# Complete Reducibility in Kac-Moody Groups

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# Abstract


If $G$ is a reductive group with Borel subgroup $B$ then the coset space $G/B$ is the chamber set of a spherical building whose Weyl distance is governed by the Bruhat order on the Weyl group. For Kac-Moody groups, the two conjugacy classes of standard Borel subgroups give rise to a twin building, with a codistance governed by the reverse Bruhat order. However, infinite root systems admit many more conjugacy classes of sets of positive roots, and these are not captured by the twin building.

This thesis defines a new object, called a citadel, designed to organise these non-standard conjugacy classes. A citadel consists of a distinguished central building, called the keep, together with further buildings associated to other conjugacy classes of positive systems. The additional buildings are related to the keep through distance functions defined by Dyer's twisted Bruhat orders. It is proved that a citadel supports notions of projections between residues, leading to a notion of convexity.

The thesis proves that a Kac-Moody group possesses two canonical citadels that are shown to form a twin citadel. The set of chambers of the twin citadel can be identified with the disjoint union of coset spaces $G/Q_\Psi$, where the $Q_\Psi$ are quasi-Borel subgroups generalising standard Borel subgroups. The construction requires the development of quasi-parabolic subgroups, generalising the parabolic subgroups, and an extension of the Bruhat-type decomposition of Billig and Dyer.

The thesis concludes by defining a notion of complete reducibility for convex subsets of twin citadels. This leads to a new notion of $G$-complete reducibility for subgroups of Kac-Moody groups, generalising existing formulations that have been derived using the twin building.

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# Contents

CHAPTER 1

# Introduction

The theory of reductive groups over algebraically closed fields is rich and pervades modern algebra and group theory. At its core, a reductive group is controlled by a finite root system that is determined by a choice of maximal torus, and many important subgroups such as the Borel subgroups and the parabolic subgroups can be characterised in this way. For example, the choice of a Borel subgroup determines a so-called standard positive system, separating the roots into a (standard) positive set and corresponding negative set, satisfying some natural axioms.

The collection of proper parabolic subgroups in a reductive group admits a surprisingly combinatorial structure called a spherical building. This object is locally isomorphic to the Coxeter complex of the finite Weyl group, equipped with a distance function that is governed by the Bruhat order. By studying the action of the group on its building, one can translate properties of subgroups into geometric questions about fixed point sets in the building. This is the basis of Serre's notion of $G$-complete reducibility which generalises the question of whether a representation

$$\rho : H \to \mathrm{GL}(V)$$

is completely reducible by replacing $\mathrm{GL}(V)$ with an arbitrary reductive group.

A Kac-Moody group is a natural generalisation of a reductive group that is typically infinite-dimensional. Nevertheless, every such group admits a maximal torus which determines a root system and Weyl group, meaning one can define notions of Borel subgroups and parabolic subgroups accordingly. The Weyl group is a crystallographic Coxeter group and the root system is often infinite. As a consequence, the geometry underlying the group is more complex. For instance, an infinite root system admits two distinct conjugacy classes of standard sets of positive roots, rather than just one in the finite case. This leads the collection of proper parabolic subgroups of a Kac-Moody group to form an object called a twin building, consisting of two buildings, one for each conjugacy class of Borel subgroup, and a codistance function between them that is governed by the reverse Bruhat order.

It has been known since at least [27] that there are non-standard sets of positive roots in infinite root systems. In affine root systems there are finitely many conjugacy classes of sets of positive roots; otherwise, there may be infinitely many. Furthermore, extensive work of Dyer [15] associates to every set of positive roots a twisted Bruhat order which recovers the usual Bruhat order and reverse Bruhat order in the situations above.

Therefore, a twin building does not capture a full range of positive systems. At the level of the Kac-Moody group, this suggests the existence of a natural class of subgroups associated to non-standard sets of positive roots, which are not encoded by the usual twin building. This observation leads to the main structure defined in this thesis, a *citadel*. A citadel generalises the notion of a building, or one half of a twin building, by adjoining extra buildings associated to conjugacy classes of non-standard sets of positive roots.

The use of the term citadel has been chosen to suggest a distinguished central building, called the *keep*, surrounded by buildings clustered according to the conjugacy class of their associated set of positive roots. The outer buildings are related to the keep via a twisted distance, but need not have a notion of distance between them.

A citadel consists of a collection of *twisted buildings* which are pairs of buildings $\mathscr{B}$ and $\mathscr{B}_\Psi$ in which $\mathscr{B}$ is a building associated to a Coxeter group $W$ and $\mathscr{B}_\Psi$ is an additional building associated to a reflection subgroup $W_\Psi$ of $W$. The pair must admit a twisted distance function in which the usual length function is replaced by Dyer's twisted length function $l_\Psi$. It is proved in Subsection 3.2.3 that there is a well-defined notion of projections between residues in twisted

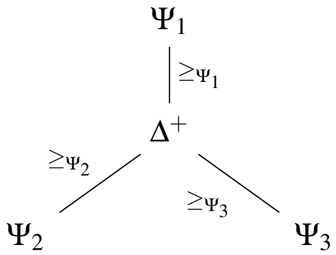


Figure 1.1: A schematic of a citadel: the keep corresponds to the standard positive system $\Delta^+$ while the surrounding components correspond to other conjugacy classes of positive systems, related to the keep via distances defined with twisted Bruhat orders.

buildings, generalising the coprojection maps found in twin buildings.

The reflection subgroups $W_\Psi$ are often proper subgroups of $W$. As a consequence, the ambient group $W$ does not generally act on the Coxeter complex of $W_\Psi$. Therefore, the choice of representative set of positive roots $\Psi$ within its orbit affects the building $\mathscr{B}_\Psi$. In order to remove this dependency, a citadel contains a family of buildings of isomorphic type for each orbit.

**Definition 1.1.** A *citadel* of type $(W,S)$ is a triple $(\mathscr{C},\mathscr{B},\delta^*)$ consisting of

- a disjoint union $\mathscr{C}$ of families of buildings, running over orbits of sets of positive roots,
- a distinguished building $\mathscr{B}$ of type $(W,S)$,
- a $W$-valued map

$$\delta^* : (\mathscr{B}\times\mathscr{C})\cup(\mathscr{C}\times\mathscr{B}) \to W,$$

such that the restriction of $\delta^*$ to any pair of buildings containing $\mathscr{B}$ defines a twisted building.

The negation of a set of positive roots is always a set of positive roots. Therefore, there is a partition of the collection of sets of positive roots induced via negation. This leads to the notion of a *twin citadel* in which one demands every building forms a twin building with its opposite. In both the citadel and the twin citadel there are notions of projection maps and convexity arising from the twisted and twin buildings inside. The main advantage of considering a twin citadel is the guarantee of a notion of opposition that can be defined at the level of the twin buildings it contains.

Let $G$ be a Kac-Moody group with maximal torus $T$ and root system $\Delta = \Delta(G,T)$. For every set of positive roots $\Psi \subseteq \Delta$ one can define a *quasi-Borel subgroup* $Q_\Psi$ in a way that generalises the standard Borel subgroup construction. In general, a quasi-Borel subgroup is of the form

$$Q_\Psi = U_\Psi N_\Psi$$

where $U_\Psi$ is generated by root groups indexed by $\Psi$ and $N_\Psi$ is a subgroup normalising $T$. The groups $Q_\Psi^0 = U_\Psi T$ appear in the work of Billig and Dyer [7, Thm. 1, Cor. 1'] where the following result, which will be called the *Billig-Dyer decomposition*, was established.

**Theorem 1.2.** *Let $B$ be a standard Borel subgroup of $G$ relative to $T$ and write $Q^0 = Q_\Psi^0$. Then*

$$G = \bigsqcup_{w \in W} BwQ^0.$$

*Moreover, for every simple reflection $s_\alpha \in S$ and $w \in W$, one has*

$$BswQ^0 \subseteq BsBwQ^0 \subset BwQ^0 \cup BswQ^0$$

*Furthermore, if $l_\Psi(sw) > l_\Psi(w)$ then $BsBwQ^0 = BswQ^0$.*

It is often the case that a Bruhat-type decomposition suggests the existence of a $W$-valued map between buildings. In this context, the Billig-Dyer decompositions define twisted buildings and lead to the construction of a twin citadel structure in Chapter 4. The asymmetry of the decomposition is precisely what distinguishes the keep from the rest of the citadel. The first result in this direction is the following.

**Proposition 1.3.** *The coset space $G/Q_\Psi$ is a family of buildings.*

The families one obtains include the standard flag variety, which will become the keep, the semi-infinite flag variety when $G$ is a loop group, and a spectrum of other twisted analogues in general.

The Billig-Dyer decomposition defines most of the twisted building structure but misses an important component associated to right multiplication of certain double cosets. This is resolved by considering a wider class of subgroups called the *quasi-parabolic subgroups* that extend the notion of a parabolic subgroup. It is proved that such subgroups have a pleasant decomposition into factors partitioned by the standard set of positive roots.

**Proposition 1.4.** *A quasi-parabolic subgroup is of the form* $Q_\theta = (U_\theta \cap U^+)(U_\theta \cap U^-)N_\theta$.

The decomposition of Proposition 1.4 provides an effective way to study quasi-parabolic subgroups and leads to the following extension of the Billig-Dyer decomposition.

**Proposition 1.5.** *Let* $B$ *be a standard Borel subgroup of* $G$ *relative to* $T$ *and write* $Q^0 = Q^0_\Psi$. *For every reflection* $t \in S_\Psi$ *and* $w \in W$*, one has*

$$BwtQ^0 \subseteq BwQ^0tQ^0 \subseteq BwQ^0 \cup BwtQ^0.$$

*Furthermore, if* $l_\Psi(wt) > l_\Psi(w)$ *then* $BwQ^0tQ^0 = BwtQ^0$.

From this point, everything is in place to define a citadel structure out of the coset spaces of Proposition 1.3. In fact, there is a natural twin citadel structure which contains the usual twin building of the group as the keep.

**Theorem 1.6.** *The disjoint union* $\mathscr{C}_G$ *of*

$$\mathscr{C}^+ = \bigsqcup_{\Omega} G/Q_\Psi \quad \text{and} \quad \mathscr{C}^- = \bigsqcup_{\Omega} G/Q_{-\Psi}$$

*admits a twin citadel structure of type* $(W,S)$.

The group $G$ acts naturally on $\mathscr{C}_G$ and identifies the stabilisers of chambers with quasi-Borel subgroups and the stabilisers of residues with *facial* quasi-parabolic subgroups. The extent to which facial quasi-parabolic subgroups describe all quasi-parabolic subgroups is unclear.

The initial motivation for this work was to study complete reducibility in Kac-Moody groups. After establishing the framework of citadels and constructing a twin citadel associated to a Kac-Moody group, the central definition and first properties of the theory are immediate.

**Definition 1.7.** A convex subset $\mathbf{M}$ of a twin citadel $\mathscr{C}$ is *$\mathscr{C}$-completely reducible* (*$\mathscr{C}$-cr* for short) if every residue in $\mathbf{M}$ has an opposite in $\mathbf{M}$.

The notion of complete reducibility relies on a global notion of convexity. However, the opposition relation is local in the sense that it is defined at the level of the twin buildings. This allows one to apply results from the theory of complete reducibility in twin buildings [12] to obtain the following result which generalises a classical result of Serre [41].

**Theorem 1.8.** *Let* $\mathbf{M}$ *be a convex subset of a twin citadel* $\mathscr{C}$*. Suppose that* $\mathbf{M}$ *contains a Levi sphere* $S$ *in every twin building of* $\mathscr{C}$ *that it meets. Then* $\mathbf{M}$ *is* $\mathscr{C}$*-cr if and only if* $\mathbf{M}_S$ *is* $\mathscr{C}_S$*-cr for all* $S$*.*

Roughly speaking, this means that one can reduce the theory of complete reducibility to certain spherical subbuildings, provided one deals with sufficiently well-behaved convex sets.

At the level of the Kac-Moody group, complete reducibility in the twin citadel can be translated into a group-theoretic definition of $G$-complete reducibility ($G$-cr for short) in the group. In particular, one has

$$H \text{ is } G\text{-cr} \iff \mathscr{C}_G^H \text{ is } \mathscr{C}_G\text{-cr}$$

for any subgroup $H$ of $G$. This defines a new notion of complete reducibility for subgroups of Kac-Moody groups, extending an existing notion due to Caprace [9] that is defined analogously using the twin building. The equivalence above can be used to obtain a group-theoretic version of Theorem 1.8.

**Corollary 1.9.** *Let* $H$ *be a subgroup of* $G$*. Suppose that whenever* $H$ *is contained in a maximal facial quasi-parabolic subgroup* $P_\Psi$ *of* $G$ *then* $H$ *is contained in a finite type quasi-Levi subgroup* $L_\Psi$ *of some facial quasi-parabolic subgroup* $P'_\Psi \subseteq P_\Psi$*. Then* $H$ *is* $G$*-cr if and only if* $H$ *is* $L_\Psi$*-cr for all such* $L_\Psi$*.*

## Organisation

The thesis is organised as follows. In Chapter 2 the necessary background of Coxeter theory, buildings, and Kac-Moody theory is recalled, placing some emphasis on an exposition of the recent realisation of Kac-Moody groups as ind-varieties. In Chapter 3 the abstract theory of citadels and twin citadels is introduced. This begins with a review of Dyer's twisted Bruhat orders and ends by defining a notion of a twin citadel for any crystallographic Coxeter system. In Chapter 4 a twin citadel is constructed out of a Kac-Moody group, providing a motivating example in the wild. In order to do this, a generalisation of parabolic subgroups is defined and studied in a style similar to the theory of parahoric subgroups for Kac-Moody groups over valued fields. Finally, Chapter 5 defines a notion of complete reducibility in twin citadels which leads to a definition of $G$-complete reducibility for subgroups of Kac-Moody groups, generalising the existing notions in the literature.

CHAPTER 2

# Preliminaries

This chapter describes the prerequisites required for the rest of the thesis. Throughout, $k$ will denote an algebraically closed field of characteristic $p \geq 0$. A ring will be assumed to be commutative with identity. If $R$ is a ring then write $R^\times$ for the unit group of $R$. Write $\mathbb{N} = \{1, 2, \cdots\}$ and $\mathbb{Z}_{\geq 0} = \mathbb{N} \cup \{0\}$.

## 2.1 Coxeter theory

In this section the basics of Coxeter groups and Coxeter systems are recalled. These are objects that are intimately related to reflection groups and appear at the heart of the theory of reductive groups and Kac-Moody groups. In particular, every reductive or Kac-Moody group possesses a certain reflection subgroup called the *Weyl group* which governs its behaviour. A comprehensive reference for this material is [1].

A *Coxeter group* is a group of the form

$$W = \langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \rangle$$

where every $s_i$ is an involution, and if $i \neq j$ then either $m_{ij} \geq 2$ or $m_{ij} = \infty$ if $(s_i s_j)^m \neq 1$ for any $m \in \mathbb{Z}$. If $S = (s_i)_{i \in I}$ is a generating set for $W$, then say $(W, S)$ is a *Coxeter system*. The elements of $S$ are called the *simple reflections*, while the *reflections* are given by $R_W = \bigcup_{w\in W} wSw^{-1}$. Throughout this thesis, assume that $S$ is finite and that $(W,S)$ is *crystallographic*, i.e. $m_{ij} = 2,3,4,6$, or $\infty$. These Coxeter systems are precisely those arising in Kac-Moody theory. If $(W,S)$ is irreducible then it is said to be *spherical*, *affine*, or *indefinite* type according to the signature of the symmetric bilinear form introduced below. In particular, the group $W$ is finite precisely when $(W,S)$ is spherical type.

**Example 2.1.**
1. The symmetric groups $S_n$ are spherical Coxeter groups. A set of simple reflections for $S_n$ is given by the transpositions $s_i = (i\, i+1)$ for $1 \le i \le n-1$.
2. The affine symmetric groups $\tilde{S}_n \cong S_n \ltimes \mathbb{Z}^{n-1}$ are affine Coxeter groups. A set of simple reflections for $\tilde{S}_n$ is the set of simple reflections $s_i$ in the previous example, along with a transposition $s_0$ satisfying
$$(s_0 s_1)^3 = 1 = (s_0 s_{n-1})^3.$$
3. The infinite dihedral group $D_\infty$ is the only infinite Coxeter group of rank two. A set of simple reflections is given by two elements $s$ and $t$ that do not commute.
4. The modular group $\mathrm{PGL}_2(\mathbb{Z})$ is a Coxeter group of indefinite type. A set of simple reflections is given by the elements
$$s_1 = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \quad s_2 = \left[\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\right] \quad s_3 = \left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$
with $m_{12} = 3$, $m_{13} = 2$, and $m_{23} = \infty$ in the notation from before.

The elements of $W$ can be represented as *words* of the form $w = s_1 \cdots s_d$ where $s_i \in S$. This defines a *(standard) length function* $l : W \to \mathbb{Z}_{\ge 0}$ on $W$, where $l(w) = d$ if the expression $w = s_1 \cdots s_d$ is minimal. In this case, the expression $w = s_1 \cdots s_d$ is called *reduced*. By convention, $l(w) = 0$ if $w$ is the identity. Otherwise, the length of an element is a positive integer. An immediate consequence of the definition is that $l(w) = l(w^{-1})$ for all $w \in W$. If $s \in S$ then $l(sw) = l(w) \pm 1$ in general. This follows from the *deletion condition*, which will not be stated here (cf. [1, §2.1]).

If $W$ is spherical then the length function on $W$ is bounded. In particular, there is a unique *longest element* $w_0$ of $W$, for which $l(w_0)$ is maximal. This element is an involution, and has the property that

$$l(ww_0) = l(w_0) - l(w) = l(w_0 w)$$

for all $w \in W$. By contrast, the length function of a non-spherical Coxeter group is unbounded and there is no such longest element.

For every subset $J \subseteq S$ one defines the *standard parabolic subgroup* $W_J = \langle J \rangle$ in $W$. A standard parabolic subgroup $W_J$ is a Coxeter system with generating set $J$. Moreover, the standard length function on $W_J$ coincides with the length function induced from $W$. If $W_J$ is finite then write $w_0(J)$ for the longest element of $W_J$.

It is possible to define a partial order on $W$ using the length function. Given $v, w \in W$, write $v \leq w$ if and only if there is a sequence $v = v_0, v_1, \ldots, v_n = w$ of elements in $W$ with $l(v_i) = l(v_{i-1}) + 1$ and $v_i v_{i-1}^{-1} \in R_W$. This order is called the *Bruhat order* and will play an important role in the sequel.

Let $V = \mathbb{R}^{|I|}$ be the real vector space with basis consisting of symbols $(e_i)_{i \in I}$. Then $W$ defines a symmetric bilinear form on $V$ by

$$B(e_i, e_j) = -\cos \frac{\pi}{m_{ij}}$$

for all $i, j \in I$. Here, one sets $B(e_i, e_j) = -1$ if $m_{ij} = \infty$. Then, for all $i \in I$ and $x \in V$, the equations

$$s_i(x) = x - 2B(e_i, x)e_i$$

define an action of $W$ on $V$ by linear reflections. In particular, there is a linear representation $\rho : W \to \mathrm{GL}(V)$ which is faithful. Hence, every Coxeter group can be realised as a real reflection group.

The basis vectors $e_i$ of $V$ are called the *simple roots* of $W$, and more commonly denoted by $\alpha_i = e_i$. A *root* of $W$ is a vector $\alpha \in V$ of the form $\alpha = w\alpha_i$ for some $w \in W$ and $i \in I$. The set of roots $\Delta$ is called the *root system* associated to $(W, S)$. The reflection corresponding to a root $\alpha = w\alpha_i$ is defined to be the conjugate $s_\alpha = ws_i w^{-1} \in W$. It is precisely the reflection through the hyperplane orthogonal to $\alpha$ in $V$.

The roots of $\Delta$ whose coordinates are non-negative with respect to the simple roots $(\alpha_i)_{i\in I}$ form the *standard set of positive roots*, denoted $\Delta^+$. In general, any set of the form $w\Delta^+$ will form a *set of positive roots*. The sets of positive roots in $\Delta$ will play a fundamental role in the sequel. The *inversion set* of an element $w \in W$ is defined to be

$$\mathscr{N}(w) = \{s_\alpha \in R_W \mid \alpha \in \Delta^+ \cap w(-\Delta^+)\}.$$

The set $\mathscr{N}(w)$ records the reflections associated with those positive roots $\alpha$ that are sent to negative roots by $w^{-1}$. The longest element $w_0$ of a finite Coxeter group $W$ is the unique element for which $\mathscr{N}(w_0) = R_W$. For infinite Coxeter groups, no such element exists. The positive roots associated to the inversion set are denoted by

$$\overline{\mathscr{N}}(w) = \{\alpha \in \Delta^+ \mid \alpha \in w(-\Delta^+)\}.$$

In the literature, it is not uncommon to define the inversion set of $w$ to be $\overline{\mathscr{N}}(w)$ but this is not the convention taken in this thesis.

It is usually helpful to study Coxeter systems in terms of words, and also in terms of the reflection representation described above. Indeed, many geometric notions have useful algebraic descriptions. For instance, one can show that $l(w) = |\mathscr{N}(w)|$ for any $w \in W$. Similarly, one has $l(ws_i) > l(w)$ if and only if $w\alpha_i \in \Delta^+$, and

$$\mathscr{N}(w) = \{t \in R_W \mid l(tw) < l(w)\}.$$

## 2.2 Algebraic groups

This section reviews the theory surrounding reductive groups. Much of this is what inspires the theory of Kac-Moody groups and therefore most of the thesis to come. For this reason, it is necessary to describe the main features of the theory, even when the full generality given here is largely unnecessary. The language of schemes via their functor of points is used as it enables one to hide some of the more technical details. This point of view will also be useful in the sequel to describe the constructions of Kac-Moody groups as certain representable functors.

### 2.2.1 Schemes and group schemes

In this subsection the notions of affine schemes and affine group schemes are defined, along with some prerequisites for the rest of the thesis. Roughly speaking, these are objects which capture the geometric properties of solutions to polynomial equations. The material of this subsection can be found in most standard textbooks on the subject, e.g. [34] or [45].

An *affine scheme* $X$ over a ring $R$, or *affine* $R$*-scheme*, is a representable functor $X$ from the category of commutative $R$-algebras to the category of sets. This means there is an $R$-algebra $A$ such that

$$X(B) \cong \mathrm{Hom}_{R\text{-alg}}(A, B)$$

for all $R$-algebras $B$. In this case, one writes $X = \mathrm{Spec}(A)$. If $B$ is an $R$-algebra, then the $B$*-points* of $X$ are the elements of the set $X(B)$. Hence a $B$-point can be thought of as an $R$-algebra homomorphism $A \to B$.

If the algebra $A$ can be presented as a quotient of a polynomial ring in finitely many variables, one says that $X$ is of *finite type*; otherwise, $X$ is of *infinite type*. When $X$ is of finite type the $B$-points of $X$ correspond to the solutions of the defining polynomial equations of $A$ with coordinates taken in $B$. If the ideal of such polynomial equations is itself finitely generated, then $X$ is said to be of *finite presentation*. Notably, if $R$ is a *Noetherian* ring, for example when $R = F$ is a field or when $R = \mathbb{Z}$, then the Hilbert basis theorem [45, A.5] implies that every affine scheme of finite type is of finite presentation. The sequel will involve schemes of finite type and infinite type.

There is a well-known equivalence of categories from the category of affine $R$-schemes to the (opposite) category of commutative $R$-algebras [34, A.33]. This means that the algebra $A$ determines $X$ uniquely up to isomorphism and vice versa. This algebra is called the *coordinate ring* of $X$ for the reasons described above.

At the moment, the affine $R$-scheme $X$ is just a functor. In order to study geometric and topological notions on $X$ one must associate to it a topological space. This space is the set of prime ideals of $A$, justifying the labelling $X = \mathrm{Spec}(A)$. The link between $X$ and the space $\mathrm{Spec}(A)$ of prime ideals is natural. Because $X$ is representable, the set of $F$-points $X(F)$, for any field $F$, corresponds to the $R$-algebra homomorphisms from $A$ to $F$. The kernel of such a map is a prime ideal of $A$. Therefore, evaluating the functor $X$ at fields naturally produces points in the set of prime ideals.

The set $\mathrm{Spec}(A)$ can be endowed with the *Zariski topology*. In this topology, the closed sets are defined as

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{a} \subseteq \mathfrak{p}\}$$

for all ideals $\mathfrak{a} \subseteq A$. By definition, the topological properties of $X$ can be rephrased into properties of the coordinate ring $A$. For instance, one has:

$$\begin{aligned} X \text{ connected} &\longleftrightarrow A \text{ contains no non-trivial idempotents,} \\ X \text{ irreducible} &\longleftrightarrow \text{nilradical of } A \text{ is a prime ideal,} \\ X \text{ reduced} &\longleftrightarrow A \text{ contains no non-zero nilpotents.} \end{aligned}$$

When $R = F$ is a field, a reduced affine $F$-scheme of finite type is called an *affine variety* over $F$, or an *affine $F$-variety*. If $k$ is an algebraically closed field, then an affine $k$-variety is fully determined by its $k$-points. As a result, one often identifies $X$ with $X(k)$.

A *morphism* $f : X \to Y$ of affine $R$-schemes is a natural transformation of functors, meaning a collection of maps

$$f_B : X(B) \to Y(B)$$

for all $R$-algebras $B$ which commute with changes of coefficient algebra $B$. The affine $R$-scheme $X$ is a *subfunctor* of $Y$ if $f_B : X(B) \to Y(B)$ is injective for every $R$-algebra $B$. If $X$ is a subfunctor of $Y$ and there is an ideal $I_Y \subset A_Y$ of the coordinate ring of $Y$ such that

$$X(B) = \{\varphi \in Y(B) \mid \varphi(I_Y) = 0\}$$

for every $R$-algebra $B$, then $f$ is a *closed immersion* and $X$ is a *closed $R$-subscheme* of $Y$. Roughly speaking, this means that closed subschemes are defined by the vanishing of polynomials. On the other hand, if $X$ is a subfunctor of $Y$ and there exists a collection of elements $g_i \in A_Y$ such that

$$X(B) = \{\varphi \in Y(B) \mid \varphi(g_i) \text{ generate } B \text{ as a } B\text{-module}\}$$

for every $R$-algebra $B$, then $f$ is an *open immersion* and $X$ is an *open $R$-subscheme* of $Y$. Note that an open subscheme of an irreducible scheme is dense.

The Zariski topology on $X$ is often insufficient when $R$ is not a field. Instead, one uses the *structure morphism* $X \to \operatorname{Spec}(R)$ and studies topological notions fiberwise. This structure morphism arises from the fact that the coordinate ring $A$ of $X$ is an $R$-algebra, hence there is an $R$-homomorphism $R \to A$.

If $\mathfrak{p}$ is a prime ideal of $R$, the *fiber over* $\mathfrak{p}$ of $X$ is the fiber product

$$X_{\mathfrak{p}} = X \times_{\operatorname{Spec}(R)} \operatorname{Spec}(\kappa(\mathfrak{p}))$$

of $R$-schemes, where $\kappa(\mathfrak{p})$ is the residue field of $\mathfrak{p}$ in $R$. This is often called the *base change* of $X$ to $\kappa(\mathfrak{p})$. An affine $R$-scheme is *(fiberwise) connected* if, for all $\mathfrak{p} \in \operatorname{Spec}(R)$, the fiber $X_{\mathfrak{p}}$ is connected with respect to the Zariski topology. It is *(fiberwise) geometrically connected* if, for all $\mathfrak{p} \in \operatorname{Spec}(R)$, the *geometric fiber*

$$X_{\overline{\mathfrak{p}}} = X \times_{\operatorname{Spec}(R)} \operatorname{Spec}(\overline{\kappa(\mathfrak{p})})$$

is connected, where $\overline{\kappa(\mathfrak{p})}$ is the algebraic closure of $\kappa(\mathfrak{p})$.

An *affine group scheme $G$* over $R$, or just *affine $R$-group*, is a group object in the category of affine $R$-schemes. By the equivalence of categories mentioned above, the group operations on $G$ correspond to $R$-algebra homomorphisms on the coordinate ring $A$. This endows $A$ with the structure of a *Hopf algebra*. In this way, the category of affine $R$-groups is equivalent to the (opposite) category of commutative Hopf algebras over $R$. Importantly, the set of $B$-points $G(B)$

form a group in this sense, even when the underlying topological space of $G$ need not have a group structure.

**Example 2.2.** The *multiplicative group* $\mathbb{G}_{m,R}$ is a fiberwise connected affine $R$-group of finite type. If $B$ is an $R$-algebra then $\mathbb{G}_{m,R}(B)$ is the group invertible elements of $B$, denoted $B^\times$. The $R$-scheme $\mathbb{G}_{m,R}$ has the coordinate ring $A = R[x,t]/(xt-1)$.

One can define a *tangent space* at every $R$-point $x \in X(R)$ as follows. Write $R[\varepsilon] = R[t]/(t^2)$ for the ring of dual numbers over $R$. The map $\pi : R[\varepsilon] \to R$ defined by sending $\varepsilon$ to zero induces a map

$$X(\pi) : X(R[\varepsilon]) \to X(R)$$

by functoriality of $X$. The *tangent space of $X$ at $x$* is then defined to be

$$T_xX = \{v \in X(R[\varepsilon]) \mid X(\pi)(v) = x\}.$$

If $X = G$ is an affine $R$-group then the tangent space of $G$ at the identity is equipped with an $R$-bilinear map called a *Lie bracket*. This means that it has the structure of a *Lie algebra* over $R$ and one denotes it by $\operatorname{Lie} G$. Furthermore, the group multiplication operation on $G$ induces isomorphisms $T_x(G) \cong \operatorname{Lie} G$ of $R$-modules for all $x \in G(R)$.

One more notion that will be of importance is that of *smoothness*. If $F$ is a field then an affine $F$-group $G$ is smooth if and only if its *Krull dimension* equals the dimension of its Lie algebra. For the most part, the $R$-groups in this thesis will be smooth. For example, $\mathbb{G}_{m,R}$ is a smooth $R$-group. Moreover, if $F$ is a field of characteristic zero then every affine $F$-group is smooth by a theorem of Cartier. Similarly, smoothness in affine $F$-groups when $F$ is perfect is equivalent to the coordinate ring being reduced. This is guaranteed when $F$ is algebraically closed.

**Example 2.3.** The *additive group* $\mathbb{G}_{a,R}$ is a smooth fiberwise connected affine $R$-group of finite type. If $B$ is an $R$-algebra then $\mathbb{G}_{a,R}(B)$ is the underlying additive group of $B$. It has the coordinate ring $A = R[x]$ and $\mathbb{G}_{a,R}(B) = \operatorname{Hom}_{R\text{-alg}}(R[x], B)$ can be thought of as evaluating polynomials in $x$ at values in $B$. The $R$-scheme structure on $\mathbb{G}_{a,R}$ is the *affine line* denoted by $\mathbb{A}^1_R$. Moreover, the Lie algebra $\operatorname{Lie} \mathbb{G}_{a,R}$ is naturally isomorphic to $R$ equipped with the trivial Lie bracket.

**Example 2.4.** The *general linear group* $\mathrm{GL}_{n,R}$ is a smooth fiberwise connected affine $R$-group of finite type. If $B$ is an $R$-algebra then $\mathrm{GL}_{n,R}(B)$ is the group of $n \times n$ invertible matrices over $B$. It has coordinate ring $A = R[x_{11}, \dots, x_{nn}, t]/(t\det(x_{ij}) - 1)$. The Lie algebra $\operatorname{Lie}\mathrm{GL}_{n,R}$ is naturally isomorphic to the $R$-module $\mathrm{M}_n(R)$ of $n \times n$ matrices over $R$ equipped with the commutator Lie bracket.

In general, there is a wider class of objects called *schemes* that need not be affine. This level of generality is unnecessary for this thesis, and every scheme encountered will be affine with one exception in §2.4.3. Henceforth, affine $R$-schemes, affine $F$-varieties, and affine $R$-groups will be abbreviated to *$R$-schemes*, *$F$-varieties*, and *$R$-groups* respectively. Similarly, a *closed subgroup* will refer to a closed $R$-subscheme that is also a subgroup $R$-scheme.

### 2.2.2 Linear algebraic groups

This subsection describes the structure of a specific subclass of $R$-groups that can be viewed as matrix groups over a field. As a consequence, it becomes easier to use group-theoretic results alongside geometric arguments, leveraging the concrete tools from linear algebra to obtain a remarkable structure theory. A standard reference for this subsection is [25].

A $k$-group $G$ of finite type is called a *linear algebraic group*. In this case, there is a $k$-homomorphism

$$G \hookrightarrow \mathrm{GL}(V) \cong \mathrm{GL}_{n,k}$$

that is a closed immersion, where $V$ is a $k$-vector space of dimension $n$. Since $k$ is algebraically closed, both $G$ and $\mathrm{GL}_{n,k}$ are $k$-varieties, meaning one can safely identify $G$ with $G(k)$ and $\mathrm{GL}_{n,k}$ with $\mathrm{GL}_n(k)$. In particular, $G$ is a group of matrices over $k$.

Suppose that $G = G(k)$ is a linear algebraic group. The embedding of $G$ into $\mathrm{GL}_n(k)$ allows one to define a *Jordan decomposition* for elements of a linear algebraic group. An element $g \in G$ is *semisimple* if it is diagonalisable as a matrix, and *unipotent* if its eigenvalues are all 1. Then every element $g \in G$ can be written uniquely as

$$g = su$$

where $s$ is semisimple, $u$ is unipotent, and $su = us$. Remarkably, $s, u \in G$ and the decomposition is independent of the choice of embedding of $G$ into $\mathrm{GL}_n(k)$.

A connected linear algebraic group is a *torus* if it consists only of semisimple elements. These $k$-groups are isomorphic to a product of copies of the multiplicative group $\mathbb{G}_{m,k}$. The maximal tori of a linear algebraic group $G$ are all conjugate by elements of $G$. Similarly, a linear algebraic group consisting of only unipotent elements is called *unipotent*. These groups can be conjugated into an upper uni-triangular subgroup whenever $G$ is embedded into some $\mathrm{GL}_n(k)$. As the latter group is nilpotent, every unipotent subgroup is nilpotent and hence soluble. For example, the additive group $\mathbb{G}_a(k)$ is a 1-dimensional unipotent subgroup via the map

$$\mathbb{G}_{a,k} \to \mathrm{GL}_2(k) : x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

Relatedly, the Lie-Kolchin theorem states that every connected soluble linear algebraic group can be conjugated into the upper-triangular subgroup whenever $G$ is embedded into some $\mathrm{GL}_n(k)$. A maximal connected soluble subgroup of $G$ is called a *Borel subgroup*. The Borel subgroups of $G$ are pairwise conjugate by elements of $G$ and cover the group. If $B$ is a Borel subgroup then one has the semi-direct decomposition $B = T \ltimes U$ where $T$ is a maximal torus of $B$ and $U$ is a connected unipotent normal subgroup. As a variety, $B$ is isomorphic to $T \times U$.

An important feature of solubility and unipotence is that they are closed under multiplication of normal subgroups. That is to say, a group generated by two normal soluble (resp. unipotent) groups is again soluble (resp. unipotent) and normal. Therefore, there is a unique maximal connected soluble normal subgroup $R(G)$, called the *radical*, and a unique maximal connected unipotent normal subgroup $R_u(G)$, called the *unipotent radical*. In fact, one can show that the unipotent radical is precisely the set of unipotent elements of the radical.

### 2.2.3 Reductive groups

In order to study linear algebraic groups in more detail, it is often useful to quotient out the normal unipotent subgroups described at the end of the previous subsection. This isolates a natural class of groups that includes the so-called *classical groups*. A standard reference for this section is again [25].

A linear algebraic group $G = G(k)$ is called *reductive* if its unipotent radical $R_u(G)$ is trivial, and *semisimple* if its radical $R(G)$ is trivial. For the remainder of the subsection, assume that $G$ is connected and reductive.

The structure of a semisimple or reductive linear algebraic group is rigidified by its maximal tori. A maximal torus $T$ of $G$ acts on $\operatorname{Lie} G$ via the adjoint action, producing a decomposition

$$\operatorname{Lie} G = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where $\mathfrak{h} = \operatorname{Lie} T$ is the zero weight-space, the *Cartan subalgebra*, and

$$\mathfrak{g}_\alpha = \{x \in \operatorname{Lie} G \mid \operatorname{Ad}(t)x = \alpha(t)x \text{ for all } t \in T\}$$

for every non-trivial weight $\alpha$. The non-trivial weights are called the *roots* of $G$ relative to $T$, and the set $\Delta = \Delta(G,T)$ is a finite root system. Accordingly, the weight-spaces $\mathfrak{g}_\alpha$ are called *root spaces*. Every root space is 1-dimensional and differentiating the equation

$$\operatorname{Ad}(t)x = \alpha(t)x$$

yields

$$[h,x] = d\alpha(h)x$$

for all $h \in \mathfrak{h}$ and $x \in \mathfrak{g}_\alpha$. In other words, the Cartan subalgebra acts via the Lie bracket on the root spaces. Since maximal tori are conjugate in $G$, the choice of $T$ does not play an important role once it is fixed.

The root decomposition of $\operatorname{Lie} G$ is reflected in $G$, and one has

$$G = \langle T, U_\alpha \mid \alpha \in \Delta \rangle$$

where each *root group* $U_\alpha$ is a copy of the additive group $\mathbb{G}_{a,k}$ that is normalised by $T$. Moreover, one has $\operatorname{Lie} U_\alpha = \mathfrak{g}_\alpha$ for all $\alpha \in \Delta$.

**Example 2.5.** In $G = \mathrm{SL}_2(k)$ there is a maximal torus

$$T = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \middle| \, c \in k^\times \right\}$$

which admits a character $\chi(\mathrm{diag(c,c^{-1})}) = c$. The roots of $\Delta(G,T)$ are given by $\alpha = 2\chi$ and $-\alpha = -2\chi$. Then

$$U_\alpha = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| \, a \in k \right\} \quad \text{and} \quad U_{-\alpha} = \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \middle| \, a \in k \right\}$$

are the corresponding root groups.

There is a finite *Weyl group* $W(G,T) = N_G(T)/C_G(T)$ relative to $T$. The Weyl group is, up to isomorphism, independent of the choice of maximal torus $T$, so one typically writes $W = W(G,T)$. This is a crystallographic Coxeter group whose root system coincides with $\Delta$. Therefore, $W$ permutes the root spaces and root groups of $G$ via representatives in $N_G(T)$.

A choice of a set of positive roots $\Delta^+ \subseteq \Delta$ determines a base of simple roots $\Pi \subseteq \Delta^+$, as well as a Borel subgroup $B$ that contains $T$. Indeed, one has an isomorphism of $k$-varieties

$$R_u(B) \cong \prod_{\alpha \in \Delta^+} U_\alpha$$

where the product may be taken in any fixed order [25, Proposition 28.1]. This provides a normal form for elements of $B = T \ltimes R_u(B)$. The choice of $\Pi$ defines a set of simple reflections $S = \{s_\alpha \mid \alpha \in \Pi\}$ which makes $(W,S)$ into a Coxeter system.

A subgroup $P$ of $G$ is called a *parabolic subgroup* if it contains a Borel subgroup of $G$. Every parabolic subgroup is connected and self-normalising. If $B \supseteq T$ is the Borel subgroup corresponding to $\Delta^+$ then the parabolic subgroups of $G$ containing $B$ are enumerated by the subsets of simple roots in $\Pi$. Moreover, every parabolic subgroup admits a *Levi decomposition* given by

$$P = L \ltimes R_u(P)$$

where the *Levi factor* $L$ is a connected reductive linear algebraic group. If $P'$ is a second parabolic subgroup then say that $P$ is *opposite* to $P'$ if $P \cap P'$ is a common Levi factor of $P$ and $P'$. This will play an important role in the sequel.

**Example 2.6.** In $G = \mathrm{GL}_n(k)$ the Borel subgroups are the subgroups that can be put in upper-triangular form. Similarly, the parabolic subgroups are those subgroups that can be put in block upper-triangular form. Therefore, up to conjugation, the Levi decomposition of a parabolic subgroup is roughly of the form

$$\begin{pmatrix} A & * \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \ltimes \begin{pmatrix} I & * \\ 0 & I \end{pmatrix}$$

where in general there are more blocks, all of which are copies of $\mathrm{GL}_{n_i}(k)$ where $\sum_i n_i = n$.

For a fixed Borel subgroup $B \supseteq T$, the group $G$ possesses a *Bruhat decomposition*

$$G = \bigsqcup_{w \in W} BwB$$

where $w \in W$ is represented by a lift in $N_G(T)$. The double cosets possess a rich structure, with

$$BswB \subseteq BsBwB \subset BwB \cup BswB$$

for all $s \in S$. Moreover, if $l(sw) > l(w)$ then $BsBwB = BswB$. These relations describe an underlying combinatorial structure on $G/B$ that will be defined in 2.5.

