagi–Sugeno–Kang
MCFRn	N-dimensional MCFR

1 Introduction

Fuzzy set theory, introduced by Zadeh (1965), allows the modeling of concepts under uncertainty, providing inference tools to manipulate them. This theory remains relevant nowadays because of its effectiveness in modeling human reasoning, gaining importance in applications in the context of explainable artificial intelligence (AI) (Moral et al. 2021; Cao et al. 2024, 2021), due to its high level of interpretability.

As properly discussed recently by Cao et al. (2024), interpretable or explainable AI (XAI) is indispensable in establishing trustable AI for bench-to-bedside translation, highly influencing human well-being. However, the majority of existing research in this area are dedicated to designing complex and sophisticated methods, without taking into account the matter of interpretability. As a consequence, the main prerequisite for implementing trustworthy AI, for example in medical domains, has not been met. Among various explanation methods for interpretable AI, fuzzy rules embedded in a fuzzy inference system (FIS) have emerged as a novel and powerful tool to bridge the communication gap between humans and advanced AI machines.

The inference mechanism of a fuzzy rule-based system (fuzzy system, for short) is represented by a collection of *if-then* rules and an inference rule. A widely-used inference rule is the Generalized Modus Ponens (GMP) (Díaz-Montarroso et al. 2024; Zadeh 1973), which consists of an *if-then* rule and a fact (fuzzy set(s) given as input to the fuzzy system) as premises, and a fuzzy set as the conclusion (also called output). So, for simplicity, the GMP can be expressed as follows:

$$\begin{array}{lll} \textit{if-then}\text{ rule:} & \textit{if } A \textit{ then } B \\ \text{Fact:} & A' & \\ \hline \text{Conclusion:} & & B' \end{array} \tag{1}$$

Zadeh proposed the Compositional Rule of Inference (CRI) (1973) and the interpolation method (1983) as mechanisms to apply the GMP and to produce an output in the form of a fuzzy set. Since then, several generalized methods have been proposed, such as the Approximate Modus Ponens defined in Modal Logic systems interpreted in the rough set language introduced by Chakraborty et al. (2025), and the one presented by Garcia-Jimenez et al. (2017), defined in terms of different aggregation functions, like t-norms (Klement et al. 2000) or overlap functions (Bedregal et al. 2013; Bustince et al. 2010), and t-conorms (Klement et al. 2000) or grouping functions (Bedregal et al. 2013; Bustince et al. 2012), in place of the minimum and the maximum, respectively, which were the aggregation functions employed originally by Zadeh. Therefore, several operators can be selected to tune the behavior of such mechanisms. The result is that different aggregation functions lead to distinct inference methods, and then different outputs may be generated.[1]

In this context, a challenge remains:

1. *Which operators would be more adequate to implement a fuzzy inference system based on the GMP?* (Konguetsof et al. 2020; Pagouropoulos et al. 2020)

[1] See Li and Zhu (2019) for a similar approach in the interval-valued context.

On one hand, an interesting way to answer this question is to propose a decision-making system to select the most adequate mechanism and the respective aggregation functions. This decision can be made by taking into account the results of the inference system, obtained from the set of *if-then* rules and the system's input, and by using a fuzzy rule comparison measure.

Notice that the comparison of fuzzy rules, analyzing their similarity[2]. This is a hot topic concerning the improvement of the interpretability of fuzzy systems with a large number of fuzzy inference rules, such as in Adaptive Neuro-Fuzzy Inference Systems (Rajab 2019).

In the literature, the terms "comparison" and "similarity" of fuzzy rules have mostly been employed, taking into account only the antecedents of the rules (see, e.g., Chen and Linkens 2004; Li et al. 2020; Rajab 2019). The motivation for such approaches was to merge similar rules in the sense of similar antecedents, thus obtaining new rules in the same antecedents' space and with corresponding consequents.

Nevertheless, several studies have explored comparison measures considering the rules' antecedents and consequents (see, e.g., Bellaaj et al. 2013; Chao et al. 1996; Garcia-Garcia et al. 2016; Jin et al. 2024) in order to simplify Takagi-Sugeno-Kang (TSK) systems (Takagi and Sugeno 1985). For this purpose, the comparison method is divided into (i) antecedent and (ii) consequent comparisons; the former is performed by aggregating the Jaccard index (Jaccard 1908, 1912) of the antecedent pairs of fuzzy sets, while the latter considers different functions for the parameters of the TSK consequent part, which are not provided by fuzzy sets. Finally, the similarity of the antecedents and the similarity of the consequents are aggregated to obtain the similarity of the whole rules. Observe that due to the different representations of antecedents and consequents in TSK systems, their comparison should be treated differently.

To the best of our knowledge, in the literature, the methods that consider both antecedents and consequents are used for comparing fuzzy rules of TSK inference systems. Thus, *none of them considers the consequents as fuzzy sets*. So, another question arises:

2. *For the fuzzy rule comparison measure that shall be applied to support solving question **(1)**, it is not necessary to consider both the antecedents and consequents of the fuzzy rules, since we are dealing with GMP?*

Once we received an affirmative answer to the above question, it became clear that such a definition of the comparison measure of two fuzzy rules should be given by a symmetric fusion function (Mesiar et al. 2016), which returns a value of 1 whenever their antecedents are equal as well as their consequents, and results in a value of 0 if either the supports of the antecedents or the supports of the consequents are disjoint. In this paper, *we present a general definition of such a fuzzy rule comparison measure*.

On the other hand, having this axiomatic definition in mind, one possible construction method would be based on the Jaccard index (mentioned earlier in this section), since it is one of the most widely employed methods to determine the similarity between (fuzzy) sets (Cross and Sudkamp 2013; Petković et al. 2021). Jaccard similarity, being simple in nature and user-friendly in implementation, has been applied in various domains and has generally performed adequately (Bag et al. 2019; Petković et al. 2021). For example, in machine learning, it is typically used to measure the stability of feature rankings (Khoshgoftaar et al. 2013; Saeys et al. 2008), or in heuristics for feature selection (Zou et al. 2016).

[2] In this paper, the concept of "similarity" is used only for "fuzzy rules". For other types of similarity, e.g., in the context of interval-valued spherical fuzzy sets, see (Khan et al. 2025c)

Now, let us define the Jaccard index of two fuzzy subsets A and B of a referential set U as: (Jaccard 1908, 1912)

$$J(A, B) = \frac{Card(A \cap B)}{Card(A \cup B)}, \tag{2}$$

where $\cap$ and $\cup$ are the fuzzy intersection and fuzzy union, respectively. Several authors have argued that the above definition of the Jaccard index is not flexible enough, as discussed by Bag et al. (2019), in the context of recommendation generation (see also Hamedani and Kim 2017; Patra et al. 2015), is often unstable and lacks a mechanism for correcting chance, as pointed out by Petković et al. in (2021). Since the Jaccard index relies only on the ordering of the items and ignores the actual score values, it is unstable in many scenarios and unable to accurately detect apparent similarities (and dissimilarities) (Petković et al. 2021, Sect. 2.4). In computing the similarity of intervals, the Jaccard index is sometimes not sensitive to changes in the width of intervals, producing equal similarities for substantially different pairs of intervals (Kabir et al. 2017).

Then, some generalizations of the Jaccard index were proposed in the literature, generally for specific aims, such as the Fuzzy Jaccard index (FUJI) (Petković et al. 2021)—a scale-invariant score for the similarity assessment of ordered lists for comparing feature-ranking lists relevant to a given machine learning task. For interval-valued data, Kabir et al. (2017) introduced a similarity measure based on the overlapping ratio. More recently, Bazhenov and Telnova (2023) proposed a generalization for both joint and non-joint interval data analysis. and the generalized Jaccard index for the signal's similarity (Fu et al. 2024).

In view of the lack of a general axiomatic definition of a generalized Jaccard-like index, we pose our final research question:

3. *For the purpose of providing construction methods for our fuzzy-rule comparison measure, what would a flexible and general axiomatization of the Jaccard index be?*

Then, in this paper, we not only study such axiomatization but also provide the construction of the generalized Jaccard index of two fuzzy subsets A and B of a referential set U in terms of their overlap (Garcia-Jimenez et al. 2015; Hierro et al. 2023) and grouping indices (Asmus et al. 2025; Dimuro et al. 2024), the former providing a measure of the proximity of the fuzzy intersection of A and B to the empty set, whereas the latter is a measure of the proximity of the fuzzy union of A and B to the referential set U.

Then, we state the following goals:

4. *The objectives of this paper are as follows:*

1. To provide an axiomatic definition of a fuzzy rule comparison measure that considers both the antecedent and consequent parts of the rule.
2. To provide an axiomatic definition of a generalization of the Jaccard index, with a construction method based on overlap and grouping indices, to be used in the construction method of the fuzzy-rule comparison measure.
3. To develop an algorithm that employs the fuzzy-rule comparison measure to select the best inference method for a given application and dataset.

Finally, to demonstrate the validity of our proposal, we illustrate our decision-making system within a fuzzy inference system for estimating life expectancy in gastric cancer based on the patient's age and the value of C-reactive protein (CRP).

5. *Structure of the paper:*

This paper is organized as follows. Section 2 shows the basic concepts that will be used in the rest of the work. Section 3 presents the classification of overlap/grouping functions

with respect to diagonal, which will be necessary for the development of the next section. In Sect. 4, we study the properties of the Jaccard index and present a generalization based on overlap and grouping indices. Section 5 introduces the concept of the rule comparison measure, presenting its desired properties. In Sect. 6, we develop an algorithm that employs the measure to choose the best inference method for the GMP in one rule fuzzy systems. Section 7 extends the concept of the measure to fuzzy rule systems with multiple antecedents. Section 8 generalizes the algorithm to choose the best method for multiple-antecedents and multiple-rules. In Sect. 9, the proposed measures and methods are used to obtain the best inference method in a gastric cancer patient survival length prediction system. Finally, in Sect. 10, the final remarks and future lines of research are presented.

2 Preliminaries

In this section, we establish the theoretical foundations and mathematical operators required for the development of our proposal. We begin by recalling the fundamental definitions of aggregation functions, fuzzy negations, and implications, which provide the basic logic for manipulating membership degrees. After that, we present overlap and grouping functions, as well as their related indices, which are their extension to fuzzy sets.

An n-ary aggregation function is a mapping $M : [0,1]^n \to [0,1]$ that is increasing in each variable and satisfies $M(0,\ldots,0) = 0$ and $M(1,\ldots,1) = 1$.

A triangular norm (t-norm) T is a commutative, associative, and increasing binary aggregation function $T : [0,1]^2 \to [0,1]$ with identity element 1 (i.e., $T(x,1) = x$). Its dual, a triangular conorm (t-conorm) S is a commutative, associative, and increasing binary aggregation function with identity element 0 (i.e., $S(x,0) = x$).

The set of all fuzzy subsets of a non-empty set U is denoted by $FS(U)$. Intersection and union of fuzzy sets $A, B \in FS(U)$ is defined by, for all $u \in U$, $(A \cap B)(u) = \min\{A(u), B(u)\}$ and $(A \cup B)(u) = \max\{A(u), B(u)\}$, respectively. We say that a fuzzy set $A \in FS(U)$ is normal if there exists $u \in U$ such that $A(u) = 1$. We write $A \leq B$ whenever $A(u) \leq B(u)$ for all $u \in U$. From now on, the set U is finite and we denote $U = \{u_1, \ldots, u_n\}$.

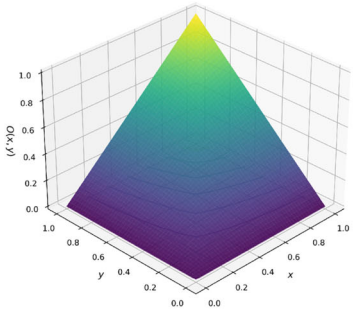
A fuzzy negation is a decreasing mapping $N : [0,1] \to [0,1]$ such that $N(0) = 1$ and $N(1) = 0$. A continuous and strictly decreasing fuzzy negation N is called a strict negation, and an involutive strict negation (i.e. a strict negation such that $N(N(x)) = x$ for all $x \in [0,1]$) is called a strong negation.