To classify semisimple and reductive groups one looks to the root system. The finite root systems are classified up to isomorphism by *Cartan matrices* that encode how a fixed choice of simple roots interact. A Cartan matrix $C = (c_{ij}) \in \mathrm{M}_n(\mathbb{Z})$ can be characterised as a diagonalisable matrix $C = DS$, where $D$ is diagonal and $S$ is positive-definite symmetric, such that $c_{ii} = 2$, the

$c_{ij}$ are non-positive integers when $i \neq j$, and $c_{ij} = 0$ implies $c_{ji} = 0$.

However, a Cartan matrix only classifies the root system and does not see the center $Z(G)$ of $G$. As a consequence, the Cartan matrices classify semisimple groups up to *isogeny* and do even less to distinguish connected reductive groups. For example, $\mathrm{SL}_n(k)$ and $\mathrm{PGL}_n(k)$ share the same Cartan matrix yet are not isomorphic as semisimple groups. Further still, both of these semisimple groups share their Cartan matrix with $\mathrm{GL}_n(k)$, a connected reductive group of higher dimension. Instead, one requires more combinatorial data that encodes the size and structure of the maximal tori in the group.

Write $X_T = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{G}_{m,k})$ for the *character lattice* of $T$ and $Y_T = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{G}_{m,k}, T)$ for the *cocharacter lattice* of $T$. If $\lambda \in Y_T$ and $\chi \in X_T$, the composition

$$\chi \circ \lambda : \mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$$

is an endomorphism $t \mapsto t^n$ for some $n \in \mathbb{Z}$. This defines a perfect pairing

$$\langle -, - \rangle : Y_T \times X_T \to \mathbb{Z}$$

where $\langle \lambda, \chi \rangle = n$ is as above. In particular, for every root $\alpha \in X_T$ there is a unique *coroot* $\alpha^\vee \in Y_T$ determined by the root system of $G$, for which $\langle \alpha^\vee, \alpha \rangle = 2$. The *(based) root datum* of $G$ with respect to $B \supseteq T$ is the tuple

$$\mathscr{D}_G = (X_T, \Pi, Y_T, \Pi^\vee)$$

where $\Pi$ is the set of simple roots associated to $B \supseteq T$, and $\Pi^\vee$ is the corresponding set of coroots.

A foundational result of Chevalley says that the connected reductive $k$-groups are determined up to unique isomorphism by their root data, and for every root datum and every algebraically closed field $k$ one can associate a connected reductive $k$-group.

### 2.2.4 Reductive group schemes

The structure theory of reductive $k$-groups can be extended to include group schemes over rings and even group schemes relative to other schemes. For a comprehensive account, see [10].

In order to extend geometric notions over a field to more general settings, one typically looks at geometric fibers. For instance, an $R$-group $G$ is *unipotent* if its geometric fibers are unipotent linear algebraic groups. Similarly, a *reductive $R$-group* is a smooth $R$-group whose geometric fibers are connected reductive groups.

The theory becomes more tractable when $G$ is *split*. That is to say, $G$ contains a *split $R$-torus* $T \cong \mathbb{G}_{m,R}^n$ for some $n \in \mathbb{N}$. In particular, the existence and uniqueness theorems of Chevalley extend to reductive $\mathbb{Z}$-groups that admit a maximal $\mathbb{Z}$-torus [10, Proposition 6.4.1]. Note that in this case, every such torus is necessarily split. Specifically, the same root data as before classifies these groups up to unique isomorphism. In the sequel, a reductive $\mathbb{Z}$-group that admits a maximal $\mathbb{Z}$-torus will be called a *Chevalley group*.

## 2.3 Abstract Kac-Moody theory

This section reviews the basics of Kac-Moody algebras and Kac-Moody groups. It should be understood by analogy to the classification of reductive groups, but in reverse order: starting from a generalisation of a (based) root datum and building a generalisation of a reductive group. The material covered in this section can be found in [31] and references therein.

### 2.3.1 Root data

The classification of reductive group schemes is, in large part, determined by the Cartan matrices that control much of a based root datum. A key insight of Kac and Moody was to observe that the condition $C = DS$ where $C$ is a Cartan matrix, $D$ a diagonal matrix, and $S$ a positive-definite symmetric matrix, could be weakened and still produce a similar theory.

A *generalised Cartan matrix*, or *GCM* for short, is a matrix $C = (c_{ij}) \in \mathrm{M}_n(\mathbb{Z})$ for which $C = DS$, where $D$ is diagonal and $S$ is symmetric, such that $c_{ii} = 2$, the $c_{ij}$ are non-positive integers when

$i \neq j$, and $c_{ij} = 0$ implies $c_{ji} = 0$. In particular, the GCM that can be written as $C = DS$ with $S$ positive definite, are precisely the Cartan matrices.

A *(based) Kac-Moody root datum* is a quadruple

$$\mathscr{D} = (X, \Pi, X^\vee, \Pi^\vee)$$

where $X$ is a free $\mathbb{Z}$-module of finite rank and $X^\vee = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ is its $\mathbb{Z}$-dual, and $\Pi$ and $\Pi^\vee$ are finite subsets of $X$ and $X^\vee$, respectively, indexed by a finite set $I$ of size $n$, along with a bijection $\alpha_i \mapsto \alpha_i^\vee$ of $\Pi$ onto $\Pi^\vee$ such that $C = (\langle \alpha_j, \alpha_i^\vee \rangle)$ is a GCM. For example, the root datum of a reductive group is a Kac-Moody root datum for which $C$ is a Cartan matrix. Henceforth, simply refer to $\mathscr{D}$ as a *root datum*. The convention for root data here is closer to the approach taken in [8], this is to align with the content of 2.2.

The *dimension* of a root datum $\mathscr{D}$ is the rank of $X$, denoted by $\dim \mathscr{D}$. Similarly, the *rank* of $\mathscr{D}$ is the cardinality of $\Pi$, denoted by $\mathrm{rk}\,\mathscr{D}$. If $C$ is a Cartan matrix, then the $\alpha \in \Pi$ (resp. $\alpha^\vee \in \Pi^\vee$) are $\mathbb{Z}$-linearly independent in $X$ (resp. $X^\vee$). In general, neither of these properties may be true. Say that $\mathscr{D}$ is *free* (resp. *cofree*) if the $\alpha \in \Pi$ (resp. $\alpha^\vee \in \Pi^\vee$) are $\mathbb{Z}$-linearly independent in $X$ (resp. $X^\vee$). The root datum $\mathscr{D}$ is *cotorsion-free* if $X^\vee / \mathbb{Z}\Pi^\vee$ is torsion-free.

**Example 2.7.** The generalised Cartan matrix $C = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$ defines a root datum by setting $X = \mathbb{Z}$ and $\alpha = -\alpha_0$. Then $\alpha^\vee = -\alpha_0^\vee$. Moreover, one has $\alpha + \alpha_0 = 0$ and similarly $\alpha^\vee + \alpha_0^\vee = 0$. It follows that $C$ is neither free nor cofree.

Let $\mathscr{D}_1 = (X_1, \Pi_1, X_1^\vee, \Pi_1^\vee)$ and $\mathscr{D}_2 = (X_2, \Pi_2, X_2^\vee, \Pi_2^\vee)$ be root data with generalised Cartan matrices $C_1$ and $C_2$ respectively. A *morphism of root data* $\varphi : \mathscr{D}_1 \to \mathscr{D}_2$ consists of a linear map $\varphi : X_1^\vee \to X_2^\vee$ and an injection $i : \Pi_1 \to \Pi_2$ such that $C_1 = C_2|_{\Pi_1 \times \Pi_1}$, $\varphi(\alpha^\vee) = i(\alpha)^\vee$, and $i(\alpha) \circ \varphi = \alpha$ for all $\alpha \in \Pi$.

If $\varphi : \mathscr{D}_1 \to \mathscr{D}_2$ is a morphism of root data, then:

- $\mathscr{D}_1$ is a *central toric extension* of $\mathscr{D}_2$ if $\varphi$ is surjective and $i$ is a bijection.
- $\mathscr{D}_1$ is a *subroot datum* of $\mathscr{D}_2$ if $\varphi$ is injective and $X_2^\vee / \varphi(X_1^\vee)$ is torsion-free.

- $\mathscr{D}_2$ is a *semidirect extension* of $\mathscr{D}_1$ if $\mathscr{D}_1$ is a subroot datum of $\mathscr{D}_2$ and $i$ is a bijection.

The next result is due to Rousseau [38, Proposition 1.3].

**Proposition 2.8.** *Let $\mathscr{D}$ be a root datum.*

1. *There exists a central toric extension $\mathscr{D}^{\mathrm{cof}}$ of $\mathscr{D}$ that is both cofree and cotorsion-free.*
2. *There exists a free semidirect extension $\mathscr{D}^{\mathrm{f}}$ of $\mathscr{D}$. Moreover, if $\mathscr{D}$ is cofree then so is $\mathscr{D}^{\mathrm{f}}$.*
3. *If $\mathscr{D}$ is free, cofree, and cotorsion-free then $\mathscr{D}$ is a semidirect extension of a root datum $\mathscr{D}^m$ which is free, cofree, cotorsion-free, and of dimension* $\operatorname{rk}\mathscr{D} + \operatorname{corank} C$.

The extensions $\mathscr{D}^{\mathrm{cof}}$, $\mathscr{D}^{\mathrm{f}}$, and $\mathscr{D}^{\mathrm{m}}$ of $\mathscr{D}$ are called the *cofree*, *free*, and *minimal free extensions*.

In general, one has $(\mathscr{D}^{\mathrm{cof}})^{\mathrm{f}} = (\mathscr{D}^{\mathrm{f}})^{\mathrm{cof}}$ by the first of the remarks following [38, Prop. 1.3]. This motivates the *Kac extension* $\mathscr{D}^{\mathrm{Kac}} := (\mathscr{D}^{\mathrm{cof}})^{\mathrm{f}}$. It is often useful to use Proposition 2.8 to reduce to the case that $\mathscr{D} = \mathscr{D}^{\mathrm{Kac}}$ because then the groups and algebras that will be constructed will possess enough cocharacters and nonzero dominant integral weights (cf. [31, Example 7.23]).

### 2.3.2 Kac-Moody Lie algebras

This subsection associates a Lie algebra to every root datum $\mathscr{D}$. The idea is to generalise a construction of Serre that associates to any Cartan matrix a finite-dimensional semisimple Lie algebra. The standard references for this subsection are [28] and [31].

The *complex Kac-Moody algebra* $\mathfrak{g}_{\mathscr{D}}$ associated to a root datum $\mathscr{D}$ is the Lie algebra generated by the *Cartan subalgebra* $\mathfrak{h}_{\mathscr{D}} = X^\vee \otimes_{\mathbb{Z}} \mathbb{C}$ and the *Chevalley generators* $(e_i, f_i)_{i\in I}$ subject to the relations:

$$[\mathfrak{h}_{\mathscr{D}}, \mathfrak{h}_{\mathscr{D}}] = 0, \quad [h, e_i] = \alpha_i(h) e_i, \quad [h, f_i] = -\alpha_i(h) f_i \quad \text{for } h \in \mathfrak{h}_{\mathscr{D}},$$
$$[e_i, f_j] = -\delta_{ij}\alpha_i^\vee,$$
$$(\operatorname{ad} e_i)^{1-c_{ij}} e_j = (\operatorname{ad} f_i)^{1-c_{ij}} f_j = 0,$$

where $\delta_{ij}$ is the Kronecker delta symbol. By construction, if $\mathscr{D}$ is the root datum of a reductive group then $\mathfrak{g}_{\mathscr{D}}$ is a finite-dimensional reductive Lie algebra. When the root datum $\mathscr{D}$ is clear, write

$\mathfrak{g}$ and $\mathfrak{h}$ for $\mathfrak{g}_{\mathscr{D}}$ and $\mathfrak{h}_{\mathscr{D}}$ respectively. This thesis follows the *Tits convention* that $[e_i, f_j] = -\delta_{ij}\alpha_i^\vee$ rather than $[e_i, f_j] = \delta_{ij}\alpha_i^\vee$ which is equivalent but removes minus signs later in the construction.

Let $Q_{\mathrm{rt}} = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ be a free $\mathbb{Z}$-module on the basis $\Pi$. One defines a $Q_{\mathrm{rt}}$-gradation of $\mathfrak{g}$ by the assignments $\deg \mathfrak{h} = 0$ and $\deg e_i = \alpha_i = -\deg(f_i)$ for all $i \in I$. Therefore

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$$

where $\Delta \subseteq Q_{\mathrm{rt}} \setminus \{0\}$ is the *root system* of $\mathfrak{g}$. The elements $\alpha_i$ are called *simple roots*, and the $\alpha$ are called *roots*. When $C$ is not a Cartan matrix, the root system $\Delta$ is no longer finite. Similarly, if $\mathscr{D}$ is not free then

$$\mathfrak{g}_\alpha \subseteq \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$$

is a proper inclusion. These are two of the major differences that can occur when one considers arbitrary GCMs. In this instance, the linear map $Q_{\mathrm{rt}} \to X$ defined by $\alpha_i \mapsto \alpha_i$ need not be injective.

**Example 2.9.** Let $\mathscr{D}$ be the root datum as defined in Example 2.7. The root system is infinite and given by

$$\Delta = \{\alpha + n\delta,\ \alpha_0 + n\delta,\ m\delta \mid n \in \mathbb{Z},\ m \in \mathbb{Z}\setminus\{0\}\}$$

where $\delta = \alpha + \alpha_0$. The element $\delta \in Q_{\mathrm{rt}}$ lies in the kernel of the map $Q_{\mathrm{rt}} \to X$. It follows that

$$\begin{aligned}\mathfrak{g}_{\alpha+n\delta} &\subseteq \{x \in \mathfrak{g} \mid [h,x] = (\alpha + n\delta)(h)x \text{ for all } h \in \mathfrak{h}\},\\ &\subseteq \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}\end{aligned}$$

for all $n \in \mathbb{Z}$. But $\mathfrak{g}_\alpha \neq \mathfrak{g}_{\alpha+n\delta}$ for any non-zero $n$, hence the inclusion

$$\mathfrak{g}_\alpha \subseteq \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$$

is proper.

The map defined by the assignment $\mathscr{D} \mapsto \mathfrak{g}_{\mathscr{D}}$ is functorial. In particular, the extensions of

Proposition 2.8 yield morphisms of the corresponding Kac-Moody algebras. The cofree extension $\mathscr{D}^{\mathrm{cof}}$ induces a central extension $\mathfrak{g}_{\mathscr{D}^{\mathrm{cof}}}$ of $\mathfrak{g}_{\mathscr{D}}$, while the free extension $\mathscr{D}^{\mathrm{f}}$ induces a semidirect extension $\mathfrak{g}_{\mathscr{D}^{\mathrm{f}}}$ of $\mathfrak{g}_{\mathscr{D}}$ in degree zero. Similarly, any free, cofree, and cotorsion-free datum induces a trivial central extension of $\mathfrak{g}_{\mathscr{D}^m}$. It follows that the extension $\mathfrak{g}_{\mathscr{D}^{\mathrm{Kac}}}$ of $\mathfrak{g}_{\mathscr{D}}$ is simply an enlargement of the Cartan subalgebra $\mathfrak{h}_{\mathscr{D}}$. As a consequence, despite the discrepancy between the $Q_{\mathrm{rt}}$-gradation and the adjoint weight decomposition, the root system $\Delta$ and the root spaces $\mathfrak{g}_\alpha$ depend only on $C$, and not on the ambient datum $\mathscr{D}$.

The partition $Q_{\mathrm{rt}}^+ = \sum_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i$ of $Q_{\mathrm{rt}}$ induces a partition of the root system $\Delta$. In particular, one has $\Delta = \Delta^+ \sqcup \Delta^-$ where $\Delta^+ = \Delta \cap Q_{\mathrm{rt}}^+$ and $\Delta^- = -\Delta^+$. The corresponding subalgebras

$$\mathfrak{n}^+ = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n}^- = \bigoplus_{\alpha\in\Delta^-} \mathfrak{g}_\alpha$$

depend only on $C$ and produce a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$.

The generalised Cartan matrix $C$ can be used to construct a Coxeter system. First, define the indices $m_{ij}$ by the rule: $m_{ij} = 2,3,4,6$ or $\infty$ according to whether $c_{ij}c_{ji} = 0,1,2,3$ or $\geq 4$. Then the *Weyl group* associated to $C$ is the group

$$W = \langle s_i \mid (s_i s_j)^{m_{ij}} = 1 \rangle$$

where $s_i(\alpha) = \alpha - \alpha(\alpha_i^\vee)\alpha_i$ is an automorphism of $Q_{\mathrm{rt}}$. By construction, the indices $m_{ij}$ force $W$ to be a crystallographic Coxeter group. The pair $(W,S)$ is a Coxeter system, where $S = (s_i)_{i\in I}$.

By construction, the Weyl group $W$ associated to $C$ acts on $Q_{\mathrm{rt}}$. In fact, $W$ acts by the same formula on $X$, and dually on $X^\vee$. Moreover, the action stabilises $\Delta$ in $Q_{\mathrm{rt}}$. A root $\alpha \in \Delta$ is *real* if $\alpha = w\alpha_i$ for some $w \in W$ and $\alpha_i \in \Pi$. As soon as $\Delta$ is infinite, there are *imaginary* roots that do not lie in the $W$-orbit of a simple root $\alpha_i$. Moreover, there is no element of the Weyl group that swaps $\Delta^+$ and $\Delta^-$. This reflects the fact that $W$ does not contain a longest element when it is infinite.

The real roots of $\Delta$ are denoted by $\Delta_{\mathrm{re}}$ and the imaginary roots by $\Delta_{\mathrm{im}} = \Delta \setminus \Delta_{\mathrm{re}}$. The partition

$\Delta = \Delta^+ \sqcup \Delta^-$ induces partitions

$$\Delta_{\text{re}} = \Delta_{\text{re}}^+ \sqcup \Delta_{\text{re}}^- \quad \text{and} \quad \Delta_{\text{im}} = \Delta_{\text{im}}^+ \sqcup \Delta_{\text{im}}^-$$

accordingly. Note that $\Delta_{\text{im}}^+$ and $\Delta_{\text{im}}^-$ are stable under the action of $W$ [28, Proposition 5.2].

An equivalent characterisation of the real roots can be obtained via the adjoint action of the Kac-Moody algebra [28, Chapter 3]. In particular, a root $\alpha \in \Delta$ is real if, for all $x \in \mathfrak{g}_\alpha$, the automorphism $\operatorname{ad} x \in \operatorname{Aut}(\mathfrak{g})$ is *locally nilpotent*. That is to say, for each $x \in \mathfrak{g}_\alpha$ and every $y \in \mathfrak{g}$ one has $(\operatorname{ad} x)^n(y) = 0$ for some $n \in \mathbb{N}$. Moreover, a real root space is always 1-dimensional, in contrast with imaginary root spaces which can have arbitrarily large dimension. This implies that, for every $i \in I$, the subalgebra $\langle f_i, \alpha_i^\vee, e_i \rangle \subseteq \mathfrak{g}$ is type $A_1$, and

$$s_i^* = \exp \operatorname{ad} f_i \cdot \exp \operatorname{ad} e_i \cdot \exp \operatorname{ad} f_i = \exp \operatorname{ad} e_i \cdot \exp \operatorname{ad} f_i \cdot \exp \operatorname{ad} e_i$$

is an automorphism of $\mathfrak{g}$.

The subgroup $W^*$ of $\operatorname{Aut}(\mathfrak{g})$ generated by the $s_i^*$ admits a surjective homomorphism $\nu : W^* \to W$ given by $s_i^* \mapsto s_i$. The action of $W^*$ stabilises $\mathfrak{h}$ and permutes the root spaces $\mathfrak{g}_\alpha$ according to the action of $W$ on $\Delta$. Further still, for every $i \in I$ and $w^* \in W^*$, the *double base* $E_i = \{w^* e_i, -w^* e_i\}$ depends only on the root $\nu(w^*)\alpha_i$. Therefore, there is a choice of root vector $e_\alpha$ in every real root space $\mathfrak{g}_\alpha$ such that, for all $\alpha \in \Delta_{\text{re}}$, $i \in I$, and $w^* \in W^*$, one has

$$e_{\alpha_i} = e_i, \quad [e_\alpha, f_\alpha] = -\alpha^\vee \quad \text{and} \quad w^* e_\alpha = \pm e_{w\alpha}$$

where $f_\alpha = e_{-\alpha}$ for all $\alpha \in \Delta_{\text{re}}^+$. This will be of use in the following subsection.

At present, the Kac-Moody algebra $\mathfrak{g}$ is defined over the complex numbers. This can be extended to $\mathbb{Z}$ using the double bases of the previous subsection. The following construction is due to Tits [44, §4].

Associated to the Lie algebra $\mathfrak{g}$ is its *universal enveloping algebra* $\mathscr{U}(\mathfrak{g})$. Let $\mathscr{U}_{\mathbb{Z}}(\mathfrak{g})$ be the

$\mathbb{Z}$-subalgebra of $\mathscr{U}(\mathfrak{g})$ generated by the *divided powers*

$$e_i^{(n)} = \frac{e_i^n}{n!}, \quad f_i^{(n)} = \frac{f_i^n}{n!}, \quad \text{and} \quad \binom{h}{n} = \frac{h(h-1)\cdots(h-n+1)}{n!}$$

for all $h \in X^\vee$, $i \in I$, and $n \in \mathbb{Z}_{\geq 0}$. The subalgebra $\mathscr{U}_\mathbb{Z}(\mathfrak{g})$ is a $\mathbb{Z}$*-form* of $\mathscr{U}(\mathfrak{g})$. Specifically, one has $\mathscr{U}(\mathfrak{g}) = \mathscr{U}_\mathbb{Z}(\mathfrak{g}) \otimes_\mathbb{Z} \mathbb{C}$. This in turn defines a $\mathbb{Z}$-form $\mathfrak{g}_\mathbb{Z} = \mathfrak{g} \cap \mathscr{U}_\mathbb{Z}(\mathfrak{g})$ of $\mathfrak{g}$. In particular,

$$\mathfrak{g}_{\alpha\mathbb{Z}} = \mathfrak{g}_\alpha \cap \mathfrak{g}_\mathbb{Z} = \mathbb{Z}e_\alpha \quad \text{and} \quad \mathscr{U}(\mathfrak{g}_\alpha) \cap \mathscr{U}(\mathfrak{g})_{n\alpha} \cap \mathscr{U}_\mathbb{Z}(\mathfrak{g}) = \mathbb{Z}e_\alpha^{(n)}$$

for all $\alpha \in \Delta_{\text{re}}$ and $n \in \mathbb{N}$.

After defining $\mathfrak{h}_\mathbb{Z}$, $\mathfrak{n}_\mathbb{Z}^+$, and $\mathfrak{n}_\mathbb{Z}^-$ in the same way, one obtains an integral triangular decomposition

$$\mathfrak{g}_\mathbb{Z} = \mathfrak{n}_\mathbb{Z}^- \oplus \mathfrak{h}_\mathbb{Z} \oplus \mathfrak{n}_\mathbb{Z}^+$$

of the $\mathbb{Z}$-form $\mathfrak{g}_\mathbb{Z}$. For any ring $R$, the $R$-algebras $\mathscr{U}_R(\mathfrak{g})$, $\mathfrak{g}_R$, $\mathfrak{h}_R$, $\mathfrak{n}_R^+$, and $\mathfrak{n}_R^-$ are defined by tensoring over $\mathbb{Z}$ the respective $\mathbb{Z}$-forms with $R$.

### 2.3.3 The constructive Tits functor

This subsection recalls the construction of a functor which associates to any root datum $\mathscr{D}$ and ring $R$, a group that acts on the $R$-algebra $\mathfrak{g}_R$ in a reasonable way. Specifically, when $R = F$ is a field, the kernel of the action is contained in a canonical $F$-torus. The construction is due to Tits [44, §3.6]

A collection of roots $\Sigma \subseteq \Delta$ is *prenilpotent* if $v(\Sigma) \subseteq \Delta^+$ and $w(\Sigma) \subseteq \Delta^-$ for some $v, w \in W$. As the sets $\Delta_{\text{im}}^+$ and $\Delta_{\text{im}}^-$ are $W$-stable, a prenilpotent set of roots $\Sigma$ is contained in $\Delta_{\text{re}}$. A prenilpotent set $\Sigma \subseteq \Delta \cup \{0\}$ that is *closed*, i.e.

$$\alpha, \beta \in \Sigma \quad \text{and} \quad \alpha + \beta \in \Delta \cup \{0\} \implies \alpha + \beta \in \Sigma$$

is called *nilpotent*. Importantly, if the pair $\{\alpha,\beta\}$ is prenilpotent then the intervals

$$[\alpha,\beta] = \{i\alpha + j\beta \mid i,j \geq 0\} \cap \Delta \quad \text{and} \quad (\alpha,\beta) = \{i\alpha + j\beta \mid i,j \geq 1\} \cap \Delta$$

are both nilpotent sets of roots. If $\alpha \neq \pm\beta$ then $\{\alpha,\beta\}$ is prenilpotent if and only if $[\alpha,\beta] \subseteq \Delta_{\mathrm{re}}$.

The nilpotent sets of roots are of particular importance for constructing unipotent $\mathbb{Z}$-groups. Recall from Example 2.3 that $\operatorname{Lie}\mathbb{G}_{a,\mathbb{Z}}$ is isomorphic as a $\mathbb{Z}$-module to $\mathbb{Z}$. Since $\mathfrak{g}_{\alpha\mathbb{Z}} = \mathbb{Z}e_\alpha$ is a free $\mathbb{Z}$-module of rank 1, one can associate to every real root $\alpha \in \Delta_{\mathrm{re}}$ a unique smooth connected unipotent $\mathbb{Z}$-group $U_\alpha$ with Lie algebra $\mathfrak{g}_{\alpha\mathbb{Z}}$. This defines an isomorphism $x_\alpha : \mathbb{G}_{a,\mathbb{Z}} \xrightarrow{\sim} U_\alpha$ which is determined at the level of $R$-points. For any ring $R$ and element $r \in R$, denote the image of $r$ under this isomorphism by $x_\alpha(r)$. If $\alpha = \alpha_i \in \Pi$ then write $x_i = x_\alpha$.

Moreover, for any nilpotent set $\Sigma \subseteq \Delta \cup \{0\}$, there is a unique unipotent $\mathbb{Z}$-group $U_\Sigma$ containing every $U_\alpha$, $\alpha \in \Sigma$, such that, for a fixed order on $\Sigma$, the product morphism

$$\prod_{\alpha \in \Sigma} U_\alpha \to U_\Sigma$$

is an isomorphism of schemes [44, Proposition 1]. Hence, for any prenilpotent pair $\{\alpha,\beta\}$, there are integer *Chevalley structure constants* $C_{ij}^{\alpha\beta}$ that determine the commutation relations between $U_\alpha$ and $U_\beta$ in $U_{(\alpha,\beta)}$. If $\Delta$ is infinite then the constants $C_{ij}^{\alpha\beta}$ may take arbitrarily large integer values.

The isomorphism above defines the normal form of the unipotent radical in 2.2.2. However, the set $\Delta_{\mathrm{re}}^+$ is nilpotent if and only if $C$ is a Cartan matrix. As a result, the unipotent $\mathbb{Z}$-group $U_\Sigma$ is insufficient in describing the positive roots $\Delta_{\mathrm{re}}^+$ when $\Delta$ is infinite. Instead, one defines a functor manually, using the commutation relations described above when they are available.

The *Steinberg functor* is the group functor $St_C : \mathbb{Z}\text{-alg} \to \mathrm{Grp}$ defined such that, for every ring $R$, the group $St_C(R)$ is the free product of the groups $U_\alpha(R)$, for $\alpha \in \Delta_{\mathrm{re}}$, subject to the relations

$$[x_\alpha(t), x_\beta(u)] = \prod_\gamma x_\gamma(C_{ij}^{\alpha\beta} t^i u^j)$$

for all $t,u \in R$, and all prenilpotent pairs $\{\alpha,\beta\}$ running through $\gamma = i\alpha + j\beta \in (\alpha,\beta)$, with $C_{ij}^{\alpha\beta}$ as above. Note that $St_C$ is not representable in general.

Similarly, define the functor $T_X : \mathbb{Z}\text{-alg} \to \mathrm{Grp}$ that sends a ring $R$ to the group $T_X(R) = X^\vee \otimes_{\mathbb{Z}} R^\times$. This is the split $\mathbb{Z}$-torus with coordinate ring $A = \mathbb{Z}[X]$, whose character group $X_{T_X}$ is canonically isomorphic to $X$. In particular, $T_X$ is canonically isomorphic to $\mathbb{G}_{m,\mathbb{Z}}^{\dim \mathscr{D}}$.

The action of $W$ on $X^\vee$ induces an action of $W$ on $T_X(R)$ for every ring $R$. Likewise, the action of $W^*$ on the double bases $E_i$ induces an action of $W^*$ on $St_C(R)$ defined by

$$s_i^*(x_\alpha(r)) = x_{s_i(\alpha)}(\pm r)$$

for all $\alpha \in \Delta_{\mathrm{re}}$, $i \in I$, and $r \in R$, where the sign is determined by the sign of $s_i^*(e_\alpha) = \pm e_{s_i(\alpha)}$.

Finally, everything is ready to define the group functor of [44] associated to an arbitrary root datum $\mathscr{D}$. The *constructive Tits functor* is the group functor $G_{\mathscr{D}}^{\mathrm{Tits}} : \mathbb{Z}\text{-alg} \to \mathrm{Grp}$ defined such that, for every ring $R$, one has the group

$$G_{\mathscr{D}}^{\mathrm{Tits}}(R) = T_X(R) * St_C(R)/(\text{relations } (\mathbf{R1}) \text{ to } (\mathbf{R4}))$$

where for all $\alpha \in \Delta_{\mathrm{re}}$, $i \in I$, $r \in R^\times$, and $t \in T_X(R)$, one has

**(R1)** $t \cdot x_i(r) \cdot t^{-1} = x_i(\alpha_i(t)r)$,

**(R2)** $\tilde{s}_i \cdot t \cdot \tilde{s}_i^{-1} = s_i(t)$,

**(R3)** $\tilde{s}_i(r^{-1}) = \tilde{s}_i \cdot \alpha_i^\vee(r)$,

**(R4)** $\tilde{s}_i \cdot x_\alpha(r) \cdot \tilde{s}_i^{-1} = s_i^*(x_\alpha(r))$,

where $\tilde{s}_i(r) = x_i(r)x_{-i}(r^{-1})x_i(r)$ and $\tilde{s}_i = \tilde{s}_i(1)$.

For every ring $R$, the canonical homomorphism $T_X(R) \to G_{\mathscr{D}}^{\mathrm{Tits}}(R)$ is injective. Similarly, the canonical homomorphism $U_\Sigma(R) \to G_{\mathscr{D}}^{\mathrm{Tits}}(R)$ is injective for any nilpotent subset $\Sigma \subseteq \Delta \cup \{0\}$. Henceforth, identify $T_X$ and every such $U_\Sigma$ with the corresponding subgroup functors of $G_{\mathscr{D}}^{\mathrm{Tits}}$. For instance, there is a subgroup functor $U_\alpha$ for every $\alpha \in \Delta_{\mathrm{re}}$. The subgroup $T_X(R)$ is called the *canonical maximal torus* of $G_{\mathscr{D}}^{\text{Tits}}$, while the groups $U_\alpha(R)$ are called *root subgroups* of $G_{\mathscr{D}}^{\text{Tits}}(R)$. Note that root subgroups are only associated to real roots.

There are subgroup functors $U_{\text{ab}}^+, U_{\text{ab}}^-, B_{\text{ab}}^+$ and $B_{\text{ab}}^-$ of $G_{\mathscr{D}}^{\text{Tits}}$ defined by

$$U_{\text{ab}}^+(R) = \langle U_\alpha(R) \mid \alpha \in \Delta_{\text{re}}^+ \rangle,$$
$$U_{\text{ab}}^-(R) = \langle U_\alpha(R) \mid \alpha \in \Delta_{\text{re}}^- \rangle,$$
$$B_{\text{ab}}^+(R) = T_X(R) U_{\text{ab}}^+(R),$$
$$B_{\text{ab}}^-(R) = T_X(R) U_{\text{ab}}^-(R),$$

for any ring $R$. Note that these are abstract group functors that play the role of unipotent subgroups and Borel subgroups but need not be representable in the previous sense. The subgroups $B_{\text{ab}}^+(R)$ and $B_{\text{ab}}^-(R)$ are called the *canonical positive Borel subgroup* and the *canonical negative Borel subgroup* of $G_{\mathscr{D}}(R)$ respectively. Moreover, there is a subgroup functor $N$ of $G_{\mathscr{D}}^{\text{Tits}}$ defined by

$$N(R) = \langle \tilde{s}_i, T_X(R) \mid i \in I \rangle$$

for any ring $R$. There is a homomorphism $N(R) \to W$ defined by the mapping $\tilde{s}_i \mapsto s_i$ whose kernel is $T_X(R)$. In particular, if $F$ is a field with $|F| \geq 3$ then

$$N(F) = N_{G_{\mathscr{D}}^{\text{Tits}}(F)}(T_X(F))$$

by [35, §8.4.1].

The mapping $\mathscr{D} \mapsto G_{\mathscr{D}}^{\text{Tits}}$ is functorial. If $\mathscr{D}_1 = (X_1, \Pi_1, X_1^\vee, \Pi_1^\vee)$ and $\mathscr{D}_2 = (X_2, \Pi_2, X_2^\vee, \Pi_2^\vee)$ are root data, then two important cases arise:

- If $\mathscr{D}_2$ is a central toric extension of $\mathscr{D}_1$ then $T_{X_2}$ surjects onto $T_{X_1}$ with kernel $T_{X_3}$, where $X_3$ is the $\mathbb{Z}$-dual of $\ker\varphi$. This implies that $G_{\mathscr{D}_2}^{\text{Tits}}$ is a central extension of $G_{\mathscr{D}_1}^{\text{Tits}}$ by $T_{X_3}$.
- If $\mathscr{D}_2$ is a semidirect extension of $\mathscr{D}_1$ then $T_{X_2} = T_{X_1} \times T_{X_3}$ for some $X_3 \subseteq X_2$. This implies that $G_{\mathscr{D}_2}^{\text{Tits}}$ is the semidirect product of $G_{\mathscr{D}_1}^{\text{Tits}}$ by the torus $T_{X_3}$. For a fixed ring $R$, the action

of $T_{X_3}(R)$ on $G_{\mathscr{D}_1}^{\mathrm{Tits}}(R)$ is given by

$$t \cdot t' \cdot t^{-1} = t' \quad \text{and} \quad t \cdot x_\alpha(r) \cdot t^{-1} = x_\alpha(\alpha(t)r)$$

for all $t \in T_{X_3}(R)$, $t' \in T_{X_1}(R)$, and $\alpha \in \Delta_{\mathrm{re}}$.

In particular, at the level of functors, the Kac extension of a root datum induces a central extension by a torus followed by a semidirect extension by another torus, or vice versa.

This subsection concludes by completing the running example of this section.

**Example 2.10.** The generalised Cartan matrix

$$C = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

defines the Weyl group $W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1 \rangle$ which is isomorphic to $D_\infty$.

The simple roots are $\alpha$ and $\alpha_0$. Set $\delta = \alpha + \alpha_0$. Then the associated root system

$$\begin{aligned} \Delta &= \{\alpha + n\delta,\ \alpha_0 + n\delta,\ m\delta \mid n \in \mathbb{Z},\ m \in \mathbb{Z} \setminus \{0\}\}, \\ &= \{\beta + n\delta \mid \beta \in \dot{\Delta},\ n \in \mathbb{Z}\} \cup \{m\delta \mid m \in \mathbb{Z} \setminus \{0\}\} \end{aligned}$$

is infinite, where $\dot{\Delta}$ is the root system $\{\alpha, -\alpha\}$ of type $A_1$.

Furthermore, one has the partition $\Delta = \Delta_{\mathrm{re}} \cup \Delta_{\mathrm{im}}$ where

$$\Delta_{\mathrm{re}} = \{\alpha + n\delta, \alpha_0 + n\delta \mid n \in \mathbb{Z}\} \quad \text{and} \quad \Delta_{\mathrm{im}} = \{m\delta \mid m \in \mathbb{Z} \setminus \{0\}\}.$$

There is also a partition $\Delta = \Delta^+ \cup \Delta^-$ where

$$\Delta^+ = \{\alpha + n\delta, \alpha_0 + n\delta \mid n \in \mathbb{Z}_{\geq 0}\} \cup \{m\delta \mid m \in \mathbb{N}\}.$$

Let $\mathscr{D}$ be the root datum associated to $C$ defined in Example 2.7. Then $G_{\mathscr{D}}^{\mathrm{Tits}}(k)$ is isomorphic to the polynomial loop group $\mathrm{SL}_2(k[t,t^{-1}])$. The corresponding canonical subgroups are

$$
\begin{aligned}
T(k) &= \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \,\middle|\, c \in k^\times \right\}, \\
U_{\alpha+n\delta}(k) &= \left\{ \begin{pmatrix} 1 & at^n \\ 0 & 1 \end{pmatrix} \,\middle|\, a \in k \right\}, \\
U_{\alpha_0+n\delta}(k) &= \left\{ \begin{pmatrix} 1 & 0 \\ at^{n+1} & 1 \end{pmatrix} \,\middle|\, a \in k \right\}, \\
B^+_{\mathrm{ab}}(k) &= \mathrm{SL}_n(k[t]) \cap \begin{pmatrix} k[t] & k[t] \\ tk[t] & k[t] \end{pmatrix}, \\
B^-_{\mathrm{ab}}(k) &= \mathrm{SL}_n(k[t^{-1}]) \cap \begin{pmatrix} k[t^{-1}] & t^{-1}k[t^{-1}] \\ k[t^{-1}] & k[t^{-1}] \end{pmatrix}.
\end{aligned}
$$

The root datum $\mathscr{D}$ is neither free nor cofree. By passing to the minimal free extension of $\mathscr{D}$, one obtains a group isomorphic to $\mathrm{SL}_2(k[t,t^{-1}]) \rtimes k^\times$ where the action is induced by $c : t \mapsto ct$.

### 2.3.4 The structure of Kac-Moody groups

In this thesis, a *Kac-Moody group* is a group of the form $G = G^{\mathrm{Tits}}_{\mathscr{D}}(k)$ where $k$ is an algebraically closed field. By construction, a Kac-Moody group is similar in many ways to a reductive group. Indeed, if the underlying matrix of $\mathscr{D}$ is a Cartan matrix then $G$ is a reductive group. In this subsection, some basic properties of Kac-Moody groups will be listed. For ease of notation, the subgroups of $G$ defined in the previous subsection will often omit reference to the datum and other subscripts. The content of this subsection is essentially [31, Appendix B.4]

The constructive Tits functor is not representable, meaning $G$ has no topology or algebro-geometric structure *a priori*. One way to sidestep this problem is to consider a certain *adjoint representation*

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathscr{U}_k(\mathfrak{g}))$$

which was defined and studied in [35, Chapter 9]. The kernel of Ad is contained in the canonical torus $T$.

A subgroup $H$ of $G$ is Ad-*diagonalisable* if $\mathscr{U}_k(\mathfrak{g})$ can be decomposed into a direct sum of 1-dimensional subspaces that are stable under $\mathrm{Ad}(H)$. Then the maximal Ad-diagonalisable subgroups of $G$ are all conjugate to $T$. This is analogous to the fact that the maximal tori of a reductive group are conjugate.

The tuples $(B^+,N)$ and $(B^-,N)$ are *saturated BN-pairs* which means that

$$G = \langle B^+,N\rangle = \langle B^-,N\rangle$$

and

$$T = B^+ \cap N = B^- \cap N = \bigcap_{w\in W} wB^+w^{-1} = \bigcap_{w\in W} wB^-w^{-1}.$$

is normal in $N$, with $W = N/T$. Moreover, there is a generating set $S$ of $W$ such that, for all $s \in S$ and $w \in W$ one has

$$BsBwB \subseteq BwB \cup BswB \quad \text{and} \quad sBs^{-1} \not\subseteq B.$$

This implies the existence of *Bruhat decompositions*

$$G = \bigsqcup_{w\in W} B^+wB^+ = \bigsqcup_{w\in W} B^-wB^-$$

that are typically distinct. Indeed, when $W$ is infinite the Borel subgroups $B^+$ and $B^-$ are not conjugate in $G$. This reflects the fact that there is no element of $W$ that swaps $\Delta^+$ and $\Delta^-$.

The BN-pairs $(B^+,N)$ and $(B^-,N)$ are *twinned*. This implies the *Birkhoff decompositions*

$$G = \bigsqcup_{w\in W} B^+wB^- = \bigsqcup_{w\in W} B^-wB^+$$

which are not equivalent to the Bruhat decompositions when $W$ is infinite. These decompositions will play an important role in the sequel.

Further still, the triple $(U^+,U^-,N)$ is a *refined* BN-pair. This is an even stronger condition implying

$$NU^+ \cap U^- = U^+ \cap NU^- = U^+ \cap N = U^- \cap N = \{1\}.$$

This leads to the right-hand sides of the equations

$$B^+ w B^+ = B^+ w U_w^+ \quad \text{and} \quad B^- w B^- = B^- w U_w^-$$

having a uniqueness of expression, where $U_w^+ = U^+ \cap w^{-1} U^- w$ and $U_w^- = U^- \cap w^{-1} U^+ w$.

A similar result holds for uniqueness of expressions in the Birkhoff decompositions. In particular, one has

$$G = \bigsqcup_{n \in N} U^+ n U^- = \bigsqcup_{n \in N} U^- n U^+$$

which have uniqueness of expression for elements of $U^+ T U^-$ and $U^- T U^+$ respectively.

The BN-pairs $(B^+, N)$ and $(B^-, N)$ define a notion of a parabolic subgroup in $G$. The *standard parabolic subgroups* of type $J \subseteq I$, relative to $T$, of *positive* and *negative sign* are the subgroups

$$P_J^+ = B^+ W_J B^+ \quad \text{and} \quad P_J^- = B^- W_J B^-$$

respectively, where $W_J$ is the parabolic subgroup of $W$ generated by $s_i$ for $i \in J$. More generally, a conjugate of $P_J^+$ is a *parabolic subgroup* of type $J$ of *positive sign*, while a conjugate of $P_J^-$ is a *parabolic subgroup* of type $J$ of *negative sign*. These are the subgroups of $G$ which contain a conjugate of $B^+$ or $B^-$ respectively.

The tuple

$$(G, T, (U_\alpha)_{\alpha \in \Delta_{\mathrm{re}}})$$

satisfies the axioms of a *linear root group datum* of type $(W, S)$ in the sense of [31, Appendix B.4]. The linearity of the root group datum implies the existence of *Levi decompositions*

$$P_J^+ = L_J \ltimes U_J^+ \quad \text{and} \quad P_J^- = L_J \ltimes U_J^-$$

where $L_J = G^{\mathrm{Tits}}_{\mathscr{D}_J}(k)$ is the Kac-Moody group associated to $\mathscr{D}(J) = (X, \Pi|_J, X^\vee, \Pi^\vee|_J)$, and

$$U_J^+ = \langle U_\alpha \mid \alpha \in \Delta_{\mathrm{re}}^+ \setminus W_J \Pi_J \rangle,$$
$$U_J^- = \langle U_\alpha \mid \alpha \in \Delta_{\mathrm{re}}^- \setminus W_J \Pi_J \rangle.$$

As is familiar in the theory of reductive groups, these Levi decompositions extend to all parabolic subgroups [35, Theorem 6.2.2].

The normal subgroups $U_J^+$ and $U_J^-$ are called the *unipotent radicals* of the respective parabolic subgroups, while the complement $L_J$ is called a *Levi factor* of the parabolic subgroup. If the Levi factor $L$ of a parabolic subgroup $P$ is abstractly isomorphic to a reductive group then $L$ and $P$ are said to be of *finite type*. A pair of parabolic subgroups $P^+$ and $P^-$ are *opposite* if they are of the same type and their intersection is a common Levi factor.

If a subgroup $H$ of $G$ is contained in the intersection of two parabolic subgroups of finite type and opposite sign, then $H$ is called *bounded*. There is a Levi decomposition for bounded subgroups. In particular, a bounded subgroup $H$ is of the form $H = L \ltimes U$ where $L$ is a Levi factor of finite type and $U$ is *bounded unipotent* in the sense of [9]. The Zariski closure of the image of a bounded subgroup in the adjoint representation is a connected algebraic group that is typically not reductive [35, §10.3].

## 2.4 Kac-Moody groups as ind-groups

The abstract Kac-Moody theory of the previous section aims to generalise the combinatorial and group theoretic aspects of reductive groups to infinite dimensions. However, it does not itself realise Kac-Moody groups as certain group objects of an algebro-geometric category, analogous to the realisation of linear algebraic groups as group schemes. For Kac-Moody groups, an appropriate generalisation of a scheme appears to be an ind-scheme. This is evidenced by the construction of Kac-Moody groups as ind-schemes, first by Mathieu's formal group [33] and more recently by Bouthier and Vasserot's minimal group [8]. In this section, the basic theory of ind-schemes is described without delving too far into the technical details which are beyond the scope of this thesis. Following this, there is an exposition of the various Kac-Moody group functors one can associate to an arbitrary root datum.

### 2.4.1 Ind-schemes and ind-groups

This subsection reviews the basic properties of ind-schemes and group ind-schemes that are required for the theory of Kac-Moody groups. The material exists in far greater generality than is given here. For more details, the author suggests [36] and references therein.

An *ind-affine ind-scheme* $X$ over a ring $R$, or an *ind-affine $R$-ind-scheme*, is a functor from the category of commutative $R$-algebras to the category of sets which admits a presentation

$$X \cong \operatorname{colim}_i X_i$$

as the filtered colimit of a directed system of affine $R$-schemes $X_i$ where the transition maps $X_i \hookrightarrow X_j$ are closed immersions for all natural numbers $i \leq j$. In particular, if $B$ is an $R$-algebra then $X(B) \cong \bigcup_i X_i(B)$. Henceforth, an ind-scheme will mean an ind-affine ind-scheme.

The underlying topological space of an ind-scheme $X$ is given by the colimit

$$|X| = \bigcup_i |X_i|$$

where $|X_i|$ is the underlying topological space of the $R$-scheme $X_i$. In particular,

$$Y \subseteq |X| \text{ closed } \iff Y \cap |X_i| \text{ closed for all } i$$

with a similar characterisation for open sets. This will be called the *ind-Zariski topology*.

If $P$ is a property of $R$-schemes then an ind-scheme $X$ is *ind-$P$* if there is a presentation $X \cong \operatorname{colim}_i X_i$ where each $X_i$ is $P$. The examples used in this thesis are connected, finitely presented, finite type, and reduced. In this vein, an *$F$-ind-variety* is an ind-(finite type reduced) $F$-ind-scheme for some field $F$.

As is the case for $R$-schemes, a *morphism* $f : X \to Y$ of $R$-ind-schemes is a natural transformation of functors. If $X = \mathrm{colim}_i X_i$ and $Y = \mathrm{colim}_j Y_j$ then the morphism $f$ factors as

$$f_i : X_i \to Y_{m(i)} \hookrightarrow Y$$

where $X_i \to Y_{m(i)}$ is a morphism of $R$-schemes and $m(i)$ is an index of $Y$ dependent on $i$. In particular,

$$\mathrm{Hom}_{\mathrm{IndSch}}(X,Y) = \lim_i \mathrm{colim}_j \mathrm{Hom}_{\mathrm{Sch}}(X_i, Y_j)$$

for all $R$-ind-schemes $X$ and $Y$. Therefore, perhaps after a suitable re-indexing, a morphism $f$ has a presentation as a system of morphisms $f_{ij} : X_i \to Y_j$ of $R$-schemes.

The category of $R$-ind-schemes is closed under fiber products [36, Lemma 1.10]. This allows one to define *embeddings* of $R$-ind-schemes. A morphism $f : X \to Y$ is *schematic* if for every $R$-scheme $T$ with a morphism $T \to Y$, the fiber product $X \times_Y T$ is an $R$-scheme. A schematic morphism $f : X \to Y$ is $P$ if the induced morphism

$$f \times_Y T : X \times_Y T \to T$$

of $R$-schemes is $P$ for all $R$-schemes $T$. For example, one obtains notions of *open immersions* and *closed immersions* of $R$-ind-schemes, defining *open $R$-ind-subschemes* and *closed $R$-ind-subschemes* respectively.

On the other hand, if $P$ is a property of a morphism of schemes then a morphism $f : X \to Y$ is *ind-$P$* if it admits a presentation in which each $f_{ij}$ is $P$. Hence, it makes sense to talk of ind-closed embeddings and there will be several important examples in the sequel. In general, a closed embedding is an ind-closed embedding but the converse need not be true.

A *group ind-scheme* $G$ over $R$, or just an *$R$-ind-group*, is a group object in the category of $R$-ind-schemes. In particular, an $R$-ind-group admits a presentation $G = \mathrm{colim}_i G_i$ where each $G_i$ is an $R$-scheme. Note that the $R$-schemes $G_i$ are typically not $R$-groups. A *closed subgroup* of $G$ will refer to a closed $R$-ind-subscheme $H$ that is also a subgroup $R$-ind-scheme.

**Example 2.11.** A countable group, such as the integers $\mathbb{Z}$, has the structure of an $R$-ind-group of ind-finite type. The presentation is given by an ascending chain of finite subsets, each of which is viewed as a disjoint union of copies of $\mathrm{Spec}(R)$.

**Example 2.12.** The functor $\mathrm{GL}_{\infty,R} = \mathrm{colim}_n \mathrm{GL}_{n,R}$, with transition maps defined by the block diagonal embedding at the level of $B$-points for every $R$-algebra $B$, is an ind-(connected finite type) $R$-ind-group.

As for $R$-schemes, one can also define a *tangent space* for an $R$-ind-scheme $X = \mathrm{colim}_i X_i$ as

$$T_x X = \mathrm{colim}_i T_x X_i$$

where $T_x X_i$ is the tangent space of $X_i$ at $x$. Note that $x \in |X_i|$ eventually. If $X = G$ is an $R$-ind-group with identity element $e$, then $\mathrm{Lie}\, G = T_e G$ has the structure of an $R$-Lie algebra [29, Proposition 4.2.2].

### 2.4.2 Attractors in ind-schemes

In this subsection we describe a powerful geometric technique sometimes known as the *dynamic approach*. The material presented here can be found in [11] and [23]. In the theory of reductive groups, this method provides a way to define parabolic subgroups without reference to an ambient root system or by using the classical characterisation involving quotients of varieties. Therefore, it is an effective way to study similar categories of groups that may have harder structure theories, such as $k$-groups [5], reductive group schemes [10], and pseudo-reductive groups [11].

The theory has been generalised significantly in recent years and the construction extends to a far wider class of geometric objects. In particular, there is a well-defined dynamic approach for group ind-schemes. Throughout, assume that $R = F$ a field or $R = \mathbb{Z}$.

Let $X$ be an $R$-ind-scheme of ind-finite type, equipped with a $\mathbb{G}_m$-action that is trivial on $R$. The action of $\mathbb{G}_m$ on $X$ is *admissible* [37, Lemma A.3.5] which implies that there is a presentation $X = \mathrm{colim}_i X_i$ that is $\mathbb{G}_m$-stable. Therefore, if $B$ is an $R$-algebra and $x \in X(B)$ then the orbit map factors as

$$\mathbb{G}_{m,B} \to (X_i)_B \to X_B$$

where $x \in X_i(B)$. Then $\lim_{a\to 0} a \cdot x$ *exists* if the $B$-morphism

$$\mathbb{G}_{m,B} \to (X_i)_B, \quad a \mapsto a \cdot x$$

extends to a $B$-morphism $\mathbb{A}^1_B \to (X_i)_B$.

The *attractor* $X^+$ is the subfunctor defined by

$$B \mapsto \{x \in X(B) \mid \lim_{a\to 0} a \cdot x \text{ exists}\}$$

for any $R$-algebra $B$, whilst the *fixed point locus* $X^0$ is the subfunctor defined by

$$B \mapsto \{x \in X(B) \mid \lim_{a\to 0} a \cdot x = x\}$$

for any $R$-algebra $B$.

The next result follows from [23, Theorem 2.1] and its proof.

**Theorem 2.13.** *Let $X = \mathrm{colim}_i X_i$ be an $R$-ind-scheme of ind-finite type. Then*

1. *The functors $X^0$ and $X^+$ are representable by closed ind-subschemes of $X$ of ind-finite type. In particular, one has*

$$X^0 = \mathrm{colim}_i X_i^0 \quad \textit{and} \quad X^+ = \mathrm{colim}_i X_i^+.$$

2. *The map $X^+ \to X^0$ defined at the level of $B$-points by*

$$x \mapsto \lim_{a\to 0} a \cdot x$$

*for every $R$-algebra $B$, is ind-affine with geometrically connected fibers.*

3. *If $Y$ is an $R$-ind-scheme equipped with a $\mathbb{G}_m$-action that is trivial on $R$, and $f : X \to Y$ is*

*schematic, then there are schematic maps*

$$X^0 \to Y^0 \quad \textit{and} \quad X^+ \to Y^+$$

*of fixed point loci and attractors respectively. Furthermore, if $f$ is a closed immersion then so are the induced maps on the fixed point loci and attractors.*

Let $G$ be an $R$-ind-group admitting a cocharacter $\lambda \in Y_G$. Then $\lambda$ defines a $\mathbb{G}_m$-action on $G$ by conjugation. The corresponding attractor is denoted by $P_\lambda(G)$ and the fixed point locus is denoted $L_\lambda(G)$.

The next result is a rephrasing of Theorem 2.13 in conjunction with the results of [11, §2.1].

**Proposition 2.14.** *Let $G = \operatorname{colim}_i G_i$ be an $R$-ind-group of ind-finite type with $\lambda \in Y_G$. Then:*

1. *$P_\lambda(G) = \operatorname{colim}_i G_i^+$ and $L_\lambda(G) = \operatorname{colim}_i G_i^0$ are closed subgroups of $G$.*
2. *There is an $R$-homomorphism $c_\lambda : P_\lambda(G) \to L_\lambda(G)$ defined at the level of $B$-points by*

$$g \mapsto \lim_{a \to 0} \lambda(a) g \lambda(a)^{-1}$$

*for any $R$-algebra $B$, whose kernel is an ind-connected subgroup, denoted by $U_\lambda(G)$.*

3. *The multiplication map $L_\lambda(G) \ltimes U_\lambda(G) \to P_\lambda(G)$ is an isomorphism of $R$-ind-groups.*
4. *If $H$ is a closed subgroup of $G$ and $\lambda \in Y_H$ then*

$$P_\lambda(H) = P_\lambda(G) \cap H, \quad L_\lambda(H) = L_\lambda(G) \cap H, \quad \textit{and} \quad U_\lambda(H) = U_\lambda(G) \cap H.$$

5. *If $B$ is an $R$-algebra then $P_{g\cdot\lambda}(G) = gP_\lambda(G)g^{-1}$ for any $g \in G(B)$. Similarly for $L_\lambda(G)$ and $U_\lambda(G)$.*

To avoid confusion, $G^0$ will always denote the *identity component* of an $R$-ind-group $G$. That is, the connected component of $G^0$ containing the identity element. In Proposition 2.14 above, note that the $G_i^0$ are the fixed point loci of the $R$-ind-*schemes* $G_i$.

Write $P_\lambda$ (resp. $L_\lambda$ and $U_\lambda$) when the ambient group is clear. Following [6], the $R$-ind-group $P_\lambda$ is called an *R-parabolic subgroup* of $G$. If $P$ is an R-parabolic subgroup of $G$ then $L = L_\lambda$ is an *R-Levi subgroup* of $P$ if $P = P_\lambda$ for some $\lambda \in Y_G$. Note that $L_\lambda = P_\lambda \cap P_{-\lambda}$ is the ind-scheme theoretic centraliser of $\lambda$ in $G$.

### 2.4.3 The formal Kac-Moody group ind-scheme

This subsection describes Mathieu's construction of a group ind-scheme associated to every root datum $\mathscr{D}$. For more details, see [33, 38]. The main idea is to construct a $\mathbb{Z}$-group associated to any closed set of roots in $\Delta$, thereby generalising the $\mathbb{Z}$-group $U_\Sigma$ defined for $\Sigma$ nilpotent. The new $\mathbb{Z}$-group, which is typically of infinite type, is then used to produce a group ind-scheme.

There is a Lie subalgebra

$$\mathfrak{g}_\Sigma = \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha \subseteq \mathfrak{g}$$

defined for any closed set of roots $\Sigma \subseteq \Delta \cup \{0\}$. Accordingly, one may define the $\mathbb{Z}$-subalgebra

$$\mathscr{U}_{\mathbb{Z}}(\Sigma) = \mathscr{U}(\mathfrak{g}_\Sigma) \cap \mathscr{U}_{\mathbb{Z}}(\mathfrak{g}) \subseteq \mathscr{U}_{\mathbb{Z}}(\mathfrak{g})$$

where $\mathscr{U}(\mathfrak{g}_\Sigma)$ is the universal enveloping algebra of $\mathfrak{g}_\Sigma$. If $\Sigma \subseteq \Delta^+ \cup \{0\}$ then $\mathfrak{g}_\Sigma$ admits a $Q_{\mathrm{rt}}^+$-grading which extends to $\mathscr{U}(\mathfrak{g}_\Sigma)$ and *a fortiori* to $\mathscr{U}_{\mathbb{Z}}(\Sigma)$.

Suppose that $\Sigma \subseteq \Delta^+ \cup \{0\}$. In order to define a scheme $U_\Sigma^{\mathrm{ma}+}$ one needs to build a coordinate ring which will represent it. Specifically, one requires a $\mathbb{Z}$-algebra $A_\Sigma$ which will become the coordinate ring, defining the scheme via the mapping $U_\Sigma^{\mathrm{ma}+}(R) = \mathrm{Hom}_{\mathbb{Z}-\mathrm{alg}}(A_\Sigma, R)$ for every ring $R$. This is realised by the *restricted dual* of $\mathscr{U}_{\mathbb{Z}}(\Sigma)$ defined as the $\mathbb{Z}$-module

$$A_\Sigma = \bigoplus_{\alpha\in\mathbb{Z}_{\geq 0}\Sigma} \mathscr{U}_{\mathbb{Z}}(\Sigma)^*_\alpha$$

where $\mathscr{U}_{\mathbb{Z}}(\Sigma)^*_\alpha = \mathrm{Hom}_{\mathbb{Z}}(\mathscr{U}_{\mathbb{Z}}(\Sigma)_\alpha, \mathbb{Z})$ is the $\mathbb{Z}$-dual of $\mathscr{U}_{\mathbb{Z}}(\Sigma)_\alpha$. In particular, $A_\Sigma$ is a commutative Hopf algebra, therefore it represents a $\mathbb{Z}$-group $U_\Sigma^{\mathrm{ma}+}$ which can be shown to be split pro-unipotent.

There is a $\mathbb{Z}_{\geq 0}$-grading of $\mathscr{U}_{\mathbb{Z}}(\Sigma)$ obtained from the *height* of elements with respect to the $Q^+_{\mathrm{rt}}$-grading. The completion of $\mathscr{U}_{\mathbb{Z}}(\Sigma)$ with respect to this $\mathbb{Z}_{\geq 0}$-grading is given by

$$\widehat{\mathscr{U}_{\mathbb{Z}}}(\Sigma) = \prod_{\alpha \in Q^+_{\mathrm{rt}}} \mathscr{U}_{\mathbb{Z}}(\Sigma)_\alpha.$$

The completed algebra $\widehat{\mathscr{U}_{\mathbb{Z}}}(\Sigma)$ can be identified with the $\mathbb{Z}$-dual of $A_\Sigma$. As a consequence, one may identify $U^{\mathrm{ma}+}_\Sigma$ with a multiplicative subgroup of $\widehat{\mathscr{U}_R}(\Sigma) = \widehat{\mathscr{U}_{\mathbb{Z}}}(\Sigma) \otimes_{\mathbb{Z}} R$ by [38, Proposition 3.2].

The construction above allows one to define a $\mathbb{Z}$-group $U^{\mathrm{ma}+}$ associated to the positive set of roots $\Delta^+ \subseteq \Delta$. Note that $U^{\mathrm{ma}+}$ depends only on the generalised Cartan matrix $C$ and not on $\mathscr{D}$. In order to make it clear in the sequel that $U^{\mathrm{ma}+}_\Sigma$ is contained in $U^{\mathrm{ma}+}$ one will often write

$$U^{\mathrm{ma}+}_\Sigma = U^{\mathrm{ma}+}(\Sigma).$$

If $\alpha \in \Delta^+$ then the set

$$\Sigma_\alpha = \mathbb{N}\alpha \cap \Delta \subseteq \Delta^+$$

is closed. In particular, if $\alpha$ is real then $\Sigma_\alpha = \{\alpha\}$ and otherwise $\Sigma_\alpha = \mathbb{N}\alpha$. Then a $\mathbb{Z}$-group $U^{\mathrm{ma}+}(\Sigma_\alpha)$ can be defined for each $\alpha \in \Delta^+$. If $\alpha$ is real, or $\Sigma$ is finite, then $U_\alpha = U^{\mathrm{pm}+}(\Sigma_\alpha)$ and $U^+_\Sigma = U^{\mathrm{ma}+}(\Sigma)$ are the unique smooth connected unipotent $\mathbb{Z}$-groups from 2.3.3 with Lie algebras $\mathfrak{g}_{\alpha\mathbb{Z}}$ and $\bigoplus_\Sigma \mathfrak{g}_{\alpha\mathbb{Z}}$ respectively.

The group of $R$-points $T_X(R)$ acts on $\mathfrak{g}_R$ via the adjoint map, and more broadly on $\widehat{\mathscr{U}^+_R} = \widehat{\mathscr{U}_R}(\Delta^+)$. Specifically, the action of an element $t \in T_X(R)$ on $\mathscr{U}_R(\mathfrak{g})_\alpha$ is multiplication by $\alpha(t) \in R^\times$. Therefore, after identifying $U^{\mathrm{ma}+}(R) \subseteq \widehat{\mathscr{U}^+_R}$, one obtains a conjugation action of $T_X$ on $U^{\mathrm{ma}+}$. Furthermore, every $U^{\mathrm{ma}+}(\Sigma)$ is stable under the action of $T_X$.

From now on, assume that $\mathscr{D} = \mathscr{D}^m$. This hypothesis will be removed later.

The conjugation action of $T_X$ on $U^{\mathrm{ma}+}$ defines the *canonical positive Borel $\mathbb{Z}$-group*

$$B^{\mathrm{ma}+} = T_X \ltimes U^{\mathrm{ma}+}$$

which mimics the Borel subgroup functor in the preceding section. Note that $B^{\mathrm{ma}+}$ is of finite type if and only if $C$ is a Cartan matrix. Similarly, there is a *canonical positive minimal parabolic $\mathbb{Z}$-group*

$$P_\alpha^{\mathrm{ma}+} = G_\alpha \ltimes U^{\mathrm{ma}+}(\Delta^+ \setminus \alpha)$$

associated to every simple root $\alpha \in \Pi$, where $G_\alpha$ is the unique Chevalley group defined by the root datum $(X, \alpha, X^\vee, \alpha^\vee)$. In particular, there is a closed $\mathbb{Z}$-group $U_{-\alpha} \subseteq G_\alpha$ acting on $U^{\mathrm{ma}+}(\Delta^+ \setminus \alpha)$ that is isomorphic to $G_{a,\mathbb{Z}}$. As the name suggests, the $\mathbb{Z}$-group $P_\alpha^{\mathrm{ma}+}$ corresponds to a minimal parabolic subgroup. Note that $T_X, U_\alpha$, and $U_{-\alpha}$ are closed in $G_\alpha$, which in turn is closed, along with $B^{\mathrm{ma}+}$, in $P_\alpha^{\mathrm{ma}+}$.

The $\mathbb{Z}$-group $B^{\mathrm{ma}+}$ acts by left and right multiplication on every $P_\alpha^{\mathrm{ma}+}$. Therefore, one can define the $\mathbb{Z}$-scheme

$$P_\alpha^{\mathrm{ma}+} \times^{B^{\mathrm{ma}+}} P_\beta^{\mathrm{ma}+} = (P_\alpha^{\mathrm{ma}+} \times P_\beta^{\mathrm{ma}+})/B^{\mathrm{ma}+}$$

which is the *fppf sheafification* of the presheaf quotient

$$R \mapsto (P_\alpha^{\mathrm{ma}+}(R) \times P_\beta^{\mathrm{ma}+}(R))/\sim$$

where $(p, p') \sim (pb^{-1}, bp')$ for all $b \in B^{\mathrm{ma}+}(R),\ p \in P_\alpha^{\mathrm{ma}+}(R)$, and $p' \in P_\beta^{\mathrm{ma}+}(R)$. For details of quotients of schemes by group actions, see [34, Chapter 7].

If $w = s_{i_1} \cdots s_{i_d}$ is a reduced decomposition in $W$, then define the $\mathbb{Z}$-scheme

$$E(w) = P_{\alpha_{i_1}}^{\mathrm{ma}+} \times^{B^{\mathrm{ma}+}} P_{\alpha_{i_2}}^{\mathrm{ma}+} \times^{B^{\mathrm{ma}+}} \cdots \times^{B^{\mathrm{ma}+}} P_{\alpha_{i_d}}^{\mathrm{ma}+}.$$

In general, $E(w)$ need not be an affine. As a remedy, one considers its *affinisation*

$$B(w) = \mathrm{Spec}(\mathbb{Z}[E(w)])$$

which is independent of the choice of reduced decomposition for $w$ [33, p. 40].

If $v \leq w$ in the Bruhat order for some $v \in W$, then $v$ is a subword of $w$ for some reduced decompositions of $v$ and $w$. Hence, after replacing $P^{\mathrm{ma}+}_{\alpha_{i_1}}$ with $B^{\mathrm{ma}+}$ for all letters $s_i$ in $w$ outside of $v$, one obtains a closed subscheme of $E(w)$ isomorphic to $E(v)$. The corresponding closed immersion $E(v) \hookrightarrow E(w)$ induces a closed immersion $B(v) \hookrightarrow B(w)$ which is independent of the choice of reduced decompositions of $v$ and $w$ [32, p. 255].

The *formal Kac-Moody $\mathbb{Z}$-group* is the functor $G^{\mathrm{ma}+}_{\mathscr{D}} : \mathbb{Z}\text{-alg} \to \mathrm{Grp}$ defined as the inductive limit

$$G^{\mathrm{ma}+}_{\mathscr{D}} = \mathrm{colim}_{w \in W} B(w)$$

in the category of presheaves, with respect to the Bruhat order on $W$. Therefore $G^{\mathrm{ma}+}_{\mathscr{D}}$ is an ind-scheme over $\mathbb{Z}$. The *opposite formal Kac-Moody $\mathbb{Z}$-group* $G^{\mathrm{ma}-}_{\mathscr{D}}$ can be defined in the same way after swapping $\Delta^+$ with $\Delta^-$.

If $v = s_{i_1} \cdots s_{i_j}$ and $w = s_{i_l} \cdots s_{i_m}$ are reduced decompositions then the morphism

$$P^{\mathrm{ma}+}_{\alpha_{i_1}} \times \cdots \times P^{\mathrm{ma}+}_{\alpha_{i_j}} \times P^{\mathrm{ma}+}_{\alpha_{i_l}} \times \cdots \times P^{\mathrm{ma}+}_{\alpha_{i_m}} \to B(v) \times B(w)$$

factors through $B(\phi(v,w))$ where $\phi(v,w) \leq vw$. That is to say, $\phi(v,w)$ admits a reduced decomposition which is a subword of a reduced decomposition of $vw$. This induces a morphism

$$G^{\mathrm{ma}+}_{\mathscr{D}} \times G^{\mathrm{ma}+}_{\mathscr{D}} \to G^{\mathrm{ma}+}_{\mathscr{D}}$$

of ind-schemes. Similarly, one obtains morphisms

$$B(w) \to B(w^{-1}) \quad \text{and} \quad \mathrm{Spec}\,\mathbb{Z} \to B^{\mathrm{ma}+} \to B(w)$$

which in turn induce morphisms

$$G^{\mathrm{ma}+}_{\mathscr{D}} \to G^{\mathrm{ma}+}_{\mathscr{D}} \quad \text{and} \quad \mathrm{Spec}\,\mathbb{Z} \to G^{\mathrm{ma}+}_{\mathscr{D}}$$

of ind-schemes, respectively [33, p. 45].

It follows that $G^{\mathrm{ma}+}_{\mathscr{D}}$ is a $\mathbb{Z}$-ind-group. Moreover, $B^{\mathrm{ma}+}, P^{\mathrm{ma}+}_{\alpha}, T_X$, and $U^{\mathrm{ma}+}$, are closed subgroups of $G_{\mathscr{D}}^{\mathrm{ma}+}$. The construction above can be modified to define an $R$-ind-group $G_{\mathscr{D}R}^{\mathrm{ma}+}$ for any ring $R$. However, the formation of $G_{\mathscr{D}}^{\mathrm{ma}+} = G_{\mathscr{D}\mathbb{Z}}^{\mathrm{ma}+}$ commutes with base change, so this is not strictly necessary (cf. [8, Proposition 2.3.3]).

As one might hope, there is a homomorphism $G_{\mathscr{D}}^{\mathrm{Tits}} \to G_{\mathscr{D}}^{\mathrm{ma}+}$ of group functors. Moreover, the Lie algebra of $G_{\mathscr{D}}^{\mathrm{ma}+}$ is isomorphic as a $\mathbb{Z}$-Lie algebra to the $\mathbb{Z}$-form of the *formal Kac-Moody algebra*

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}^{+} \oplus \mathfrak{h} \oplus \mathfrak{n}^{-}$$

where $\widehat{\mathfrak{n}}^{+} = \prod_{\alpha \in \Delta^{+}} \mathfrak{g}_{\alpha}$ is the completion of $\mathfrak{n}^{+}$. That is,

$$\operatorname{Lie} G_{\mathscr{D}}^{\mathrm{ma}+} \cong \widehat{\mathfrak{g}}_{\mathbb{Z}} = \widehat{\mathfrak{n}}_{\mathbb{Z}}^{+} \oplus \mathfrak{h}_{\mathbb{Z}} \oplus \mathfrak{n}_{\mathbb{Z}}^{-}$$

This also holds true after replacing $\mathbb{Z}$ with any ring $R$ [8, Theorem 2.4.7].

Mathieu's construction is not dependent on $\dim \mathscr{D}$ and passes through identically for any free, cofree, and cotorsion-free root datum. By Proposition 2.8, any such datum $\mathscr{D}$ is a semidirect extension of the minimal free datum $\mathscr{D}^{m}$. The result is a functor

$$G_{\mathscr{D}}^{\mathrm{ma}+} = G_{\mathscr{D}^{m}}^{\mathrm{ma}+} \rtimes T_{3}$$

where $T_3$ is as in §2.3.3. If $\mathscr{D}$ is arbitrary, then $G_{\mathscr{D}\mathrm{Kac}}^{\mathrm{Tits}}$ can be constructed via a central extension $G_{\mathscr{D}\mathrm{cof}}^{\mathrm{Tits}}$ of $G_{\mathscr{D}}^{\mathrm{Tits}}$ by a torus $T_1$, followed by a semidirect extension of $G_{\mathscr{D}\mathrm{cof}}^{\mathrm{Tits}}$ by a torus $T_2$. The homomorphism $G_{\mathscr{D}\mathrm{Kac}}^{\mathrm{Tits}} \to T_2$ with kernel $G_{\mathscr{D}\mathrm{cof}}^{\mathrm{Tits}}$ extends to a homomorphism $G_{\mathscr{D}\mathrm{Kac}}^{\mathrm{ma}+} \to T_2$ whose kernel is denoted by $G_{\mathscr{D}\mathrm{cof}}^{\mathrm{ma}+}$. Finally, one defines $G_{\mathscr{D}}^{\mathrm{ma}+}$ to be the quotient of $G_{\mathscr{D}\mathrm{cof}}^{\mathrm{ma}+}$ by $T_1$ as a group functor. Strictly speaking, this defines two functors for a free, cofree, and cotorsion-free datum, but one can be identified with the other [38, §3.19].

When $\mathscr{D} = \mathscr{D}^{m}$ and $C$ is a Cartan matrix, $G_{\mathscr{D}}^{\mathrm{ma}+}$ is a semisimple *simply-connected* Chevalley group. The discussion above implies that the functors $G_{\mathscr{D}}^{\mathrm{ma}+}$ are precisely the Chevalley groups when $C$ is a Cartan matrix.

In the sequel, the notation for $G_{\mathscr{D}}^{\mathrm{ma}+}, G_{\mathscr{D}}^{\mathrm{ma}-}$, and $T_X$ will be abbreviated to $G^{\mathrm{ma}+}, G^{\mathrm{ma}-}$, and $T$

respectively.

### 2.4.4 The minimal Kac-Moody group ind-scheme

As mentioned previously, the main obstruction preventing a Kac-Moody group possessing a geometric structure is the fact that the group functor $U^+_{\mathrm{ab}}$ is not representable. In Mathieu's construction of the formal group, one instead considers a certain completion of this functor, defining a scheme $U^{\mathrm{ma}+}$ of infinite type. This makes the resulting group ind-scheme $G^{\mathrm{ma}+}$ rather unwieldy. Another consequence of the construction is that $\mathrm{Lie}\, G^{\mathrm{ma}+}$ is a choice of completion of the desired Kac-Moody algebra and therefore symmetry is lost between positive and negative directions.

However, the formal groups $G^{\mathrm{ma}+}$ and $G^{\mathrm{ma}-}$ provide a space in which one can define a more suitable closed group ind-scheme $G$. This is achieved by considering a certain ind-subscheme of $U^{\mathrm{ma}+}$ of ind-finite type. Roughly speaking, this amounts to the fact that, over an algebraically closed field, the abstract Kac-Moody group defined in 2.3.3 can be endowed with an ind-variety structure. This section outlines the definition of $G$ which is due to Bouthier and Vasserot [8]. The construction is technical, so the exposition will be brief.

The first step towards the construction of an ind-subscheme of $G^{\mathrm{ma}+}$ that is ind-finite type is to construct an analogue of $U^{\mathrm{ma}+}$ that is ind-finite type. This is achieved using the formalism of attractors. In particular, since $T \subseteq B^{\mathrm{ma}+}$ the presentation of $G^{\mathrm{ma}+}$ as an inductive limit of the $\mathbb{Z}$-schemes $B(w)$ is $T$-stable under the conjugation action. Therefore, one can define the R-parabolic subgroup $P_\lambda(G^{\mathrm{ma}+})$ of $G^{\mathrm{ma}+}$ for any cocharacter $\lambda \in T$. Strictly speaking, this is not covered by §2.4.2 since $G^{\mathrm{ma}+}$ is not of ind-finite type. However, the details of the construction can be found in the proof of [22, Lemma 4.1] whose statement is recorded here.

**Proposition 2.15.** *Let $J \subseteq I$. Suppose that $\lambda \in T$ with $\langle \lambda, \alpha_i \rangle = 0$ for all $i \in J$, and $\langle \lambda, \alpha_i \rangle > 0$ for all $i \in I \setminus J$. Then the group ind-scheme $U_{-\lambda}(G^{\mathrm{ma}+})$ is of ind-finite presentation. Moreover, the multiplication morphism*

$$U_{-\lambda}(G^{\mathrm{ma}+}) \times P_\lambda(G^{\mathrm{ma}+}) \to G^{\mathrm{ma}+}$$

*is a quasi-compact open immersion.*

In particular, if $\lambda$ is *generic* in the sense that $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in \Pi$, then there are closed subgroups

$$U^+ = U_\lambda(G^{\text{ma}-}) \subseteq G^{\text{ma}-} \quad \text{and} \quad U^- = U_{-\lambda}(G^{\text{ma}+}) \subseteq G^{\text{ma}+}$$

of ind-finite presentation, where

$$U^+(F) = U^+_{\text{ab}}(F) = \langle U_\alpha(F) \mid \alpha \in \Delta^+_{\text{re}} \rangle$$

for any field $F$.

By Proposition 2.15, the product

$$U^- \times P_\lambda(G^{\text{ma}+}) = U^- \times T \times U^{\text{ma}+}$$

is a dense open ind-subscheme of $G^{\text{ma}+}$. The idea is to construct a similar ind-subscheme, but now of ind-finite presentation, by replacing $U^{\text{ma}+}$ with $U^+$. Thus far, $U^+$ is a closed subgroup of $G^{\text{ma}-}$ but not of $G^{\text{ma}+}$ *a priori*. This can be resolved and, as one might expect, there is an ind-closed embedding

$$U^+ \hookrightarrow U^{\text{ma}+} \subseteq G^{\text{ma}+}$$

of group ind-schemes, defined over $\mathbb{Z}$ [8, Proposition 2.4.4]. Therefore, there is an ind-closed embedding

$$U^- \times T \times U^+ \hookrightarrow U^- \times T \times U^{\text{ma}+} \subseteq G^{\text{ma}+}$$

defined over $\mathbb{Z}$, where $U^+ \times T \times U^-$ is an ind-scheme of ind-finite presentation.

Taking inspiration from the Birkhoff decomposition of a Kac-Moody group, it remains to build an ind-scheme that is covered by translates of the ind-scheme above, at least at the level of $F$-points, and then show such a functor is well-behaved. This is done formally as follows.

The *minimal Kac-Moody $\mathbb{Z}$-group* is the functor $G_{\mathscr{D}} : \mathbb{Z}\text{-alg} \to \text{Grp}$ defined as the *Zariski* *sheafification* of the presheaf given by

$$R \mapsto \bigcup_{w \in W} w(U^+(R) \times T(R) \times U^-(R))$$

for any ring $R$. For details on the process of sheafification, see [42, 007X].

There is a canonical isomorphism

$$G_{\mathscr{D}} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(R) \cong G_{\mathscr{D}R}$$

for any ring $R$, where $G_{\mathscr{D}R}$ is the functor obtained via the same construction, using instead $T_R, U_R^+$, and $U_R^-$. In other words, the construction of $G_{\mathscr{D}}$ commutes with base-change [8, Proposition 2.5.2(a)]. In the sequel, the notation will often be abbreviated to $G$ and $G_R$ respectively.

The functor $G$ is representable as an ind-finitely presented $\mathbb{Z}$-ind-scheme. In particular, there is an ind-closed embedding

$$G \hookrightarrow G^{\mathrm{ma}+}$$

of group ind-schemes over $\mathbb{Z}$. Note that this does not imply that $G$ is a closed subgroup of $\widehat{G}$ as it may not be schematic.

Moreover, the group of $F$-points $G(F)$ is canonically isomorphic to $G^{\mathrm{Tits}}(F)$ for any field $F$. Further, one has an isomorphism of $R$-Lie algebras

$$\mathrm{Lie}\, G_R \cong \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$$

for any ring $R$, where $\mathfrak{g}_{\mathbb{Z}}$ is the $\mathbb{Z}$-form of the Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}_{\mathscr{D}}$. In particular, the Lie algebras of the minimal Kac-Moody groups coincide with the Kac-Moody algebras, in contrast to the formal Kac-Moody groups. This is [8, Proposition 2.5.3] and its proof.

There is a *canonical positive Borel $\mathbb{Z}$-ind-group* and a *canonical negative Borel $\mathbb{Z}$-ind-group*

$$B^+ = T \ltimes U^+ \quad \text{and} \quad B^- = T \ltimes U^-$$

respectively. These are closed subgroups of $G$ and one has

$$U^+ = U^{\mathrm{ma}+} \cap G \quad \text{and} \quad B^+ = B^{\mathrm{ma}+} \cap G$$

from which it follows that $T$ is a closed subgroup of $G$ also [8, Proposition 2.5.2(d)].

In fact, $G_R$ is *ind-normal* over $R$ in the sense of [8]. As a consequence, the base change $G_F$ over a field $F$ is ind-reduced. It follows that, for any algebraically closed field $k$, the minimal Kac-Moody $k$-group $G_k$ is an ind-variety, and can be identified with its set of $k$-points $G_k(k)$ endowed with the ind-Zariski topology. Henceforth, a Kac-Moody group of 2.3.3 may be endowed with the ind-variety structure of $G_k$.

## 2.5 Groups and buildings

This section describes the concept of a building associated to a Coxeter system. These objects were originally defined by Tits due to their relation to semisimple algebraic groups [43]. Specifically, any semisimple linear algebraic group admits an action on an associated spherical building. In general, buildings can be defined axiomatically with no reference to a group action. However, the various generalisations of buildings provide effective ways to study more exotic structures, such as reductive groups over local fields and Kac-Moody groups. For a comprehensive survey on the subject, see [1].

### 2.5.1 Tits buildings

There are many equivalent approaches that one can take when defining buildings. In this subsection, a building is defined as a *$W$-metric space*. This is a relatively modern approach that is suited to introducing *projection maps* quickly. The goal of §3.2 will be to generalise this framework and define a new structure.

Let $(W,S)$ be a Coxeter system. A *building* of type $(W,S)$ is a pair $(\mathscr{B},\delta)$ consisting of a non-empty set $\mathscr{B}$, along with a $W$-valued map

$$\delta : \mathscr{B} \times \mathscr{B} \to W,$$

such that, for all $b, c \in \mathscr{B}$ with $\delta(b,c) = w$, and all $s \in S$, the following properties hold:

**(B1)** $\delta(b,c) = 1$ if and only if $b = c$.

**(B2)** If $b' \in \mathscr{B}$ satisfies $\delta(b',b) = s$, then $\delta(b',c) \in \{w, sw\}$. Moreover, one has $\delta(b',c) = sw$ if $l(sw) > l(w)$.