A mapping $I : [0,1]^2 \to [0,1]$ is a fuzzy implication if it is decreasing in the first variable, increasing in the second variable, and satisfies $I(1,0) = 0$ and $I(0,0) = I(1,1) = 1$.

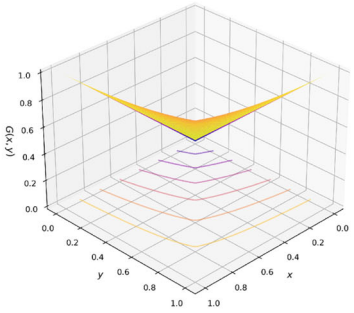
2.1 Overlap functions and overlap indexes

In this section, we briefly review overlap functions and overlap indices. Overlap functions (Bustince et al. 2010) were proposed to model the degree of overlapping between two degrees of membership, offering a more flexible alternative to continuous t-norms by omitting the associativity requirement. Similarly, overlap indices (Dubois et al. 2000) quantify the global degree of intersection between two fuzzy sets A and B, measuring how much both concepts share across the entire universe of discourse.

Definition 2.1 (Bustince et al. 2010) An overlap function is a mapping $O : [0,1]^2 \to [0,1]$ satisfying the following conditions for all $x, y \in [0,1]$:



(a) OB Overlap



(b) GB Grouping

Fig. 1 Visual representation of the behaviour of the OB overlap function ($O(x, y) = \sqrt{x \cdot y \cdot min\{x, y\}}$) and its corresponding dual grouping function OG ($G(x, y) = 1 - \sqrt{((1-x)\cdot(1-y)) \cdot min\{(1-x), (1-y)\}}$)

($O1$) $O(x, y) = O(y, x)$;
($O2$) $O(x, y) = 0$ if and only if $xy = 0$;
($O3$) $O(x, y) = 1$ if and only if $x = y = 1$;
($O4$) O is increasing in each variable;
($O5$) O is continuous.

Remark 2.2 To provide an intuitive understanding of overlap functions, consider a classification problem where an object must be assigned to one of two classes, A or B. Let $x, y \in [0, 1]$ be the degrees of membership to classes A and B, respectively. Overlap functions $O(x, y)$ measures the degree to which an element belongs to both classes simultaneously. Their main defining characteristics, visualized in Fig. 1a, are: the property ($O2$), the zero-divisor property, stating that if either input is zero, the output is strictly zero, and vice-versa, which reflects the logical condition that an intersection cannot exist if one of the components is entirely absent; and the property ($O3$), stating that the results is equal to 1 if and only if both inputs are equal to 1, stating the strong condition of the full intersection.

Overlap indices were proposed as an axiomatization of Zadeh's consistency index (Zadeh 1978), which is a measure of the proximity of the fuzzy intersection of two fuzzy subsets A and B to the empty set. From an intuitive perspective, while the overlap functions discussed previously are applied over two membership degrees, overlap indices provide a measure of the intersection of two fuzzy subsets. See Fig. 2 for a visual representation.

Definition 2.3 (Dubois et al. 2000) An overlap index is a mapping $\mathcal{O} : FS(U) \times FS(U) \to [0, 1]$ satisfying the following conditions for all $A, B, C \in FS(U)$:

($\mathcal{O}1$) $\mathcal{O}(A, B) = \mathcal{O}(B, A)$;
($\mathcal{O}2$) $\mathcal{O}(A, B) = 0$ if and only if $A \cap B = \emptyset$;
($\mathcal{O}3$) If $B \subseteq C$, then $\mathcal{O}(A, B) \leq \mathcal{O}(A, C)$.

We say that an overlap index $\mathcal{O}$ is normal if it holds:

$$\text{If there exists } u \in U \text{ such that } A(u) \cdot B(u) = 1, \text{ then } \mathcal{O}(A, B) = 1.$$

An in-depth study of overlap indices can be found in (Garcia-Jimenez et al. 2015), the same work shows how to construct these indices in terms of aggregations of overlap functions as shown below.

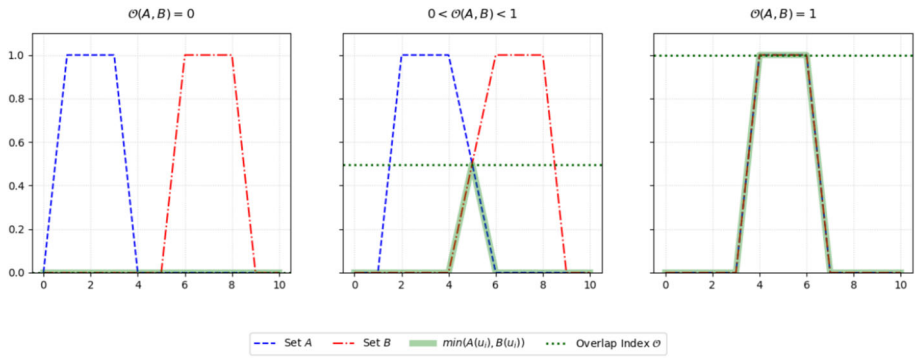


Fig. 2 Intuitive representation of Overlap Indices between two fuzzy sets A and B. **a** When sets are disjoint, the overlap index is minimal (zero). **b** Partial intersection yields an intermediate index value. **c** When the sets are coincident at a point where both membership degrees are 1, the overlap index reaches its maximum value of 1

Proposition 2.4 (Garcia-Jimenez et al. 2015) *Let* $M : [0,1]^n \to [0,1]$ *be an aggregation function such that* $M(x_1,\ldots,x_n) = 0$ *if and only if* $x_1 = \cdots = x_n = 0$. *Let* $O : [0,1]^2 \to [0,1]$ *be an overlap function. Then the mapping* $\mathcal{O} : FS(U) \times FS(U) \to [0,1]$ *defined, for all* $A, B \in FS(U)$, *by*

$$\mathcal{O}(A,B) = M\Big(O\Big(A(u_1),B(u_1)\Big),\ldots,O\Big(A(u_n),B(u_n)\Big)\Big) \tag{3}$$

is an overlap index.

2.2 Grouping functions and grouping index

In this section, we recall the concept of grouping index introduced by (2024). A grouping index provides a measure of the proximity of the fuzzy union of two fuzzy subsets A and B to the referential set. Similar to how overlap indices are constructed from overlap functions, these indices can be built from grouping functions, which were originally proposed in (Bustince et al. 2012) as a complementary notion to the latter.

Definition 2.5 (Bustince et al. 2012) A grouping function is a mapping $G : [0,1]^2 \to [0,1]$ satisfying the following conditions for all $x, y \in [0,1]$:

$(G1)$ $G(x,y) = G(y,x)$;
$(G2)$ $G(x,y) = 1$ if and only if $x = 1$ or $y = 1$;
$(G3)$ $G(x,y) = 0$ if and only if $x = y = 0$;
$(G4)$ G is increasing in each variable;
$(G5)$ G is continuous.

A relation between grouping functions and overlap functions was proved in (Bustince et al. 2012).

Remark 2.6 Consider the same classification problem presented in Remark 2.2, where $x, y \in [0,1]$ are the degrees of membership of an object to classes A and B, respectively. Observe that grouping functions $G(x,y)$, serve as the dual concept to overlap functions, measure the degree of evidence supporting at least one of the classes. As shown in Fig. 1b, their main

defining characteristics, are: the 1-boundary property $(G2)$: if either input is fully supported ($x = 1$ or $y = 1$), the output is 1, and vice-versa; and the property $(G3)$, stating that the results is equal to 0 if and only if both inputs are equal to 0.

Proposition 2.7 (Bustince et al. 2012) *Let $N : [0, 1] \to [0, 1]$ be a strict negation. If O is an overlap function, then*

$$G(x, y) = N\Big(O\Big(N(x), N(y)\Big)\Big) \tag{4}$$

is a grouping function. Reciprocally, if G is a grouping function, then

$$O(x, y) = N\Big(G\Big(N(x), N(y)\Big)\Big) \tag{5}$$

is an overlap function.

Subsequently, we find it beneficial that the idempotency property is maintained in the relationship between O and G.

Proposition 2.8 *Let $N : [0, 1] \to [0, 1]$ be a strong negation. If O is idempotent, then G obtained by* Eq. (4) *is idempotent too. And vice versa for* Eq. (5).

Now, we present the grouping index (Dimuro et al. 2024) between fuzzy sets, which is a complementary notion to Dubois's (Dubois et al. 2000) overlap index.

Definition 2.9 A grouping index is a mapping $\mathcal{G} : FS(U) \times FS(U) \to [0, 1]$ satisfying the following conditions for all $A, B, C \in FS(U)$:

($\mathcal{G}1$) $\mathcal{G}(A, B) = \mathcal{G}(B, A)$;
($\mathcal{G}2$) $\mathcal{G}(A, B) = 1$ if and only if, $A \cup B = U$;
($\mathcal{G}3$) If $B \subseteq C$, then $\mathcal{G}(A, B) \leq \mathcal{G}(A, C)$.

We say that a grouping index $\mathcal{G}$ is anti-normal if it holds: (Dimuro et al. 2024)

$$\text{If there exists } u \in U \text{ such that } A(u) = B(u) = 0, \text{ then } \mathcal{G}(A, B) = 0.$$

The grouping index is a measure of the proximity of the fuzzy union of two fuzzy subsets A and B to the referential universe. As visualized in Fig. 3, the index quantifies the global span of the concepts. Rather than evaluating their overlap, it assesses how well the sets together map the entire universe, penalizing regions with low membership degrees and reaching its maximum only when their combined union completely covers the referential space.

Example 2.10 Overlap index was introduced as a generalization of Zadeh's consistency index (Zadeh 1978) between two fuzzy sets $A, B \in FS(U)$:

$$\mathcal{O}(A, B) = \max_{i=1,\ldots,n} \min(A(u_i), B(u_i)). \tag{6}$$

Similarly, a natural example of grouping index is

$$\mathcal{G}(A, B) = \min_{i=1,\ldots,n} \max(A(u_i), B(u_i)). \tag{7}$$

It is worth pointing out that the overlap index given by Eq. (6) can be obtained by Prop. 2.4 taking $M = \max$, and the dual grouping index given by Eq. (7) by Prop. 2.11 taking $M = \min$.

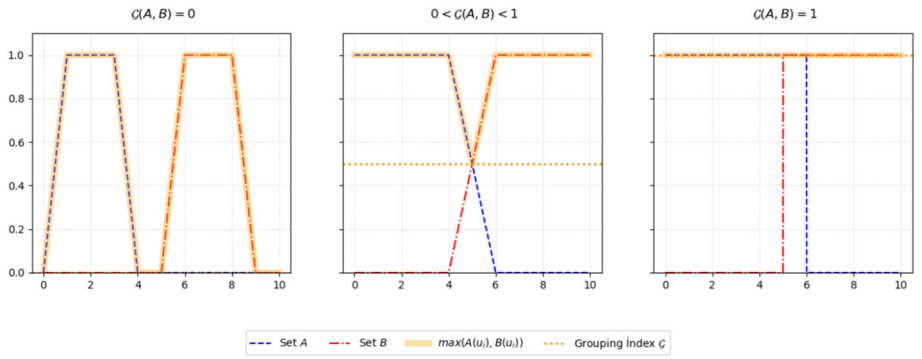


Fig. 3 Intuitive representation of Groping Indices between two fuzzy sets A and B. **a** Disjoint fuzzy sets where the lack of coverage at certain points yield $G = 0$. **b** Two sets covering the entire universe but dipping in their intersection without ever reaching zero, resulting in a partial grouping value. **c** Two sets that together provide full coverage, ensuring that for every point in the universe at least one set has a membership degree of 1, leading to the maximum grouping value $G = 1$

Grouping index can be obtained by aggregation of grouping functions as in the following proposition, which is a particular case of (Dimuro et al. 2024, Theorem 1):

Proposition 2.11 *Let $M : [0,1]^n \to [0,1]$ be an aggregation function such that $M(x_1, \ldots, x_n) = 1$ if and only if $x_1 = \cdots = x_n = 1$. Let $G : [0,1]^2 \to [0,1]$ be a grouping function. Then the mapping $\mathcal{G} : FS(U) \times FS(U) \to [0,1]$ defined, for all $A, B \in FS(U)$, by*

$$\mathcal{G}(A,B) = M\Big(G\Big(A(u_1), B(u_1)\Big), \ldots, G\Big(A(u_n), B(u_n)\Big)\Big) \tag{8}$$

is a grouping index.

3 Classification of overlap/grouping functions with respect to diagonal

In this section, we are going to deal with the diagonals of overlap and grouping functions, i.e. the differentiable functions $\delta : [0,1] \to [0,1]$ such that $\delta(x) = O(x,x)$ or $\delta(x) = G(x,x)$ for all $x \in [0,1]$. In particular, we show that this view leads to a partition of the set of all overlap functions as well as to a partition of the set of all grouping functions. Our motivation arises from the need of pairs O, G with the same diagonal (for Construction of Jaccard index in Sect. 4).

Henceforth, we will use $\mathcal{F}$ to denote a class of functions that is of special interest for our further development:

$$\mathcal{F} = \{f : [0,1] \to [0,1] \mid f \text{ is increasing, continuous, } f(x) = 0 \Leftrightarrow x = 0, \text{ and } f(x) = 1 \Leftrightarrow x = 1\}.$$

Proposition 3.1 *The following assertions are equivalent:*

(i) $\delta \in \mathcal{F}$.
(ii) There exists an overlap function O such that $\delta(x) = O(x,x)$ for all $x \in [0,1]$.
(iii) There exists a grouping function G such that $\delta(x) = G(x,x)$ for all $x \in [0,1]$.

Proof (ii) $\Rightarrow$ (i) directly follows from Def. 2.1 and (iii) $\Rightarrow$ (i) from Def. 2.5.

For (i) $\Rightarrow$ (ii) observe that the mapping $O : [0, 1]^2 \to [0, 1]$ defined, for all $x, y \in [0, 1]$, by $O(x, y) = \delta\big(\min(x, y)\big)$ is an overlap function.

Finally, with respect to (i) $\Rightarrow$ (iii) observe that the mapping $G : [0, 1]^2 \to [0, 1]$ defined, for all $x, y \in [0, 1]$, by $G(x, y) = \delta\big(\max(x, y)\big)$ is a grouping function. □

Remark 3.2 (i) A classification (partition) of the set of all overlap functions, denoted by $All(O)$, based on diagonal functions follows from Prop. 3.1. Let, for fixed $\delta : [0, 1] \to [0, 1]$, the class of all overlap functions with given diagonal δ be denoted by:

$$\mathcal{D}_O^{\delta} = \Big\{ O \in All(O) \mid O(x, x) = \delta(x) \text{ for all } x \in [0, 1] \Big\}. \tag{9}$$

By Prop. 3.1, $\mathcal{D}_O^{\delta}$ is non-empty if and only if $\delta \in \mathcal{F}$. So, the partition of the set of all overlap functions with respect to diagonals can be expressed:

$$All(O) = \bigcup_{\delta \in \mathcal{F}} \mathcal{D}_O^{\delta} \tag{10}$$

where $\mathcal{D}_O^{\delta_1}$ and $\mathcal{D}_O^{\delta_2}$ are disjunctive non-empty sets for any pair $\delta_1, \delta_2 \in \mathcal{F}$.

(ii) Similarly, if the set of all grouping functions is denoted by $All(G)$ and

$$\mathcal{D}_G^{\delta} = \Big\{ G \in All(G) \mid G(x, x) = \delta(x) \text{ for all } x \in [0, 1] \Big\}, \tag{11}$$

then the partition of the set of all grouping functions can be expressed:

$$All(G) = \bigcup_{\delta \in \mathcal{F}} \mathcal{D}_G^{\delta} \tag{12}$$

where $\mathcal{D}_G^{\delta_1}$ and $\mathcal{D}_G^{\delta_2}$ are disjunctive non-empty sets for any pair $\delta_1, \delta_2 \in \mathcal{F}$.

In the following proposition, a construction method for some subsets of $\mathcal{D}_O^{\delta}$ and $\mathcal{D}_G^{\delta}$, for a fixed diagonal δ, is proposed.

Proposition 3.3 *(i) Let $\delta, f \in \mathcal{F}$ be such that $\delta(x) \le f(x) \le \delta(x) + (1 - x)\delta'(x)$ for all $x \in [0, 1]$. Then the following is an overlap function:*

$$O^{\delta, f}(x, y) = \delta\big(\min(x, y)\big) + \frac{\max(x, y) - \min(x, y)}{1 - \min(x, y)} \cdot \Big(f\big(\min(x, y)\big) - \delta\big(\min(x, y)\big) \Big) \tag{13}$$

with the convention $\frac{0}{0} = 1$.

(ii) Let $g, \delta \in \mathcal{F}$ be such that $\delta(y) - y\delta'(y) \le g(y) \le \delta(y)$ for all $y \in [0, 1]$. Then the following is a grouping function:

$$G^{g, \delta}(x, y) = g\big(\max(x, y)\big) + \frac{\min(x, y)}{\max(x, y)} \cdot \Big(\delta\big(\max(x, y)\big) - g\big(\max(x, y)\big) \Big) \tag{14}$$

with the convention $\frac{0}{0} = 1$.

Proof (i) The properties (O1), (O2), (O3) and (O5) directly follows from Eq. (13). With respect to (O4): let $x \le y$ (the case $y \le x$ follows from the symmetry), then

$$O^{\delta, f}(x, y) = \delta(x) + \frac{y - x}{1 - x} \cdot \Big(f(x) - \delta(x) \Big).$$

Clearly, $O^{\delta,f}$ is increasing with respect to variable y, so we need to prove increasingnes with respect to variable x. For a fixed $y \in]0,1[$ we have for all $x \in [0,y]$:

$$\begin{aligned} f(x) &\leq \delta(x) + (1-x)\delta'(x) \\ &= \delta(x) + \frac{(1-x)(1-y)}{1-y} \cdot \delta'(x) \\ &\leq \delta(x) + \frac{(1-x)(1-y)}{1-y} \cdot \delta'(x) + \frac{(1-x)(y-x)}{1-y} \cdot f'(x) \end{aligned}$$

hence

$$0 \leq \delta(x) - f(x) + \frac{(1-x)(1-y)}{1-y} \cdot \delta'(x) + \frac{(1-x)(y-x)}{1-y} \cdot f'(x)$$

and finaly, by multiplying the previous inequality with $\frac{1-y}{(1-x)^2}$ it follows:

$$0 \leq \frac{1-y}{(1-x)^2} \cdot \delta(x) - \frac{1-y}{(1-x)^2} \cdot f(x) + \frac{1-y}{1-x} \cdot \delta'(x) + \frac{y-x}{1-x} \cdot f'(x) = \frac{\partial O^{\delta,f}}{\partial x}(x,y).$$

(ii) The proof is similar to that of item (i). We only show the increasingness of $G^{g,\delta}$ with respect to variable y for the case $x \leq y$. For a fixed $x \in]0,1[$ we have for all $y \in [x,1]$:

$$g(y) \geq \delta(y) - y\delta'(y) \geq \delta(y) - y\delta'(y) - \frac{y(y-x)}{x} \cdot g'(y)$$

thus

$$0 \leq g(y) - \delta(y) + y\delta'(y) + \frac{y(y-x)}{x} \cdot g'(y)$$

and by multiplying the previous inequality with $\frac{x}{y^2}$ it follows:

$$0 \leq \frac{x}{y^2} \cdot g(y) - \frac{x}{y^2} \cdot \delta(y) + \frac{x}{y} \cdot \delta'(y) + \frac{y-x}{y} \cdot g'(y) = \frac{\partial G^{g,\delta}}{\partial y}(x,y).$$

□

Remark 3.4 (i) Note that putting $\delta(x) = f(x) = g(x) = x$, we obtain $O^{\delta,f} = \min$ and $G^{g,\delta} = \max$. So we received the pair of dual t-norm/t-conorm which are idempotent.

(ii) Taking $\delta(x) = x^2$ and $f(x) = x$, we have $O^{\delta,f}(x,y) = xy$. On the other hand, taking $\delta(x) = 2x - x^2$ and $g(x) = x$, we have $G^{g,\delta}(x,y) = x + y - xy$. So we again received the pair of dual t-norm/t-conorm, however now they are not idempotent. It is worth pointing out that the functions $\delta_1(x) = x^2$ and $\delta_2(x) = 2x - x^2$ are dual in the sense $\delta_2(x) = 1 - \delta_1(1-x)$ which corresponds to the duality of overlap and grouping functions described in Prop. 2.7. And the same duality (although trivial) can be seen in item (i) of this Remark between $\delta_1(x) = x$ and $\delta_2(x) = x$.

Remark 3.5 (i) The geometrical interpretation of $O^{\delta,f}$ given in Prop. 3.3 (i) can be seen in Fig. 4 (on the left). The construction is as follows: besides given boundary conditions $O^{\delta,f}(0,y) = O^{\delta,f}(x,0) = 0$ and $O^{\delta,f}(1,1) = 1$, we fix diagonal $O^{\delta,f}(x,x) = \delta(x)$; and upper and right side of the unit square $O^{\delta,f}(x,1) = f(x)$, $O^{\delta,f}(1,y) = f(y)$. We fill in the remaining points of the unit square $[0,1]^2$ using linear interpolation according to the dashed lines in Fig. 4 (on the left).

(ii) The construction of $O^{\delta,f}$ given in Prop 3.3 (i) implies another partition of the set of all overlap functions with respect to diagonal δ and upper side f of the unit square (or right side - from symmetry of overlap functions it follows that they have to be the same). It can be expressed as:

$$All(O) = \bigcup_{\delta, f \in \mathcal{F},\, \delta(x) \le f(x) \le \delta(x) + (1-x)\delta'(x)} \mathcal{D}_O^{\delta,f} \tag{15}$$

where

$$\mathcal{D}_O^{\delta,f} = \Big\{ O \in All(O) \mid O(x,x) = \delta(x) \text{ and } O(x,1) = f(x) \text{ for all } x \in [0,1] \Big\}. \tag{16}$$

Note that, by item (i) of this Remark, classes $\mathcal{D}_O^{\delta_1,f_1}$ and $\mathcal{D}_O^{\delta_2,f_2}$ are non-empty disjunctive sets for any pair of pairs $(\delta_1, f_1) \in \mathcal{F}^2$ and $(\delta_2, f_2) \in \mathcal{F}^2$ with $\delta_i(x) \le f_i(x) \le \delta_i(x) + (1-x)\delta_i'(x)$ for all $x \in [0,1]$, $i = 1, 2$.

(iii) The geometrical interpretation of $G^{g,\delta}$ can be seen in Fig. 4 (on the right). The construction is similar to that in item (i): besides given boundary conditions $G^{g,\delta}(1, y) = G^{g,\delta}(x, 1) = 1$ and $G^{g,\delta}(0, 0) = 0$, we fix diagonal $G^{g,\delta}(x, x) = \delta(x)$; and lower and left side of the unit square $G^{g,\delta}(x, 0) = g(x)$, $G^{g,\delta}(0, y) = g(y)$. We fill in the remaining points of the unit square $[0, 1]^2$ using linear interpolation according to the dashed lines in Fig. 4 (on the right).