**(B3)** There is $b' \in \mathscr{B}$ with $\delta(b',b) = s$ and $\delta(b',c) = sw$.

This is a combinatorial object which encodes the standard length function of $(W,S)$. The map $\delta$ will often be omitted from notation. The elements of $\mathscr{B}$ are called *chambers*.

**Example 2.16.** A Coxeter group $W$ is a building of type $(W,S)$, where

$$\delta : W \times W \to W$$

is defined by $\delta(v,w) = v^{-1}w$. Then **(B1)** and **(B3)** are immediate, and observe that if $\delta(b',b) = (b')^{-1}b = s$ and $\delta(b,c) = b^{-1}c = w$ then

$$\delta(b',c) = (b')^{-1}c = (b')^{-1}bb^{-1}c = sw$$

regardless of the extra condition in **(B2)**.

A pair of chambers $b$ and $c$ in $\mathscr{B}$ are called *s-adjacent* if $\delta(b,c) = s$ and *s-equivalent* if $\delta(b,c) \in \{1,s\}$. The $s$-adjacency relation is symmetric while $s$-equivalence is an equivalence relation. Call the equivalence classes *s-panels*, or simply *panels*. The building $\mathscr{B}$ is *thick* if every panel contains at least three chambers and *thin* otherwise, e.g. the Coxeter groups of Example 2.16.

If $b$ and $c$ are $s$-adjacent for some $s \in S$ then call them *adjacent*. Then a *gallery* in $\mathscr{B}$ is a sequence $b_0, \dots, b_n$ of chambers where each $b_i$ and $b_{i-1}$ are adjacent. The *distance* $d(b_0, b_n)$ between $b_0$ and $b_n$ is the minimal length gallery between them. Then $b_0, \dots, b_n$ is a *minimal gallery* if

$d(b_0,b_n)=n$. It is a basic fact that $d(b,c)=l(\delta(b,c))$ for all $b,c\in\mathscr{B}$. In particular, there is a gallery between every pair of chambers. Moreover, one can show that $\delta(b,c)=\delta(c,b)^{-1}$ and therefore $d(b,c)=d(c,b)$.

If $W$ is finite then the distance between a pair of chambers is bounded. In particular, it is at most $l(w_0)$ where $w_0$ is the longest element of $W$. A pair of chambers $b$ and $c$ are *opposite* if $d(b,c)=l(w_0)$. In general, a chamber will have many opposite chambers. This opposition relation exists only in spherical buildings. An analogous property for pairs of non-spherical buildings will be defined in 3.2.1.

If $J\subseteq S$ then say $b$ and $c$ are *$J$-equivalent* if $\delta(b,c)\in W_J$. This is a direct generalisation of $s$-equivalence, and is also an equivalence relation. A *$J$-residue* is an equivalence class under this relation. It follows that the $J$-residue containing $b$ is given by

$$R_J(b)=\{c\in\mathscr{B}\mid\delta(b,c)\in W_J\}.$$

A subset of $\mathscr{B}$ is called a *residue* of *type* $J$ if it is a $J$-residue for some $J\subseteq S$. For instance, an $s$-panel is a residue of type $J=\{s\}$. The $J$-residues of $\mathscr{B}$ are themselves buildings of type $(W_J,J)$.

For every residue $R_J$ and every chamber $c\in\mathscr{B}$, there is a unique chamber $b\in R_J$ for which $d(b,c)$ is minimal [1, Proposition 5.34]. This chamber $b$ is called the *projection* of $c$ onto $R_J$ and its existence is intimately linked to the fact that every coset of $W_J$ in $W$ has a unique element of minimal length. This can be extended further to the notion of a projection of a residue onto another residue. If the residues are of the same type, then the projection maps between them define isomorphisms between the corresponding buildings [1, Proposition 5.37].

A subset $M\subseteq\mathscr{B}$ is *convex* if it contains a minimal gallery for every pair of chambers $b,c\in M$. For example, every residue is convex. Equivalently, $M$ is convex if and only if it is closed under projections. This means that convex subsets are a particularly useful class of subsets that one can work with. A countable intersection of convex sets is again convex, provided it is non-empty. The smallest convex set containing a given non-empty subset of $\mathscr{B}$ is called the *convex hull*.

An arbitrary subset of $\mathscr{B}$ is called a *subbuilding* if it is a building of type $(W,S)$, along with the restriction of $\delta$. An *apartment* of $\mathscr{B}$ is a thin subbuilding. Equivalently, a non-empty subset of $\mathscr{B}$ is an apartment if it is thin and convex. It is a fundamental property of buildings that the apartments are *isometric* to the Coxeter complexes of Example 2.16. These subbuildings give a local model for $\mathscr{B}$ in the sense that any pair of chambers of $\mathscr{B}$ share a common apartment. Moreover, when $W$ is finite, there is precisely one opposite to every chamber in an apartment.

### 2.5.2 Buildings associated to groups

This subsection briefly outlines the relationship between a BN-pair and a building. This is a particularly important idea that motivates much of the rest of the thesis. In particular, the existence of twinned BN-pairs in Kac-Moody groups implies the existence of an associated *twin building*. The results in the sequel will establish a far greater class of buildings associated to a Kac-Moody group, which may imply the existence of a new generalisation of a BN-pair. This will be discussed in §6.

Suppose that a group $G$ acts on a building $\mathscr{B}$. The action is *strongly transitive* if it is transitive on pairs $(\mathscr{A},b)$ where $\mathscr{A}$ is an apartment and $b \in \mathscr{A}$ is a chamber. In this case, one can fix a distinguished pair $(\mathscr{A}_0,b_0)$ consisting of the *fundamental apartment* $\mathscr{A}_0$ and the *fundamental chamber* $b_0$. The following result is due to Tits [1, Theorem 6.56].

**Theorem 2.17.** *Let $G$ be a group with a BN-pair $(B,N)$. Then there is a thick building $\mathscr{B}$ that admits a strongly transitive action of $G$, where $B$ is the stabiliser of the fundamental chamber and $N$ is the stabiliser of the fundamental apartment.*

To construct the building $\mathscr{B}$ explicitly, one considers coset space $G/B$ equipped with the $W$-valued map

$$\delta : G/B \times G/B \to W$$

defined by $\delta(gB,hB) = w$ if $g^{-1}h \in BwB$, where $g,h \in G$.

The fact that $\delta$ satisfies **(B1)** is immediate, while the axioms **(B2)** and **(B3)** follow from the property

$$BswB \subseteq BsBwB \subset BwB \cup BswB$$

of the Bruhat decomposition. Similarly, the condition

$$sBs^{-1} \not\subseteq B$$

implies that the building is thick.

**Example 2.18.** 1. Let $G$ be a reductive group with a maximal torus $T$. Suppose that $B$ is a Borel subgroup containing $T$, and set $N = N_G(T)$. Then $(B,N)$ is a BN-pair for $G$ and $G/B$ is a spherical building. The apartments of $G/B$ correspond to maximal tori of $G$. If $S$ is a maximal torus of $G$ then the chambers of the associated apartment correspond to Borel subgroups containing $S$. The parabolic subgroups of $G$ correspond to residues in $G/B$.

2. Let $G$ be a Kac-Moody group. Then $(B^+,N)$ and $(B^-,N)$ are BN-pairs and therefore $G$ admits two thick (often non-spherical) buildings. The fact that the BN-pairs are twinned implies that the two buildings form a *twin building* which will be defined in §3.2.1.

CHAPTER 3

# Citadels

In this chapter a generalisation of a twin building is defined, in which one component is replaced by a building associated to an arbitrary system of positive roots. The codistance that relates the two components of a twin building is then replaced by a distance function governed by an appropriate twisted Bruhat order. It is then shown that projection maps exist between components of the new structure, subject to natural restraints on residues that one finds in twin building theory. The existence of projections defines a notion of convexity in a way that is in line with the analogous notions in buildings and twin buildings.

One feature of the new pairs of buildings is that their local models are not stable under the action of the ambient Weyl group. This introduces an element of choice when picking a representative of the conjugacy class of positive system. The solution provided in this chapter is to consider all choices at once and glue the resulting object appropriately. It is straightforward to show that the distance functions on the subobjects do not degenerate.

Finally, a citadel is defined. This is an object that contains all the conjugacy classes of positive roots at once. It should be thought of as a constellation of buildings, with enclaves of buildings corresponding to the conjugacy class of their associated positive system. There is a single central building, called the keep, corresponding to the usual standard system of positive roots, and every building has a well-defined distance to and from it. The other buildings do not relate to one another *a priori*. However, there is a natural partition of the space induced from taking negations of systems of positive roots. Therefore, every building has a twin, lying on the other side of the partition, and the resulting pairs can possess a twin building structure.

## 3.1 Preliminaries

This section records some generalisations of standard results for Coxeter systems. The material can all be attributed to Dyer [14, 17] and precise references will be given where appropriate. Throughout, let $(W,S)$ be a Coxeter system with length function $l : W \to \mathbb{Z}_{\geq 0}$. Moreover, assume that $W$ is crystallographic and $S$ is finite.

### 3.1.1 Twisted length functions

In this subsection several length functions on a Coxeter group are introduced, generalising the well-known standard length function. The material presented here can be found in greater generality in [15].

Let $V$ be the faithful representation of $W$ defined in 2.1. Suppose that $\Delta$ is the corresponding root system. The following definition appears as early as [26].

**Definition 3.1.** A subset $\Psi \subset \Delta$ is a *set of positive roots* if it is closed, $\Psi \cap (-\Psi) = \emptyset$, and $\Psi \cup (-\Psi) = \Delta$.

The standard example of a set of positive roots comes from a *root basis*, i.e. a collection of roots $\{\alpha_1, \ldots, \alpha_n\}$ such that for every $\beta \in \Delta$, one can write $\beta = \sum_{i=1}^{n} c_i \alpha_i$ where the $c_i \in \mathbb{Z}$ are either all non-negative or all non-positive. Then the set of roots for which the $c_i$ are all non-negative forms a set of positive roots. These sets of positive roots are often called *standard*, e.g. the standard set of positive roots $\Delta^+$ associated to the simple roots. This approach yields every set of positive roots precisely when $W$ is finite. In general, there are finitely many distinct $W$-orbits of sets of positive roots when $W$ is affine, and possibly infinitely many when $W$ is of indefinite type [27, 19].

**Example 3.2.** Let $(W,S)$ be a Coxeter system of affine type that is *untwisted* in the sense of [28, Chapter 7]. Then $\Delta$ is a real affine root system with roots of the form $\alpha + n\delta$ for some $n \in \mathbb{Z}$, where $\alpha$ is a root of the underlying finite root system $\dot{\Delta}$ and $\delta$ is the indecomposable imaginary root. The standard set of positive roots is

$$\Delta^+ = \dot{\Delta}^+ \cup \{\alpha + n\delta \in \Delta \mid n \in \mathbb{N}\}.$$

Suppose that $J$ is a subset of the simple roots associated to $\dot{\Delta}^+$. Then

$$\Psi_J = \{\alpha + n\delta \in \Delta \mid \alpha \in \dot{\Delta}^+ \setminus \mathbb{Z}J,\ n \in \mathbb{Z}\} \cup \dot{\Delta}^+ \cup \{\alpha + n\delta \in \Delta \mid \alpha \in \dot{\Delta} \cap \mathbb{Z}J,\ n \in \mathbb{N}\}.$$

is a set of positive roots and every set of positive roots is of this form, up to conjugation by $W \times \{\pm 1\}$. This classification is due to [27]. If $J = \emptyset$ then

$$\Psi_J = \{\alpha + n\delta \in \Delta \mid \alpha \in \dot{\Delta}^+,\ n \in \mathbb{Z}\}$$

is often called the *natural partition*.

The next result gives a recipe to construct all sets of positive roots, due to Billig-Dyer [7, Proposition 1].

**Proposition 3.3.** *Fix an ordered basis of $V$. Then the set of roots $\Psi \subset \Delta$ whose first non-zero coordinate is positive with respect to this basis is a set of positive roots. Moreover, every set of positive roots can be obtained this way.*

Note that multiple orderings may give rise to the same set of positive roots. Roughly speaking, Proposition 3.3 says that every set of positive roots arises by taking successive hyperplanes in $V$ along with choices of positive and negative halves. In contrast to finite root systems, the number of hyperplanes required in infinite root systems may be greater than one.

Let $\Psi$ be a set of positive roots in $\Delta$. Set $A_\Psi = \{s_\alpha \mid \alpha \in \Delta^+ \cap (-\Psi)\}$. Then $A_\Psi$ is an initial section of a reflection order in the sense of [16]. This can be thought of in the following way. A *reflection order* on $\Delta^+$ is a total order such that, for all $\alpha, \beta \in \Delta^+$, the restriction of the order to the plane $\mathbb{R}\alpha + \mathbb{R}\beta$ is either the order defined by a clockwise or anticlockwise sweep through the plane, beginning at a negative root. An *initial section* of a reflection order $\preceq$ is a set of reflections $A \subseteq R_W$ for which $s \preceq t$ for all $s \in A$ and $t \in R_W \setminus A$.

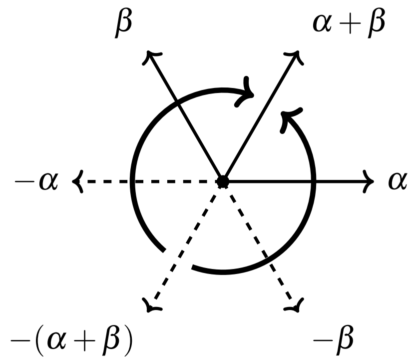


Figure 3.1: The possible restrictions of a reflection order to $\mathbb{R}\alpha+\mathbb{R}\beta$

Importantly, there is a length function on $W$ associated to any initial section [15]. The *twisted length function* $l_\Psi : W \to \mathbb{Z}$ associated to $A_\Psi$ is defined by

$$l_\Psi(w) = l(w) - 2|(\Delta^+ \setminus \Psi) \cap w^{-1}(-\Delta^+)|.$$

In words, $l_\Psi(w)$ counts (with multiplicity) the number of roots in $\Delta^+$ sent to $-\Delta^+$ by $w$, where a root is counted with multiplicity 1 if it lies in $\Psi$ and with multiplicity $-1$ otherwise. If $\Psi = \Delta^+$ then one retrieves the standard length function $l_+ = l$ while $\Psi = -\Delta^+$ returns the opposite length function $l_- = -l$.

The inversion sets $\mathscr{N}(w)$ have an equivalent characterisation of a more algebraic flavour which is described in [13, Lemma 1.3]. In particular, there is an *inversion function* $\mathscr{N} : W \to \mathscr{P}(R_W)$ that is characterised by $\mathscr{N}(s) = \{s\}$ for all $s \in S$, and $\mathscr{N}(vw) = \mathscr{N}(v) + v\mathscr{N}(w)v^{-1}$ for all $v, w \in W$. Here, the addition sign refers to the operation of symmetric difference. This immediately defines an action of $W$ on the collection of sets $A_\Psi$ given by $wA_\Psi = \mathscr{N}(w) + wA_\Psi w^{-1}$ for all $w \in W$ and $A \subseteq R_W$. As one might hope, $wA_\Psi = A_{w\Psi}$ for all $w \in W$.

The following proposition records basic properties of twisted length functions [15].

**Proposition 3.4.** *Let $\Psi \subset \Delta$ be a set of positive roots. Then:*

1. *If $w \in W$ is the identity then $l_\Psi(w) = 0$.*

2. *For all $w \in W$, we have $l_{-\Psi}(w) = -l_{\Psi}(w)$.*

3. *For all $t \in R_W$ and $w \in W$, we have $l_{\Psi}(tw) > l_{\Psi}(w)$ if and only if $t \notin wA_{\Psi}$.*

4. *For all $s \in S$ and $w \in W$, we have $l_{\Psi}(sw) = l_{\Psi}(w) \pm 1$.*

Alongside the length function, there is a *twisted Bruhat order* $\leq_{\Psi}$ on $W$ such that $v \leq_{\Psi} w$ if and only if there is a sequence $v = v_0, v_1, \ldots, v_n = w$ of elements in $W$ with $l_{\Psi}(v_i) = l_{\Psi}(v_{i-1}) + 1$ and $v_i v_{i-1}^{-1} \in R_W$. Therefore, Proposition 3.4(4) can be written as the following:

5. For all $s \in S$ and $w \in W$, we have

$$l_{\Psi}(sw) - l_{\Psi}(w) = \begin{cases} +1 & \text{if } sw >_{\Psi} w, \\ -1 & \text{if } sw <_{\Psi} w. \end{cases}$$

*Remark* 3.5. The analogue of Proposition 3.4(3) for right multiplication can be deduced as follows:

$$\begin{aligned} l_{\Psi}(wt) > l_{\Psi}(w) &\iff l_{\Psi}((wtw^{-1})w) > l_{\Psi}(w), \\ &\iff wtw^{-1} \notin wA_{\Psi}, \\ &\iff wtw^{-1} \notin \mathscr{N}(w) + wA_{\Psi}w^{-1}, \\ &\iff t \notin w^{-1}\mathscr{N}(w)w + A_{\Psi}, \\ &\iff t \notin \mathscr{N}(w^{-1}) + A_{\Psi}. \end{aligned}$$

The twisted Bruhat order when $\Psi = \Delta^+$ is simply the Bruhat order, while $\Psi = -\Delta^+$ induces the reverse Bruhat order. This is part of a more general pattern. Indeed, the definition of $\Psi$ implies that $A_{-\Psi} = R_W \setminus A_{\Psi}$ for any $\Psi \subset \Delta$. It follows that $\leq_{-\Psi}$ is the reverse of the order $\leq_{\Psi}$ for any set of positive roots [15, Remark 1.6]. Note that $(W, \leq_{\Psi})$ is order isomorphic to $(W, \leq_{w\Psi})$ for all $w \in W$. For more detail on this topic, see [17, §1.12].

**Example 3.6.** Let $(W,S)$ be an untwisted affine Coxeter system. Suppose that $\Psi$ is the natural partition of Example 3.2. Then $A_{\Psi} = \{s_{n\delta-\alpha} \mid \alpha \in \dot{\Delta}^+,\ n \in \mathbb{N}\}$. By [17, Proposition 1.16] the associated partial order is isomorphic to the order on the set of alcoves in the natural Euclidean reflection representation studied by Lusztig [30]. This is sometimes referred to as the *semi-infinite Bruhat order*. Here, the Euclidean reflection representation is the standard realisation of an affine Coxeter group by reflections in affine hyperplanes in Euclidean space, see [1, §10.1]. Note that it is distinct from the faithful linear representation defined in 2.1.

### 3.1.2 Reflection subgroups

This subsection presents material on certain subgroups of Coxeter groups that are generated by reflections. An important class of such subgroups arise from sets of positive roots in the ambient root system. The content can be found in [13] or [15].

A subgroup $W'$ of $W$ is a *reflection subgroup* if it is generated by the reflections of $W$ that it contains. If $W'$ is a reflection subgroup of $W$ then it is shown in [13, Theorem 1.8] that $(W', S')$ is a Coxeter system, where

$$S' = S(W') = \{t \in R_W \mid \mathscr{N}(t) \cap W' = \{t\}\}$$

is called a *set of Coxeter generators* for $W'$. The pair $(W', S')$ is called a *reflection subsystem* of $(W, S)$ with an inversion function $\mathscr{N}' : W' \to \mathscr{P}(R_W \cap W')$ given by $\mathscr{N}'(w) = \mathscr{N}(w) \cap W'$. It is important to note that $S'$ need not be finite, even when $S$ is finite.

In the ambient root system $\Delta$ one can define

$$\Pi_{W'} = \{\alpha \in \Delta^+ \mid s_\alpha \in S'\} \quad \text{and} \quad \Delta_{W'} = \{\alpha \in \Delta \mid s_\alpha \in W'\}.$$

Then

$$\Pi_{W'} = \{\alpha \in \Delta^+ \mid \overline{\mathscr{N}}(s_\alpha) \cap \Delta_{W'} = \{\alpha\}\} \quad \text{and} \quad \Delta_{W'} = W'\Pi_{W'}$$

by [13, Lemma 2.4(ii)(a)] and [13, Theorem 1.8] respectively.

The property of being a reflection subsystem is transitive [13, Corollary 1.9]. That is, if $W''$ is a subgroup of $W'$ then

$$(W'',S'') \text{ reflection subsystem of } (W',S') \iff (W'',S'') \text{ reflection subsystem of } (W,S)$$

where $S''=S(W'')$. This is particularly useful when studying parabolic subgroups of reflection subgroups.

A set of positive roots $\Psi \subseteq \Delta$ defines a reflection subsystem of $(W,S)$ in the following way. First, define $S_\Psi = \{t \in R_W \mid tA_\Psi = \{t\}+A_\Psi\}$. Then $W_\Psi = \langle S_\Psi \rangle$ is a Coxeter group and $(W_\Psi, S_\Psi)$ is a Coxeter system. The corresponding inversion function $\mathscr{N}_\Psi : W_\Psi \to \mathscr{P}(R_W \cap W_\Psi)$ satisfies the property that $\mathscr{N}_\Psi(w) = A_\Psi + wA_\Psi$. If $\Psi = \Delta^+$ then $\mathscr{N}(s) = \{s\}$ for $s \in S$ implies $S = S_\Psi$ and $W = W_\Psi$. The next example illustrates that $W_\Psi$ is a proper reflection subgroup of $W$ in general.

**Example 3.7.** Let $(W,S)$ be a Coxeter system of type $\tilde{A}_2$. Write $\dot{\Delta} = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$ for the underlying $A_2$ root system of $\Delta$. Suppose that $\Psi$ is a set of positive roots of $\Delta$.

1. If $\Psi$ is the natural partition then $S_\Psi$ is empty and $W_\Psi$ is trivial.
2. Set $J=\{\beta\}$ and $\Psi = \Psi_J$. Then $A_\Psi = \{s_{\alpha-n\delta}, s_{\alpha+\beta-n\delta},\ n \in \mathbb{N}\}$. A straightforward calculation proves that
$$S_\Psi = \{s_\beta, s_{\delta-\beta}\}.$$
This can be verified as follows:
   - Suppose $t = s_\beta$. Then $tA_\Psi = \mathscr{N}(s_\beta) + s_\beta A_\Psi s_\beta^{-1} = \{s_\beta\} + A_\Psi$.
   - Suppose $t = s_{\delta-\beta}$. Then
$$\begin{aligned} tA_\Psi &= \mathscr{N}(s_{\delta-\beta}) + s_{\delta-\beta}A_\Psi s_{\delta-\beta}^{-1}, \\ &= \{s_\alpha, s_{\delta-\alpha-\beta}, s_{\delta-\beta}\} + \{s_{\alpha+\beta-(n+1)\delta}, s_\alpha, s_{\alpha-n\delta} \mid n \in \mathbb{N}\}, \\ &= \{s_{\delta-\beta}\} + A_\Psi. \end{aligned}$$

   This exhausts the reflections contained in $S_\Psi$. It follows that $W_\Psi$ is type $\tilde{A}_1$.

The next result should be compared with Proposition 3.4. The proof can be found in [15, §1.8].

**Proposition 3.8.** *Let $\Psi \subseteq \Delta$ be a set of positive roots. Then:*

1. *For all $t \in S_\Psi$ and $w \in W$, one has*

$$l_\Psi(wt) - l_\Psi(w) = \begin{cases} +1 & \text{if } wt >_\Psi w, \\ -1 & \text{if } wt <_\Psi w. \end{cases}$$

2. *If $sw >_\Psi w$ and $swt <_\Psi wt$ then $wt >_\Psi w$ and $sw = wt$.*

Before continuing, some notation will be set. Suppose that $W'$ is a reflection subgroup of $W$. There is a standard length function on $W'$ which will be denoted by $l'$. For any set of positive roots $\Psi \subseteq \Delta$ there is also a length function on $W'$ defined using $W' \cap A_\Psi$. This length function will be denoted $l'_\Psi$. Note that the length function $l'_\emptyset$ on $W'$ induced from the standard length function $l$ on $W$ need not coincide with $l'$. This is illustrated in the following example.

**Example 3.9.** Let $W = \langle s_1, s_2 \mid (s_1s_2)^4 = 1\rangle$ be the dihedral group of a square. Consider the reflection subgroup $W' = \langle s_1, s_2s_1s_2\rangle$ of $W$. Observe that $s_1(s_2s_1s_2)$ has order 2 in $W$, so $W' \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. It follows that $S' = \{s_1, s_2s_1s_2\}$. But then $l'(s_2s_1s_2) = 1 \neq 3 = l(s_2s_1s_2)$. This example can also be used to show that the Bruhat order on a reflection subgroup $W' \subseteq W$ does not coincide with the order induced as a subset of $W$.

### 3.1.3 Cosets

In the sequel it will be important to have knowledge of coset representatives of reflection subgroups that are minimal with respect to different length functions. This is intimately linked to the notion of a projection in building theory and will play an analogous role in the objects defined later. The results presented are due to Dyer in full generality. However, some statements are difficult to reference precisely or are currently unpublished. For this reason, the full proofs will be given here for completeness. The unpublished proofs were communicated to the author by Dyer privately, and are reproduced here with permission. Specifically, the proofs of Lemma 3.11, Lemma 3.12, and Corollary 3.13 are due to Dyer.

Fix a set of positive roots $\Psi \subseteq \Delta$. Set $W' = W_\Psi$, $S' = S_\Psi$, $R' = R_{W_\Psi}$, and $\mathcal{N}' = \mathcal{N}_\Psi$. Therefore

$W'_K$ is the standard parabolic subgroup of $W'$ of type $K \subseteq S'$.

Consider the set

$$X_{W'} = \{w \in W \mid l(tw) > l(w) \text{ for all } t \in R'\} = \{w \in W \mid \mathcal{N}(w) \cap R' = \emptyset\}$$

of *minimal length left coset representatives* for $W'$ in $W$. This name is justified in the following result which is well-known for standard parabolic subgroups. The general case is due to Dyer, e.g. [13, Proposition 1.13] and [14, Corollary 3.4].

**Proposition 3.10.** *Suppose $w \in W$ and write $w = uv$ where $u \in W'$ and $v \in X_{W'}$. Then:*

1. *$X_{W'}$ is the set of elements of minimal length in their left coset of $W'$.*
2. *$X_{W'} = \{w \in W \mid l(tw) > l(w)$ for all $t \in S'\}$.*
3. *The expression $w = uv$ is unique.*
4. *$l(tw) < l(w)$ if and only if $l'(tu) < l'(u)$ for all $t \in R'$.*

In particular, there is a unique element $x_0$ in each coset $W'w$ of minimal length $l(x_0)$. Moreover, $x_0$ is contained in $X_{W'}$. There is a right analogue of Proposition 3.10 obtained by inverting elements. This form will be used often in the sequel.

The next two results relate different twisted length functions in $W$.

**Lemma 3.11.** *Let $J \subseteq S$. Suppose $w \in W$ is of the form $w = uv$ where $u \in W_J$ and $v \in W$. Then*

$$l_\Psi(w) = l_{vA_\Psi \cap W_J}(u) + l_\Psi(v).$$

*In particular, the map $W_J \to W_J v$ defined by $u \mapsto uv$ is an isomorphism of posets where $W_J$ has order $\leq_{vA_\Psi \cap W_J}$ and $W_J v$ has the order $\leq_\Psi$ induced from $W$.*

*Proof.* Let $t$ be a reflection of $W_J$. By Proposition 3.4(3), one has

$$l_\Psi(tw) < l_\Psi(w) \iff t \in wA_\Psi.$$

for all $w \in W$. But

$$wA_\Psi = u(A_{v\Psi}) = \mathscr{N}(u) + uA_{v\Psi}u^{-1},$$

from which it follows that

$$\begin{aligned} l_\Psi(tw) < l_\Psi(w) &\iff t \in \mathscr{N}(u) + uA_{v\Psi}u^{-1}, \\ &\iff u^{-1}tu \in u^{-1}\mathscr{N}(u)u + A_{v\Psi}, \\ &\iff u^{-1}tu \in \mathscr{N}(u^{-1}) + A_{v\Psi}. \end{aligned}$$

Observe that $u^{-1}tu \in W_J$, so the latter condition can be rewritten as

$$u^{-1}tu \in (\mathscr{N}(u^{-1}) \cap W_J) + (A_{v\Psi} \cap W_J).$$

Altogether, one has

$$\begin{aligned} l_\Psi(tw) < l_\Psi(w) &\iff u^{-1}tu \in (\mathscr{N}(u^{-1}) \cap W_J) + (A_{v\Psi} \cap W_J). \\ &\iff t \in (\mathscr{N}(u) \cap W_J) + u(A_{v\Psi} \cap W_J)u^{-1}. \end{aligned}$$

Since $W_J$ is a standard parabolic subgroup, the standard length function on $W_J$ coincides with the standard length function on $W$. In particular, $\mathscr{N}(u) \cap W_J$ is the inversion set of $u$ in $W_J$. Therefore

$$(\mathscr{N}(u) \cap W_J) + u(A_{v\Psi} \cap W_J)u^{-1} = u(A_{v\Psi} \cap W_J)$$

and thus $l_\Psi(tw) < l_\Psi(w)$ if and only if $l_{v\Psi \cap W_J}(tu) < l_{v\Psi \cap W_J}(u)$ by Proposition 3.4(3).

The length additivity equality now follows from induction on $l(u)$. For the base case, if $l(u) = 0$ then $u$ is the identity in $W$ and the equality holds by Proposition 3.4(1). Assume the equality is true for $l(u) = n$. Suppose $t \in J$ with $l(tu) = l(u) + 1$. By Proposition 3.4(4), if $t \in J \subseteq S$ then $l_{v\Psi \cap W_J}(tu) = l_{v\Psi \cap W_J}(u) \pm 1$, from which it follows that $l_\Psi(tw) = l_\Psi(w) \pm 1$ (with compatible same signs) by the discussion above. Therefore

$$
\begin{aligned}
l_\Psi(tw) &= l_\Psi(w) \pm 1, \\
&= l_{vA_\Psi \cap W_J}(u) + l_\Psi(v) \pm 1, \\
&= l_{vA_\Psi \cap W_J}(tu) + l_\Psi(v),
\end{aligned}
$$

where the induction hypothesis has been used in the second equality. This proves the length additivity equation. The second claim follows from this. □

**Lemma 3.12.** *Suppose $w \in W$ and write $w = uv$ where $u \in X_{W'}$ and $v \in W'$. Then*

$$l_\Psi(w) = l_\Psi(u) + l'_\Psi(v^{-1}).$$

*In particular, the map $W' \to wW'$ defined by $v^{-1} \mapsto uv$ is an isomorphism of posets where $W'$ has order $\leq_{W' \cap A_\Psi}$ and $wW'$ has the order $\leq_\Psi$ induced from $W$.*

*Proof.* Let $t$ be a reflection of $W'$. By Proposition 3.4(3), one has

$$l'_\Psi(tv^{-1}) < l'_\Psi(v^{-1}) \iff t \in v^{-1}A_\Psi.$$

for all $v \in W'$. The latter inclusion can be written as $t \in \mathscr{N}'(v^{-1}) + v^{-1}(A_\Psi \cap W')v$. By viewing $W'$ as an arbitrary reflection subgroup of $W$, one has $\mathscr{N}'(v^{-1}) = \mathscr{N}(v^{-1}) \cap W'$. However, since $W' = W_\Psi$ one also has

$$
\begin{aligned}
\mathscr{N}'(v^{-1}) &= A_\Psi + v^{-1}A_\Psi, \\
&= A_\Psi + \mathscr{N}(v^{-1}) + v^{-1}A_\Psi v, \\
&= (A_\Psi + \mathscr{N}(v^{-1}) + v^{-1}A_\Psi v) \cap W', \\
&= (A_\Psi \cap W') + (\mathscr{N}(v^{-1}) \cap W') + v^{-1}(A_\Psi \cap W')v.
\end{aligned}
$$

Hence

$$l'_\Psi(tv^{-1}) < l'_\Psi(v^{-1}) \iff t \in (A_\Psi \cap W') + (\mathscr{N}(v^{-1}) \cap W').$$

However, $u \in X_{W'}$ implies $\mathcal{N}(u^{-1}) \cap W' = \emptyset$ by definition, hence $v^{-1}\mathcal{N}(u^{-1})v \cap W' = \emptyset$. It follows that

$$\begin{aligned} \mathcal{N}(v^{-1}) \cap W' &= (\mathcal{N}(v^{-1}) + v^{-1}\mathcal{N}(u^{-1})v) \cap W', \\ &= \mathcal{N}(v^{-1}u^{-1}) \cap W'. \end{aligned}$$

Altogether, using the fact that $t \in W'$ and $w = uv$, one has

$$\begin{aligned} l'_\Psi(tv^{-1}) < l'_\Psi(v^{-1}) &\iff t \in A_\Psi + \mathcal{N}(v^{-1}u^{-1}), \\ &\iff l_\Psi(wt) < l_\Psi(w) \end{aligned}$$

where the final equivalence is an application of Remark 3.5. The length additivity equality now follows from induction on $l'(v^{-1})$ using the same argument as Lemma 3.11. The second statement follows from the length additivity. □

In the sequel only special cases of Lemmas 3.11 and 3.12 will be used. Suppose that $J \subseteq S$ and $K \subseteq S'$. Define

$$\begin{aligned} {}^J W_\Psi &= \{w \in W \mid l_\Psi(sw) > l_\Psi(w) \text{ for all } s \in J\}, \\ &= \{w \in W \mid wA_\Psi \cap W_J = \emptyset\}, \text{ and} \\ W_\Psi^K &= \{w \in W \mid l_\Psi(wt) > l_\Psi(w) \text{ for all } t \in K\}, \\ &= \{w \in W \mid (\mathcal{N}(w^{-1}) + A_\Psi) \cap W'_K = \emptyset\}. \end{aligned}$$

For simplicity, when $\Psi = \Delta^\pm$ write ${}^J W_\Psi = {}^J W_\pm$ and $W_\Psi^K = W_\pm^K$. Note that ${}^J W_+$ is the standard set of minimal left coset representatives, often denoted by ${}^J W$. The content of the following result implies that the elements $w \in {}^J W_\Psi$ (resp. $w \in W_\Psi^K$) are those that are of minimal twisted length $l_A(w)$ in their coset $W_J w$ (resp. $wW'_K$).

**Corollary 3.13.** *1. Let $J \subseteq S$. Suppose $w \in W$ is of the form $w = uv$ where $u \in W_J$ and $v \in {}^J W_\Psi$. Then*

$$l_\Psi(w) = l(u) + l_\Psi(v). \tag{3.1}$$

*In particular, the map $W_J \to W_J v$ defined by $u \mapsto uv$ is an isomorphism of posets where $W_J$ has the Bruhat order induced from $W$ and $W_J v$ has the order $\leq_\Psi$ induced from $W$.*

2. *Let $K \subseteq S'$. Suppose $w \in W$ is of the form $w = uv$ where $u \in W_\Psi^K$ and $v \in W'_K$. Then*

$$l_\Psi(w) = l_\Psi(u) + l'(v). \tag{3.2}$$

*In particular, the map $W'_K \to uW'_K$ defined by $v^{-1} \mapsto uv$ is an isomorphism of posets where $W'_K$ has the Bruhat order induced from $W'$ and $uW'_K$ has the order $\leq_\Psi$ induced from $W$.*

*Proof.* For (1) note that $vA_\Psi \cap W_J = \emptyset$ by definition of ${}^J W_\Psi$. The result then follows from Lemma 3.11. The proof of (2) is more involved. Write $u = xy^{-1}$ where $y \in W'$ and $x^{-1} \in X_{W'}$. Then $\mathcal{N}(u^{-1}) \cap W' = \mathcal{N}(y) \cap W'$. Observe that

$$\begin{aligned}
\mathcal{N}'(y) &= A_\Psi + yA_\Psi, \\
&= A_\Psi + \mathcal{N}(y) + yA_\Psi y^{-1}, \\
&= (A_\Psi + \mathcal{N}(y) + yA_\Psi y^{-1}) \cap W', \\
&= (A_\Psi \cap W') + (\mathcal{N}(y) \cap W') + (yA_\Psi y^{-1} \cap W'), \\
&= (A_\Psi \cap W') + (\mathcal{N}(y) \cap W') + y(A_\Psi \cap W')y^{-1}.
\end{aligned}$$

It follows that

$$\begin{aligned}
(\mathcal{N}(u^{-1}) + A_\Psi) \cap W' &= (\mathcal{N}(y) + A_\Psi) \cap W', \\
&= (\mathcal{N}(y) \cap W') + (A_\Psi \cap W'), \\
&= \mathcal{N}'(y) + y(A_\Psi \cap W')y^{-1},
\end{aligned}$$

where the last expression is simply $y(A_\Psi \cap W')$ in $W'$. The hypothesis that $u \in W_\Psi^K$ implies $(\mathcal{N}(u^{-1}) + A_\Psi) \cap W'_K = \emptyset$, hence also $y(A_\Psi \cap W') \cap W'_K = \emptyset$. This is analogous to the condition used in the proof of (1).

To obtain the result, first write $uv = xy^{-1}v = x(v^{-1}y)^{-1}$. Then, apply Lemma 3.12 to $w = uv$, followed by (1) in $W'$, and then Lemma 3.12 once more. In particular, one has

$$\begin{aligned} l_\Psi(w) &= l_\Psi(x) + l'_\Psi(v^{-1}y), \\ &= l_\Psi(x) + l'(v^{-1}) + l'_\Psi(y), \\ &= l_\Psi(xy^{-1}) + l'(v^{-1}), \\ &= l_\Psi(u) + l'(v), \end{aligned}$$

as required. The final equality makes use of the fact that $l'(v^{-1}) = l'(v)$ for the standard length function $l'$ of $W'$. □

If $W_J$ is finite or $\Psi = \Delta^+$, then every element $w \in W$ can be written uniquely as $w = uv$ where $u \in W_J$ and $v \in {}^J W_\Psi$. That is to say, every coset $W_J w$ has a coset representative in ${}^J W_\Psi$ with minimal twisted length. The analogous statement is true for the cosets $wW'_K$ if $W'_K$ is finite or $\Psi = \Delta^+$.

**Example 3.14.** Let $J \subseteq S$.

1. If $\Psi = \Delta^+$ then (3.1) becomes the so-called *gate property*. That is, for all $v \in {}^J W_+$ one has

$$l(w_J v) = l(w_J) + l(v)$$

   for all $w_J \in W_J$. For more on this, see [1, Figure 1.6, Remark 2.22].

2. If $\Psi = \Delta^-$ and $W_J$ is finite, then one obtains the well-known equality

$$l(w_J w_0) = l(w_0) - l(w_J)$$

   for all $w_J \in W_J$, where $w_0$ is the longest element in $W_J$. Indeed, $w_0 \in {}^J W_-$ has the minimal twisted length $l_-(w_0)$. But $l_-(w) = -l(w)$ for all $w \in W$. Therefore $l(w_0)$ is maximal.

### 3.1.4 Double cosets and intersections

This subsection uses the results of the previous subsection to obtain generalisations of well-known results due to Kilmoyer and Howlett, respectively. The case of $\Psi = \Delta^+$ can be found in standard textbooks, e.g. [1, Proposition 2.23, Corollary 2.25]. The proofs presented below were communicated to the author by Dyer privately, and are reproduced here with permission.