(iv) The construction of $G^{g,\delta}$ given in Prop. 3.3 (ii) implies another partition of the set of all grouping functions with respect to diagonal δ and left side g of the unit square (or lower side):

$$All(G) = \bigcup_{g, \delta \in \mathcal{F},\, \delta(y) - y\delta'(y) \le g(y) \le \delta(y)} \mathcal{D}_G^{g,\delta} \tag{17}$$

where

$$\mathcal{D}_G^{g,\delta} = \Big\{ G \in All(G) \mid G(x,x) = \delta(x) \text{ and } G(0,y) = g(y) \text{ for all } x, y \in [0,1] \Big\}. \tag{18}$$

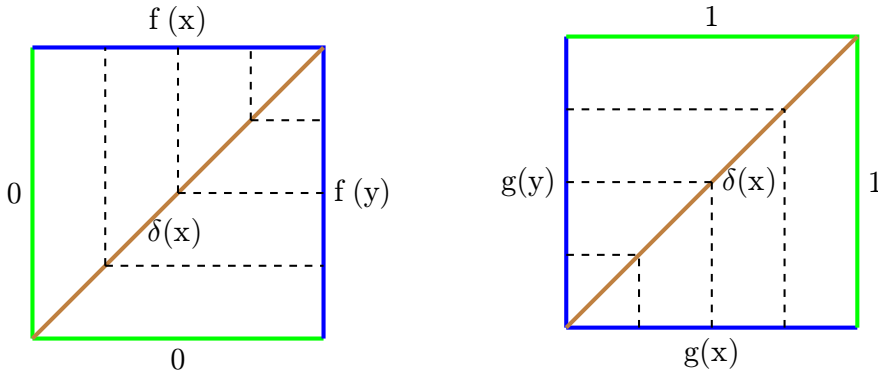
From item (iii) it follows that the classes $\mathcal{D}_G^{g_1,\delta_1}$ and $\mathcal{D}_G^{g_2,\delta_2}$ are non-empty disjunctive sets for any pair of pairs $(g_1, \delta_1) \in \mathcal{F}^2$ and $(g_2, \delta_2) \in \mathcal{F}^2$ with $\delta_i(y) - y\delta_i'(y) \le g_i(y) \le \delta_i(y)$ for all $y \in [0, 1]$, $i = 1, 2$.

Example 3.6 We give some examples of functions $\delta \in \mathcal{F}$ and corresponding possible functions $f \in \mathcal{F}$ satisfying the assumptions of Prop. 3.3 (1) and $g \in \mathcal{F}$ satisfying the assumptions of Prop. 3.3 (2):

- Let $\delta(x) = x$, then $x \le f(x) \le 1$ and $0 \le g(x) \le x$.
- Let $\delta(x) = x^2$, then $x^2 \le f(x) \le 2x - x^2$ and $0 \le g(x) \le x^2$.
- Let $\delta(x) = 2x - x^2$, then $2x - x^2 \le f(x) \le 1$ and $x^2 \le g(x) \le 2x - x^2$.
- Let $\delta(x) = \sqrt{x}$, then $\sqrt{x} \le f(x) \le 1$ and $\frac{\sqrt{x}}{2} \le g(x) \le \sqrt{x}$.

3.1 Idempotent overlap and grouping functions

In this subsection we study the classification and construction of idempotent overlap and grouping functions, i.e., functions with fixed diagonal $\delta(x) = x$, we will write $\delta = Id$. Hence, considering the partition of $All(O)$ (or $All(G)$) given by Eq. (10) (or Eq. (12)), we focus on the class $\mathcal{D}_O^{Id}$ (or $\mathcal{D}_G^{Id}$).



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Fig. 4 On the left: The construction of $O^{\delta,f}$ according to Prop. 3.3 (i) and Rem. 3.5 (i). On the right: The construction of $G^{g,\delta}$ according to Prop. 3.3 (ii) and Rem. 3.5 (iii). In both cases, the dashed lines indicate how the linear interpolation is provided

The analytical expressions presented in the following Proposition are obtained by fixing the diagonal section $\delta(x) = x$ and the boundary conditions of the operator, and then extending these values to the entire unit square using linear interpolation along different directions, as illustrated by the dashed lines in Fig. 5.

Proposition 3.7 *(i) Let $f \in \mathcal{F}$ be such that $f(x) \geq xf'(x)$ for all $x \in [0,1]$. Then the following are overlap functions:*

$$O^{Id,f}(x,y) = \max(x,y) \cdot f\left(\frac{\min(x,y)}{\max(x,y)}\right) \tag{19}$$

and

$$O^{Id,f}(x,y) = \frac{\min(x,y)}{1+\min(x,y)-\max(x,y)} \cdot f\Big(1+\min(x,y)-\max(x,y)\Big) \tag{20}$$

with the convention $\frac{0}{0} = 1$.

(ii) Let $g \in \mathcal{F}$ be such that $g(x) \leq 1-(1-x)g'(x)$ for all $x \in [0,1]$. Then the following are grouping functions:

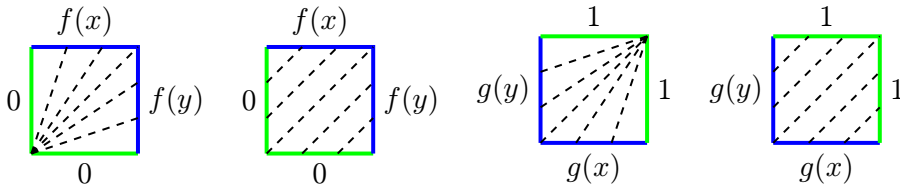
$$G^{g,Id}(x,y) = \min(x,y) + \Big(1-\min(x,y)\Big) \cdot g\left(\frac{\max(x,y)-\min(x,y)}{1-\min(x,y)}\right) \tag{21}$$

and

$$G^{g,Id}(x,y) = \frac{\min(x,y)}{1+\min(x,y)-\max(x,y)} + \frac{1-\max(x,y)}{1+\min(x,y)-\max(x,y)} \cdot f\Big(\max(x,y)-\min(x,y)\Big) \tag{22}$$

with the convention $\frac{0}{0} = 1$.

Proof The properties $(O1)$, $(O2)$, $(O3)$ and $(O5)$ directly follows from Eqs. (19)–(20). Similarly, the properties $(G1)$, $(G2)$, $(G3)$ and $(G5)$ directly follows from Eqs. (21)–(22).



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Fig. 5 The construction of $O^{Id,f}$ and $G^{g,Id}$ according to Prop. 3.7. From the left: $O^{Id,f}$ by Eq. (19), $O^{Id,f}$ by Eq. (20), $G^{g,Id}$ by Eq. (21), and $G^{g,Id}$ by Eq. (22)

The proof of monotonicity, i.e. property *(O4)* for Eqs. (19)–(20) and *(G4)* for Eqs. (21)–(22), in all four cases, follows from the partial derivatives of the functions. We only show the case for Eq. (21) (the other cases are similar): Let $x \leq y$, then

$$G^{g,Id}(x, y) = x + (1 - x) \cdot g\left(\frac{y - x}{1 - x}\right)$$

Clearly, $G^{g,Id}$ is increasing with respect to variable y, so we only need to prove the increasingness with respect to variable x. We have:

$$\frac{\partial G^{g,Id}}{\partial x}(x, y) = 1 - g(z) - \frac{1 - y}{1 - x} \cdot \frac{\partial g(z)}{\partial z} = 1 - g(z) - (1 - z) \cdot \frac{\partial g(z)}{\partial z}$$

where $z = \frac{y-x}{1-x}$. Thus, $\frac{\partial G^{g,Id}}{\partial x}(x, y) \geq 0$ for all $x \leq y$, $x, y \in [0, 1]$ whenever

$$g(z) \leq 1 - (1 - z) \cdot \frac{\partial g(z)}{\partial z}$$

for all $z \in [0, 1]$. □

Remark 3.8 (i) The geometrical interpretations of the overlap functions $O^{Id,f}$ given in Prop. 3.7 (i) and grouping functions $G^{g,Id}$ given in Prop. 3.7 (ii) is similar to that described in Rem. 3.5. However, the linear interpolation is made according to the dashed lines in Fig. 5.

(ii) Prop. 3.7 implies partition of the set of all idempotent overlap functions, $\mathcal{D}_O^{Id}$, and grouping functions, $\mathcal{D}_G^{Id}$. It can be expressed:

$$\mathcal{D}_O^{Id} = \bigcup_{f \in \mathcal{F},\, f(x) \geq x f'(x)} \mathcal{D}_O^{Id,f} \tag{23}$$

where

$$\mathcal{D}_O^{Id,f} = \left\{ O \in All(O) \mid O(x, x) = x \text{ and } O(x, 1) = f(x) \text{ for all } x \in [0, 1] \right\} \tag{24}$$

and

$$\mathcal{D}_G^{Id} = \bigcup_{g \in \mathcal{F},\, g(x) \leq 1 - (1-x) f'(x)} \mathcal{D}_G^{g,Id} \tag{25}$$

where

$$\mathcal{D}_G^{g,Id} = \left\{ G \in All(G) \mid G(x, x) = x \text{ and } G(0, y) = g(y) \text{ for all } x, y \in [0, 1] \right\}. \tag{26}$$

Example 3.9 The condition $f(x) \geq xf'(x)$ of Prop. 3.7 (i) is satisfied, e.g., for $f(x) = x$, $f(x) = 2x - x^2$ and $f(x) = \sqrt{x}$, however it is not satisfied for $f(x) = x^2$. So the functions given by Eq. (19) and Eq. (20) are idempotent overlap functions for $f(x) = x$, $f(x) = 2x - x^2$ and $f(x) = \sqrt{x}$, moreover both equations give minimum as result in the case of $f(x) = x$.

Similarly, the condition $g(x) \leq 1 - (1 - x)g'(x)$ of Prop. 3.7 (ii) is satisfied, e.g., for $g(x) = x$ and $g(x) = x^2$, however it is not satisfied for $g(x) = 2x - x^2$ and $g(x) = \sqrt{x}$. So the functions given by Eqs. (21) and (22) are idempotent grouping functions for $g(x) = x$ and $g(x) = x^2$, moreover, both equations result in maximum in the case of $g(x) = x$.

4 Generalized Jaccard index

The Jaccard index (Baioletti et al. 2012; Jaccard 1908) is a widely used similarity measure for binary data in computer science. It has been applied in several contexts, such as conceptual distance (Yan et al. 2025), fuzzy validity–probability for decision making (Sabahi and Akbarzadeh-T 2014), and analysis of inconsistency in a knowledge base (Mu 2024).

Despite its widespread use, the original formulation presents certain drawbacks, as discussed in the Introduction, like lack of flexibility and stability, absence of a mechanism for chance correction, and, mainly it is not able to detect apparent similarities (and dissimilarities). Therefore, in this section, we develop a more flexible and general version of this index.

Let us first recall the original definition of the Jaccard index. If A and B denote the sets of positive attributes, then the Jaccard similarity index is given by

$$J(A, B) = \frac{Card(A \cap B)}{Card(A \cup B)}. \tag{27}$$

In this section, we study an axiomatic definition of the Jaccard index for fuzzy sets and its construction in terms of overlap and grouping indices.

Definition 4.1 A Jaccard index is a mapping $J : FS(U) \times FS(U) \to [0, 1]$ satisfying the following conditions for all $A, B \in FS(U)$:

(J1) $J(A, B) = J(B, A)$;
(J2) $J(A, B) = 0$ if and only if $A \cap B = \emptyset$;
(J3) $J(A, B) = 1$ whenever $A = B$.

Example 4.2 Let $U = \{u_1, \ldots, u_n\}$ and consider the condition:

(J3*) $J(A, B) = 1$ if and only if $A = B$.

Then the following mappings are Jaccard indices satisfying 4.2:

$$J_1(A, B) = \begin{cases} 0, & \text{if } A = B = \emptyset; \\ \dfrac{\sum_{i=1}^{n} (A(u_i) \wedge B(u_i))}{\sum_{i=1}^{n} (A(u_i) \vee B(u_i))}, & otherwise. \end{cases} \tag{28}$$

$$J_2(A, B) = \begin{cases} 0, & \text{if } A = B = \emptyset; \\ \dfrac{\sum_{i=1}^{n} \sqrt{A(u_i) \cdot B(u_i)}}{\sum_{i=1}^{n} (1 - \sqrt{(1 - A(u_i)) \cdot (1 - B(u_i))})}, & otherwise. \end{cases} \tag{29}$$

Considering only normal fuzzy subsets of a universe U, i.e. the set $NFS(U)$, we can obtain a Jaccard index in means of normal overlap index as shown in the following theorem.

Theorem 4.3 *Let $\mathcal{O} : FS(U) \times FS(U) \to [0, 1]$ be a normal overlap index. Let $J^N : NFS(U) \times NFS(U) \to [0, 1]$ be the mapping defined, for all $A, B \in NFS(U)$, by:*

$$J^N(A, B) = \mathcal{O}(A, B). \tag{30}$$

Then J^N satisfies the conditions (J1), (J2) and (J3) for all $A, B \in NFS(U)$.

The construction method for Jaccard indices in terms of overlap and grouping functions is proposed in the following theorem.

Theorem 4.4 *Let $\mathcal{O}, \mathcal{G} : FS(U) \times FS(U) \to [0, 1]$ be an overlap index and grouping index, respectively, where, for all $A, B \in FS(U)$, it holds:*

(i) $\mathcal{O}(A, A) = \mathcal{G}(A, A)$;
(ii) *if* $\mathcal{G}(A, B) = 0$, *then* $A \cap B = \emptyset$;

Then a mapping $J : FS(U) \times FS(U) \to [0, 1]$ defined, for all $A, B \in FS(U)$, by the formula:

$$J(A, B) = \min\left(1, \frac{\mathcal{O}(A, B)}{\mathcal{G}(A, B)}\right), \tag{31}$$

with the convention $\frac{0}{0} = 0$, is a Jaccard index.

Proof First, observe that the mapping J is well-defined. Property (J1) follows from ($\mathcal{O}1$) and ($\mathcal{G}1$), property (J2) from ($\mathcal{O}2$), (ii) and the convention $\frac{0}{0} = 0$; and finally property (J3) from (i). □

Example 4.5 The Jaccard indices in Example 4.2 can be obtained by applying Theorem 4.4 taking $\mathcal{O}(A, B) = \frac{1}{n}\sum_{i=1}^{n}(A(u_i) \wedge B(u_i))$ and $\mathcal{G}(A, B) = \frac{1}{n}\sum_{i=1}^{n}(A(u_i) \vee B(u_i))$ for J_1; and $\mathcal{O}(A, B) = \frac{1}{n}\sum_{i=1}^{n}\sqrt{A(u_i) \cdot B(u_i)}$ and $\mathcal{G}(A, B) = \frac{1}{n}\sum_{i=1}^{n}\left(1 - \sqrt{(1 - A(u_i)) \cdot (1 - B(u_i))}\right)$ for J_2.

Remark 4.6 The assumptions on $\mathcal{O}$ and $\mathcal{G}$ in Theorem 4.4 look rather restrictive, however, there are more possibilities for how to obtain appropriate overlap and grouping indices. Clearly, $\mathcal{O}$ and $\mathcal{G}$ should be somehow connected. First of all, intuitively, it is reasonable to use pairs of overlap and grouping indices that are related, for instance, according to Proposition 2.7 and the construction methods given by Propositions 2.4 and 2.11. Secondly, from the technical point of view, we need pairs that satisfy assumption (i) of the theorem; and this again can be obtained, for example, by the same construction. We give two examples of the construction of Jaccard index according to Theorem 4.4

Construction 1 (idempotent overlap/grouping functions):

Step 1: Let N be a strong fuzzy negation and O an idempotent overlap function (or G an idempotent grouping function). By Propositions 2.7 and 2.8 we obtain an idempotent grouping function G (or an idempotent overlap function O).

Step 2: Let $M : [0,1]^n \to [0,1]$ be an aggregation function such that $M(x_1, \ldots, x_n) = 0$ if and only if $x_1 = \cdots = x_n = 0$; and $M(x_1, \ldots, x_n) = 1$ if and only if $x_1 = \cdots = x_n = 1$. Then, by Prop. 2.4, taking M and O from Step 1, we obtain overlap index $\mathcal{O}$. Similarly, by Prop. 2.11, taking M and G from Step 1, we obtain grouping index $\mathcal{G}$. It is easy to check that the obtained $\mathcal{O}$ and $\mathcal{G}$ satisfy assumptions of Theorem 4.4.

Step 3: Finally, a Jaccard index J is given by Theorem 4.4 using overlap index $\mathcal{O}$ and grouping index $\mathcal{G}$ from Step 2.

Construction 1 can be summarized as follows:

Inputs:

- a strong fuzzy negation N;
- $O \in \mathcal{D}_O^{Id}$ or $G \in \mathcal{D}_G^{Id}$;
- an n-ary aggregation function M such that $M(x_1, \ldots, x_n) = 0$ if and only if $x_1 = \cdots = x_n = 0$ and $M(x_1, \ldots, x_n) = 1$ if and only if $x_1 = \cdots = x_n = 1$.

Outputs:

$$J^{O,N,M}(A,B) = \min\left(1, \frac{M\Big(O\Big(A(u_1),B(u_1)\Big),\ldots,O\Big(A(u_n),B(u_n)\Big)\Big)}{M\Big(N\Big(O\Big(N(A(u_1)),N(B(u_1))\Big)\Big),\ldots,N\Big(O\Big(N(A(u_n)),N(B(u_n))\Big)\Big)\Big)}\right) \tag{32}$$

and

$$J^{G,N,M}(A,B) = \min\left(1, \frac{M\Big(N\Big(G\Big(N(A(u_1)),N(B(u_1))\Big)\Big),\ldots,N\Big(G\Big(N(A(u_n)),N(B(u_n))\Big)\Big)\Big)}{M\Big(G\Big(A(u_1),B(u_1)\Big),\ldots,G\Big(A(u_n),B(u_n)\Big)\Big)}\right) \tag{33}$$

are classes of Jaccard indices.

The second construction is a generalization of the first one in the sense that except considering idempotent overlap/grouping functions in Step 1, we consider any pair of O, G with the same diagonal, i.e. such that $O(x,x) = G(x,x)$ for all $x \in [0,1]$. The construction of such pairs is described in Sect. 3.

Construction 2 (overlap/grouping functions with the same diagonal):

Step 1: Let O and G be an overlap function and grouping function, respectively, with the same diagonal.

Step 2: The same as Step 2 of Construction 1.

Step 3: The same as Step 3 of Construction 1.

Construction 2 can be summarized as follows:

Inputs:

- $\delta \in \mathcal{F}$, $O \in \mathcal{D}_O^{\delta}$ and $G \in \mathcal{D}_G^{\delta}$;
- an n-ary aggregation function M such that $M(x_1, \ldots, x_n) = 0$ if and only if $x_1 = \cdots = x_n = 0$ and $M(x_1, \ldots, x_n) = 1$ if and only if $x_1 = \cdots = x_n = 1$.

Output:

$$J^{O,G}(A, B) = \min\left(1, \frac{M\Big(O\Big(A(u_1), B(u_1)\Big), \ldots, O\Big(A(u_n), B(u_n)\Big)\Big)}{M\Big(G\Big(A(u_1), B(u_1)\Big), \ldots, G\Big(A(u_n), B(u_n)\Big)\Big)}\right) \tag{34}$$

is a classe of Jaccard indices.

The composition of Jaccard index with a function with appropriate properties is again Jaccard index. Thus, based on the following proposition, the easy construction of more extensive classes of Jaccard indices is given.

Proposition 4.7 *Let $f : [0, 1] \to [0, 1]$ be an increasing function such that $f(x) = 0$ if and only if $x = 0$ and $f(x) = 1$ if and only if $x = 1$. If $J : FS(U) \times FS(U) \to [0, 1]$ is a Jaccard index, then $f(J)$ is also a Jaccard index.*

5 Measure for comparing fuzzy rules

As highlighted in the Introduction, there are currently no rule comparison methods that simultaneously evaluate both antecedents and consequents when both are defined as fuzzy sets. To address this gap, in this section, we propose a new index that expresses the degree to which we can consider two fuzzy rules to be equal.

Let $\mathcal{R}(U, V)$ denote the set of all fuzzy rules with single input and single output given as follows:

$$R_i : \qquad \text{If } u \text{ is } A_i, \text{ then } v \text{ is } B_i$$

where U, V are non-empty sets, $A_i \in FS(U)$, $B_i \in FS(V)$ and u, v are linguistic variables.

To compare fuzzy rules, we must compare the fuzzy sets which form them. In more detail, for the rules $R_1 = (A_1, B_1)$ and $R_2 = (A_2, B_2)$, the comparison measure should hold the following properties:

1. The measure is symmetric. The comparison measure of rule R_1 and rule R_2 should be the same as the comparison measure of rule R_2 and rule R_1.
2. The comparison measure is equal to zero if, and only if, the supports of the antecedents A_1 and A_2 do not overlap (Fig. 6a), or the supports of the consequents B_1 and B_2 do not overlap or neither the antecedents nor the consequents overlap. In the case of the antecedents, is experimentally evidenced that the outcome of the GMP with very different antecedents is not accurate. Additionally, regarding the consequents, two rules with similar antecedents and different consequents can be seen as a contradiction. These two reasons support this property.
3. The comparison measure is equal to one, whenever each pair of fuzzy sets in the antecedents and consequents are equal (Fig. 6b).

From those properties, we introduce the axiomatic definition of the fuzzy rule comparison measure.

Definition 5.1 A measure for comparing fuzzy rules (MCFR) is a mapping $\mathcal{C} : \mathcal{R}(U, V) \times \mathcal{R}(U, V) \to [0, 1]$ satisfying the following conditions for all $R_1, R_2 \in \mathcal{R}(U, V)$:

$(\mathcal{C}1)$ $\mathcal{C}(R_1, R_2) = \mathcal{C}(R_2, R_1)$;

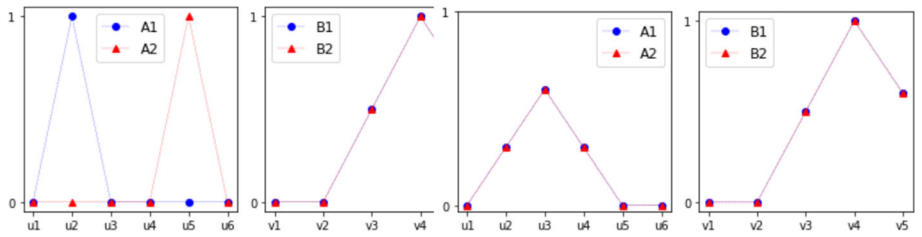


Fig. 6 Fuzzy rules: **a** completely disjoint antecedents, **b** equal antecedent and consequent

($\mathcal{C}2$) $\mathcal{C}(R_1, R_2) = 0$ if and only if $A_1 \cap A_2 = \emptyset$ or $B_1 \cap B_2 = \emptyset$;
($\mathcal{C}3$) $\mathcal{C}(R_1, R_2) = 1$, whenever $A_1 = A_2$ and $B_1 = B_2$.

The class of all measures for comparing fuzzy rules on $\mathcal{R}(U, V)$ is denoted by $MCFR(\mathcal{R}(U, V))$.