For $J \subseteq S$ and $K \subseteq S'$, define the set

$$^{J}W_{\Psi}^{K} = {}^{J}W_{\Psi} \cap W_{\Psi}^{K}.$$

**Proposition 3.15.** *Let $J \subseteq S$ and $K \subseteq S'$. Suppose $w \in {}^{J}W_{\Psi}^{K}$. Then*

$$W_J \cap wW'_K w^{-1} = W_L$$

*where $L = J \cap wKw^{-1}$.*

*Proof.* The inclusion $W_L \subseteq W_J \cap wW'_K w^{-1}$ is immediate. Suppose $u \in W_J \cap wW'_K w^{-1}$. So $u \in W_J$ and $u = wvw^{-1}$ for some $v \in W'_K$. It follows that $uw = wv$. As an element of $W_J$ one can write $u$ using reflections $s_i \in J$. Let $s = s_i$ be one such reflection. Then $1 \leq s \leq u$ in the Bruhat order. By Corollary 3.13(2) one has $w \leq_{\Psi} sw \leq_{\Psi} uw$ since $w \in {}^{J}W_{\Psi}^{K} \subseteq {}^{J}W_{\Psi}$. Moreover, $l_{\Psi}(sw) = l_{\Psi}(w) + 1$. Substituting $uw = wv$ gives $w \leq_{\Psi} sw \leq_{\Psi} wv$. But $w \in W_{\Psi}^{K}$ also. Therefore Corollary 3.13(2) implies that $sw$ lies between $w$ and $wv$ in the coset $wW'_K$. Hence $sw = wt$ for some $t \in W'_K$. Furthermore,

$$l_{\Psi}(wt) = l_{\Psi}(sw) = l_{\Psi}(w) + 1$$

which implies $l'(t) = 1$ by Corollary 3.13(2). In other words, $t \in K$ and thus $s = wtw^{-1} \in W_L$. Altogether, this implies that $u \in W_L$ as required. □

**Proposition 3.16.** *Let $J \subseteq S$ and $K \subseteq S'$. Suppose $w \in {}^{J}W_{\Psi}^{K}$. Then every element $w'$ of the double* *coset $W_J w W'_K$ can be written uniquely as*

$$w' = abwc$$

*where $a \in W_J \cap W_+^{J \cap wKw^{-1}}$, $b \in W_{J \cap wKw^{-1}}$, and $c \in W'_K \cap {}^{K \cap w^{-1}Jw}W'_+$. Moreover, one has*

$$l_\Psi(w') = l(a) + l(b) + l_\Psi(w) + l'(c)$$

*with $wc \in {}^J W_\Psi$, $aw \in W_\Psi^K$, and $w' = awb'c$ where $b' = w^{-1}bw \in W'_{K \cap w^{-1}Jw}$ and $l(b) = l'(b')$.*

*Proof.* Let $w' \in W_J w W'_K$, so $w' = a'wc'$ for some $a' \in W_J$ and $c' \in W'_K$. As an element of $W'_K$, one can write $c' = c''c$ where $c'' \in W'_{K \cap w^{-1}Jw}$ and $c \in W'_K \cap {}^{K \cap w^{-1}Jw}W'_+$. Then

$$a'wc' = a'(wc''w^{-1})wc$$

and moreover $a'' = a'(wc''w^{-1}) \in W_J$. If $a''wc = a''_1 wc_1$ for some $a''_1 \in W_J$ and $c_1 \in W'_K \cap {}^{K \cap w^{-1}Jw}W'_+$ then $a'' = a''_1$ and $c = c_1$. Indeed, since $c, c_1 \in W'_K$ and $a'', a''_1 \in W_J$, one has

$$cc_1^{-1} = w^{-1}((a'')^{-1}a''_1)w \in W'_K \cap w^{-1}W_J w = W_{K \cap w^{-1}Jw}$$

where the final equality is a result of Proposition 3.15. But $c$ and $c_1$ are unique minimal left coset representatives for $W_{K \cap w^{-1}Jw}$ in $W'_K$. Then $cc_1^{-1} \in W_{K \cap w^{-1}Jw}$ implies that $c$ and $c_1$ are contained in the same left coset of $W_{K \cap w^{-1}Jw}$. It must therefore follow that $c = c_1$, then also $a'' = a''_1$. To complete the uniqueness of expression statement, notice that the decomposition $a'' = ab$ is unique, where $a \in W_J \cap W_+^{J \cap wKw^{-1}}$ and $b \in W_{J \cap wKw^{-1}}$.

Now, observe that

$$
\begin{aligned}
wcA_\Psi \cap W_J &= (\mathscr{N}(w) + wA_{c\Psi}w^{-1}) \cap W_J,\\
&= (\mathscr{N}(w) + w(\mathscr{N}'(c) + A_\Psi)w^{-1}) \cap W_J,\\
&= (\mathscr{N}(w) + w\mathscr{N}'(c)w^{-1} + wA_\Psi w^{-1}) \cap W_J,\\
&= (wA_\Psi + w\mathscr{N}'(c)w^{-1}) \cap W_J,\\
&= (wA_\Psi \cap W_J) + (w\mathscr{N}'(c)w^{-1} \cap W_J).
\end{aligned}
$$

Since $w \in {}^JW_\Psi$ one has $wA_\Psi \cap W_J = \emptyset$. Moreover, $\mathscr{N}'(c) \subseteq W'_K$ because $c \in W'_K$. This implies that

$$\mathscr{N}'(c) \cap w^{-1}W_Jw = \mathscr{N}'(c) \cap W_{K\cap w^{-1}Jw} = \emptyset$$

where the first equality is an application of Proposition 3.15 and the second equality follows from the fact that $c \in {}^{K\cap w^{-1}Jw}W'_+$. Altogether, this implies $wcA_\Psi \cap W_J = \emptyset$ and thus $wc \in {}^JW_\Psi$. Then, applying Corollary 3.13 three times, yields

$$
\begin{aligned}
l_\Psi(a''wc) &= l(a'') + l_\Psi(wc),\\
&= l(a) + l(b) + l_\Psi(wc),\\
&= l(a) + l(b) + l_\Psi(w) + l'(c).
\end{aligned}
$$

Observe that $abwc = aw(w^{-1}bw)c$ where $b' = w^{-1}bw \in W'_{K\cap w^{-1}Jw}$. By Proposition 3.15, conjugation by $w^{-1}$ induces a length-preserving isomorphism $W_{wKw^{-1}\cap J} \simeq W'_{K\cap w^{-1}Jw}$ of Coxeter groups, whence $l(b) = l'(b')$. Finally, notice that

$$
\begin{aligned}
l_\Psi(awb'c) &= l_\Psi(abwc),\\
&= l(a) + l(b) + l_\Psi(w) + l'(c),\\
&= l(a) + l'(b') + l_\Psi(w) + l'(c),\\
&= l_\Psi(aw) + l'(b'c),
\end{aligned}
$$

which, because $b'c \in W'_K$, implies that $aw \in W^K_\Psi$. $\square$

In particular, if $W_J$ and $W'_K$ are finite or $\Psi = \Delta^+$, then every double coset $W_J v W'_K$ has a unique element $w$ of minimal twisted length $l_\Psi(w)$ and $w \in {}^J W_\Psi^K$.

**Example 3.17.** If $\Psi = \Delta^+$ then Proposition 3.16 retrieves the fact that every double coset $W_J v W_K$ has a unique representative $w$ of minimal length $l(w)$, every element $w'$ of $W_J w W_K$ can be written uniquely as

$$w' = w_J w w_K$$

where $w_J \in W_J$ and $w_K \in W_K$, and

$$l(w') = l(w_J) + l(w) + l(w_K).$$

## 3.2 Pairs of buildings

Let $(W,S)$ be a Coxeter system with root system $\Delta$. When $W$ is infinite there is more than one conjugacy class of sets of positive roots in $\Delta$, leading one to consider pairs of buildings associated to the standard sets of positive roots $\Delta^+$ and $\Delta^-$ respectively. Often, one can define a certain $W$-valued map between these pairs of buildings, producing so-called twin buildings. These objects were developed by Ronan and Tits and have been studied extensively since. For more on this topic, see [1].

This subsection defines and studies other pairs of buildings that arise naturally from $(W,S)$ and admit similar $W$-valued maps between them. This includes twin buildings associated to reflection subsystems of the form $(W_\Psi, S_\Psi)$ for some set of positive roots $\Psi \subseteq \Delta$, but also generalisations of a twin buildings in which one component is replaced by a building of type $(W_\Psi, S_\Psi)$. In both of these instances, Dyer's theory of twisted Bruhat orders and reflection subsystems plays an essential role.

### 3.2.1 Twin buildings

In this subsection the notion of a twin building is recalled. In particular, one can associate such an object to any reflection subsystem of the form $(W_\Psi, S_\Psi)$ where $\Psi$ is a set of positive roots in

$\Delta$.

Let $\Psi \subseteq \Delta$ be a set of positive roots. It is straightforward to deduce that

$$(W_\Psi, S_\Psi) = (W_{-\Psi}, S_{-\Psi})$$

and therefore $l^\Psi$ will denote their common standard length function. Suppose that $\mathscr{B}_\Psi$ and $\mathscr{B}_{-\Psi}$ are buildings of type $(W_\Psi, S_\Psi)$ and write $\delta_\Psi$ and $\delta_{-\Psi}$ for the respective $W_\Psi$-distance functions.

A *twinning* of $\mathscr{B}_\Psi$ and $\mathscr{B}_{-\Psi}$ is a $W_\Psi$-valued map

$$\delta^*_\Psi : (\mathscr{B}_\Psi \times \mathscr{B}_{-\Psi}) \cup (\mathscr{B}_{-\Psi} \times \mathscr{B}_\Psi) \to W_\Psi$$

such that, for all $b \in \mathscr{B}_{\varepsilon\Psi}$ and $c \in \mathscr{B}_{-\varepsilon\Psi}$ with $\delta^*_\Psi(b,c) = w$ and $\varepsilon \in \{+,-\}$, and all $s \in S_\Psi$, the following properties hold:

**(T1)** $\delta^*_\Psi(b,c) = \delta^*_\Psi(c,b)^{-1}$.

**(T2)** If $b' \in \mathscr{B}_{\varepsilon\Psi}$ satisfies $\delta_{\varepsilon\Psi}(b',b) = s$, then $\delta^*_\Psi(b',c) = sw$ if $l^\Psi(sw) < l^\Psi(w)$.

**(T3)** There is $b' \in \mathscr{B}_{\varepsilon\Psi}$ with $\delta_{\varepsilon\Psi}(b',b) = s$ and $\delta^*_\Psi(b',c) = sw$.

A triple $(\mathscr{B}_\Psi, \mathscr{B}_{-\Psi}, \delta^*_\Psi)$ satisfying **(T1)-(T3)** is called a *twin building* of type $(W_\Psi, S_\Psi)$.

It will be important for the sequel to note that the condition

$$l^\Psi(sw) < l^\Psi(w)$$

found in **(T2)** is analogous to the condition

$$l^\Psi(sw) > l^\Psi(w)$$

found in **(B2)** for the buildings $\mathscr{B}_{\varepsilon\Psi}$. In particular, the former condition is equivalent to the latter condition if one replaces the standard length function $l^\Psi$ on $W_\Psi$ with its reverse length function given by $-l^\Psi$. In this way, the twin building structure encodes the Bruhat order within each building, and the reverse Bruhat order is encoded in how the buildings interact.

### 3.2.2 Twisted buildings

This subsection builds upon the observation that the $W$-metric on a building is governed by the standard length function, while the twinning map between buildings is governed by the reverse length function.

**Definition 3.18.** An $\varepsilon$-*twisted building* of type $(W,S,\Psi)$ is a triple $(\mathscr{B},\mathscr{B}_\Psi,\delta^*)$ where $(\mathscr{B},\delta)$ is a building of type $(W,S)$ and $(\mathscr{B}_\Psi,\delta_\Psi)$ is a building of type $(W_\Psi,S_\Psi)$, along with a $W$-valued map

$$\delta^* : (\mathscr{B}\times\mathscr{B}_\Psi)\cup(\mathscr{B}_\Psi\times\mathscr{B})\to W$$

such that, for all $b\in\mathscr{B}$ and $c\in\mathscr{B}_\Psi$ with $\delta^*(b,c)=w$, and all $s\in S$ and $t\in S_\Psi$, the following properties hold:

**(T'1)** $\delta^*(b,c)=\delta^*(c,b)^{-1}$.

**(T'2)** (a) If $b'\in\mathscr{B}$ satisfies $\delta(b',b)=s$, then $\delta^*(b',c)=sw$ if $l_{\varepsilon\Psi}(sw)>l_{\varepsilon\Psi}(w)$.

(b) If $c'\in\mathscr{B}_\Psi$ satisfies $\delta_\Psi(c,c')=t$, then $\delta^*(b,c')=wt$ if $l_{\varepsilon\Psi}(wt)>l_{\varepsilon\Psi}(w)$.

**(T'3)** (a) There is $b'\in\mathscr{B}$ with $\delta(b',b)=s$ and $\delta^*(b',c)=sw$.

(b) There is $c'\in\mathscr{B}_\Psi$ with $\delta_\Psi(c,c')=t$ and $\delta^*(b,c')=wt$.

In this case, the $W$-valued map $\delta^*$ is called an $\varepsilon$-*twisting* of $\mathscr{B}$ and $\mathscr{B}_\Psi$.

To the author's knowledge, twisted buildings are not defined in the literature beyond twin buildings. When $\varepsilon=+$ the twisted building $(\mathscr{B},\mathscr{B}_\Psi,\delta^*)$ will be called *positively twisted*, and *negatively twisted* otherwise. Similarly, $\delta^*$ will be called a *positive twisting* or a *negative twisting* respectively.

**Example 3.19.** Let $(\mathscr{B},\delta)$ be a building of type $(W,S)$ associated to $\Delta^+$. Then the ordinary $W$-metric on $\mathscr{B}$ is a positive twisting. On the other hand, if $\delta^*$ is a negative twisting then $\delta^*$ is a twinning, and $(\mathscr{B},\mathscr{B},\delta^*)$ is a twin building of type $(W,S)$.

**Example 3.20.** Let $(W,S)$ be a Coxeter system with root system $\Delta$. Suppose that $\Psi\subseteq\Delta$ is a set of positive roots. Then $W$ and $W_\Psi$ are canonically thin buildings of type $(W,S)$ and $(W_\Psi,S_\Psi)$

respectively. Moreover, the disjoint union $W \sqcup W_\Psi$ is a twisted building with twisting map $\delta^* : W \times W_\Psi \to W$ defined by

$$\delta^*(v,w) = v^{-1}w$$

for all $v \in W$ and $w \in W_\Psi$. In fact, $\delta^*$ is both a positive twisting and a negative twisting. This twisted building is the *thin twisted building* associated to $W$ and $W_\Psi$.

It will be helpful to assume the following technical assumption on sets of positive roots. This will have important implications for the corresponding twisted buildings.

**Definition 3.21.** A set of positive roots $\Psi \subseteq \Delta$ is *aligned* if $l_\Psi(t)$ is constant for all $t \in S_\Psi$.

In the sequel, if $S_\Psi$ is empty then $\Psi$ will be aligned. It is straightforward to see that $\Psi$ is aligned if and only if $-\Psi$ is aligned. If $\Psi$ is aligned then a twisted building of the form $(\mathscr{B}, \mathscr{B}_\Psi, \delta^*)$ will be called an *aligned twisted building*.

**Example 3.22.** The sets $\Psi_J$ described in Example 3.2 are aligned and $l_{\Psi_J}(t) = 1$ for all $t \in S_{\Psi_J}$.

**Example 3.23.** If $\Psi = w\Delta^+$ for some non-trivial $w \in W$ then $\Psi$ need not be aligned. Indeed, one has $S_\Psi = wSw^{-1}$ and therefore $l_\Psi(t) = \pm 1$ for all $t = wsw^{-1}$ with $s \in S$. In particular, $l_\Psi(t) = 1$ if $t \in A_\Psi$, and $l_\Psi(t) = -1$ otherwise. But

$$A_\Psi = A_{w\Delta^+} = \{s_\alpha \mid \alpha \in \Delta^+ \cap (-w\Delta^+)\}$$

is simply the inversion set $\mathscr{N}(w)$ of $w$. Set $t = ws_\alpha w^{-1}$ for some simple root $\alpha \in \Pi$. Hence $t = s_{w(\alpha)} \in A_\Psi$ if and only if $w(\alpha) \in -\Delta^+$. It follows that $\Psi$ is not aligned if one can find simple roots $\alpha, \beta \in \Delta^+$ for which $w(\alpha) \in \Delta^+$ and $w(\beta) \notin \Delta^+$.

An aligned set of positive roots $\Psi$ is of *positive sign*, written $\operatorname{sgn}(\Psi) = +1$, if $l_\Psi(t)$ is positive for all $t \in S_\Psi$ and of *negative sign*, so $\operatorname{sgn}(\Psi) = -1$, if $l_\Psi(t)$ is negative for all $t \in S_\Psi$. Importantly, if $\Psi$ is of positive sign then $\Pi_\Psi \subseteq \Psi$, and if $\Psi$ is of negative sign then $-\Pi_\Psi \subseteq \Psi$.

This partitions the aligned sets of positive roots for which $S_\Psi$ is non-empty. When $S_\Psi$ is empty we do not assign a sign to $\Psi$. An aligned set of positive roots $\Psi$ is of positive sign if and only if it lies in the same partition as $\Delta^+$. Moreover, $\Psi$ is of positive sign if and only if $-\Psi$ is of negative sign.

*Remark* 3.24. Let $\Psi$ be a set of positive roots. If $S_\Psi$ is empty, then any building of type $(W_\Psi, S_\Psi)$ will be a single point. When $W$ is of affine type, this point should be thought of as lying at infinity. As a consequence, the corresponding twisted buildings behave in slightly different ways. For the purpose of the later chapters, this case will not be excluded.

Henceforth, fix an $\varepsilon$-twisted building $(\mathscr{B}, \mathscr{B}_\Psi, \delta^*)$ of type $(W, S, \Psi)$.

### 3.2.3 Projections

Essential tools in the theory of buildings are the projection maps. It is well-known that there are projection maps onto residues in a twin building, provided the residues are spherical. In this subsection it is shown that there are well-defined projection maps between components of twisted buildings subject to similar constraints. The approach taken here is inspired by the theory of twin buildings, e.g. [1, §5.8].

The simplest case of a projection is given by the following result, whose proof is essentially [1, Lemma 5.139].

**Lemma 3.25.** *Let $b \in \mathscr{B}$ and $c \in \mathscr{B}_\Psi$. Suppose that $\delta^*(b,c) = w$. Then, for all $s \in S$ and $t \in S_\Psi$, one has the following:*

1. *If $b' \in \mathscr{B}$ with $\delta(b',b) = s$ then $\delta^*(b',c) \in \{w, sw\}$.*
2. *If $c' \in \mathscr{B}_\Psi$ with $\delta_\Psi(c,c') = t$ then $\delta^*(b,c') \in \{w, wt\}$.*

*Furthermore, if $l_{\varepsilon\Psi}(sw) < l_{\varepsilon\Psi}(w)$ then there is a unique chamber $b' \in \mathscr{B}$ with $\delta(b',b) = s$ and $\delta^*(b',c) = sw$. Similarly, if $l_{\varepsilon\Psi}(wt) < l_{\varepsilon\Psi}(w)$ then there is a unique chamber $c' \in \mathscr{B}_\Psi$ with $\delta_\Psi(c,c') = t$ and $\delta^*(b,c') = wt$.*

*Proof.* If $l_{\varepsilon\Psi}(sw) > l_{\varepsilon\Psi}(w)$ then $\delta^*(b',c) = sw$ by **(T'2)**. So assume otherwise. By **(T'3)** there is some $b'' \in \mathscr{B}$ with $\delta(b'',b) = s$ and $\delta^*(b'',c) = sw$. If $b' = b''$ then (1) is satisfied, so assume otherwise. Since $\delta(b'',b) = s$, and $\delta(b',b) = s$ by the hypothesis, one can deduce that $\delta(b',b'') \in \{1, s\}$. But $b' \neq b''$ by assumption, so $\delta(b',b'') = s$. One also has $l_{\varepsilon\Psi}(s(sw)) >$

$l_{\varepsilon\Psi}(sw)$ by assumption, therefore **(T'2)** implies $\delta^*(b',c) = s(sw) = w$. This proves (1). The argument for (2) is the same. The final statement is a repackaging of this argument. $\square$

This gives a notion of projection onto panels in twisted buildings.

**Definition 3.26.** Let $P$ be a panel of $\mathscr{B}$ and $Q$ a panel of $\mathscr{B}_\Psi$. Suppose that $b \in P$ and $c \in Q$. The unique chamber $b'$ of Lemma 3.25 is the *projection* of $c$ onto $P$, denoted by $\mathrm{proj}_P c$. The analogous definition is made for the *projection* of $b$ onto $Q$, denoted by $\mathrm{proj}_Q b$.

The remainder of the subsection is spent defining projections onto larger residues. This begins with the next result which is an analogue of [1, Lemma 5.140].

**Lemma 3.27.** *Let $b, b' \in \mathscr{B}$ and $c, c' \in \mathscr{B}_\Psi$. Suppose that $u = \delta(b,b')$, $v = \delta^*(b',c)$, and $w = \delta_\Psi(c,c')$. Then:*

1. *If $u = s_1 \cdots s_l$ where $s_i \in S$ then $\delta^*(b,c) = s_{i_1} \cdots s_{i_k} v$ for some subword $s_{i_1} \cdots s_{i_k}$ of $s_1 \cdots s_l$.*
2. *If $w = t_1 \cdots t_l$ where $t_i \in S_\Psi$ then $\delta^*(b',c') = v t_{i_1} \cdots t_{i_k}$ for some subword $t_{i_1} \cdots t_{i_k}$ of $t_1 \cdots t_l$.*
3. *If $l_{\varepsilon\Psi}(uv) = l(u) + l_{\varepsilon\Psi}(v)$ then $\delta^*(b,c) = uv$.*
4. *If $l_{\varepsilon\Psi}(vw) = l_{\varepsilon\Psi}(v) + l'(w)$ then $\delta^*(b',c') = vw$.*

*Proof.* For (1) choose a gallery in $\mathscr{B}$ from $b$ to $b'$ described by $s_1 \cdots s_l$. The result follows from repeated applications of Lemma 3.25(1). Similarly, one deduces (2) using a gallery in $\mathscr{B}_\Psi$ from $c$ to $c'$ described by $t_1 \cdots t_l$, along with repeated applications of Lemma 3.25(2). The case of (3) is split into two cases for clarity:

- Suppose that $\varepsilon = +$. Let $u = s_1 \cdots s_d$ be a reduced decomposition. Then $l_\Psi(sx) = l_\Psi(x) \pm 1$ for all $s \in S$ and $x \in W$ by Proposition 3.4(3.1.1). The hypothesis implies that

  $$l_\Psi(s_i \cdots s_d v) > l_\Psi(s_{i+1} \cdots s_d v) \text{ and } l_\Psi(s_d v) > l_\Psi(v)$$

  for all $1 \leq i \leq d-1$. The result now follows from repeated applications of **(T'2)** on the gallery from $b$ to $b'$ described by $s_1 \cdots s_d$.

- Suppose that $\varepsilon = -$. Then the hypothesis now implies that

$$l_\Psi(s_i \cdots s_d v) < l_\Psi(s_{i+1} \cdots s_d v) \text{ and } l_\Psi(s_d v) < l_\Psi(v)$$

for all $1 \leq i \leq d-1$. This is equivalent to

$$l_{-\Psi}(s_i \cdots s_d v) > l_{-\Psi}(s_{i+1} \cdots s_d v) \text{ and } l_{-\Psi}(s_d v) > l_{-\Psi}(v)$$

for all $1 \leq i \leq d-1$. The same argument as before yields the result.

An analogous argument is made for (4). Indeed, $l_\Psi(xt) = l_\Psi(x) \pm 1$ for all $t \in S_\Psi$ and $x \in W$ by Proposition 3.8(1). If $w = t_1 \cdots t_d$ is a reduced decomposition in $W_\Psi$ then the argument follows identically. $\square$

Suppose that $R$ is a $J$-residue of $\mathscr{B}$ and $R_\Psi$ a $K$-residue of $\mathscr{B}_\Psi$ where $J \subseteq S$ and $K \subseteq S_\Psi$. Set

$$\delta^*(R, R_\Psi) = \{\delta^*(b,c) \mid b \in R,\ c \in R_\Psi\}.$$

The next result describes $\delta^*(R, R_\Psi)$ as a double coset. The proof follows that of [1, Lemma 5.29].

**Lemma 3.28.** *Let $R$ be a $J$-residue of $\mathscr{B}$ and $R_\Psi$ a $K$-residue of $\mathscr{B}_\Psi$ where $J \subseteq S$ and $K \subseteq S_\Psi$. Then $\delta^*(R, R_\Psi) = W_J \delta^*(b,c)(W_\Psi)_K$ for any $b \in R$ and $c \in R_\Psi$.*

*Proof.* Suppose that $b \in \mathscr{B}$ and $c \in \mathscr{B}_\Psi$ with $\delta^*(b,c) = v$. If $b' \in R$ and $c' \in R_\Psi$ then $\delta(b', b) \in W_J$ and $\delta_\Psi(c, c') \in (W_\Psi)_K$. Set $u = \delta(b', b)$ and $w = \delta_\Psi(c, c')$. Applying Lemma 3.27(1) and Lemma 3.27(2) in succession, one can deduce that $\delta^*(b', c') = u'vw'$ for some subwords $u'$ of $u$ and $w'$ of $w$. Therefore $\delta^*(R, R_\Psi) \subseteq W_J \delta^*(b,c)(W_\Psi)_K$. This becomes an equality with **(T'3)**. $\square$

In light of Proposition 3.16, the content of Lemma 3.28 implies that $\delta^*(R, R_\Psi)$ contains a unique element $w$ whose twisted length $l_\Psi(w)$ is minimal, provided $W_J$ and $W'_K$ are finite. This element will be denoted by $w = \min(\delta^*(R, R_\Psi))$. It will now be shown that if $R$ is a chamber $b$ then there is a unique chamber $c \in R_\Psi$ for which $\delta^*(b,c) = w$, and vice versa.

**Definition 3.29.** For all $b \in \mathscr{B}$ and $c \in \mathscr{B}_\Psi$, define the *distance*

$$d^\Psi(b,c) = \begin{cases} l_{\varepsilon\Psi}(\delta^*(b,c)) & \text{if } S_\Psi = \emptyset \\ \operatorname{sgn}(\Psi) \cdot l_\Psi(\delta^*(b,c)), & \text{otherwise.} \end{cases}$$

The reason one sees the sign factor in the definition above can be attributed to the fact that the projection will be the chamber for which the twisted length $l_{\varepsilon\Psi}$ is minimal (cf. Lemma 3.25). Therefore, the chamber $b'$ from Lemma 3.25 can be characterised as the unique chamber in the $s$-panel of $b$ whose distance to $c$ is minimal if $\varepsilon$ and $\Psi$ are of the same sign, or if $S_\Psi$ is empty, and maximal otherwise. The analogous characterisation holds true for $c'$ of Lemma 3.25. The next result extends this phenomenon from panels to spherical residues.

**Example 3.30.** The definition of distance in a twisted building is a simple generalisation of the notion of codistance $d^*(b,c) = l(b,c)$ in twin buildings. In particular, observe that

$$d^{-\varepsilon}(b,c) = l(b,c) = d^*(b,c).$$

The proof is identical to [1, Lemma 5.149].

**Proposition 3.31.** *Let $R$ be a spherical $J$-residue of $\mathscr{B}$ and $R_\Psi$ a spherical $K$-residue of $\mathscr{B}_\Psi$ where $J \subseteq S$ and $K \subseteq S_\Psi$. Suppose that $b \in \mathscr{B}$ and $c \in \mathscr{B}_\Psi$. Then:*

1. *There is a unique chamber $b' \in R$ whose distance to $c$ is minimal if $\varepsilon$ and $\Psi$ are of the same sign, or if $S_\Psi$ is empty, and maximal otherwise.*
2. *There is a unique chamber $c' \in R_\Psi$ whose distance to $b$ is minimal if $\varepsilon$ and $\Psi$ are of the same sign, or if $S_\Psi$ is empty, and maximal otherwise.*

*Moreover, the chambers $b'$ and $c'$ satisfy:*

1. *For all $b \in R$, one has $\delta^*(b,c) = \delta(b,b')\delta^*(b',c)$ and*

$$d^\Psi(b,c) = \begin{cases} d^\Psi(b',c) - d(b,b') & \textit{if } \operatorname{sgn}(\Psi) = -\varepsilon, \\ d(b,b') + d^\Psi(b',c) & \textit{otherwise.} \end{cases}$$

2. *For all $c \in R_\Psi$, one has $\delta^*(b,c) = \delta^*(b,c')\delta_\Psi(c',c)$ and*

$$d^\Psi(b,c) = \begin{cases} d^\Psi(b,c') - d_\Psi(c',c) & \text{if } \operatorname{sgn}(\Psi) = -\varepsilon, \\ d^\Psi(b,c') + d_\Psi(c',c) & \text{otherwise.} \end{cases}$$

*Proof.* Assume that $\varepsilon = +$ and that $\Psi$ is of positive sign. Suppose that $b' \in R$ such that $d^\Psi(b',c)$ is minimal among chambers of $R$. Then, by (3.1) one has

$$l_\Psi(w_J\delta^*(b',c)) = l(w_J) + l_\Psi(\delta^*(b',c))$$

for all $w_J \in W_J$. In particular,

$$l_\Psi(\delta(b,b')\delta^*(b',c)) = l(\delta(b,b')) + l_\Psi(\delta^*(b',c)) \tag{3.3}$$

for all $b \in R$. By Lemma 3.27(3), this implies $\delta^*(b,c) = \delta(b,b')\delta^*(b',c)$. The case of $\varepsilon = -$ and $\Psi$ of negative sign requires the assumption that $d^\Psi(b',c) = l_{-\Psi}(\delta^*(b',c))$ is minimal. By (3.1) one has

$$l_{-\Psi}(w_J\delta^*(b',c)) = l(w_J) + l_{-\Psi}(\delta^*(b',c)) \tag{3.4}$$

for all $w_J \in W_J$. Whence $\delta^*(b,c) = \delta(b,b')\delta^*(b',c)$ for all $b \in R$ by the same argument. A similar argument is used in the case that $S_\Psi$ is empty. The remaining cases, when $\varepsilon$ and $\Psi$ are opposite sign, follow one of the arguments above after assuming $d^\Psi(b',c)$ is maximal: if $\operatorname{sgn}(\Psi) = -1$ then apply the first argument; otherwise, apply the second argument. For the final part, assume again $\varepsilon = +$ and that $\Psi$ is of positive sign with $b' \in \mathscr{B}$ as above. Then

$$\begin{aligned} d^\Psi(b,c) &= l_\Psi(\delta(b,b')\delta^*(b',c)), \\ &= l(\delta(b,b')) + l_\Psi(\delta^*(b',c)), \\ &= d(b,b') + d^\Psi(b',c) \end{aligned}$$

for all $b \in R$, by Lemma 3.27(3). The case of $\varepsilon = -$ and $\Psi$ of negative sign is obtained in a similar fashion from (3.4) using the fact that $l_{-\Psi}(x) = -l_\Psi(x)$ for all $x \in W$. If $\varepsilon$ and $\Psi$ are opposite sign, then applying $\operatorname{sgn}(\Psi)$ to (3.3) or (3.4) yields

$$d^{\Psi}(b,c) = -d(b,b') + d^{\Psi}(b',c).$$

These equations also prove the uniqueness of $b'$. The analogous arguments are made for $c' \in \mathscr{B}_{\Psi}$. $\square$

**Example 3.32.** In the case of the twin building, the sign of $\varepsilon$ and $\Psi$ are opposite by definition. Then Proposition 3.31 describes the existence of co-projections and gives the analogue

$$d^{*}(b,c) = d^{*}(b',c) - d(b,b')$$

of the gate property for all $b \in R$.

It is now possible to extend the notion of a projection from panels to spherical residues.

**Definition 3.33.** Let $R$ be a spherical residue of $\mathscr{B}$ and $R_{\Psi}$ a spherical residue of $\mathscr{B}_{\Psi}$. Suppose that $b \in R$ and $c \in R_{\Psi}$. The unique chamber $b'$ of Proposition 3.31 is the *projection* of $c$ onto $R$, denoted by $\operatorname{proj}_R c$. The analogous definition is made for the *projection* of $b$ onto $R_{\Psi}$, denoted by $\operatorname{proj}_{R_{\Psi}} b$.

In particular, the projection is characterised as the unique chamber in the residue at minimal (resp. maximal) distance from the source, depending on the sign of $\Psi$ relative to $\varepsilon$. This subsection concludes by providing a description of projections of residues onto residues.

**Definition 3.34.** Let $R$ be a spherical residue of $\mathscr{B}$ and $R_{\Psi}$ a residue of $\mathscr{B}_{\Psi}$. The *projection* of $R_{\Psi}$ onto $R$ is the set

$$\operatorname{proj}_R R_{\Psi} = \{\operatorname{proj}_R c \mid c \in R_{\Psi}\}.$$

The analogous definition is made for the *projection* of $R$ onto $R_{\Psi}$, denoted $\operatorname{proj}_{R_{\Psi}} R$, provided $R_{\Psi}$ is spherical.

The next result is an analogue of [1, Lemma 5.36, Exercise 5.168].

**Proposition 3.35.** *Let $R$ be a spherical $J$-residue of $\mathscr{B}$ and $R_{\Psi}$ a $K$-residue of $\mathscr{B}_{\Psi}$ where $J \subseteq S$*

*and $K \subseteq S_\Psi$. Suppose that $w = \min(\delta^*(R, R_\Psi))$. Then:*

*1.*

$$\operatorname{proj}_R R_\Psi = \begin{cases} \{b \in R \mid w \in \delta^*(b, R_\Psi)\} & \text{if } \varepsilon = +, \\ \{b \in R \mid w_0(J)w \in \delta^*(b, R_\Psi)\} & \text{otherwise.} \end{cases}$$

*Moreover, if $b \in R$ and $c \in R_\Psi$ where $w = \delta^*(b,c)$ then $b = \operatorname{proj}_R c$.*

*2.*

$$\operatorname{proj}_{R_\Psi} R = \begin{cases} \{c \in R_\Psi \mid w \in \delta^*(R, c)\} & \varepsilon = +, \\ \{c \in R_\Psi \mid ww_0(K) \in \delta^*(R, c)\} & \text{otherwise.} \end{cases}$$

*Moreover, if $b \in R$ and $c \in R_\Psi$ where $w = \delta^*(b,c)$ then $c = \operatorname{proj}_{R_\Psi} b$.*

3. *If $\varepsilon$ is positive then $\operatorname{proj}_R R_\Psi$ is a residue of type $J \cap wKw^{-1}$; otherwise, $\operatorname{proj}_R R_\Psi$ is a residue of type $J \cap w_0(J)wKw^{-1}w_0(J)^{-1}$.*

4. *If $\varepsilon$ is positive then $\operatorname{proj}_{R_\Psi} R$ is a residue of type $K \cap w^{-1}Jw$; otherwise, $\operatorname{proj}_{R_\Psi} R$ is a residue of type $K \cap w_0(J)^{-1}w^{-1}Jww_0(J)$.*

*Proof.* If $S_\Psi$ is empty then the result follows from Proposition 3.31. So assume $\varepsilon$ and $\Psi$ are of positive sign. Set $P = \{b \in R \mid w \in \delta^*(b, R_\Psi)\}$. If $b \in R$ and $c \in R_\Psi$ where $w = \delta^*(b,c) \in \delta^*(b, R_\Psi)$ then $b = \operatorname{proj}_R c \in \operatorname{proj}_R R_\Psi$ by definition. Hence $P \subseteq \operatorname{proj}_R R_\Psi$. For the converse, suppose $b = \operatorname{proj}_R c'$ for some $c' \in R_\Psi$. Set $w' = \delta^*(b, c')$, so $w' \in \delta^*(R, R_\Psi) = W_J w W'_K$ by Lemma 3.28. By the definition of projection, $w' = \delta^*(b,c') = \min(\delta^*(R, c'))$. Using Proposition 3.16, one may write $w' = abwc$ where $ab \in W_J$ and $c \in W'_K$, and

$$l_\Psi(w') = l(a) + l(b) + l_\Psi(w) + l'(c).$$

But $w'$ is of minimal twisted length $l_\Psi(w')$ in $\delta^*(R, c') = W_J w'$, whence $l(a) = l(b) = 0$ and $w' = wc$. In other words, $w \in w'W'_K = \delta^*(b, R_\Psi)$ and therefore $b \in P$. This proves the first case of (1) when $\varepsilon$ and $\Psi$ are of the same sign. If $\varepsilon$ and $\Psi$ are of negative sign, then (1) follows from the same argument using $w_0(J)w$, since $l_{-\Psi}(w_0(J)w)$ is minimal in $W_J w$. The analogous cases of (2) follow similarly. For the remaining cases, first assume $\varepsilon$ and $\Psi$ are opposite sign. If $\varepsilon$ is positive then set $P' = \{b \in R \mid w \in \delta^*(b, R_\Psi)\}$. If $b \in R$ and $c \in R_\Psi$ where $w = \delta^*(b,c) \in \delta^*(b, R_\Psi)$

then $b = \operatorname{proj}_R c \in \operatorname{proj}_R R_\Psi$ because $d^\Psi(b,c) = -l_\Psi(w)$ is maximal. Therefore $P' \subseteq \operatorname{proj}_R R_\Psi$. Now suppose $b = \operatorname{proj}_R c'$ for some $c' \in R_\Psi$. Set $w' = \delta^*(b,c')$, so $w' \in \delta^*(R,R_\Psi) = W_J w W'_K$ by Lemma 3.28. Then $w' = abwc$ where $ab \in W_J$ and $c \in W'_K$, and

$$l_\Psi(w') = l(a) + l(b) + l_\Psi(w) + l'(c)$$

by Proposition 3.16. It follows from $b = \operatorname{proj}_R c'$ that $w' = \min(\delta^*(R,c'))$. Hence $w' = wc$ as required. The final case can be constructed as a combination of the previous two cases. Again, the analogous cases of (2) follow similarly.

Suppose that $\varepsilon$ is positive and $b' \in \operatorname{proj}_R R_\Psi$. Using (1) choose $c' \in R_\Psi$ with $w = \delta^*(b',c')$. In particular, $b' = \operatorname{proj}_R c'$. If $b \in R$, with $u = \delta(b,b')$, then

$$\delta^*(b,c') = \delta(b,b')\delta^*(b',c') = uw$$

by Proposition 3.31. Observe that $b \in \operatorname{proj}_R R_\Psi$ if $u \in W_J$ and $w \in \delta^*(b,R_\Psi)$. But

$$w = \delta^*(b,R_\Psi) = \delta^*(b,c')W'_K = uwW'_K$$

where the second equality follows from Lemma 3.28. Altogether, one has

$$b \in \operatorname{proj}_R R_\Psi \iff u \in W_J \cap wW'_K w^{-1} \iff u \in W_L$$

where the second equivalence is an application of Proposition 3.16, where $L = J \cap wKw^{-1}$. If $\varepsilon$ is negative then the same argument yields

$$b \in \operatorname{proj}_R R_\Psi \iff u \in W_J \cap w_0(J)wW'_K w^{-1} w_0(J)^{-1} \iff u \in W_L.$$

This proves (3) and (4) follows by an analogous argument. $\square$

*Remark* 3.36. The analogue of Proposition 3.35 in twin buildings is the fact that

$$\mathrm{proj}_R S = \{b \in R \mid w_0(J)w \in \delta^*(b,S)\}$$

for two spherical residues $R$ and $S$, of opposite sign, where $w$ is the element of $\delta^*(R,S)$ with $l(w)$ minimal. The difference between this and the statement of Proposition 3.35 is superficial. Indeed, note that the element $w$ of the proposition is of minimal *twisted* length $l_\Psi(w)$ in $\delta^*(R,R_\Psi)$. As a consequence, one is always led to the second option in Proposition 3.35(1,2).

*Remark* 3.37. In twin buildings, the hypothesis that the source residue in Proposition 3.35 is spherical can be removed. This does not appear to be the case in general, where sphericity ensures $w$ is well-defined.

### 3.2.4 Convexity

In buildings and twin buildings there are well-established notions of convexity. In a twin building, one can no longer approach convexity via minimal galleries. Instead, a subcomplex of a twin building is *convex* if it is closed under projections. That is to say, closed under projections between residues in opposing buildings, and also closed under projections within the individual buildings themselves. One also assumes that any convex subcomplex meets both components of the twin building. This implies that the intersection of a convex subcomplex with each building is convex in that building.

The discussion above motivates the following generalisation to twisted buildings.

**Definition 3.38.** A pair $(M,M_\Psi)$ of non-empty subsets $M \subseteq \mathscr{B}$ and $M_\Psi \subseteq \mathscr{B}_\Psi$ is called *convex* if $\mathrm{proj}_P b \in M \cup M_\Psi$ for every chamber $b \in M \cup M_\Psi$ and every panel $P \in \mathscr{B} \cup \mathscr{B}_\Psi$ that meets $M \cup M_\Psi$.

Just as for twin buildings, the definition can be simplified in twisted buildings.

**Lemma 3.39.** *A pair $(M,M_\Psi)$ of non-empty subsets $M \subseteq \mathscr{B}$ and $M_\Psi \subseteq \mathscr{B}_\Psi$ is convex if and only if it is closed under projections.*

*Proof.* If $b$ and $P$ lie in the same building, then the result is well-known [1, Lemma 5.45]. The remaining case is essentially [1, Exercise 5.169]. $\square$

If $(M, M_\Psi)$ is convex then $M$ is convex in $\mathscr{B}$ and $M_\Psi$ is convex in $\mathscr{B}_\Psi$. This is immediate from the definition. The converse is not necessarily true, this already fails for twin buildings. Another similarity is the fact that the intersection (taken component-wise) of a family of convex pairs is again convex, provided the intersections remain non-empty.

**Definition 3.40.** The *convex hull* $\mathrm{Con}(M, M_\Psi)$ of a pair $(M, M_\Psi)$ of non-empty subsets $M \subseteq \mathscr{B}$ and $M_\Psi \subseteq \mathscr{B}_\Psi$ is the intersection of all convex pairs containing $(M, M_\Psi)$.

In other words, the convex hull of a pair $(M, M_\Psi)$ is the smallest (ordered component-wise) convex pair containing $(M, M_\Psi)$. It follows that $\mathrm{Con}(M, M_\Psi) \cap \mathscr{B}$ contains the convex hull of $M$ in $\mathscr{B}$, and similarly for $M_\Psi$, but the converse need not be true.