An MCFR can be obtained in terms of two Jaccard indices J_1, J_2 (possibly $J_1 = J_2$) and a binary fusion function F with convenient properties, as described in the following theorem.

Theorem 5.2 *Let $J_1 : FS(U) \times FS(U) \to [0, 1]$ and $J_2 : FS(V) \times FS(V) \to [0, 1]$ be Jaccard indices and let $F : [0, 1]^2 \to [0, 1]$ be a function such that:*

(F1) $F(1, 1) = 1$;
(F2) $F(x, y) = 0$ *if and only if* $x \cdot y = 0$.

Then a function $\mathcal{C} : \mathcal{R}(U, V) \times \mathcal{R}(U, V) \to [0, 1]$ defined, for all $R_1, R_2 \in \mathcal{R}(U, V)$, by

$$\mathcal{C}(R_1, R_2) = F\Big(J_1(A_1, A_2), J_2(B_1, B_2)\Big) \tag{35}$$

is a measure for comparing fuzzy rules (i.e. $\mathcal{C}$ satisfies axioms ($\mathcal{C}1$), ($\mathcal{C}2$) and ($\mathcal{C}3$)).

Proof ($\mathcal{C}1$) directly follows from (J1) and Eq. (35). ($\mathcal{C}3$) follows from (J3), Eq. (35) and the assumption $F(1, 1) = 1$. Finally, with respect to ($\mathcal{C}2$), observe that $F\big(J_1(A_1, A_2), J_2(B_1, B_2)\big) = 0$ if and only if $J_1(A_1, A_2) = 0$ or $J_2(B_1, B_2) = 0$, which holds if and only if $A_1 \cap A_2 = \emptyset$ or $B_1 \cap B_2 = \emptyset$. □

Corollary 5.3 *Following* Theorem 5.2 *and* 4.4

$$MCFR(R_1, R_2) = F\left(\min\left(1, \frac{\mathcal{O}(A_1, A_2)}{\mathcal{G}(A_1, A_2)}\right), \min\left(1, \frac{\mathcal{O}(B_1, B_2)}{\mathcal{G}(B_1, B_2)}\right)\right) \tag{36}$$

Example 5.4 (i) In order to obtain $\mathcal{C} \in MCFR(\mathcal{R}(U, V))$, the function F should only satisfy boundary conditions specified in Theorem 5.2. In these cases, for example, the choice of F as any binary overlap function is natural (however, it is not the only possibility). Hence, for

$$\mathcal{C}(R_1, R_2) = O\Big(J_1(A_1, A_2), J_2(B_1, B_2)\Big), \tag{37}$$

where O is an overlap function, we have $\mathcal{C}(R_1, R_2) \in MCFR(\mathcal{R}(U, V))$.

(ii) Copulas (Sklar 1959) can be another choice for the F function. Thus, for

$$\mathcal{C}(R_1, R_2) = CO\Big(J_1(A_1, A_2), J_2(B_1, B_2)\Big), \tag{38}$$

where CO is a copula satisfying (F1) and (F2), we have $\mathcal{C}(R_1, R_2) \in MCFR(\mathcal{R}(U, V))$. An example of such copula is the function $C : [0, 1]^2 \to [0, 1]$, given by $C(x, y) = xy + x^2y(1 - x)(1 - y)$, for all $x, y \in [0, 1]$ (Marco-Detchart et al. 2021).

(iii) Pseudo-overlaps (Zhang et al. 2022) and quasi-overlaps (Paiva et al. 2021) can also be selected as the function F. Then, constructing

$$\mathcal{C}(R_1, R_2) = PQOF\Big(J_1(A_1, A_2), J_2(B_1, B_2)\Big), \tag{39}$$

where $PQOF$ is a pseudo-overlap or a quasi-overlap function satisfying (F1) and (F2), we have $\mathcal{C}(R_1, R_2) \in MCFR(\mathcal{R}(U, V))$. An example of such pseudo-overlap function is the function $POF_m : [0, 1]^2 \to [0, 1]$, given by $POF_m(x, y) = min(x^p, y^p)$, with $p, q > 0$, for all $x, y \in [0, 1]$.

(iv) Aggregation and pre-aggregation (Bustince et al. 2020, 2016; Dimuro et al. 2016; Lucca et al. 2016) functions satisfying (F2) are a possible choice for the function F. Example of such aggregation and pre-aggregation functions functions are, respectively: (Lucca et al. 2018, 2019)

$$F_{RS}(x, y) = \min\left(\frac{(x + 1)\sqrt{y}}{2}, y\sqrt{x}\right)$$

$$F_{NA}(x, y) = \begin{cases} x & \text{if } x \leq y \\ \min(\frac{x}{2}, y) & \text{otherwise.} \end{cases}$$

6 Algorithm to select the best inference method

Fuzzy rules-based systems can be used in expert systems to deal with the natural uncertainty and imprecise information of the human knowledge. One form of inference used by this systems is the GMP presedented by Zadeh (1973), which is a generalization of the traditional modus ponens. The GMP is expressed as follows:

$$\begin{array}{ll} \text{Rule:} & \text{if } x \text{ is } A \text{ then } y \text{ is } B \\ \text{Fact:} & x \text{ is } A' \\ \hline \text{Conclusion:} & y \text{ is } B' \end{array} \tag{40}$$

where $A, A' \in FS(U)$, $B, B' \in FS(V)$, and x and y are variables that take values in U and V respectively.

In order to form the conclusion $B' = \{(v, B'(v))|v \in V\}$ different mechanisms can be used to obtain the elements $B'(v)$, such as:

- **Zadeh's Compositional Rule of Inference** (Zadeh 1973) can be used to calculate each $B'(v)$, given the t-norm T and the t-conorm S and a relation R:

$$B'(v) = \underset{u \in U}{S}(T(A'(u), \mathsf{R}(u, v))) \tag{41}$$

$$\text{Where } \mathsf{R}(u, v) = \underset{\substack{u \in U \\ v \in V}}{I}(A(u), B(v)) \tag{42}$$

- **Zadeh's Interpolation method** (Zadeh 1983), which was generalized by Garcia-Jimenez et al. (2017), obtains each element $B'(v)$ through an overlap function O and an overlap

index $\mathcal{O}$:

$$B'(v) = O(B(v), \mathcal{O}(A, A')) \tag{43}$$

Given the various mechanisms available to infer a conclusion from a rule and a fact under the GMP, and considering that multiple operators must be selected for each mechanism, a formal selection framework is required to identify the optimal configuration. In this context, to determine the most suitable inference method for a system consisting of a single rule R and a fact A', we apply a selection process based on the comparison measure $\mathcal{C}$ (Definition 5.1). For each candidate inference method m_i in a set of available methods $\mathcal{M}$, the procedure is as follows:

1. A conclusion B'_i is derived using the i-th method. $B'_i = m_i(R, A')$
2. A rule is constructed using the provided A' fact as the antecedent and the inferred conclusion B' as the consequent. $R'_i = (A', B'_i)$.
3. The consistency between the reconstructed rule R' and the original rule R is computed as $C_i = C(R'_i, R)$.

Once the comparison values have been computed for all candidate methods, the process concludes by selecting the optimal method:

4. Finally, the chosen method m_j is the one that satisfies $j = argmax_i\{C_i | m_i \in \mathcal{M}\}$.

7 Measure for comparing n-dimensional fuzzy rules

In this section, we extend the measure developed in Sect. 5 to be able to compare fuzzy rules with multiple antecedents.

Let $\mathcal{R}(U^j, V)$ denote the set of all fuzzy rules with multiple input and single output given as follows:

$$R_i : \text{If } u^1 \text{ is } A_i^1 \text{ and } \ldots \text{ and } u^n \text{ is } A_i^n \text{ , then } v \text{ is } B_i$$

where, for $j \in \{1, \ldots, n\}$, U^j and V are non-empty sets, $A_i^j \in FS(U^j)$, $B_i \in FS(V)$ and u^j, v are linguistic variables.

Definition 7.1 A measure for comparing n-dimensional fuzzy rules (MCFRn) is a mapping $\mathcal{C}_n : \mathcal{R}(U^j, V) \times \mathcal{R}(U^j, V) \to [0, 1]$ satisfying the following conditions for all $R_1, R_2 \in (U^j, V)$:

(C_n1) $\mathcal{C}_n(R_1, R_2) = \mathcal{C}_n(R_2, R_1)$;

(C_n2) $\mathcal{C}_n(R_1, R_2) = 0$ if and only if there exists $j \in \{1, \ldots, n\}$ such that $\left(A_1^j\right) \cap \left(A_2^j\right) = \emptyset$ or $B_1 \cap B_2 = \emptyset$;

(C_n3) $\mathcal{C}_n(R_1, R_2) = 1$, whenever $A_1^j = A_2^j$ for all $j \in \{1, \ldots, n\}$ and $B_1 = B_2$.

The class of all measures for comparing fuzzy rules on $\mathcal{R}(U^j, V)$ is denoted by $MCFRn(\mathcal{R}(U^j, V))$.

An MCFRn can be obtained in terms of n+1 Jaccard indices $J_1, \ldots, J_{n+1}$ and an (n+1)-ary fusion function F with convenient properties, as described in the following theorem.

Theorem 7.2 *Let $J_j : FS(U^j) \times FS(U^j) \to [0, 1]$ for $j \in \{1, \ldots, n\}$ and $J_{n+1} : FS(V) \times FS(V) \to [0, 1]$ be Jaccard indices and let $F : [0, 1]^{n+1} \to [0, 1]$ be a function such that:*

- $F(1, \ldots, 1) = 1$;

- $F(x_1, \ldots, x_{n+1}) = 0$ *if and only if there exists* $i \in \{1, \ldots, n+1\}$ *such that* $x_i = 0$.

Then a function $\mathcal{C}_n : \mathcal{R}(U^j, V) \times \mathcal{R}(U^j, V) \to [0, 1]$ *defined, for all* $R_1, R_2 \in \mathcal{R}(U^j, V)$, *by*

$$\mathcal{C}_n(R_1, R_2) = F\Big(J_1(A_1^1, A_2^1), \ldots, J_n(A_1^n, A_2^n), J_{n+1}(B_1, B_2)\Big) \tag{44}$$

is a measure for comparing fuzzy rules (i.e. $\mathcal{C}$ *satisfies axioms* (C_n1), (C_n2) *and* (C_n3)).

Proof (C_n1) directly follows from (J1) and Eq. (44). (C_n3) follows from (J3), Eq. (44) and the assumption $F(1, \ldots, 1) = 1$. Finally, with respect to (C_n2), observe that $F\big(J_1(A_1^1, A_2^1), \ldots, J_n(A_1^n, A_2^n), J_{n+1}(B_1, B_2)\big) = 0$ if and only if $J_2(B_1, B_2) = 0$ or there exists $j \in \{1, \ldots, n\}$ such that $J_j(A_1^j, A_2^j) = 0$, which holds if and only if $B_1 \cap B_2 = \emptyset$ or there exists $j \in \{1, \ldots, n\}$ such that $A_1^j \cap A_2^j = \emptyset$. □

8 Algorithm to select the best inference method in multi-dimensional fuzzy rule bases

We have previously studied how to apply our comparison measure to select the best method for a given input and a rule, in this section we will extend that procedure to multiple rule-based fuzzy systems. The structure of those systems is presented below:

$$\begin{array}{ll}
\text{Rule } R_1: & \text{if } x_1 \text{ is } A_1^1 \text{ and } \ldots \text{and } x_n \text{ is } A_1^n \text{ then } y \text{ is } B_1 \\
\ldots & \\
\text{Rule } R_p: & \text{if } x_1 \text{ is } A_p^1 \text{ and } \ldots \text{and } x_n \text{ is } A_p^n \text{ then } y \text{ is } B_p \\
\text{Fact:} & x_1 \text{ is } A'^1 \text{ and } \ldots \text{and } x_n \text{ is } A'^n \\
\hline
\text{Conclusion:} & y \text{ is } B'
\end{array} \tag{45}$$

where, for $j \in \{1, \ldots, p\}$ and $k \in \{1, \ldots, n\}$, A_j^k, $A'^k \in FS(U^k)$ and B_j, B' $\in FS(V)$. x_k and y are the input and output variables which take values in U^k for k $\in \{1, \ldots, n\}$ and V respectively.

To obtain a conclusion from these methods they can be extended to multiple antecedents as follows:

- **For the compositional rule of inference** (Eq. 41):

  First, a conclusion is obtained for each of the antecedents, for all $k \in \{1, \ldots, n\}$:

  $$B'^k(v) = S(T(A^{k'}(u^k), \mathsf{R}^k(u, v))),$$

  where for all $u \in U$ and for all $v \in V$, $\mathsf{R}^k(u, v) = I(A^k(u^k), B(v))$ is the relation between the k-th antecedent and the consequent of the rule, I is an implication function, S is a t-conorm and T is a t-norm.
  Finally, the obtained conclusions are aggregated with a t-norm T_2:

  $$B'(v) = T_2(B'^k(v)) \quad \text{and} \quad B' = \{(v, B'(v)) \mid v \in V\}$$

  (46)

- **For the interpolation method** (Eq. 43):

The consistency for each pair of respective antecedents is computed with an overlap index $\mathcal{O}$ and then aggregated by a t-norm T:

$$\gamma = T(\mathcal{O}(A^1, A^{1'}), \ldots, \mathcal{O}(A^n, A^{n'})) \tag{47}$$

Then, the interpolation is performed with an overlap function O over the computed consistency and the consequent of the rule:

$$B'(v) = O(B(v), \gamma) \quad \text{and} \quad B' = \{(v, B'(v)) \mid v \in V\}$$

One method that can be applied along the CRI (46) or the interpolation method (47) is called first infer then aggregate (FITA). In this method, each rule is fired individually producing a conclusion B'^j and finally those will be combined by an aggregation operator to obtain the system's output:

$$B' = S_M(B'^j) \; \forall j \in 1, \ldots, p \tag{48}$$

In this context, to determine the most suitable inference method for these more complex systems, we extend the selection framework to the case of n-dimensional rule bases. Given a fuzzy rule base (45) and the n-dimensional rule comparison measure (Definition 7.1), we generalize the selection algorithm described in Sect. 6 to the case of multiple n-dimensional rules. This approach evaluates the consistency of the inferred conclusion by comparing the reconstructed rule against every individual rule within the rule base, subsequently aggregating these results into a single global metric. Specifically, for each candidate inference method $m_i \in \mathcal{M}$, the procedure is performed as follows:

1. A conclusion B'_i is computed by applying the i-th method to the fact A' and the entire Rule Base $\mathcal{B}$: $B'_i = m_i(\mathcal{B}, A')$.
2. A rule is constructed using the provided A' fact as the antecedent and the inferred conclusion B' as the consequent: $R'_i = (A', B'_i)$
3. The reconstructed rule R'_i is compared against each individual rule R^j in the Rule Base, yielding a set of local comparison values $C_{i,j} = \mathcal{C}(R'_i, R^j)$ for $j = 1, \ldots, m$.
4. To obtain a single representative value for the performance of the method, these local results are combined using an aggregation function $\mathcal{A}$, resulting in a global comparison value $C_i = \mathcal{A}(C_{i,1}, \ldots, C_{i,m})$.

Once the comparison values have been computed for all candidate methods, the optimal inference method m_j is selected as the one that maximizes the aggregated score: $j = \arg\max_i \{C_i \mid m_i \in \mathcal{M}\}$

9 Illustrative example: selection of the best inference method for gastric cancer patients life expectancy prediction system

In this section, we apply our methodology to a fuzzy expert system designed to predict the life expectancy of gastric cancer patients. This system is based on the work of Zettervall (2011). The main goal is to employ our proposed comparison measure to select the most suitable inference method for this medical decision support system.

9.1 Problem statement and data description

A gastric cancer diagnosis impacts not only the length but also the quality of the patient's life. The progression of the disease and the aggressive nature of treatments often produce adverse effects that can worsen the patient's final days. In this context, accurate estimation of life expectancy is crucial: a longer survival prognosis might justify aggressive treatment, while a shorter one might suggest palliative care to maximize the quality of remaining time.

To aid in this complex decision, we use a fuzzy expert system based on Zettervall (2011). The system considers two input attributes and one output variable:

- **Input Attributes:**
  - **Age** (x_1)**:** The patient's age, defined in the range [0,100] years.
  - **CRP-value** (x_2)**:** The C-reactive protein level, a clinical marker defined in the interval [0,60] mg/l.
- **Output Variable:**
  - **Survival length** (y)**:** The estimated life expectancy, taking values in the interval [0, 5] years.

9.2 Fuzzy inference system and methodology

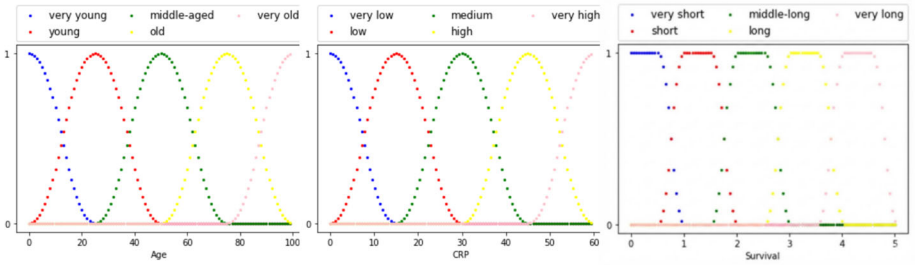
The proposed approach is designed to select the most suitable inference method for a specific input. The overall process is illustrated in Fig. 8 and consists of five main steps:

1. **Fuzzification:** The crisp input values (Age and CRP-value) are converted into fuzzy sets using the defined membership functions.
2. **Inference:** Several inference methods are applied in parallel to the fuzzified inputs.
3. **Evaluation:** With the inferred consequent, a new rule $R_i\,(A',\ B_i')$ is formed and compared against the system's rule base. This comparison is performed using the proposed fuzzy rule comparison measure (Definition 7.1), which quantifies the consistency between the method's output and the expert knowledge.
4. **Selection:** The inference method that yields the highest similarity value is selected as the optimal one for the current case.
5. **Defuzzification:** The output of the selected best method is defuzzified to provide the final estimated survival length.

The fuzzy sets for the variables were defined via parametric S-functions, following the medical criteria described in (2011). To enable the numerical computation of the inference and comparison measures, these continuously defined fuzzy sets are discretized into a universe of discourse consisting of 100 points. The graphical representation of these membership functions is shown in Fig. 7, where the subplots from left to right correspond to Age, CRP-value, and Survival length, respectively. Their linguistic terms are defined as follows:

- **Age:** Linguistic terms are Very young, Young, Middle-aged, Old, Very old.
- **CRP-value:** Linguistic terms are Very low, Low, Medium, High, Very high.
- **Survival length:** Linguistic terms are Very short, Short, Middle-long, Long, Very long.

The knowledge base consists of fuzzy rules generated from patient data. It is important to note that the construction of these rules and the membership functions strictly follows the mathematical formulation proposed by Zettervall, specifically using Equation (3.11) from her



(a) Age (b) CRP (c) Survival

Fig. 7 Membership functions of the fuzzy sets for the clinical variables. From left to right: Age (years), CRP-value (mg/L), and Survival length (years). Each variable is represented by five linguistic terms as defined in the main text

Table 1 Rule base for the fuzzy controller estimating "survival length"

				CRP		
Survival	Very low	Low	Medium	High	Very high	
	Very young					
	Young					
Age	Middle-age	Middle-Long				
	Old	Middle-long	Short	Short	Short	Very short
	Very old	Short	Short	Very short	Very short	Very short

thesis (2011). These equations model the relationship between the physiological parameters and the expected survival length.

The resulting rule base contains 11 fuzzy rules, represented as:

$$R^k : \text{If Age is } A^k \text{ and CRP is } B^k, \text{ then Survival is } C^k \tag{49}$$

where k is the rule index. The full rule base configuration is shown in Table 1.

9.3 Experimental study

The main objective of this experiment is to demonstrate how our proposed methodology identifies the most suitable inference method for specific input cases. The system works by applying multiple inference methods in parallel for a given input. It then compares the inferred rule (formed by the input and its computed output) against the entire established rule base.

To validate this approach, we conduct a comprehensive sensitivity analysis to evaluate the robustness of the selection process across three dimensions of variability: clinical scenarios (inputs), inference parameterizations (methods), and comparison measures (metrics).

Regarding the input sensitivity, to ensure the system responds adequately to diverse clinical scenarios, we selected four distinct patient profiles (input pairs of age and CRP-value):

- Middle-aged and very-low
- Old and low
- Very-old and very-low

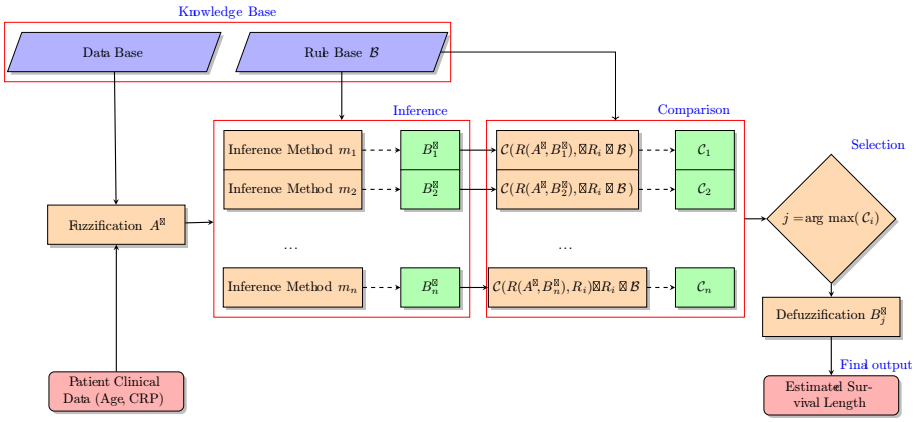


Fig. 8 Flowchart of the proposed methodology for the gastric cancer prediction system. The diagram illustrates the execution of multiple inference engines, the subsequent evaluation stage and the selection of the most logically consistent output

Table 2 Fuzzy logic operators and indices utilized in the experimental study

Operator	fName	Expression
I_q	Klir-Yuan 2 Implication	$I_q(x,y) = \begin{cases} y & \text{if } x = 1 \\ 1-x & \text{if } x \neq 1 \text{ and } y \neq 1 \\ 1 & \text{otherwise} \end{cases}$
$I_\triangle$	Goguen Implication	$I_\triangle(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise} \end{cases}$
I_r	Reichenbach Implication	$I_r(x,y) = 1 - x + xy$
T_P	Product t-norm	$T_P(x,y) = xy$
T_M	Minimum t-norm	$T_M(x,y) = \min(x,y)$
S_M	Maximum t-conorm	$S_M(x,y) = \max(x,y)$
O_Z	Zadeh's Consistency Index	$O_Z(A,B) = \max_u(\min(A(u), B(u)))$
O_P	Product Overlap Index	$O_P(A,B) = \max_u(A(u) \cdot B(u))$
O_{GM}	Geometric Mean Overlap Index	$O_{GM}(A,B) = \max_u \sqrt{A(u) \cdot B(u)}$

- Very-old and very-high

For each of these cases, we apply the selection algorithm described in Sect. 8. To address the methodological sensitivity, we evaluate 12 different parameterizations encompassing the CRI method (46) and Zadeh's interpolation method (47), detailed in Tables 3 and 4. All configurations employ the maximum t-conorm S_M in Eq. (48) to aggregate the partial conclusions into the final system output. Table 2 summarizes the specific fuzzy operators, implications, and overlap indices employed in these different inference methods.