## 3.3 The definition of a citadel

Thus far, it has been shown that one can define building structures to pairs of sets of positive roots that are either negations of one another in the twinning case, or when one set in the pair is a standard set of positive roots in the twisting case. The final section of this chapter brings together these structures to create an object that can encode every set of positive roots associated to a Coxeter system.

This is achieved in two steps. First, to every aligned set of positive roots $\Psi \subseteq \Delta$ define a disjoint union

$$\mathscr{B}(\Psi) = \bigsqcup_{i \in I_\Psi} \mathscr{B}_{\Psi,i}$$

running over a countable index set $I_\Psi$, where each $\mathscr{B}_{\Psi,i}$ is a building whose type is isomorphic to $(W_\Psi, S_\Psi)$. This should be thought of as accounting for the fact that the reflection subgroups $W_\Psi$ of $W$ may be proper, meaning apartments in $\mathscr{B}_\Psi$ need not be stable under a natural action of $W$. In this way, it will be assumed that $\mathscr{B}(\Psi) = \mathscr{B}_\Psi$ when $\Psi$ is a standard set of positive roots.

Next, let $\Omega$ be a collection of aligned representatives $\Psi \subseteq \Delta$ of distinct $W$-orbits of sets of positive roots in $\Delta$. Then set

$$\mathscr{C} = \bigsqcup_{\Omega} \mathscr{B}(\Psi)$$

using the notation above. In what follows, it will be assumed that $\Omega$ contains the orbit of the standard set of positive roots. Moreover, assume that each $\Psi$ is of the same fixed sign whenever $S_\Psi$ is non-empty.

**Definition 3.41.** A *citadel* of type $(W,S)$ is a triple $(\mathscr{C}, \mathscr{B}, \delta^*)$ where $\mathscr{C} = \bigsqcup_{\Omega} \mathscr{B}(\Psi)$ and $\mathscr{B}$ are defined as above, along with a $W$-valued map

$$\delta^* : (\mathscr{B} \times \mathscr{C}) \cup (\mathscr{C} \times \mathscr{B}) \to W,$$

such that, for every building $\mathscr{B}_{\Psi,i} \subseteq \mathscr{B}(\Psi)$, the triple

$$\left(\mathscr{B}, \mathscr{B}_{\Psi,i}, \delta^*|_{(\mathscr{B} \times \mathscr{B}_{\Psi,i}) \cup (\mathscr{B}_{\Psi,i} \times \mathscr{B})}\right)$$

is a twisted building of type isomorphic to $(W,S,\Psi)$.

A citadel is a structure that is covered by buildings. Therefore, freely take language from the buildings within the citadel. For example, a *chamber* (resp. *residue*) of a citadel $\mathscr{C}$ is a chamber (resp. residue) of some building $\mathscr{B}_{\Psi,i} \subseteq \mathscr{C}$. Similarly the *sign* of a residue in $\mathscr{C}$ is the sign of the residue in the family $\mathscr{B}(\Psi)$ that contains it. Therefore, the sign is controlled by $S_\Psi$. This presents an interesting phenomenon that will be of note in the sequel. In particular, even if $\mathscr{B}$ has no non-spherical residues, the other buildings $\mathscr{B}_{\Psi,i}$ may be, themselves, non-spherical residues that are strictly contained in $\mathscr{C}$.

The buildings that cover $\mathscr{C}$ relate to one another and form twisted buildings. The distinguished building $\mathscr{B}$ in a citadel is called the *keep*. This should be thought of as the building at the centre of the citadel, to which every other building is connected, hence the name.

The behaviour of a citadel $\mathscr{C}$ is also largely determined locally. For instance, the projections of a citadel are just the projections at the level of the various substructures within it. This means that $\mathscr{C}$ has two distinct kinds of projections, those within buildings or within twisted buildings.

$$\begin{tikzcd}
& \mathscr{B}(\Psi_\emptyset) \arrow[d, dash] & \\
& \mathscr{B} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathscr{B}(\Psi_{\{\alpha\}}) & & \mathscr{B}(\Psi_{\{\beta\}})
\end{tikzcd}$$

Figure 3.2: The citadel of type $\tilde{A}_2$.

One notion in the citadel that is defined globally is that of convexity. Say that a set $\mathbf{M} \subseteq \mathscr{C}$ is *convex* if it is closed under projections and meets the keep. Specifically, the set $\mathbf{M}$ is not required to meet every building in $\mathscr{C}$.

The motivating example for citadels arises from Kac-Moody groups. In this instance, the keep corresponds to the well-known (positive) building, hence admits a strongly transitive action of the group. The fixed point sets of $\mathscr{C}$ under the action of a subgroup should therefore always stabilise $\mathscr{B}$, but perhaps not the other buildings within the citadel.

The next example illustrates a citadel structure arising from a Coxeter system.

**Example 3.42.** Let $(W,S)$ be a Coxeter system with root system $\Delta$. For every set of positive roots $\Psi \subseteq \Delta$ the reflection subgroup $W_\Psi$ is canonically a thin building of type $(W_\Psi, S_\Psi)$ with $W_\Psi$-metric $\delta_\Psi : W_\Psi \times W_\Psi \to W_\Psi$ defined

$$\delta_\Psi(u,v) = u^{-1}v$$

for all $u, v \in W_\Psi$.

Fix a set $\Psi$ of positive roots in $\Delta$ and set $\mathscr{B}_\Psi = W_\Psi$. If $w \in W$ then $W_{w\Psi} = wW_\Psi w^{-1}$ is also a reflection subgroup and $S_{w\Psi} = wS_\Psi w^{-1}$ is its canonical set of generators by [13, Lemma 1.7]. In particular, the Coxeter systems $(W_\Psi, S_\Psi)$ and $(W_{w\Psi}, S_{w\Psi})$ are isomorphic. Denote by $\mathscr{B}_{\Psi,w}$ the corresponding canonical thin building of type $(W_{w\Psi}, S_{w\Psi})$.

The induced map from $\mathscr{B}_\Psi$ to $\mathscr{B}_{\Psi,w}$ implies

$$\delta_{w\Psi}(w \cdot u, w \cdot v) = w\delta_\Psi(u,v)w^{-1} = wu^{-1}vw^{-1}$$

where $\delta_{w\Psi}$ is the usual $W$-metric on $\mathscr{B}_{\Psi,w}$. Of course, if $\Psi$ is standard then $W_\Psi = W_{w\Psi}$ and this action is an automorphism on $\mathscr{B}_\Psi$. In general, this is only the case when $w \in W_\Psi$.

By Example 3.20 the map $\delta^* : W \times W_\Psi \to W$ defined by

$$\delta^*(u,v) = u^{-1}v$$

for all $u \in W$ and $v \in W_\Psi$ produces a twisted building structure on $W \sqcup W_\Psi$. It is straightforward to see that the induced map from $\mathscr{B}_\Psi$ to $\mathscr{B}_{\Psi,w}$ is compatible with the twisted building structure on $W \sqcup W_{w\Psi}$.

Let $H_\Psi = \mathrm{Stab}_W(\Psi)$ and $K_\Psi = H_\Psi \ltimes W_\Psi$. The construction above can be captured using the twisted product

$$\mathscr{B}(\Psi) := W \times_{K_\Psi} W_\Psi = W \times W_\Psi / \sim$$

where $(w,v) \sim (wk, k^{-1} \cdot v)$ for all $w \in W$, $v \in W_\Psi$, and $k \in K_\Psi$. Here, $k^{-1} \cdot v = u^{-1}(h^{-1}vh)$ where $k = hu$ for some $h \in H_\Psi$ and $u \in W_\Psi$. Notice that for every $w \in W$ there is a $W_{w\Psi}$-valued map

$$\delta_{w\Psi} : (w \times W_\Psi) \times (w \times W_\Psi) \to W_{w\Psi}$$

defined by $\delta_{w\Psi}([w,u],[w,v]) = wu^{-1}vw^{-1}$ for all $u, v \in W_\Psi$. This turns the space $\mathscr{B}(\Psi)$ into a disjoint union of isomorphic buildings of the form $\mathscr{B}_{\Psi,w} = W_{w\Psi}$ described above. That is, the map $[w,v] \mapsto wvw^{-1}$ where $w \in W$ and $v \in W_\Psi$, defines a bijection

$$\mathscr{B}(\Psi) \simeq \bigsqcup_{W/K_\Psi} W_{w\Psi}.$$

The set of chambers in $\mathscr{B}(\Psi)$ is canonically isomorphic to $W/H_\Psi$. Thus, there is a $W$-action

$$W \times \mathscr{B}(\Psi) \to \mathscr{B}(\Psi)$$

where $w \cdot vH_\Psi = wvH_\Psi$ for all $v, w \in W$. This allows one to pass between the buildings in the disjoint union above.

The *thin citadel* of $(W,S)$ is defined to be

$$\mathscr{C}_W = \bigsqcup_{\Omega} \mathscr{B}(\Psi)$$

where $\Omega$ runs over every $W$-orbit of aligned positive sets of roots in $\Delta$ of the same fixed sign. For example, if $\Omega$ contains $\Delta^+$ then $\mathscr{B}(\Delta^+) = W$ whereas if $W$ is affine and $\Psi = \Psi_\emptyset$ then $\mathscr{B}(\Psi) = \dot{W}$ is the underlying finite Coxeter group thought of as a collection of trivial buildings.

The definition of a citadel requires that every $\Psi$ is of the same fixed sign. This can be loosened to define the following larger object.

**Definition 3.43.** A *twin citadel* of type $(W,S)$ is a triple $(\mathscr{C}_+, \mathscr{C}_-, (\delta^*_\Psi)_\Psi)$ where $\mathscr{C}_+ = \bigsqcup_\Omega \mathscr{B}(\Psi)$ and $\mathscr{C}_- = \bigsqcup_\Omega \mathscr{B}(-\Psi)$ are citadels of type $(W,S)$, along with a $W_\Psi$-valued map

$$\delta^*_\Psi : (\mathscr{B}_\Psi \times \mathscr{B}_{-\Psi}) \cup (\mathscr{B}_{-\Psi} \times \mathscr{B}_\Psi) \to W_\Psi,$$

for every $\Psi$, such that, for every building $\mathscr{B}_{\Psi,i} \subseteq \mathscr{B}(\Psi)$, the triple

$$(\mathscr{B}_{\Psi,i}, \mathscr{B}_{-\Psi,i}, \delta^*_\Psi)$$

is a twin building of type isomorphic to $(W_\Psi, S_\Psi)$.

Suppose that $\mathscr{C} = \mathscr{C}_+ \cup \mathscr{C}_-$ is a twin citadel with keeps $\mathscr{B}_+$ and $\mathscr{B}_-$ respectively. Then $\mathscr{B} = \mathscr{B}_+ \cup \mathscr{B}_-$ forms a distinguished twin building which will be called the *twin keep*. Just as for the citadel, borrow language for the twin citadel from the buildings that cover it.

The main difference in a twin citadel is the guarantee of opposite chambers and residues, owing to the existence of twin buildings within it. Therefore, there is a third kind of projection in a twin citadel which comes from projections between twin buildings. In a similar way, one can define a subset $\mathbf{M} \subseteq \mathscr{C}$ to be *convex* in $\mathscr{C}$ if it is closed under projections and meets the twin keep.

*Remark* 3.44. The definitions above are intentionally quite weak to allow for a broader range of structures. However, one could go even further in the case of twin citadels and drop the demand that every $\mathscr{B}_{\Psi,i} \subseteq \mathscr{B}(\Psi)$ forms a twin building.

CHAPTER 4

# The citadels of a Kac-Moody group

This chapter associates to any Kac-Moody group $G$ a twin citadel structure. The construction itself is reminiscent of the construction of the twin building of $G$, where one constructs each component as a coset space $G/B$ where $B$ is a standard Borel subgroup of varying sign. In particular, one first constructs a subgroup $Q$ associated to an arbitrary set of positive roots in the root system of $G$, not simply those that arise from root bases. Then, a building is identified in the coset space $G/Q$. In contrast to the components of the twin building, these new buildings are typically proper subsets of the corresponding coset space. The entire coset space can be naturally identified with the collection of all buildings associated to the same conjugacy class of a positive set of roots. The twinning and twisting maps arise naturally from certain Bruhat-type decompositions of the group and the definition of the twin citadel follows accordingly.

The chapter makes use of two main inputs from the literature. First, the Billig-Dyer decomposition of [7] provides an analogue of the Bruhat and Birkhoff decompositions that define the usual buildings and twin building associated to $G$. To verify every axiom of a twisted building, it is proved that the Billig-Dyer decomposition also admits a right-handed multiplication property. The proof of this result rests on a detailed study of certain subgroups of $G$ that play an analogous role to the parahoric subgroups appearing in the construction of masures in [20]. This study adapts the proofs of *op. cit.* to the context of this thesis. Therefore, the reader familiar with the construction of masures for Kac-Moody groups should recognise the strategy taken here to construct citadels.

Throughout, $G$ denotes a Kac-Moody group associated to a free datum.

## 4.1 Quasi-parabolic subgroups

This section defines and studies analogues of parabolic subgroups associated to non-standard sets of positive roots. In the end, these subgroups will become the stabilisers of residues in the twin citadel, or points in its vectorial model. The approach taken here is inspired by the work of [20] and [38] when defining parahoric subgroups associated to masures. The arguments provided have analogues in *op. cit.*, but using slightly different objects. For the sake of completeness, the full proofs are detailed below.

### 4.1.1 The definition

A typical way to define and study parabolic subgroups in reductive groups is using cocharacters of the group. This is due in part to the fact that the group of cocharacters, by duality with the group of characters, defines half-spaces in the ambient finite root system associated to the group. This is the basis of the approach taken in this subsection. The main difference that arises is due to the fact that the root system of an arbitrary Kac-Moody group is infinite. In short, this means that a single cocharacter, or equivalently single half-space, is often not enough to define every partition that is of interest.

Fix a maximal torus $T$ of $G$ and write $\Delta = \Delta(G,T)$. Recall from Proposition 3.3 that every set of positive roots $\Psi \subseteq \Delta_{\mathrm{re}}$ arises as the set of roots that are positive with respect to a lexicographical ordering of the vector space $\mathbb{R}\Delta$. This can be understood via cocharacters of $T$ in the following way.

The space of *virtual cocharacters* $Y_T(\mathbb{R}) = Y_T \otimes_{\mathbb{Z}} \mathbb{R}$ of $T$ gives a faithful reflection representation for $W$. Details of this construction can be found in [39]. In particular, there is a closed convex cone, called the *Tits cone* $\tau_+$, which can be identified with the Coxeter complex of $(W,S)$. If $W$

is finite then $\tau_+ = Y_T(\mathbb{R})$. Otherwise, the *twin Tits cone* $\tau_+ \cup -\tau_+$ is typically a proper subset of $Y_T(\mathbb{R})$.

If $S$ is a subtorus of $T$ then $Y_S(\mathbb{R})$ can be identified as a linear subspace of $Y_T(\mathbb{R})$. Following [35, Chapter 10], a subtorus $S$ of $T$ is *generic* if $Y_S(\mathbb{R}) \subseteq Y_T(\mathbb{R})$ meets the interior of the Tits cone in $Y_T(\mathbb{R})$. If $Y_S(\mathbb{R})$ is generated by its intersection with the Tits cone then $S$ is *almost generic*; otherwise, $S$ is *non-generic*. These definitions extend to cover all tori in $G$, i.e. the definition is independent of the choice of $T$ containing $S$.

*Remark* 4.1. More recently, generic tori have been called *J-regular* in [22] for finite subsets $J \subseteq I$. In this language, the almost generic tori are the $J$-regular tori where $J \subseteq I$ is an arbitrary subset.

The duality between $X_T(\mathbb{R})$ and $Y_T(\mathbb{R})$ provides a correspondence between virtual cocharacters and hyperplanes of the root system $\mathbb{R}\Delta$. That is, for every $\lambda \in Y_T(\mathbb{R})$ one obtains the hyperplane $\ker\langle\lambda, -\rangle$ in $X_T(\mathbb{R})$. Therefore, if $\Psi \subseteq \Delta_{\mathrm{re}}$ is a set of positive roots then there is a tuple $\underline{\lambda} = (\lambda_1, \dots, \lambda_l) \in Y_T^l(\mathbb{R})$ for which

$$\Psi = \Delta_{\underline{\lambda}} = \{\alpha \in \Delta_{\mathrm{re}} \mid \langle \underline{\lambda}, \alpha \rangle \geq_{\mathrm{lex}} 0\}$$

by Proposition 3.3. The rank of $\Delta$ is finite, so one may assume that the tuple is also finite. Moreover, one has $-\Delta_{\underline{\lambda}} = \Delta_{-\underline{\lambda}}$. It is well-known that $\Psi$ is a standard set of positive roots if $\underline{\lambda} = \lambda_1 \in Y_T$ lies in the interior of the Tits cone.

The lexicographical condition above can be expressed algebraically via non-Archimedean field extensions. One benefit of this approach is that it keeps track of the order of cocharacters in $\underline{\lambda}$ in a concrete way. Let $\mathbb{K} = \mathbb{R}(\varepsilon)$ be a non-Archimedean ordered field with $0 < \varepsilon < r$ for every $r \in \mathbb{R}$ positive. The perfect pairing between $X_T$ and $Y_T$ extends $\mathbb{K}$-linearly to a pairing

$$\langle -, - \rangle : Y_T(\mathbb{K}) \times X_T(\mathbb{K}) \to \mathbb{K}$$

where $X_T(\mathbb{K}) = X_T \otimes_{\mathbb{Z}} \mathbb{K}$ and $Y_T(\mathbb{K}) = Y_T \otimes_{\mathbb{Z}} \mathbb{K}$. In a similar way, the action of $W$ on $X_T$ and $Y_T$

extends to an action on $X_T(\mathbb{K})$ and $Y_T(\mathbb{K})$ respectively. Then

$$\langle \underline{\lambda}, \alpha \rangle \geq_{\text{lex}} 0 \iff \langle \theta_{\underline{\lambda}}, \alpha \rangle \geq 0$$

where $\theta_{\underline{\lambda}} = \lambda_1 + \varepsilon\lambda_2 + \cdots + \varepsilon^{l-1}\lambda_l \in Y_T(\mathbb{K})$. Henceforth, write

$$\Delta_\theta = \{\alpha \in \Delta_{\text{re}} \mid \langle \theta, \alpha \rangle \geq 0\}$$

for all $\theta \in Y_T(\mathbb{K})$.

*Remark* 4.2. The construction above can be understood as a phenomenon of infinite hyperplane arrangements. In $Y_T(\mathbb{R})$ every root $\alpha \in \Delta_{\text{re}}$ defines a hyperplane $\ker\langle -, \alpha \rangle$. Fixing a set of positive roots $\Psi \subseteq \Delta_{\text{re}}$ corresponds to choosing an open half-space associated to each such hyperplane. The intersection of these half-spaces defines what is known as a *pre-region* in $Y_T(\mathbb{R})$. If $\Delta$ is a finite root system then every pre-region contains an $\mathbb{R}$-point. In other words, one can find a virtual cocharacter $\lambda \in Y_T(\mathbb{R})$ for which $\lambda(\alpha) > 0$ precisely when $\alpha \in \Psi$. In general, a pre-region may not contain any $\mathbb{R}$-points. However, since $Y_T(\mathbb{R})$ is finite dimensional, every pre-region is a *region* and also admits a point over some ordered field $\mathbb{K} = \mathbb{R}(\varepsilon)$ with $0 < \varepsilon < r$ for all $r \in \mathbb{R}$ positive [3, Lemma 2.1.7]. The former means that, for every finite subset $\Sigma \subseteq \Delta_{\text{re}}$, there is a real virtual cocharacter $\lambda_\Sigma \in Y_T(\mathbb{R})$ such that

$$\langle \lambda_\Sigma, \alpha \rangle > 0 \text{ if } \alpha \in \Sigma \cap \Psi \quad \text{and} \quad \langle \lambda_\Sigma, \alpha \rangle < 0 \text{ if } \alpha \in \Sigma \cap -\Psi.$$

The latter is equivalent to the existence of some $\theta \in Y_T(\mathbb{K})$ such that

$$\theta(\alpha) > 0 \iff \alpha \in \Psi.$$

Set

$$N_\theta = \langle n \in N_G(T) \mid n\Delta_\theta = \Delta_\theta \rangle.$$

It is straightforward to see that $T \subseteq N_\theta$. The discussion above motivates the following definition.

**Definition 4.3.** Let $\theta \in Y_T(\mathbb{K})$. A subgroup of the form

$$Q = Q_\theta := \langle N_\theta, U_\alpha \mid \alpha \in \Delta_{\mathrm{re}},\ \langle \theta, \alpha \rangle \geq 0 \rangle$$

is called a *standard quasi-parabolic subgroup* of $G$, relative to $T$. A subgroup that is conjugate to a standard quasi-parabolic subgroup is called a *quasi-parabolic subgroup*. Moreover, if $\Delta_\theta$ is a set of positive roots then $Q$ is a *standard quasi-Borel subgroup* of $G$, relative to $T$. In a similar fashion, a subgroup that is conjugate to a standard quasi-Borel subgroup is called a *quasi-Borel subgroup*.

Note that every quasi-parabolic subgroup is of the form $Q = Q_\theta$ for some $\theta \in Y_S(\mathbb{K})$ where $S$ is a maximal torus of $G$. The root groups $U_\alpha$ must be taken relative to $\Delta = \Delta(G,S)$. In the sequel, a quasi-Borel subgroup associated to a set of positive roots $\Psi = \Delta_\theta$ will often be denoted by $Q = Q_\Psi$.

The presence of $N_\theta$ in the construction of a quasi-Borel subgroup is an important detail which is not seen in Borel subgroups. In particular, if $\Delta_\theta = \Psi$ for some standard set of positive roots $\Psi \subseteq \Delta_{\mathrm{re}}$ then $N_\theta = T$. In general, $N_\theta$ can contain non-trivial lifts of elements in the Weyl group $W$ [40, Proposition 3.3.1].

**Example 4.4.** Let $G = \mathrm{GL}_3(k[t,t^{-1}]) \rtimes k^\times$ with $T$ the diagonal maximal torus in the standard copy of $\mathrm{GL}_3(k) \times k^\times$. The root system $\Delta(G,T)$ is untwisted affine type and the sets of positive roots are classified by Example 3.2. In line with the notation of Example 3.7, one has simple roots $\{\alpha, \beta\}$ of the underlying finite root system $\dot{\Delta}$ of type $A_2$. The Weyl group $W$ of $G$ is isomorphic to $S_3 \ltimes \mathbb{Z}^2$. The copy of $S_3$ is generated by elements $s_\alpha$ and $s_\beta$ that are lifted, respectively, by elements

$$n_\alpha = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in N_G(T) \quad \text{and} \quad n_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \in N_G(T).$$

Similarly, the translation subgroup $\mathbb{Z}^2$ is generated by elements $t_{\alpha^\vee}$ and $t_{\beta^\vee}$ that are lifted, respectively, by elements

$$n_{\alpha^\vee} = \begin{pmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in N_G(T) \quad \text{and} \quad n_{\beta^\vee} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-1} \end{pmatrix} \in N_G(T).$$

Then:

1. If $\Psi = \Psi_{\{\alpha,\beta\}} = \Delta^+$ then $Q_\Psi = B^+$ where $\mathrm{Stab}_W(\Psi) = 1$.
2. If $\Psi = \Psi_{\{\beta\}}$ then $Q_\Psi = \langle Q_\Psi^0, n\rangle$ where

$$Q_\Psi^0 = \begin{pmatrix} k^\times & k[t,t^{-1}] & k[t,t^{-1}] \\ 0 & k[t] & k[t] \\ 0 & tk[t] & k[t] \end{pmatrix} \rtimes k^\times \quad \text{and} \quad n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} n_{\alpha^\vee} n_\beta = \begin{pmatrix} t & 0 & 0 \\ 0 & 0 & t^{-1} \\ 0 & 1 & 0 \end{pmatrix}$$

   The element $n$ lifts $t_{\alpha^\vee} s_\beta \in W$ and $\mathrm{Stab}_W(\Psi) \cong \mathbb{Z}$. Observe that $n^2$ is a lift of the translation $t_{\alpha^\vee}^2 t_{\beta^\vee} \in W$ which generates a normal subgroup $\Lambda_\Psi \cong \mathbb{Z}$ of $\mathrm{Stab}_W(\Psi)$. Therefore, one has the non-split extension

$$1 \to \Lambda_\Psi \to \mathrm{Stab}_W(\Psi) \to C_2 \to 1$$

   where $C_2$ is the cyclic group of order 2.
3. If $\Psi = \Psi_\emptyset$ then $Q_\Psi = \langle Q_\Psi^0, n_{\alpha^\vee}, n_{\beta^\vee}\rangle$ where

$$Q_\Psi^0 = \begin{pmatrix} k^\times & k[t,t^{-1}] & k[t,t^{-1}] \\ 0 & k^\times & k[t,t^{-1}] \\ 0 & 0 & k^\times \end{pmatrix} \rtimes k^\times$$

   and $\mathrm{Stab}_W(\Psi) = \langle t_{\alpha^\vee}, t_{\beta^\vee}\rangle \cong \mathbb{Z}^2$.

Let $\theta \in Y_T(\mathbb{K})$ and define

$$U_\theta = \langle U_\alpha \mid \alpha \in \Delta_{\mathrm{re}},\ \langle \theta, \alpha\rangle \geq 0\rangle.$$

It is immediate from the definition that $N_\theta$ normalises $U_\theta$, hence

$$Q_\theta = U_\theta N_\theta.$$

Define the subgroup

$$Q_\theta^0 = \langle T, U_\alpha \mid \alpha \in \Delta_{\mathrm{re}},\ \langle \theta, \alpha \rangle \geq 0 \rangle.$$

Then $Q_\theta = Q_\theta^0 N_\theta$. As seen in Example 4.4, the inclusion $Q_\Psi^0 \subseteq Q_\Psi$ is typically proper for quasi-Borel subgroups that are not Borel subgroups. When $G$ is equipped with the ind-Zariski topology this subgroup is precisely the identity component of $Q_\theta$.

There is a notion of *sign* associated to quasi-Borel subgroups that is induced from the sign of the associated set of positive roots. Specifically, a quasi-Borel subgroup $Q = Q_\theta$ is of *positive* (resp. *negative*) *sign* if $\Psi = \Delta_\theta$ is of positive (resp. negative) sign. Note that a quasi-Borel subgroup may have no sign, for instance when $S_\Psi$ is empty.

The definition of quasi-Borel subgroups is implicit in [7] where some properties are proved in characteristic zero. The authors note that they remain true in positive characteristic by the same arguments. The next result records some of the properties established in *op. cit.* for quasi-Borel subgroups.

**Proposition 4.5.** *Let $G$ be a Kac-Moody group with maximal torus $T$. Suppose that $B$ is a standard Borel subgroup of $G$ relative to $T$. Then*

$$G = \bigsqcup_{w \in W} BwQ^0$$

*for every standard quasi-Borel subgroup $Q = Q_\Psi$ of $G$ relative to $T$. For all $\alpha \in \Pi$ and $w \in W$, one has*

$$BswQ^0 \subseteq BsBwQ^0 \subset BwQ^0 \cup BswQ^0$$

*where $s = s_\alpha$ is a simple reflection. Moreover, if $l_\Psi(sw) > l_\Psi(w)$ then $BsBwQ^0 = BswQ^0$.*

In the sequel, the first statement of Proposition 4.5 will define a twisting map and the second statement will verify **(T'2)**(a) and **(T'3)**(a). The remaining axioms follow from a right-analogue of the second statement that will be proved later.

The notion of sign for quasi-Borel subgroups extends to quasi-parabolic subgroups. Specifically, a quasi-parabolic subgroup $Q = Q_\theta$ is of *positive* (resp. *negative*) *sign* if it contains a quasi-Borel subgroup of positive (resp. negative) sign. A pair $P$ and $Q$ of quasi-parabolic subgroups will be called *opposite* if there is a maximal torus $S$ of $G$ such that $P = Q_\theta$ and $Q = Q_{-\theta}$ for some $\theta \in Y_S(\mathbb{K})$. Therefore, if a quasi-parabolic subgroup possesses a sign, then its opposite will be of opposite sign. The intersection of a quasi-parabolic subgroup with an opposite quasi-parabolic subgroup is called a *quasi-Levi subgroup*.

**Example 4.6.** Keep the notation of Example 4.4 and consider the cocharacter $\lambda \in Y_T$ defined by $\lambda(c) = \mathrm{diag}(c,1,1) \times 1$ for all $c \in k^\times$. Then

$$Q_\theta = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \rtimes k^\times.$$

In particular, if $\mu \in Y_T$ is defined by $\mu(c) = \mathrm{diag}(c, c^2, c^3) \times c$ for all $c \in k^\times$, then

$$Q_{\lambda+\varepsilon\mu} = Q_{\Psi_{\{\beta\}}} = \begin{pmatrix} k[t] & k[t,t^{-1}] & k[t,t^{-1}] \\ 0 & k[t] & k[t] \\ 0 & tk[t] & k[t] \end{pmatrix} \rtimes k^\times$$

is the quasi-Borel subgroup from before.

In general, an element $\theta \in Y_T(\mathbb{K})$ describes a *quasi-parabolic* set of roots in the sense of [18, §2.5]. That is, the set $\Delta_\theta$ is closed and $\Delta_{\mathrm{re}} = \Delta_\theta \cup (-\Delta_\theta)$. If $\Delta_\theta$ is not a set of positive roots then the set $\Phi_\theta = \Delta_\theta \cap (-\Delta_\theta)$ is non-empty. Moreover, $\Phi_\theta$ is *root generated*, and therefore a closed subroot system of $\Delta$ by [21, Theorem 1].

It follows that every quasi-parabolic subgroup contains a quasi-Borel subgroup. Indeed, pick some $\lambda \in Y_T(\mathbb{K})$ that is regular in $\Phi_\theta$ and define $\theta' = \theta + \varepsilon^{l+1}\lambda$ where $l$ is the highest degree of $\varepsilon$ in $\theta$. Then $Q_{\theta'} \subseteq Q_\theta$ is a quasi-Borel subgroup. This process is illustrated in Example 4.6 above.

### 4.1.2 A decomposition of quasi-parabolic subgroups

This subsection provides a decomposition for quasi-parabolic subgroups which gives an effective way to study them. The study of quasi-parabolic subgroups ultimately reduces to the study of the subgroups $U_\theta$. However, it is difficult to handle $U_\theta$ in $G$ due to the absence of commutation relations between root groups $U_\alpha$ and $U_\beta$ when $\alpha$ and $\beta$ do not form a prenilpotent pair. In order to resolve this issue, one can define several larger subgroups in the completions of $G$ and show that they descend to $G$ in a well-defined manner. The method is inspired by [20] and [38].

Fix a maximal torus $T$ of $G$ and write $\Delta = \Delta(G,T)$. Before passing to the maximal groups, some notation must be established. For all $\alpha \in \Delta_{\mathrm{re}}$ define

$$U_{\alpha,\theta} = \begin{cases} U_\alpha & \text{if } \langle \theta, \alpha \rangle \geq 0, \\ 1 & \text{otherwise.} \end{cases}$$

Write $U_\theta^{(\alpha)}$ for the group generated by $U_{\alpha,\theta}$ and $U_{-\alpha,\theta}$. Therefore $U_\theta^{(\alpha)}$ is either a root group or a semisimple rank 1 algebraic subgroup of $G$. Set

$$N_\theta^{(\alpha)} = N_G(T) \cap U_\theta^{(\alpha)} \quad \text{and} \quad N_\theta^u = \langle N_\theta^{(\alpha)} \mid \alpha \in \Delta_{\mathrm{re}} \rangle.$$

In particular, $N_\theta^u$ is generated by reflections arising from subgroups isogenous to $\mathrm{SL}_2$ in $U_\theta$. The next result is [20, Lemma 3.3].

**Lemma 4.7.** *Let $\theta \in Y_T(\mathbb{K})$. Then*

$$U_\theta^{(\alpha)} = U_{\alpha,\theta} U_{-\alpha,\theta} N_\theta^{(\alpha)} = U_{-\alpha,\theta} U_{\alpha,\theta} N_\theta^{(\alpha)}$$

*for all $\alpha \in \Delta_{\mathrm{re}}$.*

*Proof.* If $\langle \theta, \alpha \rangle \neq 0$ then $U_\theta^{(\alpha)}$ is either $U_{\alpha,\theta}$ or $U_{-\alpha,\theta}$ and the result is trivial. Otherwise, the result follows from the rank 1 Bruhat decomposition in $U_\theta^{(\alpha)}$ (cf. [25, §28.2]). □

Recall the pro-unipotent subgroup $U^{\mathrm{ma}+}$ of $G^{\mathrm{ma}+}$ and write $U^{\mathrm{ma}-}$ for the analogous subgroup in the opposite formal group $G^{\mathrm{ma}-}$. In particular, one has

$$U^+ \subseteq U^{\mathrm{ma}+} \quad \text{and} \quad U^- \subseteq U^{\mathrm{ma}-}.$$

In the sequel, one would like to consider similar completed subgroups associated to $\theta$. This is achieved as follows. Let $\mathfrak{g}$ be the Kac-Moody algebra of $G$. Recall that the $\mathbb{Z}$-form $\mathscr{U} = \mathscr{U}_{\mathbb{Z}}(\mathfrak{g})$ of the universal enveloping algebra of $\mathfrak{g}$ admits a $Q_{\mathrm{rt}}$-grading

$$\mathscr{U} = \bigoplus_{\alpha \in Q_{\mathrm{rt}}} \mathscr{U}_\alpha.$$

If $\Sigma \subseteq \Delta \cup \{0\}$ is closed then

$$\mathfrak{g}_\Sigma = \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$$

is a Lie algebra, and write $\mathscr{U}(\Sigma) = \mathscr{U} \cap \mathscr{U}_{\mathbb{C}}(\mathfrak{g}_\Sigma)$. These two algebras are stable by the adjoint action of $\mathscr{U}(\Sigma)$. Moreover, $\mathscr{U}(\Sigma)$ can be $\mathbb{Z}$-graded by total degree, defining the positive completion $\widehat{\mathscr{U}}^p(\Sigma)$ of $\mathscr{U}(\Sigma)$. Then $\widehat{\mathscr{U}}^p(\Sigma)$ is a submodule of $\widehat{\mathscr{U}}^p$ that is stable under its adjoint action.

If instead $\Sigma \subseteq \Delta^+$ is closed then $\mathscr{U}(\Sigma)$ admits a $Q_{\mathrm{rt}}^+$-grading and therefore one can form the completion

$$\widehat{\mathscr{U}}(\Sigma) = \prod_{\alpha \in Q_{\mathrm{rt}}^+} \mathscr{U}(\Sigma)_\alpha.$$

after grading by total degree. In particular, set

$$\mathscr{U}_\theta = \bigoplus_{\substack{\alpha \in Q_{\mathrm{rt}} \\ \langle \theta, \alpha \rangle \geq 0}} \mathscr{U}_\alpha \otimes_{\mathbb{Z}} k$$

and write $\widehat{\mathscr{U}}_\theta^p$ and $\widehat{\mathscr{U}}_\theta^n$ for its positive and negative completion with respect to the total degree and its opposite, respectively. Then define

$$\mathscr{U}_\theta(\Sigma) = \bigoplus_{\substack{\alpha \in Q_{\mathrm{rt}}^+ \\ \langle \theta, \alpha \rangle \geq 0}} \mathscr{U}(\Sigma)_\alpha \otimes_{\mathbb{Z}} k \quad \text{and} \quad \widehat{\mathscr{U}_\theta}(\Sigma) = \prod_{\substack{\alpha \in Q_{\mathrm{rt}}^+ \\ \langle \theta, \alpha \rangle \geq 0}} \mathscr{U}(\Sigma)_\alpha \otimes_{\mathbb{Z}} k.$$

It is straightforward to see that $\widehat{\mathscr{U}_\theta}(\Sigma)$ is a subalgebra of $\widehat{\mathscr{U}_\theta^p}$ and of $\widehat{\mathscr{U}_k}(\Sigma) = \widehat{\mathscr{U}}(\Sigma) \otimes_{\mathbb{Z}} k$.

In this instance, the intersection $U_\theta^{\mathrm{ma}}(\Sigma)$ of $\widehat{\mathscr{U}_\theta}(\Sigma)$ or $\widehat{\mathscr{U}_\theta^p}$ with $U_\Sigma^{\mathrm{ma}}$ is a subgroup of $U_\Sigma^{\mathrm{ma}}$. Note that $U_\theta^{\mathrm{ma}}(\Sigma)$ and $U_\theta^{\mathrm{pm}}(\Sigma) = G \cap U_\theta^{\mathrm{ma}}(\Sigma)$ stabilise $\widehat{\mathscr{U}_\theta^p}$ for the adjoint action of $U_\Sigma^{\mathrm{ma}}$ on $\widehat{\mathscr{U}_k^p}$, the positive completion of $\mathscr{U}_k$. A similar statement holds for a subgroup $U_\theta^{\mathrm{nm}}(\Sigma)$ obtained via intersecting $G$ with a negative analogue of $U_\theta^{\mathrm{ma}}(\Sigma)$.

Define the closed sets

$$\Delta_\theta^+ = \{\alpha \in \Delta^+ \mid \langle \theta, \alpha \rangle \geq 0\} \quad \text{and} \quad \Delta_\theta^- = \{\alpha \in \Delta^- \mid \langle \theta, \alpha \rangle \geq 0\}.$$

By [38, Lemme 3.3] there are pro-unipotent subgroups

$$U_\theta^{\mathrm{ma}+} = U^{\mathrm{ma}+}(\Delta_\theta^+) \subseteq G^{\mathrm{ma}+} \quad \text{and} \quad U_\theta^{\mathrm{ma}-} = U^{\mathrm{ma}-}(\Delta_\theta^-) \subseteq G^{\mathrm{ma}-}$$

that are closed in $U^{\mathrm{ma}+}$ and $U^{\mathrm{ma}-}$ respectively. The intersections of these subgroups with $G$ are given by

$$U_\theta^{\mathrm{pm}+} = G \cap U_\theta^{\mathrm{ma}+} = U^+ \cap U_\theta^{\mathrm{ma}+} \quad \text{and} \quad U_\theta^{\mathrm{nm}-} = G \cap U_\theta^{\mathrm{ma}-} = U^- \cap U_\theta^{\mathrm{ma}-}$$

where the second equalities follow from

$$U^+ = G \cap U^{\mathrm{ma}+} \quad \text{and} \quad U^- = G \cap U^{\mathrm{ma}-}$$

respectively. Importantly, it is not immediate as to whether $U_\theta^{\mathrm{pm}+}$ and $U_\theta^{\mathrm{nm}-}$ coincide with

$$U_\theta^+ = U^+ \cap U_\theta \quad \text{and} \quad U_\theta^- = U^- \cap U_\theta$$

respectively. This will be proved by the end of the subsection.

Let $\alpha$ be a simple root of $\Delta$. Again by [38, Lemme 3.3] one has

$$U^{\mathrm{ma}+} = U_\alpha \ltimes U^{\mathrm{ma}}(\Delta^+ \setminus \alpha)$$

which in turn implies that

$$U_\theta^{\mathrm{ma}+} = U_{\alpha,\theta} \ltimes U_\theta^{\mathrm{ma}}(\Delta^+ \setminus \alpha) \quad \text{and} \quad U_\theta^{\mathrm{pm}+} = U_{\alpha,\theta} \ltimes U_\theta^{\mathrm{pm}}(\Delta^+ \setminus \alpha)$$

where $U_\theta^{\mathrm{pm}}(\Delta^+ \setminus \alpha) = G \cap U_\theta^{\mathrm{ma}}(\Delta^+ \setminus \alpha)$. Similarly, one has

$$U_\theta^{\mathrm{ma}-} = U_{-\alpha,\theta} \ltimes U_\theta^{\mathrm{ma}}(\Delta^- \setminus -\alpha) \quad \text{and} \quad U_\theta^{\mathrm{nm}-} = U_{-\alpha,\theta} \ltimes U_\theta^{\mathrm{nm}}(\Delta^- \setminus -\alpha)$$

where $U_\theta^{\mathrm{nm}}(\Delta^- \setminus -\alpha) = G \cap U_\theta^{\mathrm{ma}}(\Delta^- \setminus -\alpha)$.

Define

$$U_\theta^{\mathrm{pm}} = \langle U_\theta, U_\theta^{\mathrm{pm}+} \rangle \quad \text{and} \quad U_\theta^{\mathrm{nm}} = \langle U_\theta, U_\theta^{\mathrm{nm}-} \rangle.$$

The next result is a version of [20, Proposition 3.4] that provides useful decompositions for the subgroups $U_\theta$, $U_\theta^{\mathrm{pm}}$, and $U_\theta^{\mathrm{nm}}$ in terms of positive and negative root groups.

**Proposition 4.8.** *Let $\theta \in Y_T(\mathbb{K})$. Then*

$$U_\theta = U_\theta^+ U_\theta^- N_\theta^u = U_\theta^- U_\theta^+ N_\theta^u.$$

*Moreover, one has*

$$U_\theta^{\mathrm{pm}} = U_\theta^{\mathrm{pm}+} U_\theta^- N_\theta^u \quad \textit{and} \quad U_\theta^{\mathrm{nm}} = U_\theta^{\mathrm{nm}-} U_\theta^+ N_\theta^u.$$

*Proof.* Set $U = U_\theta^{\mathrm{pm}+} U_\theta^{\mathrm{nm}-} N_\theta^u$. By the previous discussion, if $\alpha \in \Delta$ is simple then

$$\begin{aligned} U &= U_{\alpha,\theta} U_\theta^{\mathrm{pm}}(\Delta^+ \setminus \alpha) U_{-\alpha,\theta} U_\theta^{\mathrm{nm}}(\Delta^- \setminus -\alpha) N_\theta^u, \\ &= U_\theta^{\mathrm{pm}}(\Delta^+ \setminus \alpha) U_\theta^{\mathrm{nm}}(\Delta^- \setminus -\alpha) U_{\alpha,\theta} U_{-\alpha,\theta} N_\theta^u, \\ &= U_\theta^{\mathrm{pm}}(\Delta^+ \setminus \alpha) U_\theta^{\mathrm{nm}}(\Delta^- \setminus -\alpha) U_{-\alpha,\theta} U_{\alpha,\theta} N_\theta^u, \end{aligned}$$

where the second equality follows from the fact that $U_\theta^{(\alpha)}$ normalises $U_\theta^{\mathrm{pm}+}(\Delta^+ \setminus \alpha)$ and $U_\theta^{\mathrm{nm}-}(\Delta^- \setminus -\alpha)$, while the third equality follows from Lemma 4.7. Performing the procedure in reverse, one obtains

$$
\begin{aligned}
U &= U_\theta^{\mathrm{pm}}(\Delta^+ \setminus \alpha) U_\theta^{\mathrm{nm}}(\Delta^- \setminus -\alpha) U_{-\alpha,\theta} U_{\alpha,\theta} N_\theta^u, \\
&= U_{-\alpha,\theta} U_\theta^{\mathrm{pm}}(\Delta^+ \setminus \alpha) U_{\alpha,\theta} U_\theta^{\mathrm{nm}}(\Delta^- \setminus -\alpha) N_\theta^u, \\
&= U_\theta^{\mathrm{pm}}(s_\alpha \Delta^+) U_\theta^{\mathrm{nm}}(s_\alpha \Delta^-) N_\theta^u.
\end{aligned}
$$

In particular, $U$ is invariant under conjugation of $\Delta^+$ by elements of $W$. It follows that every $U_{\alpha,\theta}$ with $\alpha \in \Delta_{\mathrm{re}}$ can appear in a leftmost factor in a presentation of $U$. Therefore $U$ is stable under left multiplication by all such $U_{\alpha,\theta}$. Hence

$$U_\theta \subseteq U_\theta^{\mathrm{pm}} \subseteq U.$$

Suppose that $g \in U \cap U^{\mathrm{ma}+}$. In particular, write $g = u_+ u_- n$ where $u_+ \in U_\theta^{\mathrm{pm}+}$, $u_- \in U_\theta^{\mathrm{nm}-}$, and $n \in N_\theta^u$. Then $U_\theta^{\mathrm{pm}+} \subseteq U^{\mathrm{ma}+}$ implies that

$$u_- n = u_+^{-1} g \in U^{\mathrm{ma}+} \cap U^- N$$

where the final intersection is trivial by the Birkhoff decomposition. It follows that

$$u_-^{-1} = n \in U^- \cap N$$

which is again trivial by the same reasoning, so $u_- = n = 1$. This proves that $U \cap U^{\mathrm{ma}+} = U_\theta^{\mathrm{pm}+}$ and now $U_\theta^{\mathrm{pm}} \subseteq U$ implies that

$$U_\theta^{\mathrm{pm}+} = U_\theta^{\mathrm{pm}} \cap U^{\mathrm{ma}+} = U_\theta^{\mathrm{pm}} \cap U^+.$$

Set

$$U_\theta(\Delta^+ \setminus \alpha) = U_\theta \cap U_\theta^{\mathrm{pm}}(\Delta^+ \setminus \alpha) = U_\theta^+ \cap U_\theta^{\mathrm{pm}}(\Delta^+ \setminus \alpha)$$

where the final equality follows from the fact that $U_\theta^{\mathrm{pm}}(\Delta^+ \setminus \alpha) \subseteq U^+$. Observe that $U_\theta^{(\alpha)}$ normalises $U_\theta(\Delta^+ \setminus \alpha)$. Indeed, suppose that $g \in U_{\alpha,\theta}$ and $h \in U_\theta(\Delta^+ \setminus \alpha)$. Then $g, h \in U_\theta$ and

$g$ normalises $U_\theta^{\mathrm{pm}}(\Delta^+ \setminus \alpha)$. Therefore

$$ghg^{-1} \in U_\theta \cap U_\theta^{\mathrm{pm}}(\Delta^+ \setminus \alpha) = U_\theta(\Delta^+ \setminus \alpha)$$

as required. The same argument holds if $g \in U_{-\alpha,\theta}$ which proves the claim.

The observation above implies that

$$U_\theta^+ = U_{\alpha,\theta} \ltimes U_\theta(\Delta^+ \setminus \alpha).$$

Indeed, suppose that $u \in U_\theta^+$. Using $U_\theta^+ \subseteq U_\theta^{\mathrm{pm}+}$ one can write $u = u_\alpha u'$ where $u_\alpha \in U_{\alpha,\theta}$ and $u' \in U_\theta^{\mathrm{pm}}(\Delta^+ \setminus \alpha)$. But then

$$u' = u_\alpha^{-1} u \in U_\theta \cap U_\theta^{\mathrm{pm}}(\Delta^+ \setminus \alpha) = U_\theta(\Delta^+ \setminus \alpha).$$

The reverse inclusion is immediate, since $U_{\alpha,\theta}$ and $U_\theta(\Delta^+ \setminus \alpha)$ are contained in $U_\theta^+$. An analogous result can be obtained for $U_\theta^-$.

Finally, repeating the initial argument of the proof on

$$U' = U_\theta^+ U_\theta^- N_\theta^u \quad \text{or} \quad U' = U_\theta^{\mathrm{pm}+} U_\theta^- N_\theta^u \quad \text{or} \quad U' = U_\theta^{\mathrm{nm}-} U_\theta^+ N_\theta^u$$

respectively, using the decompositions of the previous paragraph, yields the result. □

*Remark* 4.9. In the special case when $\theta$ defines a set of positive roots $\Psi \subseteq \Delta_{\mathrm{re}}$ the decomposition of $U_\theta$ in Proposition 4.8 retrieves [7, Theorem 3].

In order to get an even better handle on $U_\theta$ one considers a large subgroup defined as the stabiliser in $G$ of certain natural modules, such as the enveloping algebra. This allows one to work effectively with individual elements in the group.

Recall that $U_\theta^{\mathrm{ma}+}$ stabilises $\widehat{\mathscr{U}_\theta^p}$ under the adjoint action. Similarly, the group $U^+$ stabilises $\mathscr{U}_k$ for the adjoint action of $G$ and $U^+$. These two actions coincide on $U^+ = G \cap U^{\mathrm{ma}+}$, meaning $U_\theta^{\mathrm{pm}+} = U^+ \cap U_\theta^{\mathrm{ma}+}$ stabilises $\mathscr{U}_\theta = \mathscr{U}_k \cap \widehat{\mathscr{U}_\theta^p}$ for the adjoint action of $G$. A similar argument shows that $U_\theta^{\mathrm{nm}-}$ stabilises $\mathscr{U}_\theta$. By construction, $\mathscr{U}_\theta$ is stable under $U_{\alpha,\theta} = U_\theta^{\mathrm{ma}}(\alpha)$ for all $\alpha \in \Delta_{\mathrm{re}}$, hence also $U_\theta$. This implies that $U_\theta^{\mathrm{pm}}$ and $U_\theta^{\mathrm{nm}}$ stabilise $\mathscr{U}_\theta$ also.

An analogous process gives the same conclusions when considering highest and lowest weight modules of $G$. Importantly, if $M_k$ is one such module over $k$ then there is a submodule $M_\theta$, defined in a similar way as above, that is also a $\widehat{\mathscr{U}_\theta^p}$-module. It can be shown that $M_\theta$ is stable under the action of $T$, $U_\theta$, $U_\theta^{\mathrm{pm}}$, and $U_\theta^{\mathrm{nm}}$. For more detail on this topic, see [38, §2.14,§4.8].

Define $\widetilde{P}_\theta$ to be the subgroup of $G$ stabilising $\mathscr{U}_\theta$ and $M_\theta$ for every highest or lowest weight module $M$. By the previous discussion $\widetilde{P}_\theta$ contains $T$, $U_\theta$, $U_\theta^{\mathrm{pm}}$ and $U_\theta^{\mathrm{nm}}$. Set $\widetilde{N}_\theta = \widetilde{P}_\theta \cap N$.

The next result is a version of [38, Lemme 4.11]. The proof given here will consider only $\mathscr{U}_\theta$ but the argument is the same for $M_\theta$ a highest or lowest weight module. In what follows, if $g \in \widetilde{P}_\theta$ then write

$$_\nu|g|_\mu$$

for the restriction of $g$ to a $\mu$-weight space followed by the projection onto a $\nu$-weight space.

**Proposition 4.10.** *Let $\theta \in Y_T(\mathbb{K})$. Then*

$$\widetilde{P}_\theta \cap U^+N = U_\theta^{\mathrm{pm}+}\widetilde{N}_\theta \quad \textit{and} \quad \widetilde{P}_\theta \cap U^-N = U_\theta^{\mathrm{nm}-}\widetilde{N}_\theta.$$

*Proof.* Let $g = un \in \widetilde{P}_\theta \cap U^+N$ where $u \in U^+$ and $n \in N$ lifts some $w \in W$. The element $n$ sends a weight $\mu$ to a weight $w\mu$, so one can write

$$n = \bigoplus_\mu {}_{w\mu}|n|_\mu.$$

However, since $u$ acts upper-triangularly

$${}_{w\mu}|un|_\mu = {}_{w\mu}|n|_\mu$$

for all $\mu$, meaning $n$ stabilises $\mathscr{U}_\theta$ and thus $n \in \widetilde{N}_\theta$. Hence

$$\widetilde{P}_\theta \cap U^+N = (\widetilde{P}_\theta \cap U^+)\widetilde{N}_\theta.$$

It remains to show that $\widetilde{P}_\theta \cap U^+ = U_\theta^{\mathrm{pm}+}$. The inclusion $U_\theta^{\mathrm{pm}+} \subseteq \widetilde{P}_\theta \cap U^+$ follows from the discussion above. For the reverse inclusion, suppose that $u \in \widetilde{P}_\theta \cap U^+$. It suffices to show that $u \in U_\theta^{\mathrm{ma}+}$, then $u \in U^+ \cap U_\theta^{\mathrm{ma}+} = U_\theta^{\mathrm{pm}+}$.

Using the fact that $u \in U^+ \subseteq U^{\mathrm{ma}+}$ one can write

$$u = \prod_{\alpha \in \Delta^+} u_\alpha \quad \text{where} \quad u_\alpha = [\exp] t_{\alpha 1} e_{\alpha 1} \cdots [\exp] t_{\alpha l} e_{\alpha l}$$

for some basis $e_{\alpha 1}, \ldots, e_{\alpha l}$ of $\mathfrak{g}_{\alpha\mathbb{Z}}$ and $t_{\alpha 1}, \ldots, t_{\alpha l} \in k$, where $[\exp]$ denotes the *twisted exponential* of [38, §2.8]. Moreover, assume that the product is ordered such that the height of $\alpha$ increases from right to left. This generalises the usual decomposition of the unipotent radical of a Borel subgroup in a reductive group via its simple root elements. The remainder of the proof does not make use of any specific properties of the decomposition, so the details are deferred to *loc. cit.*

It will be proved by induction on the height that $\langle \theta, \alpha \rangle \geq 0$ for all $u_\alpha \neq 1$ in the product. In particular, assume that $\langle \theta, \beta \rangle \geq 0$ for all $\beta \in \Delta^+$ with $\mathrm{ht}(\beta) < \mathrm{ht}(\alpha)$. If $u_\beta$ is a factor to the right of $u_\alpha$ then $u_\beta \in U_\theta^{\mathrm{pm}+}$ by assumption, so suppose that $u_\beta = 1$ without loss of generality. Henceforth, one has

$$u = \left( \prod_{\substack{\beta \neq \alpha \\ \mathrm{ht}(\beta) \geq \mathrm{ht}(\alpha)}} u_\beta \right) u_\alpha$$

and also ${}_\alpha|u|_0 = {}_\alpha|u_\alpha|_0$. Suppose for a contradiction that $\langle \theta, \alpha \rangle < 0$. Pick some $h$ in the $\mathbb{Z}$-basis of $\mathfrak{h}_\mathbb{Z}$ such that $\alpha(h) = m \neq 0$. Up to replacing $h$ with $-h$ one can assume that $m > 0$. A direct computation shows that the $\alpha$-weight component of $\mathrm{Ad}(u_\alpha)\binom{h}{n}$ is

$$\sum_{i=1}^{l} t_{\alpha i} \, \mathrm{ad}(e_{\alpha i}) \binom{h}{n} = -\sum_{i=1}^{l} t_{\alpha i} e_{\alpha i} \left( \sum_{q=0}^{n-1} \binom{h}{q} \binom{\alpha(h)}{n-q} \right)$$

where $n \in \mathbb{Z}_{\geq 0}$. For more details, see [38, Lemme 4.11].

However, the $e_{\alpha i}\binom{h}{q}$ for $1 \leq i \leq l$ and $q \in \mathbb{Z}_{\geq 0}$ belong to a $\mathbb{Z}$-basis of $\mathscr{U}_\mathbb{Z}$ which means that each such $e_{\alpha i}\binom{h}{q}$ must lie in the $\alpha$-weight component. In particular, when $n = m$ and $q = 0$ one obtains a factor

$$-t_{\alpha i}e_{\alpha i}\binom{h}{0}\binom{m}{m} = -t_{\alpha i}e_{\alpha i} \in (\mathscr{U}_\theta)_\alpha$$

But by assumption one has $\langle\theta,\alpha\rangle < 0$, meaning there is no $\alpha$-weight space in $\mathscr{U}_\theta$. It follows that $t_{\alpha i} = 0$ and therefore $u_\alpha = 1$ as required. Altogether, one obtains $u \in G \cap U_\theta^{\mathrm{ma}+} = U_\theta^{\mathrm{pm}+}$ which gives the result. An analogous argument can be made for the negations. □

Define the subgroup

$$P_\theta = \langle \widetilde{P}_\theta, N_\theta \rangle.$$

In the end, it will be shown that $P_\theta$ coincides with the standard quasi-parabolic subgroup $Q_\theta$ relative to $T$.

The next result uses the strategy of [20, Remarks 3.7] to obtain a useful decomposition of $P_\theta$, replacing the Iwasawa decomposition of *op. cit.* with the Billig-Dyer decomposition of Proposition 4.5.

**Lemma 4.11.** *Let $\theta \in Y_T(\mathbb{K})$. Then*

$$P_\theta = U_\theta^+ U_\theta^{\mathrm{nm}-} N_\theta = U_\theta^{\mathrm{pm}+} U_\theta^- N_\theta.$$

*Proof.* It follows from $U_\theta^{\mathrm{pm}+} \subseteq \widetilde{P}_\theta$ that $U_\Psi \subseteq \widetilde{P}_\theta$ for some set of positive roots $\Psi \subseteq \Delta_{\mathrm{re}}$. Therefore, the Billig-Dyer decompositions

$$G = U^+ N U_\Psi = U^- N U_\Psi$$

of Proposition 4.5 imply that

$$\widetilde{P}_\theta = (\widetilde{P}_\theta \cap U^+ N) U_\Psi = (\widetilde{P}_\theta \cap U^- N) U_\Psi.$$

By Proposition 4.10 this becomes

$$\widetilde{P}_\theta = U_\theta^{\mathrm{pm}+} \widetilde{N}_\theta U_\Psi = U_\theta^{\mathrm{nm}-} \widetilde{N}_\theta U_\Psi.$$

But $\widetilde{N}_\theta$ normalises $U_\theta \supseteq U_\Psi$ and therefore

$$\widetilde{P}_\theta = U_\theta^{\mathrm{pm}+} U_\theta^- \widetilde{N}_\theta = U_\theta^{\mathrm{nm}-} U_\theta^+ \widetilde{N}_\theta$$

where Proposition 4.8 has been used to write $U_\theta = U_\theta^+ U_\theta^- N_\theta^u = U_\theta^- U_\theta^+ N_\theta^u$. Now $N_\theta$ contains $\widetilde{N}_\theta$ and normalises $\widetilde{P}_\theta$, $U_\theta^{\mathrm{pm}+}$, $U_\theta^{\mathrm{nm}-}$, $U_\theta^+$, and $U_\theta^-$, giving the result. □

The next result follows immediately from the previous two, using an argument analogous to that of [2, §2.4].

**Proposition 4.12.** *Let $\theta \in Y_T(\mathbb{K})$. Then*

$$P_\theta = U_\theta^+ U_\theta^- N_\theta = U_\theta^- U_\theta^+ N_\theta.$$

*In particular, $U_\theta^+ = U_\theta^{\mathrm{pm}+}$ and $U_\theta^- = U_\theta^{\mathrm{nm}-}$.*

*Proof.* By Lemma 4.11 one has

$$P_\theta = U_\theta^{\mathrm{pm}+} N_\theta U_\theta^- = U_\theta^{\mathrm{nm}-} N_\theta U_\theta^+ = U_\theta^+ N_\theta U_\theta^{\mathrm{nm}-}$$

where the third equality follows from the third expression using the map $g \mapsto g^{-1}$. Then

$$P_\theta \cap U^+ T U^- = U_\theta^{\mathrm{pm}+} T U_\theta^- = U_\theta^+ T U_\theta^{\mathrm{nm}-}$$

using the second and fourth expressions above. The Birkhoff decomposition is unique on the subset $U^+TU^-$ which implies that $U_\theta^+ = U_\theta^{\mathrm{pm}+}$ and $U_\theta^- = U_\theta^{\mathrm{nm}-}$. The result now follows. □

Note that Proposition 4.8 and Proposition 4.12 imply that

$$U_\Psi = U_\Psi^+ U_\Psi^- = U_\Psi^{\mathrm{pm}+} U_\Psi^{\mathrm{nm}-}$$

for any set of positive roots $\Psi \subseteq \Delta_{\mathrm{re}}$. The first equality is precisely [7, Theorem 3]. Finally, the goal of the subsection is proved.

**Corollary 4.13.** *Let $\theta \in Y_T(\mathbb{K})$. Then $P_\theta = Q_\theta$.*

*Proof.* The inclusion $Q_\theta \subseteq P_\theta$ is immediate. For the reverse inclusion, observe that $P_\theta$ is generated by $U_\theta^+$, $U_\theta^-$, and $N_\theta$ by Proposition 4.12. Therefore $P_\theta \subseteq Q_\theta$ as required. □

The following consequence of Proposition 4.12 will be useful in the sequel.

**Corollary 4.14.** *Let $\theta \in Y_T(\mathbb{K})$. Then $N \cap Q_\theta = N_\theta$.*

*Proof.* By Proposition 4.12 and Corollary 4.13 an element $g \in N \cap Q_\theta$ can be written as

$$g = n = u_+ u_- n_\theta$$

where $n \in N$, $n_\theta \in N_\theta$, $u_+ \in U_\theta^+$, and $u_- \in U_\theta^-$. But then

$$u_+^{-1} n n_\theta^{-1} = u_- \in U^+ N \cap U^-$$

where the intersection is trivial by the Birkhoff decomposition. It follows that $u_- = 1$. The Birkhoff decomposition also implies that $U^+ \cap N = 1$ from which one deduces

$$u_+ = u_- = 1 \quad \text{and} \quad n = n_\theta$$

as required. □

*Remark* 4.15. The decomposition for the subgroup $P_\theta$ should be compared with the decompositions of parahoric subgroups associated to filters of subsets in $Y_T(\mathbb{R})$. Specifically, when masures were first defined in [20] it was unclear whether such subgroups would have nice decompositions that would allow them to be studied effectively. In other words, it was unclear whether parahoric subgroups were *good fixators* in the sense of *op. cit.* It was known in the case where the filter was a point, which is analogous to the context here. Recently, this has been resolved and it is known that all filters have good fixators [24, Theorem 6.6.7].

## 4.2 Constructing citadels

In this section, a twisted building is defined for every conjugacy class of quasi-Borel subgroup in $G$. Specifically, to every set of positive roots $\Psi \subseteq \Delta_{\mathrm{re}}$ under a certain finiteness restriction, one defines a Kac-Moody subgroup $G_\Psi$ whose Weyl group is the reflection subgroup $W_\Psi$. The $G_\Psi$ define buildings which are related to the keep $G/B$, for some fixed Borel subgroup $B$, using Billig-Dyer decompositions. Afterwards, the twisted buildings are bundled together to produce two citadels associated to $G$ which are naturally twinned.

### 4.2.1 A right-analogue of the Billig-Dyer decomposition

The aim of this subsection is to extend the work of Billig-Dyer to include a right-analogue of Proposition 4.5. As mentioned previously, this will provide the axioms **(T'2)**(b) and **(T'3)**(b) for the twisted buildings constructed in the sequel.

Let $\Pi_\Psi = \{\alpha \in \Delta^+ \mid s_\alpha \in S_\Psi\}$ be the standard set of simple roots associated to $W_\Psi$. In what follows, write

$$\alpha_t = \Psi \cap \{\alpha, -\alpha\}$$

for all $t = s_\alpha \in S_\Psi$. Therefore $\alpha_t$ is the unique root in $\Psi$ representing $t$. Note that if $\Psi$ is of positive sign then $\Pi_\Psi \subseteq \Psi$ and $\alpha_t = \alpha$ for all $t = s_\alpha \in S_\Psi$ with $\alpha \in \Pi_\Psi$.

**Lemma 4.16.** *Let $G$ be a Kac-Moody group with maximal torus $T$. Suppose that $\Psi$ is a set of positive roots of $\Delta$. Then*

$$U_\Psi = U_{\alpha_t}(U_\Psi \cap t^{-1}U_\Psi t) = (U_\Psi \cap t^{-1}U_\Psi t)U_{\alpha_t}$$

*for all $t \in S_\Psi$.*

*Proof.* By [7, Theorem 3] and Proposition 4.12 one may write

$$U_\Psi = U_\Psi^+ U_\Psi^- = U_\Psi^{\mathrm{pm}+} U_\Psi^{\mathrm{nm}-}$$

where $U_\Psi^{\mathrm{pm}+} = U_\theta^{\mathrm{pm}+}$ for some $\theta \in Y_T(\mathbb{K})$ defining $\Psi$, and similarly for $U_\Psi^{\mathrm{nm}-}$. Moreover, by Proposition 4.12

$$U_\Psi^+ = U_\Psi^{\mathrm{pm}+} \quad \text{and} \quad U_\Psi^- = U_\Psi^{\mathrm{nm}-}.$$

Suppose that $t = s_\alpha \in S_\Psi$. Without loss of generality, write $\alpha = \alpha_t$. Moreover, assume that $\alpha \in \Psi \cap \Delta^+$. The case for $\alpha \in \Psi \cap \Delta^-$ would be symmetric.

Since $\Psi$ and $\Delta^+$ are closed in $\Delta$, the intersection $A = \Psi \cap \Delta^+$ is closed in $\Delta^+$. Moreover, $A_0 = A \setminus \alpha$ is closed and an ideal in $A$. Indeed:

- To see closure, suppose that $\beta, \gamma \in A_0$ and $\beta + \gamma \in \Delta$. Then $\beta + \gamma \in A$ since $A_0 \subseteq A$ and $A$ is closed. It remains to show that $\beta + \gamma \neq \alpha$. Suppose otherwise. Then

$$t\beta, t\gamma \in \Psi \setminus \alpha \quad \text{and} \quad t\beta + t\gamma = t\alpha = -\alpha$$

  by the fact that $t(\Psi \setminus \alpha) = \Psi \setminus \alpha$ and $t(\alpha) = -\alpha$ since $t \in S_\Psi$. But then $-\alpha \in \Psi \setminus \alpha$ implies that $\alpha \notin \Psi$ which is a contradiction. Hence $\beta + \gamma \in A_0$ and thus $A_0$ is closed in $A$.

- For the ideal property, suppose that $\beta \in A_0$ and $\gamma \in A$ and $\beta + \gamma \in \Delta$. One must show that $\beta + \gamma \in A_0$. Note that $\beta + \gamma \in A$, so it suffices to show that $\beta + \gamma \neq \alpha$. Suppose otherwise. Then $\gamma \neq \alpha$ which implies $\gamma \in A_0$. The rest of the argument follows the previous bullet.

The inclusions $A_0 \subseteq A \subseteq \Delta^+$ applied to [38, Lemme 5] or [31, Lemma 8.58] give

$$U^{\mathrm{ma}+}(A) = U_\alpha \ltimes U^{\mathrm{ma}+}(A_0) = U^{\mathrm{ma}+}(A_0) \rtimes U_\alpha$$

in the maximal Kac-Moody group $G^{\mathrm{ma}+}$. Hence

$$U_\Psi^+ = U^{\mathrm{pm}+}(A) = G \cap (U_\alpha \ltimes U^{\mathrm{ma}+}(A_0)).$$

Write $g \in U_\Psi^+$ as $g = u_\alpha u'$ where $u_\alpha \in U_\alpha$ and $u' \in U^{\mathrm{ma}+}(A_0)$. Then

$$u' = u_\alpha^{-1} g \in G \cap U^{\mathrm{ma}+}(A_0).$$

But $A_0 = t\Psi \cap \Delta^+$ and therefore Proposition 4.12 implies that

$$U_\Psi^+ = U_\alpha \ltimes (G \cap U^{\mathrm{ma}+}(A_0)) = U_\alpha \ltimes U_{t\Psi}^+.$$

Substituting this into the original equation, one obtains

$$U_\Psi = U_\alpha U_{t\Psi}^+ U_\Psi^-$$

Note that $t(U_{t\Psi}^+ U_\Psi^-)t^{-1} \subseteq U_\Psi$ and

$$U_{t\Psi}^+ = U_{t\Psi}^{\mathrm{pm}+} = G \cap U^{\mathrm{ma}+}(A_0) \subseteq G \cap U^{\mathrm{ma}+}(A) = U_\Psi^{\mathrm{pm}+} = U_\Psi^+$$

therefore

$$U_{t\Psi}^+ U_\Psi^- \subseteq U_\Psi \cap t^{-1} U_\Psi t.$$

It follows that

$$U_\Psi \subseteq U_\alpha (U_\Psi \cap t^{-1} U_\Psi t)$$

and the reverse inclusion is immediate. The decomposition in which $U_\alpha$ appears on the right is proved in an analogous way. □

The following result should be considered as an extension of Proposition 4.5.

**Proposition 4.17.** *Let $G$ be a Kac-Moody group with maximal torus $T$. Suppose that $B$ is the standard positive Borel subgroup of $G$ relative to $T$ and that $Q = Q_\Psi$ is a standard quasi-Borel subgroup of $G$ relative to $T$. Then, for all $t \in S_\Psi$ and $w \in W$, one has*

$$BwtQ^0 \subseteq BwQ^0tQ^0 \subseteq BwQ^0 \cup BwtQ^0.$$

*Moreover, if $l_\Psi(wt) > l_\Psi(w)$ then $BwQ^0tQ^0 = BwtQ^0$.*

*Proof.* The proof follows standard arguments, given here for completeness. Fix $t = s_\alpha \in S_\Psi$. Without loss of generality, write $\alpha = \alpha_t \in \Psi$. By Lemma 4.16, one may write $Q^0 = U_\alpha Q'_\alpha T$

where $Q'_\alpha$ is normalised by $t$. Then

$$BwQ^0tQ^0 = BwU_\alpha tQ'_\alpha Q^0 = BwU_\alpha tQ^0 = BU_{w(\alpha)}wtQ^0.$$

If $U_{w(\alpha)} \subseteq B$ then $BwQ^0tQ^0 = BwtQ^0$. So assume otherwise and consider $u \in U_\alpha$. Then $u,t \in \langle U_\alpha, U_{-\alpha}\rangle$ and therefore $ut = u_-u_+h$ where $u_- \in U_{-\alpha}$, $u_+ \in U_\alpha$, and $h \in T$. This follows from the Bruhat decomposition in the semisimple rank 1 group $\langle U_\alpha, U_{-\alpha}\rangle$ (cf. [25, §28.2]). If $u$ is the identity element then $BwutQ^0 = BwtQ^0$, so assume otherwise. Then

$$BwutQ^0 = Bwu_-u_+hQ^0 = Bwu_-Q^0.$$

But $U_{w(\alpha)} \not\subseteq B$ by assumption, hence $U_{w(-\alpha)} \subseteq B$ and $BwutQ^0 = BwQ^0$.

For the final statement, recall from Remark 3.5 that $l_\Psi(wt) > l_\Psi(w)$ is equivalent to $t \notin \mathscr{N}(w^{-1}) + A_\Psi$. If $\alpha \in \Delta^+$ then $\alpha \in \Psi \cap \Delta^+$ and thus $t \notin A_\Psi$. It follows that $t \notin \mathscr{N}(w^{-1})$ which implies that $w(\alpha) \in \Delta^+$. On the other hand, if $\alpha \in \Delta^-$ then $-\alpha \in (-\Psi) \cap \Delta^+$ which means $t \in A_\Psi$. Therefore $t \notin \mathscr{N}(w^{-1}) + A_\Psi$ implies that $t \in \mathscr{N}(w^{-1})$. But then $w(-\alpha) \in \Delta^-$, or equivalently $w(\alpha) \in \Delta^+$, as required. □

### 4.2.2 The twisted buildings

This subsection will construct a collection of twisted buildings that arise naturally from quasi-Borel subgroups in $G$. The construction is a generalisation of the usual way in which one associates a twin building to a pair of opposite standard Borel subgroups.

Throughout, fix a maximal torus $T$ of $G$. Let $B$ be the standard positive Borel subgroup of $G$ relative to $T$. The Bruhat decomposition of $G$ with respect to $B$ defines the $W$-valued map

$$\delta : G/B \times G/B \to W, \quad \delta(gB, hB) = w \text{ if } g^{-1}h \in BwB.$$

It is well-known that $(G/B, \delta)$ is a building of type $(W,S)$. This building will be the keep of the positive citadel associated to $G$, denoted henceforth by $\mathscr{B}$.

Set $\Delta = \Delta(G,T)$ and $\Delta_\Psi = \{\alpha \in \Delta \mid s_\alpha \in W_\Psi\}$. In particular, $\Delta_\Psi$ is the root system associated to $W_\Psi$. Define the subgroups

$$G_\Psi = \langle N_\Psi, U_\alpha \mid \alpha \in \Delta_\Psi \cap \Delta_{\mathrm{re}} \rangle,$$
$$G_\Psi^0 = \langle T, U_\alpha \mid \alpha \in \Delta_\Psi \cap \Delta_{\mathrm{re}} \rangle,$$

for every set of positive roots $\Psi \subseteq \Delta_{\mathrm{re}}$. Since $N_\Psi \subseteq G_\Psi$ normalises $G_\Psi^0$ one has

$$G_\Psi = G_\Psi^0 N_\Psi.$$

From this point onwards, assume that $S_\Psi$ is finite for every set of positive roots $\Psi \subseteq \Delta$. This becomes a genuine restriction when $W$ is of indefinite type, in which case $S_\Psi$ may be infinite. The restriction implies that $G_\Psi^0$ is a Kac-Moody subgroup of $G$ with Weyl group $W_\Psi$.

*Remark* 4.18. The finiteness restriction is not necessary in the construction of abstract citadels, since the theory of buildings allows Coxeter systems of infinite rank. It is plausible that the finiteness restriction on $S_\Psi$ can be removed in the group-theoretic setting (cf. [21, Theorem 1]). However, doing so would involve a systematic treatment of groups associated to generalised Cartan matrices of infinite rank. This is beyond the scope of the thesis and will be explored in future work.

**Lemma 4.19.** *Let $\Psi$ be an aligned set of positive roots in $\Delta$. Then*

$$G_\Psi \cap Q_\Psi = (G_\Psi^0 \cap Q_\Psi^0) N_\Psi$$

*where $\overline{B}_\Psi^0 = G_\Psi^0 \cap Q_\Psi^0$ is a standard Borel subgroup of $G_\Psi^0$ relative to $T$. In particular, if $\Psi$ is of positive sign then*

$$\overline{B}_\Psi^0 = \langle T, U_\alpha \mid \alpha \in \Pi_\Psi \rangle.$$

*Proof.* If $S_\Psi$ is empty then $W_\Psi$ is trivial and $G_\Psi = N_\Psi$. In this case,

$$G_\Psi \cap Q_\Psi = N_\Psi \quad \text{and} \quad G_\Psi^0 \cap Q_\Psi^0 = T$$

by Corollary 4.14 which gives the result. Henceforth, assume that $S_\Psi$ is non-empty. Replacing $\Psi$ by $-\Psi$ and the corresponding Borel subgroup by its opposite if necessary, it suffices to treat the case that $\Psi$ is of positive sign. Hence $\Pi_\Psi \subseteq \Psi$. Define the subgroup

$$\overline{B}^0_\Psi = \langle T, U_\alpha \mid \alpha \in \Pi_\Psi \rangle.$$

Set $\overline{B}_\Psi = \overline{B}^0_\Psi N_\Psi$. The inclusion $G_\Psi \cap Q_\Psi \supseteq \overline{B}_\Psi$ is immediate. For the opposite inclusion $G_\Psi \cap Q_\Psi \subseteq \overline{B}_\Psi$ pick an element $g \in G_\Psi \cap Q_\Psi$. So $g = hn$ for some $h \in G^0_\Psi$ and $n \in N_\Psi$. Observe that $\overline{B}^0_\Psi$ is the standard positive Borel subgroup of $G^0_\Psi$ relative to $T$. Therefore, $h = bn'b'$ where $b, b' \in \overline{B}^0_\Psi$ and $n' \in N_{G^0_\Psi}(T)$ represents some $w \in W_\Psi$. By the first inclusion and the fact that $h = gn^{-1} \in Q_\Psi$, one has

$$n' = b^{-1}h(b')^{-1} \in G_\Psi \cap Q_\Psi.$$

In particular,

$$n' \in N_{G^0_\Psi}(T) \cap Q_\Psi = G^0_\Psi \cap N_\Psi$$

by Corollary 4.14. It follows that $w \in W_\Psi \cap \mathrm{Stab}_W(\Psi)$. But then

$$\mathscr{N}_\Psi(w) = A_\Psi + wA_\Psi = A_\Psi + A_\Psi = \emptyset$$

implies $w = 1$ and thus $n' \in T$. This implies $h \in \overline{B}^0_\Psi$ which concludes the proof. □

Henceforth, assume that $\Psi$ is a set of positive roots in $\Delta$ of positive sign. The set $G^0_\Psi/\overline{B}^0_\Psi$ can be endowed with a building structure of type $(W_\Psi, S_\Psi)$ in the same way as above. In particular, there is a $W_\Psi$-valued map

$$\delta_\Psi : G^0_\Psi/\overline{B}^0_\Psi \times G^0_\Psi/\overline{B}^0_\Psi \to W_\Psi, \quad \delta_\Psi(g\overline{B}^0_\Psi, h\overline{B}^0_\Psi) = w \text{ if } g^{-1}h \in \overline{B}^0_\Psi w \overline{B}^0_\Psi$$

that is well-defined by the Bruhat decomposition on $G^0_\Psi$. Note that $g, h \in G^0_\Psi$.

Set $\overline{B}_\Psi = \overline{B}^0_\Psi N_\Psi$. The inclusion $G^0_\Psi \subseteq G_\Psi$ induces a surjective map

$$G^0_\Psi/\overline{B}^0_\Psi \to G_\Psi/\overline{B}_\Psi, \quad g\overline{B}^0_\Psi \mapsto g\overline{B}_\Psi$$

using the fact that $G_\Psi = G^0_\Psi N_\Psi$ and $N_\Psi \subseteq \overline{B}_\Psi$. Moreover, the map is injective. Indeed, if $g, h \in G^0_\Psi$ and $g\overline{B}_\Psi = h\overline{B}_\Psi$ then

$$g^{-1}h \in G^0_\Psi \cap \overline{B}_\Psi = G^0_\Psi \cap \overline{B}^0_\Psi N_\Psi = \overline{B}^0_\Psi$$

where the final equality follows from $G^0_\Psi \cap N_\Psi = T \subseteq \overline{B}^0_\Psi$ which is implicit in the proof of Lemma 4.19.

The inverse bijection $\phi$ of the map above identifies the coset spaces

$$\phi : G_\Psi/\overline{B}_\Psi \xrightarrow{\sim} G^0_\Psi/\overline{B}^0_\Psi.$$

By transport of structure, $\phi$ defines a building on the set $G_\Psi/\overline{B}_\Psi$ with a $W_\Psi$-valued map

$$\delta_\Psi : G_\Psi/\overline{B}_\Psi \times G_\Psi/\overline{B}_\Psi \to W_\Psi, \quad \delta_\Psi(g\overline{B}_\Psi, h\overline{B}_\Psi) = w \text{ if } g_0^{-1}h_0 \in \overline{B}^0_\Psi w \overline{B}^0_\Psi$$

where $\phi(g\overline{B}_\Psi) = g_0\overline{B}^0_\Psi$ and $\phi(h\overline{B}_\Psi) = h_0\overline{B}^0_\Psi$ for some $g_0, h_0 \in G^0_\Psi$.

The induced building structure on $G_\Psi/\overline{B}_\Psi$ will be denoted by $\mathscr{B}_\Psi$. This construction is analogous to the convention that the building of a non-connected reductive group is the building associated to its identity component. The key difference is that $G_\Psi$, and more specifically $N_\Psi$, acts on $\mathscr{B}_\Psi$ in ways that are not *type-preserving* (cf. [6, §7.9]). In particular, $N_\Psi$ may not fix $S_\Psi$ pointwise and therefore

$$\delta_\Psi(n \cdot c, n \cdot c') \neq \delta_\Psi(c, c')$$

for pairs $c, c' \in \mathscr{B}_\Psi$ in general.

**Example 4.20.** Let $G = \mathrm{GL}_3(k[t,t^{-1}]) \rtimes k^\times$. Suppose that $\Psi = \Psi_{\{\beta\}}$ in the notation of Example 3.7. From Example 4.4, one has

$$\mathrm{Stab}_W(\Psi) = \langle x \rangle \cong \mathbb{Z}$$

where $x = t_{\alpha^\vee} s_\beta$. But $W_\Psi = \langle s_\beta, s_{\delta-\beta} \rangle$ from Example 3.7. A direct computation shows that the conjugation action of $x$ on $(W_\Psi, S_\Psi)$ coincides with the diagram automorphism swapping the two nodes of $\tilde{A}_1$. In particular, for all $c, c' \in \mathscr{B}_\Psi$, one has

$$\delta_\Psi(c,c') = s_\beta \implies \delta_\Psi(n\cdot c, n\cdot c') = s_{\delta-\beta}$$

where $n \in N_\Psi$ is a lift of $x \in W$.

Define the map

$$i_\Psi : G^0_\Psi/\overline{B}^0_\Psi \to G/Q^0_\Psi, \quad i_\Psi(g\overline{B}^0_\Psi) = gQ^0_\Psi.$$

**Lemma 4.21.** *The map $i_\Psi$ is well-defined, injective, and $G^0_\Psi$-equivariant.*

*Proof.* The fact that $i_\Psi$ is well-defined follows from $\overline{B}^0_\Psi \subseteq Q^0_\Psi$. Indeed, if $g\overline{B}^0_\Psi = h\overline{B}^0_\Psi$ then $g^{-1}h \in \overline{B}^0_\Psi$. Hence

$$i_\Psi(g\overline{B}^0_\Psi) = gQ^0_\Psi = g(g^{-1}h)Q^0_\Psi = hQ^0_\Psi = i_\Psi(h\overline{B}^0_\Psi)$$

as required. To see that $i_\Psi$ is injective, suppose that $gQ^0_\Psi = hQ^0_\Psi$ for some $g, h \in G^0_\Psi$. Then

$$g^{-1}h \in G^0_\Psi \cap Q^0_\Psi = \overline{B}^0_\Psi$$

by Lemma 4.19. It follows that $g\overline{B}^0_\Psi = h\overline{B}^0_\Psi$ and $G^0_\Psi$-equivariance is immediate from the construction. □

The map

$$\widetilde{i}_\Psi = i_\Psi \circ \phi : G_\Psi/\overline{B}_\Psi \to G/Q^0_\Psi, \quad \widetilde{i}_\Psi(h\overline{B}_\Psi) = h_0 Q^0_\Psi \text{ where } \phi(h\overline{B}_\Psi) = h_0\overline{B}^0_\Psi$$

is well-defined, as the composition of well-defined maps $\phi$ and $i_\Psi$. In particular, $\mathscr{B}_\Psi$ can be identified with the subset

$$\widetilde{i}_\Psi(G_\Psi/\overline{B}_\Psi) = \{gQ^0_\Psi \mid g \in G^0_\Psi\} \subseteq G/Q^0_\Psi.$$

This identification allows one to define a $W$-valued map

$$\delta^* : G/B \times G_\Psi/\overline{B}_\Psi \to W, \quad \delta^*(gB, h\overline{B}_\Psi) = w \text{ if } g^{-1}h_0 \in BwQ^0_\Psi$$

where $\phi(h\overline{B}_\Psi) = h_0\overline{B}^0_\Psi$ as above. The map is well-defined by the disjoint Billig-Dyer decomposition of Proposition 4.5 and the previous discussion.