9.3.1 Evaluation metrics

To evaluate and compare the performance of each inference method, we use the fuzzy rule comparison measure defined in our methodology. This metric represents the degree of similarity (ranging from 0 to 1) between the method's output and the expert knowledge base. A

Table 3 Different parametrizations of the CRI method

Method	I	T	S	T_2
1	I_q	T_P	S_M	T_M
2	$I_\triangle$	T_P	S_M	T_M
3	I_r	T_P	S_M	T_M
4	I_q	T_M	S_M	T_M
5	$I_\triangle$	T_M	S_M	T_M
6	I_r	T_M	S_M	T_M

Table 4 Different parametrizations of the interpolation method

Method	O	T	O
7	O_Z	T_M	T_P
8	O_P	T_M	T_P
9	O_{GM}	T_M	T_P
10	O_Z	T_M	T_M
11	O_P	T_M	T_M
12	O_{GM}	T_M	T_M

higher score indicates that the specific inference parameterization is more logically consistent with the reference rules for that particular input.

Finally, to analyze the metric sensitivity and the behaviour of the proposed generalized Jaccard index, we performed two parallel experiments to test different comparison measures (constructed following Eq. (44) from Theorem 7.2, selecting the minimum as the function F):

- **Experiment A (Original Jaccard):** Uses the standard Jaccard index, defined for discrete fuzzy sets as expressed in Eq. (28).
- **Experiment B (Generalized Jaccard):** Uses the proposed Generalized Jaccard index, constructed following Theorem 4.4 through Eq. (31), where:

$$\mathcal{O}(A, B) = \frac{1}{n}\sum_{i=1}^{n}\sqrt{A(u_i)\cdot B(u_i)} \tag{50}$$

$$\mathcal{G}(A, B) = \frac{1}{n}\sum_{i=1}^{n}\left(1-\sqrt{(1-A(u_i))\cdot(1-B(u_i))}\right) \tag{51}$$

9.4 Results and discussion

The comparison scores obtained for Experiment A and Experiment B are displayed in Tables 5 and 6, respectively. An in-depth examination of this numerical data reveals several key findings regarding both the selection methodology and the behaviour of the proposed metrics.

First, the proposed approach successfully identifies a best-performing method across all sensitivity test cases. In both experiments and regardless of the patient profile, Method 8 (Interpolation with $\mathcal{O} = \mathcal{O}_P$, $T = TM$ and $\mathcal{O} = T_P$) consistently yields the highest similarity score. For instance, in the 'Very-old and very-low' profile, Method 8 achieves a

Table 5 Results for experiment A—Original Jaccard index

Method	Middle-age Very low	Old Low	Very old Very low	Very old Very high
Method 1	0.20000	0.20000	0.20065	0.15509
Method 2	0.20000	0.20000	0.36056	0.15509
Method 3	0.20000	0.20000	0.20057	0.15509
Method 4	0.20000	0.20000	0.20022	0.15509
Method 5	0.20000	0.20000	0.35334	0.15509
Method 6	0.20000	0.20000	0.20022	0.15509
Method 7	0.71134	0.58552	0.70910	0.65428
Method 8	**0.82484**	**0.73057**	**0.81965**	**0.78000**
Method 9	0.69394	0.56499	0.69179	0.63535
Method 10	0.68456	0.56294	0.68335	0.62570
Method 11	0.80098	0.71003	0.79817	0.75380
Method 12	0.66735	0.54280	0.66618	0.60720

Bold values indicate the highest rule comparison measure obtained for each case. The maximum value identifies the most suitable inference method, representing the best performance among the compared methods

Table 6 Results for experiment B—Generalized Jaccard index

Method	Middle-age Very low	Old Low	Very old Very low	Very old Very high
Method 1	0.22601	0.22601	0.24008	0.23671
Method 2	0.22601	0.22601	0.43146	0.23671
Method 3	0.22601	0.22601	0.23985	0.23671
Method 4	0.22601	0.22601	0.23671	0.23671
Method 5	0.22601	0.22601	0.42439	0.23671
Method 6	0.22601	0.22601	0.23671	0.23671
Method 7	0.82306	0.73070	0.82224	0.78253
Method 8	**0.90370**	**0.84541**	**0.90197**	**0.87720**
Method 9	0.80892	0.71185	0.80811	0.76621
Method 10	0.80569	0.71717	0.80500	0.76275
Method 11	0.88919	0.83268	0.88764	0.85978
Method 12	0.79113	0.69816	0.79045	0.74614

Bold values indicate the highest rule comparison measure obtained for each case. The maximum value identifies the most suitable inference method, representing the best performance among the compared methods

score of 0.81 in Experiment A and 0.90 in Experiment B, significantly outperforming the CRI-based configurations (Methods 1-4), which generally score below 0.4.

This numerical gap is visually corroborated in Fig. 9. By examining the membership functions produced for this specific case, we can observe that interpolation-based methods, particularly Method 8 and 11, closely approximate the shape of the expected consequent, showing minimal deviation in the second half of the fuzzy set. On the other hand, the low scores for CRI methods are justified by their structural inconsistency with the expert knowledge: Method 1 produces a constant membership of 1 (losing all discriminative power), while

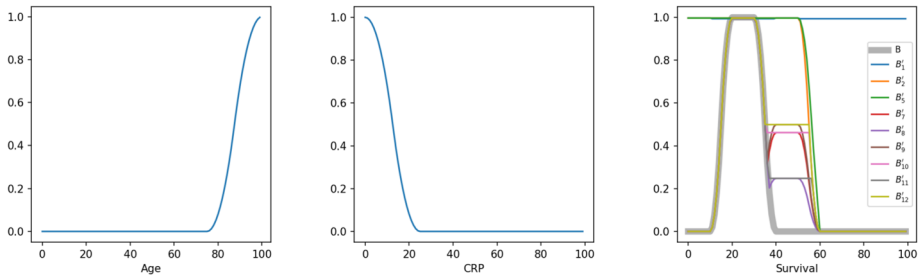


Fig. 9 Visual representation of fuzzy inference outputs for the 'Very-old and very-low' patient profile. The leftmost figures show the input fuzzy sets (Age and CRP). The survival plot compares the expected fuzzy set (expert rule consequent) against the outputs of different inference methods. Interpolation methods show high geometric overlap with the reference, while CRI methods exhibit structural deviations, such as saturation at 1 and delayed downward slopes

Methods 2 and 5 maintain a constant value of 1 and only begin their downward slope much later than the reference set. This clear visual correlation proves the algorithm's capability to unequivocally identify the inference method that produces the best logical match with the underlying knowledge base.

Second, a structural comparison between the two tables highlights the operational differences and the specific advantages of our proposed Generalized Jaccard measure. In Table 5 (Experiment A), the original Jaccard index yields generally lower comparison values (ranging roughly between 0.15 and 0.82), as the strict nature of the standard intersection heavily penalizes partial overlaps. Conversely, Table 6 (Experiment B) demonstrates that our construction based on overlap and grouping functions mitigates this penalization. As a result, the generalized index tends to assign higher similarity scores overall, shifting the values upward (approximately 0.22 to 0.90). The generalized Jaccard index allows for a more flexible interpretation of fuzzy similarity, avoiding the severe penalization of the standard minimum-based intersection when evaluating partial overlaps, while still perfectly maintaining the discriminative power necessary to select the optimal inference method. This grants greater flexibility, allowing the system designer to adjust the strictness of the similarity evaluation depending on the specific requirements of the application.

Finally, while the proposed dynamic selection methodology and the generalized metric offer significant advantages, certain limitations must be acknowledged and contextualized:

Computational Overhead: The system requires computing multiple inference parameterizations in parallel for a single input before making a selection. Although the computational cost of executing a single fuzzy inference method is not particularly high, performing multiple executions introduces an additional load. However, this parallelization is easily handled by modern computing architectures, making the approach perfectly viable for clinical decision support systems, where the accuracy of the selected rule is far more critical than strict real-time execution speeds.

Knowledge Base Dependency: The effectiveness of the dynamic selection is intrinsically tied to the quality of the expert knowledge base. Since the algorithm optimizes for logical consistency with the established reference rules, any inaccuracies or omissions in the expert knowledge will inevitably influence the selection process.

Parameter Selection and Flexibility: The Generalized Jaccard index introduces significant flexibility through its mathematical formulation (Theorem 4.4), which requires the user to select specific overlap and grouping functions. Although our experimental study focused on

evaluating two specific, representative metric configurations to demonstrate this approach, the results obtained were highly consistent and coherent in identifying the optimal inference method. While absolute coherence cannot be theoretically guaranteed for every possible, unexplored combination of functions, the empirical evidence suggests that the metric remains robust under reasonable design choices.

10 Conclusion

In this paper, we present an algorithm for a decision-making system to determine the best inference method. Thus, given a set of inference methods and a fact or input to the system, we can determine which inference method is the most suitable inference method based on the available information. This decision-making system considers the output of the inference system and constructs a new rule composed of the input fact and the output as a consequent. The system selects the inference method that produces a rule that most closely matches the given rule base.

For the development of this algorithm, we have defined a fuzzy rule comparison measure that determines the degree of similarity between two fuzzy rules. This symmetric measure returns the value 1 if both rules are equal and the value 0 if any pair of fuzzy sets of the rule components is disjoint. For the construction of the measure, we have used the Jaccard index, which is widely used in the literature to compare fuzzy sets.

In addition, we propose a generalization of the Jaccard index to overcome the rigidity of the original definition and to allow more flexibility in the definitions of the comparison measures. This new generalized definition of the Jaccard index is based on overlap indices and grouping indices. We describe the properties of the Jaccard index and the requirements that the overlap and grouping indices must fulfill to be employed in the construction methods that we propose.

Finally, we apply the algorithm to a gastric cancer patient survival length estimator expert system. We selected two comparison measures, one based on the Jaccard index and the other based on the generalized Jaccard index. We observed that, for the selected measures, the algorithm selected the same inference method. By means of this example, we showed that our methodology can be applied to select an inference method to be used in a specific fuzzy rule system.

In our experimentation with the selected operators, we observed that the measures proposed to construct the comparison measure led to the same inference methods. However, the measure based on the generalized Jaccard index provides higher comparison values and wider ranges between the most and least similar rules. Both approaches are correct and useful, but their application depends on the specific problem and requirements. An adequate selection of the grouping and overlap indices can adapt the behavior of the measure to the needs of the specific application.

Beyond the specific clinical case studied, the proposed theory has significant practical implications for XAI frameworks based on fuzzy rules. In critical domains such as healthcare or finance, fuzzy rule-based systems are often preferred due to their white-box nature and linguistic interpretability. A key advantage of the proposed methodology is that it enables the system to autonomously determine the most suitable inference result for each specific input. This eliminates the need for manual intervention to calibrate or tune the inference parameters, ensuring that the system remains highly accurate and adaptive while fully preserving the transparency of the expert-validated rule base.

In future works, we intend to extend this comparison measure to other fuzzy rule systems based on continuous fuzzy sets, as well as to fuzzy rules with different consequent types, such as classification systems or numerical outputs.

It would be interesting to extend the proposed approach to the context of intuitionistic fuzzy rough set theory, to handle complex uncertainty, inconsistent information, vagueness, and incompleteness in data (Khan et al. 2025a, b).

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Data Availability The data used for the illustrative example in Section 9 is based on the study by Zettervall (2011), as described in the text, and can be extracted from that cited work. Additionally, the expert knowledge base and the fuzzy sets required to replicate the results are fully provided within this manuscript in Table 1 and Figure 7, respectively. The authors remain at your disposal for any further clarifications regarding these data.

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