Before showing that $\delta^*$ is a twisting map one must prove the following technical lemma which allows one to pass from cosets involving $Q^0_\Psi$ to those involving $\overline{B}^0_\Psi$.

**Lemma 4.22.** *Let $\Psi$ be an aligned set of positive roots in $\Delta$. Then*

$$Q^0_\Psi t Q^0_\Psi = Q^0_\Psi \overline{B}^0_\Psi t \overline{B}^0_\Psi$$

*for all $t = s_\alpha \in S_\Psi$.*

*Proof.* Fix $t = s_\alpha \in S_\Psi$ and assume $\alpha = \alpha_t$. The inclusion $Q^0_\Psi \overline{B}^0_\Psi t \overline{B}^0_\Psi \subseteq Q^0_\Psi t Q^0_\Psi$ is immediate, since $\overline{B}^0_\Psi \subseteq Q^0_\Psi$. For the reverse inclusion, use Lemma 4.16 to write

$$Q^0_\Psi = Q'_\alpha U_\alpha T$$

where $Q'_\alpha = U_\Psi \cap t^{-1}U_\Psi t$. Then

$$\begin{aligned} Q^0_\Psi t Q^0_\Psi &= Q^0_\Psi t Q'_\alpha U_\alpha T, \\ &= Q^0_\Psi Q'_\alpha t U_\alpha T, \\ &= Q^0_\Psi t U_\alpha T, \\ &\subseteq Q^0_\Psi t \overline{B}^0_\Psi, \\ &\subseteq Q^0_\Psi \overline{B}^0_\Psi t \overline{B}^0_\Psi \end{aligned}$$

as required. □

Everything is now in place to complete the aim of this subsection and define a twisted building structure between $\mathscr{B}$ and $\mathscr{B}_\Psi$.

**Proposition 4.23.** *The triple* $(\mathscr{B}, \mathscr{B}_\Psi, \delta^*)$ *is a twisted building of type* $(W, S, \Psi)$.

*Proof.* By the discussion above, $\mathscr{B}$ and $\mathscr{B}_\Psi$ are buildings of type $(W,S)$ and $(W_\Psi, S_\Psi)$ respectively. Observe that **(T'1)** is immediate from the definition of $\delta^*$. The rest of the proof follows standard arguments, outlined here for completeness.

For **(T'2)** pick some $s \in S$ and $w \in W$. Set $b' = gB$, $b = hB$ for some $g, h \in G$. Similarly, let $c \in \mathscr{B}_\Psi$ such that $\phi(c) = k\overline{B}^0_\Psi$ and $\widetilde{\iota}_\Psi(c) = kQ^0_\Psi$ for some $k \in G^0_\Psi$. Suppose that $\delta(b', b) = s$ and $\delta^*(b, c) = w$. Therefore $g^{-1}h \in BsB$ and $h^{-1}k \in BwQ^0_\Psi$ respectively. Then

$$g^{-1}k = (g^{-1}h)(h^{-1}k) \in BsBwQ^0_\Psi.$$

If $l_\Psi(sw) > l_\Psi(w)$ then $BsBwQ^0_\Psi = BswQ^0_\Psi$ by Proposition 4.5, hence $\delta^*(b', c) = sw$. This gives the first part of **(T'2)**. The remaining case follows from the analogous argument using Proposition 4.17. In particular, pick some $t = s_\alpha \in S_\Psi$ and $w \in W$. Set $b = gB$ for some $g \in G$, and let $c \in \mathscr{B}_\Psi$ with $\phi(c) = h\overline{B}^0_\Psi$ and $\widetilde{\iota}_\Psi(c) = hQ^0_\Psi$ for some $h \in G^0_\Psi$. Suppose that $\delta^*(b, c) = w$, so $g^{-1}h \in BwQ^0_\Psi$. Let $c' \in \mathscr{B}_\Psi$ with $\phi(c') = k\overline{B}^0_\Psi$ and $\widetilde{\iota}_\Psi(c') = kQ^0_\Psi$ for some $k \in G^0_\Psi$. Suppose that $\delta_\Psi(c, c') = t$, so $h^{-1}k \in \overline{B}^0_\Psi t \overline{B}^0_\Psi$. Then

$$g^{-1}k = (g^{-1}h)(h^{-1}k) \in BwQ^0_\Psi t \overline{B}^0_\Psi \subseteq BwQ^0_\Psi t Q^0_\Psi.$$

If $l_\Psi(wt) > l_\Psi(w)$ then $BwQ^0_\Psi t Q^0_\Psi = BwtQ^0_\Psi$ by Proposition 4.17, hence $\delta^*(b, c') = wt$.

For **(T'3)**, set $b = gB$ for some $g \in G$, and let $c \in \mathscr{B}_\Psi$ with $\phi(c) = h\overline{B}^0_\Psi$ and $\widetilde{\iota}_\Psi(c) = hQ^0_\Psi$ for some $h \in G^0_\Psi$. Suppose that $\delta^*(b, c) = w$, so $g^{-1}h \in BwQ^0_\Psi$. Substituting $w \mapsto sw$ into Proposition 4.5, one deduces that $g^{-1}h = xy$ for some $x \in BsB$ and $y \in BswQ^0_\Psi$. Set $b' = gxB$. Then

$$x^{-1}g^{-1} \cdot h = y \in BswQ^0_\Psi \implies \delta^*(b', c) = sw$$

and

$$x^{-1}g^{-1} \cdot g = x^{-1} \in BsB \implies \delta(b', b) = s.$$

This gives the first case of **(T'3)**. In the latter case, one makes the substitution $w \mapsto wt$ in Proposition 4.17. Then $g^{-1}h = xy$ for some $x \in BwtQ^0_\Psi$ and $y \in \overline{B}^0_\Psi t \overline{B}^0_\Psi$ by Lemma 4.22. Similar to before, let $c' \in \mathscr{B}_\Psi$ with $\phi(c') = gx\overline{B}^0_\Psi$ and $\widetilde{\imath}_\Psi(c') = gxQ^0_\Psi$. Note that

$$gx = hy^{-1} \in G^0_\Psi \cdot \overline{B}^0_\Psi t \overline{B}^0_\Psi \subseteq G^0_\Psi$$

so this is well-defined. Then

$$g^{-1} \cdot gx = x \in BwtQ^0_\Psi \implies \delta^*(b, c') = wt$$

and

$$h^{-1} \cdot gx = y^{-1} \in \overline{B}^0_\Psi t \overline{B}^0_\Psi \implies \delta_\Psi(c, c') = t.$$

$\square$

In particular, the twisted buildings of Proposition 4.23 are positively twisted. It follows from the result that the canonical twin building of $G$ is one of a much larger class of twisted buildings that can be associated to $G$ when it is not of finite type.

### 4.2.3 The citadel

In this subsection the twisted buildings of the previous subsection will be bundled together to form a citadel structure. The construction should be compared with the description of the thin citadel associated to a Coxeter system $(W, S)$ in Example 3.42.

If $Q_\Psi$ is a standard Borel subgroup then $G_\Psi = G$ and therefore $\mathscr{B}_\Psi = \mathscr{B}$ is $G$-stable. In general, the left multiplication action of $G$ on the coset space $G/Q_\Psi$ does not preserve $\widetilde{\imath}_\Psi(\mathscr{B}_\Psi)$. Instead, if $g \notin G_\Psi Q_\Psi$ then the image of $\widetilde{\imath}_\Psi(\mathscr{B}_\Psi)$ under multiplication by $g$ is a building defined by the cosets of $gQ_\Psi g^{-1}$.

The goal is to construct a single $G$-space that one can study. This can be achieved using the quotient space

$$\mathscr{B}(\Psi)^{\text{fine}} = G \times_{G_\Psi} \mathscr{B}_\Psi = G \times \mathscr{B}_\Psi / \sim$$

defined by $(g,x) \sim (gh, h^{-1} \cdot x)$ where $g \in G$, $x \in \mathscr{B}_\Psi$, and $h \in G_\Psi$. In the case where $Q_\Psi = B$ one retrieves $G/B$. In general, there is a $G$-equivariant bijection

$$\mathscr{B}(\Psi)^{\text{fine}} \xrightarrow{\sim} G/\overline{B}_\Psi$$

defined by the mapping $[g, h\overline{B}_\Psi] = [gh\overline{B}_\Psi]$.

However, this identification is too fine when considering the action of $G$. In particular, the stabiliser in $G$ of a point in $\mathscr{B}(\Psi)^{\text{fine}}$ is a Borel subgroup of some conjugate of $G_\Psi$ in $G$. Ideally, one would like the stabilisers of points to be in one-to-one correspondence with quasi-Borel subgroups of $G$.

The solution is to consider the local chamber sets $G_\Psi/\overline{B}_\Psi$ inside the coset space $G/Q_\Psi$ with an appropriate gluing. In order to complete this goal, the following technical lemma is proved.

**Lemma 4.24.** *Let $\Psi \subseteq \Delta_{\text{re}}$ be a set of positive roots. Then $G_\Psi Q_\Psi$ is a subgroup of $G$.*

*Proof.* First, observe that Proposition 4.12 implies that

$$Q^0_\Psi t \subseteq G^0_\Psi Q^0_\Psi \tag{4.1}$$

for all $t = s_\alpha \in S_\Psi$. Indeed, fix $t = s_\alpha \in S_\Psi$ and assume that $\alpha = \alpha_t$. If $q = u_\alpha q'$ for some $q \in Q^0_\Psi$ and $q' \in Q^0_\Psi \cap t^{-1} Q^0_\Psi t$ then

$$qt = u_\alpha q' t = u_\alpha t(t^{-1} q' t) \in G^0_\Psi Q^0_\Psi$$

as required.

In order for $G_\Psi Q_\Psi$ to be a subgroup it suffices to show that $Q_\Psi G_\Psi \subseteq G_\Psi Q_\Psi$. Recall that

$$G_\Psi = G^0_\Psi N_\Psi, \quad Q_\Psi = Q^0_\Psi N_\Psi, \quad \text{and} \quad N_\Psi = G_\Psi \cap Q_\Psi.$$

Therefore, it is enough to check $Q_\Psi^0 G_\Psi^0 \subseteq G_\Psi^0 Q_\Psi^0$. By the Bruhat decomposition in $G_\Psi^0$, one has

$$G_\Psi^0 = \bigsqcup_{w \in W_\Psi} \overline{B}_\Psi^0 w \overline{B}_\Psi^0.$$

Since $\overline{B}_\Psi^0 \subseteq Q_\Psi^0$ the problem reduces to proving that $Q_\Psi^0 w \subseteq G_\Psi^0 Q_\Psi^0$ for all $w \in W_\Psi$. Suppose that $w = t_1 \cdots t_d$ is a reduced decomposition for $w \in W_\Psi$. Then repeated applications of (4.1) yields the result. □

In light of Lemma 4.24, define the subgroup $K_\Psi = G_\Psi Q_\Psi$.

**Proposition 4.25.** *The natural map*

$$G_\Psi/(G_\Psi \cap Q_\Psi) \to K_\Psi/Q_\Psi, \quad g(G_\Psi \cap Q_\Psi) \mapsto gQ_\Psi$$

*is a $G_\Psi$-equivariant bijection. In particular, there is a bijection*

$$\mathscr{B}_\Psi = G_\Psi/\overline{B}_\Psi \xrightarrow{\sim} K_\Psi/Q_\Psi.$$

*which induces an isomorphism*

$$\mathscr{B}(\Psi) = G \times_{K_\Psi} \mathscr{B}_\Psi \xrightarrow{\sim} G/Q_\Psi.$$

*Proof.* The first claim follows from (4.1)and the second follows from Lemma 4.19. □

It follows from Proposition 4.25 that the cosets $gQ_\Psi$ of $\mathscr{B}(\Psi)$ correspond to chambers in the building $\mathscr{B}_\Psi$ when $g \in K_\Psi$. The other buildings in $\mathscr{B}(\Psi)$ that correspond to different conjugates of $Q_\Psi$ in $G$, are enumerated by $G/K_\Psi$.

The $W$-metric from the keep to $\mathscr{B}(\Psi)$ is defined componentwise. Specifically, the twisting map between each building and the keep is obtained as the translate of the twisting map

$$\delta^* : \mathscr{B} \times \mathscr{B}_\Psi \to W$$

defined above.

Define the disjoint union

$$\mathscr{C}^+ = \bigsqcup_{\Omega} \mathscr{B}(\Psi)$$

running over the $W$-orbits $\Omega$ of sets of admissible positive roots $\Psi \subseteq \Delta_{\mathrm{re}}$ of positive sign. This is the underlying space for the citadel associated to $G$. Write $\mathscr{B}^+ = \mathscr{B}$. The next result is immediate.

**Proposition 4.26.** *The triple $(\mathscr{C}^+, \mathscr{B}^+, \delta^*)$ is a citadel of type $(W,S)$.*

The next example illustrates a citadel structure of an affine Kac-Moody group.

**Example 4.27.** Let $G = \mathrm{GL}_3(k[t,t^{-1}]) \rtimes k^\times$. Then $G$ is type $\tilde{A}_2$. From Example 3.2, there are four $W$-orbits of sets of positive roots of positive sign in $\Delta$.

The subgroups $Q_\Psi$ for the representatives $\Psi = \Psi_J$ are

$$B^+, \begin{pmatrix} k[t] & * & * \\ & k[t] & k[t] \\ & tk[t] & k[t] \end{pmatrix}, \begin{pmatrix} k[t] & k[t] & * \\ tk[t] & k[t] & * \\ & & k[t] \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} k[t] & & \\ & k[t] & \\ & & k[t] \end{pmatrix}.$$

The subgroups $G_\Psi$ are, respectively

$$G, \begin{pmatrix} * & & \\ & * & * \\ & * & * \end{pmatrix}, \begin{pmatrix} * & * & \\ * & * & \\ & & * \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}.$$

The subgroups $\overline{B}_\Psi$ are, respectively

$$B^+, \begin{pmatrix} k[t] & & \\ & k[t] & k[t] \\ & tk[t] & k[t] \end{pmatrix}, \begin{pmatrix} k[t] & k[t] & \\ tk[t] & k[t] & \\ & & k[t] \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} k[t] & & \\ & k[t] & \\ & & k[t] \end{pmatrix}.$$

From this, it is straightforward to see that the buildings $\mathscr{B}_\Psi$ are of type $\tilde{A}_2$, $\tilde{A}_1$, $\tilde{A}_1$, and trivial respectively. The corresponding subgroups $K_\Psi$ are, respectively

$$G, \begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix}, \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}.$$

The construction described above involves only sets of positive roots of positive sign. By an analogous process, one can define a second citadel

$$\mathscr{C}^- = \bigsqcup_{\Omega} \mathscr{B}(-\Psi)$$

running over the same collection $\Omega$ of $W$-orbits of sets of admissible positive roots $\Psi \subseteq \Delta_{\mathrm{re}}$. Therefore

$$\mathscr{B}(-\Psi) \simeq G/Q_{-\Psi}.$$

The keep of $\mathscr{C}^-$ is the typical negative building

$$\mathscr{B}^- = G/B^-$$

where $B^-$ is the Borel subgroup of $G$ opposite $B = B^+$, relative to $T$. Note that in this case, one uses an analogue of the Billig-Dyer decomposition where $B^+$ is replaced by $B^-$.

Observe that

$$W_\Psi = W_{-\Psi} \quad \text{and} \quad G_\Psi = G_{-\Psi}$$

which means that $\overline{B}^0_\Psi$ and $\overline{B}^0_{-\Psi}$ are opposite standard Borel subgroups of $G^0_\Psi$. Therefore, the Birkhoff decomposition

$$G^0_\Psi = \bigsqcup_{w \in W_\Psi} \overline{B}^0_{\varepsilon\Psi} w \overline{B}^0_{-\varepsilon\Psi}, \quad \varepsilon \in \{+,-\}$$

defines a twinning

$$\delta^*_\Psi : G^0_\Psi/\overline{B}^0_{\varepsilon\Psi} \times G^0_\Psi/\overline{B}^0_{-\varepsilon\Psi} \to W_\Psi, \quad \delta^*_\Psi(g\overline{B}^0_{\varepsilon\Psi}, h\overline{B}^0_{-\varepsilon\Psi}) = w \text{ if } g^{-1}h \in \overline{B}^0_{\varepsilon\Psi} w \overline{B}^0_{-\varepsilon\Psi}.$$

By transport of structure, $\phi$ defines a building $\mathscr{B}_{-\Psi}$ on the set $G_\Psi/\overline{B}_{-\Psi}$ with a $W_\Psi$-valued map

$$\delta^*_\Psi : G_\Psi/\overline{B}_{\varepsilon\Psi} \times G_\Psi/\overline{B}_{-\varepsilon\Psi} \to W_\Psi, \quad \delta^*_\Psi(g\overline{B}_{\varepsilon\Psi}, h\overline{B}_{-\varepsilon\Psi}) = w \text{ if } g_0^{-1}h_0 \in \overline{B}^0_{\varepsilon\Psi} w \overline{B}^0_{-\varepsilon\Psi}$$

where $\phi(g\overline{B}_{\varepsilon\Psi}) = g_0\overline{B}^0_{\varepsilon\Psi}$ and $\phi(h\overline{B}_{-\varepsilon\Psi}) = h_0\overline{B}^0_{-\varepsilon\Psi}$ for some $g_0, h_0 \in G^0_\Psi$.

In particular, the triples $(\mathscr{B}_\Psi, \mathscr{B}_{-\Psi}, \delta^*_\Psi)$ are twin buildings of type $(W_\Psi, S_\Psi)$. For example, the buildings $\mathscr{B}^+$ and $\mathscr{B}^-$ form the usual twin building associated to $G$. The next result is immediate.

**Proposition 4.28.** *The triple* $(\mathscr{C}^+, \mathscr{C}^-, (\delta^*_\Psi)_\Psi)$ *is a twin citadel of type* $(W,S)$.

In the sequel, this twin citadel will be denoted by $\mathscr{C} = \mathscr{C}_G$. Note that $G$ acts transitively on the chamber set of $\mathscr{B}(\Psi)$ for all $\Psi$. More precisely, $K_\Psi$ acts transitively on the chamber set of the distinguished building $\mathscr{B}_\Psi$, while $G/K_\Psi$ indexes the building components occurring in $\mathscr{B}(\Psi)$. Moreover, the action of $G$ on $\mathscr{C}$ permutes the building components of $\mathscr{C}$ and the $W$-metric on each component is obtained via transporting the local twisting map. In a similar fashion, the action of $K_\Psi$ on $\mathscr{B}_\Psi \cup \mathscr{B}_{-\Psi}$ preserves the $W_\Psi$-metrics $\delta_\Psi$ and $\delta^*_\Psi$.

*Remark* 4.29. The approach taken in this thesis to axiomatise the structures between coset spaces of the form $G/Q_\Psi$ has ignored maps of the form

$$G/Q_{\Psi_1} \times G/Q_{\Psi_2} \to W$$

where $\Psi_1$ and $\Psi_2$ are both non-standard and not the negations of one another. One reason for this choice is the fact that $G$ does not always possess a Bruhat decomposition with respect to $Q_{\Psi_1}$ and $Q_{\Psi_2}$. Specifically, one typically has

$$G \neq Q_{\Psi_1} W Q_{\Psi_2}.$$

Therefore, any map

$$\delta^* : G/Q_{\Psi_1} \times G/Q_{\Psi_2} \to W$$

for which $\delta^*(gQ_{\Psi_1}, hQ_{\Psi_2}) = w$ if $g^{-1}h \in Q_{\Psi_1} w Q_{\Psi_2}$ is not everywhere defined. Moreover, one cannot reduce to a smaller group, as in the case for the twinnings $\delta^*_\Psi$, because $G_{\Psi_1} \neq G_{\Psi_2}$.

This is illustrated in the following example. Let $G = \mathrm{GL}_2(k[t,t^{-1}]) \rtimes k^\times$. Suppose $Q$ is the upper-triangular subgroup of $G$. In particular, $Q$ corresponds to a chamber in $\mathscr{B}(\Psi)$ where $\Psi$ is the natural partition. A direct calculation shows

$$g = \begin{pmatrix} 1 & 0 \\ 1+t & 1 \end{pmatrix} \times 1 \notin QWQ.$$

It follows that $\delta^*(Q, gQ)$ is not defined. This can be seen as a failure of the *common apartment property* that has been observed in other building-like structures, e.g. [20] or [5]. It is reasonable to expect that one could refine the definition of a (twin) citadel to capture this phenomenon.

### 4.2.4 A dictionary between the group and the citadel

It is well-known that chambers of the twin keep $\mathscr{B}^+ \cup \mathscr{B}^-$ are in one-to-one correspondence with the Borel subgroups of $G$, and the residues correspond to parabolic subgroups of $G$. This subsection shows that, in an analogous way, the chambers of the twin citadel $\mathscr{C}_G$ correspond to quasi-Borel subgroups, while the residues correspond to quasi-parabolic subgroups.

Fix a set of positive roots $\Psi \subseteq \Delta_{\mathrm{re}}$ occurring in the construction of $\mathscr{C}_G$. Then $Q_\Psi$ is a quasi-Borel subgroup of $G$ and one has the identification

$$\mathscr{B}(\Psi) \simeq G/Q_\Psi.$$

From this, it is immediate that the stabiliser of a chamber $gQ_\Psi$ in $\mathscr{B}(\Psi)$ is the quasi-Borel subgroup $gQ_\Psi g^{-1}$. It follows that every quasi-Borel subgroup appears as the stabiliser of a chamber in $\mathscr{C}_G$.

Let $K \subseteq S_\Psi$. Suppose that $\overline{P}_K$ is the standard parabolic subgroup of $G^0_\Psi$ of type $K$ containing the Borel subgroup $\overline{B}^0_\Psi$. Then

$$Q_{\Psi,K} = \overline{P}_K Q_\Psi$$

is a subgroup of $K_\Psi$ by the argument of Lemma 4.24. In fact, it is a quasi-parabolic subgroup. It is straightforward to see that $Q_{\Psi,K}$ coincides with the stabiliser of the residue

$$R_K = \{pQ_\Psi \mid p \in \overline{P}_K\} \subseteq G/Q_\Psi$$

of type $K$ in $\mathscr{B}_\Psi$. Indeed, $R_K = \overline{P}_K Q_\Psi / Q_\Psi$ so $gR_K \subseteq R_K$ for all $g \in Q_{\Psi,K}$ is immediate, and the converse follows from considering $g^{-1}$. It follows that the stabiliser of any residue of type $K$ in $\mathscr{B}(\Psi)$ is a conjugate of $Q_{\Psi,K}$ and any such subgroup appears this way. A quasi-parabolic subgroup of this form, for an admissible $\Psi$, will be called *facial*.

The stabiliser of a chamber in the twin keep $\mathscr{B}^+ \cup \mathscr{B}^-$ is a Borel subgroup. For instance, the standard positive Borel subgroup $B$ containing $T$ corresponds to the fundamental chamber in $\mathscr{B}^+$ of the apartment associated to $T$. It is well-known that $B \cap N_G(T) = T$. In other words, if $g \in G$ fixes the fundamental chamber and the apartment associated to $T$ then it acts trivially on the apartment. This is no longer the case away from the twin keep. In particular, since

$$N \cap \overline{B}_\Psi = N_\Psi$$

by Corollary 4.14 there are non-trivial apartment automorphisms in $\mathscr{B}_\Psi$ that fix chambers.

When $G$ is reductive, the parabolic subgroups of $G$ are facial and self-normalising. This means that there is a correspondence between parabolic subgroups of $G$ and residues of the building associated to $G$. It is plausible that all quasi-parabolic subgroups of Kac-Moody groups are self-normalising and facial. If proved in the affirmative, the residues of $\mathscr{C}_G$ would be in one-to-one correspondence with quasi-parabolic subgroups.

**Conjecture 4.30.** *Let $G$ be a Kac-Moody group. If $Q$ is a quasi-parabolic subgroup of $G$ then $Q$ is facial and $N_G(Q) = Q$.*

There is a partial order on parabolic subgroups of $G$, defined by reverse inclusion. This induces a partial order $\preceq$ on the set of facial quasi-parabolic subgroups of $G$. In particular, if $Q_1$ and $Q_2$ are facial quasi-parabolic subgroups then write $Q_1 \preceq Q_2$ if $Q_1 = \overline{P}_1 Q_1'$ and $Q_2 = \overline{P}_2 Q_2'$ for some parabolic subgroups $\overline{P}_1$ and $\overline{P}_2$ of $G_\Psi$ with $\overline{P}_2 \subseteq \overline{P}_1$. As a poset, it is the disjoint union of the posets of parabolic subgroups, ordered by reverse inclusion, of each $G_\Psi$. The quasi-Borel subgroups of $G$ are the maximal elements in the poset. The minimal elements are facial quasi-parabolic subgroups of type $S_\Psi$ for some set of positive roots $\Psi$ in $\Delta$. As is the convention for buildings, one allows $S = S_\Psi$, meaning $G$ is a minimal element too.

The preceding discussion describes a simplicial approach to citadels. That is to say, a description of the (twin) citadel as the disjoint union of simplicial complexes. The maximal simplices correspond to chambers and quasi-Borel subgroups, while the minimal simplices correspond to the buildings $\mathscr{B}_\Psi$. The families $\mathscr{B}(\Psi)$ here correspond to orbits of minimal simplices under the action of $G$.

*Remark* 4.31. There is an obvious way in which one could define a vectorial model $\mathscr{V}_G$ of the twin citadel $\mathscr{C}_G$. The vectorial apartments would be the $\mathbb{K}$-vector spaces $Y_T(\mathbb{K})$ running over the maximal tori $T$ of $G$. The gluing operation would involve the quasi-parabolic subgroups $Q_\theta$, or perhaps a slightly smaller subgroup $Q'_\theta$ for which $N \cap Q'_\theta$ was the point stabiliser of $\theta \in Y_T(\mathbb{K})$. By construction every quasi-parabolic subgroup would be associated to a point of $\mathscr{V}_G$ in a natural way, not just those which are facial or associated to admissible sets of positive roots. This will be the focus of future work.

CHAPTER 5

# Complete reducibility

This chapter defines and studies the notion of complete reducibility in citadels. This leads to a new notion of complete reducibility for subgroups of Kac-Moody groups, distinct from current definitions in the literature, that faithfully generalises existing notions for connected and non-connected reductive algebraic groups. Once this is done, some basic consequences of the definition are established.

## 5.1 Background

Let $X$ be a spherical building. A convex subset $Y$ of $X$ is *completely reducible* if every residue in $Y$ has an opposite in $Y$. This definition is due to Serre [41]. Strictly speaking, the assumption of convexity is not necessary, but it will be true in every case in the sequel.

Suppose that $X = X(G)$ is the building associated to some connected reductive group $G$. Then $G$ acts strongly transitively on $X$. If $H$ is a subgroup of $G$ then the set of residues in $X$ stabilised by $H$ is denoted $X^H$. It is well-known that $X^H$ is convex. This fact enables one to study the complete reducibility of $X^H$ safely. In particular, $X^H$ is completely reducible in $X$ if and only if

$$H \subseteq P \text{ a parabolic subgroup} \implies H \subseteq L \text{ a Levi subgroup of } P. \tag{5.1}$$

A subgroup $H$ of $G$ with the property (5.1) is called *G-completely reducible*, or *G-cr* for short. Otherwise, say that $H$ is *non-G-cr*.

In his seminal paper, Serre proves a number of equivalent characterisations of complete reducibility. Most of these are recorded in the next result.

**Proposition 5.1.** *Let $X$ be a spherical building and $Y$ a convex subcomplex of $X$. Then the following are equivalent:*

1. *$Y$ is completely reducible in $X$.*
2. *$Y$ contains a pair of opposite residues with the same dimension as $Y$.*
3. *$Y$ contains a Levi sphere $S$ with the same dimension as $Y$.*
4. *Every vertex in $Y$ has an opposite in $Y$.*

Here, a *Levi sphere* is the convex hull of a pair of opposite residues in $X$. A helpful way to think about such objects comes from the geometric realisation of $X$. First, note that a Levi sphere is contained in some apartment of $X$, by minimality of the convex hull. When this apartment is viewed as a sphere in some Euclidean space, the Levi sphere is an intersection of this sphere with a vector subspace.

Let $S$ be the Levi sphere of $X$ defined by the residues $R$ and $R'$. It is well-known that there is an isomorphism $R \to R'$ of buildings given by the projection maps [1, Proposition 5.37]. Write $X_S$ for the building that $R$ defines in $X$. If $Y$ is a subcomplex of $X$ then write $Y_S$ for the corresponding subcomplex in $X_S$. The following result is due to Serre [41, Proposition 2.5].

**Theorem 5.2.** *Let $X$ be a spherical building and $Y$ a convex subcomplex of $X$. Suppose that $Y$ contains a Levi sphere $S$. Then $Y$ is completely reducible in $X$ if and only if $Y_S$ is completely reducible in $X_S$.*

In particular, the Levi spheres in $X$ provide a way to study complete reducibility locally. The corresponding result for subgroups of $G$ is recorded in the following corollary.

**Corollary 5.3.** *Let $G$ be a reductive group and $H$ a subgroup of $G$. Suppose $H$ is contained in a Levi subgroup $L$ of $G$. Then $H$ is $G$-cr if and only if $H$ is $L$-cr.*

A natural generalisation of a spherical building that preserves an opposition relation is a twin building. The corresponding definition for completely reducible subcomplexes of twin buildings follows the original definition verbatim. That is, a convex subcomplex $Y$ of $X$ is *completely reducible* if every residue in $Y$ contains an opposite in $Y$. This definition was first written down by Dawson [12]. Note that opposite residues in twin buildings lie in different components.

Let $X = X(G)$ be the twin building associated to a Kac-Moody group. Then $X^H$ is completely reducible in $X$ if and only if whenever $H$ is contained in a parabolic subgroup $P$ of $G$ then $H$ is contained in a Levi subgroup of $P$. A subgroup $H$ of $G$ is *basically G-completely reducible*, or *basically G-cr* for short, if this property is satisfied. Otherwise, say that $H$ is *basically non-G-cr*. The name is justified by the fact that the parabolic subgroups in $G$ are those quasi-parabolic subgroups that arise from root bases.

Prior to the work of Dawson, there was a definition of *completely reducible* subgroups in groups with a twin BN-pair, e.g. Kac-Moody groups, due to Caprace [9]. This definition assumes that $H$ is *bounded*, i.e. contained in the intersection of two finite type parabolic subgroups of opposite sign. It follows from [35, §10.3] that $H$ must be an algebraic subgroup of $G$. The remaining condition in the definition requires that one check (5.1) using only parabolic subgroups of finite type. In the sequel, a subgroup of this form will be called *boundedly G-completely reducible*, or *boundedly G-cr* for short. Otherwise, say that $H$ is *boundedly non-G-cr*.

A number of classical results concerning complete reducibility in spherical buildings can be extended to twin buildings. The next result records these results, the proofs of which are due to [12]. In order to state the result, generalise the notion of a *Levi sphere* to mean the convex hull of a pair of opposite spherical residues, now in a possibly non-spherical twin building $X$.

**Proposition 5.4.** *Let $X = (X_+, X_-)$ be a twin building and $Y$ a convex subcomplex of $X$. Suppose that $Y$ contains a spherical residue in each component of $X$. Then the following are equivalent:*

1. *$Y$ is completely reducible in $X$.*

2. *$Y$ contains a pair of opposite (spherical) residues with the same dimension as $Y$.*

3. *$Y$ contains a Levi sphere $S$ with the same dimension as $Y$.*

4. *Every spherical vertex in $Y$ has an opposite in $Y$.*

The assumption that $Y$ contains a spherical simplex in each component is quite restrictive. This is illustrated clearly in the consequence it has for subgroups of Kac-Moody groups. Indeed, if $H$ is a subgroup of a Kac-Moody group $G$ and $X = X(G)$, then $X^H$ contains a spherical residue in each component if and only if $H$ is bounded. This should be compared with the hypotheses of bounded $G$-complete reducibility.

Let $S$ be the Levi sphere of $X$ defined by the spherical residues $R$ and $R'$. Just as for spherical buildings, there is an isomorphism $R \to R'$ of buildings given by the co-projection maps [1, Proposition 5.152]. Again, write $X_S$ for the building that $R$ defines in $X$ and similarly $Y_S$ for the subcomplex of $X_S$ defined by some subcomplex $Y$ in $X$. Note that $X_S$ is contained in only one of the components of $X$. The next result is [12, Proposition 28].

**Theorem 5.5.** *Let $X = (X_+, X_-)$ be a twin building and $Y$ a convex subcomplex of $X$. Suppose that $Y$ contains a Levi sphere $S$. Then $Y$ is completely reducible in $X$ if and only if $Y_S$ is completely reducible in $X_S$.*

Note that, by the hypothesis, the subcomplex $Y$ of Theorem 5.5 contains a spherical residue in each component.

**Corollary 5.6.** *Let $G$ be a Kac-Moody group and $H$ a subgroup of $G$. Suppose $H$ is contained in a finite type Levi subgroup $L$ of $G$. Then $H$ is basically $G$-cr if and only if $H$ is $L$-cr.*

Importantly, finite type Levi subgroups of Kac-Moody groups are reductive subgroups. Therefore, the content of Corollary 5.6 shows that basic complete reducibility, of *a priori* algebraic subgroups of Kac-Moody groups, can be checked at the level of the reductive subgroups that contain them. This provides an easy way to generate basically $G$-cr or basically non-$G$-cr subgroups in Kac-Moody groups.

There are limitations to the notions of bounded and basic $G$-complete reducibility. For a start, consider a Kac-Moody group $G$ of affine type. Then every proper parabolic subgroup of $G$ is finite type. This causes a peculiarity if $H$ is not contained in any proper parabolic subgroups of $G$. In particular, if $H$ is unbounded then $H$ is trivially boundedly non-$G$-cr whereas $H$ is trivially basically $G$-cr.

On the technical side, one might also expect that key results from $G$-complete reducibility in reductive groups held in the greater generality of Kac-Moody groups, at least when $G$ is linear. For example, the fact that the property of $G$-complete reducibility in reductive groups is closed under taking centralisers and normalisers [6]. The next example illustrates how this fails for boundedly $G$-cr subgroups.

**Example 5.7.** Let $G = \mathrm{SL}_n(k[t,t^{-1}]) \rtimes k^*$. Fix an embedding $\mathrm{SL}_n(k) \subseteq G$ and choose a maximal torus $T = S \times k^*$ of $G$ where $S$ is a maximal torus of $\mathrm{SL}_n(k)$. It is straightforward to see that $C_G(T) = T$ is boundedly $G$-cr. However, recall that $C_G(S) \cong (S \times \mathbb{Z}) \rtimes k^*$ is unbounded. Therefore $C_G(S)$ must be boundedly non-$G$-cr. In particular, the centraliser of a boundedly $G$-cr subgroup may not be boundedly $G$-cr. This phenomenon is not restricted to maximal tori. In fact, if $L$ is Levi subgroup of $G$ properly contained in $\mathrm{SL}_n(k)$ then $C_G(L)$ is also unbounded.

A second property that is fundamental to the theory of $G$-cr in reductive groups is that every unipotent subgroup $U \subseteq G$ is non-$G$-cr. This can be deduced from the fact that $U \subseteq B$ for some Borel subgroup, hence if $U$ is $G$-cr then $U$ is contained in some maximal torus of $B$, which is impossible. The next example explores cases of 'unipotent' subgroups of Kac-Moody groups for which this fact breaks down.

**Example 5.8.** Let $G$ be a Kac-Moody group. Suppose that $B_1$ and $B_2$ are Borel subgroups of $G$ with opposite sign. Set $U = R_u(B_1 \cap B_2)$. Then $U$ is *bounded unipotent* in the sense of [9]. It is proved in *op. cit.* that $U \cap B \subseteq R_u(B)$ for all Borel subgroups $B$ of $G$. Therefore $U$ is not boundedly $G$-cr, as one would expect.

On the other hand, let $G = \mathrm{SL}_n(k[t,t^{-1}]) \rtimes k^*$. Suppose that $U'$ is the upper uni-triangular matrices in $G$. Then $U'$ is a soluble ind-group generated by unipotent algebraic groups. It is straightforward to see that $U'$ is not bounded, hence $U'$ is (trivially) boundedly non-$G$-cr. However, $U'$ is not contained in *any* proper parabolic subgroup of $G$. This implies that $U'$ is (trivially) basically $G$-cr.

The problems that arise in the examples above come from quasi-parabolic subgroups that are not parabolic. In particular, keeping the same notation, note that $C_G(S) \ltimes U'$ is just the subgroup of upper-triangular matrices in $G$. Therefore, the natural setting in which to extend the notion of complete reducibility is to that of the twin citadel. This will be achieved in the following sections.

## 5.2 Completely reducible sets in citadels

In order to define any notion of complete reducibility one requires a notion of opposition. This leads us to consider subcomplexes in twin citadels, rather than citadels alone. Throughout this section, let $\mathscr{C} = (\mathscr{C}_+, \mathscr{C}_-)$ be a twin citadel.

**Definition 5.9.** A convex subset $\mathbf{M} \subseteq \mathscr{C}$ is *$\mathscr{C}$-completely reducible* (*$\mathscr{C}$-cr* for short) if every residue in $\mathbf{M}$ has an opposite in $\mathbf{M}$.

Note that opposition is only defined between the twin buildings in a twin citadel. Therefore, a convex subset is $\mathscr{C}$-cr if and only if its intersection with every twin building $\mathscr{B}$ it meets is $\mathscr{B}$-cr. This means that complete reducibility in $\mathscr{C}$ can be checked at the level of the twin buildings $\mathscr{B}$ in $\mathscr{C}$. It follows that the definition is equivalent to Dawson's notion of complete reducibility for convex subsets of the twin keep. In general, one can apply Proposition 5.4 to each $\mathscr{B}$ in $\mathscr{C}$.

An advantage of studying complete reducibility using the entire twin citadel, and not only the twin keep, is that residues within the twin citadel are often non-standard. Therefore, the space of convex sets that one can study is far richer. This is visible already in the thin twin citadel $\mathscr{C}_W$ of Example 3.42 when $W$ is affine. In this example, the proper residues of the thin twin keep $\mathscr{B}_W = W \times W$ are finite parabolic subgroups of the form $W_J$ for some $J \subseteq S$. On the other hand, proper residues of $\mathscr{C}_W$ may be type $K = S_\Psi$ for some $W_\Psi \subsetneq W$, which will often be non-spherical.

The rest of the section records a generalisation of Theorem 5.5 for twin citadels. First, one must define the analogue of a Levi sphere in a twin citadel.

**Definition 5.10.** A *Levi sphere* in $\mathscr{C}$ is the convex hull of a pair of opposite spherical residues.

By the same reasoning as before, the Levi spheres in $\mathscr{C}$ are precisely the Levi spheres of each twin building in $\mathscr{C}$. This follows from the fact that opposition is defined between twin buildings of $\mathscr{C}$, which are convex.

Suppose $S = (R, R')$ is a Levi sphere in $\mathscr{C}$, where $R$ and $R'$ are opposite spherical residues in some twin building $\mathscr{B} \subseteq \mathscr{C}$. As before, there is a building $\mathscr{B}_S$ associated to $S$ in $\mathscr{B}$. Denote this building in $\mathscr{C}$ by $\mathscr{C}_S$. Similarly, if $\mathbf{M} \subseteq \mathscr{C}$ then write $\mathbf{M}_S$ for the subset of $\mathscr{C}_S$ defined by $\mathbf{M}$. Note that $\mathbf{M}_S$ is empty if $\mathbf{M}$ does not meet $\mathscr{B}$. The next result is an immediate consequence of [12, Proposition 28].

**Theorem 5.11.** *Let* $\mathbf{M}$ *be a convex subset of* $\mathscr{C}$. *Suppose that* $\mathbf{M}$ *contains a Levi sphere* $S$ *in every twin building of* $\mathscr{C}$ *that it meets. Then* $\mathbf{M}$ *is* $\mathscr{C}$*-cr if and only if* $\mathbf{M}_S$ *is* $\mathscr{C}_S$*-cr for all* $S$.

## 5.3 $G$-complete reducibility in Kac-Moody groups

This section defines a notion of $G$-complete reducibility for Kac-Moody groups, using the notion of complete reducibility in the citadel. Some examples are given to illustrate the differences between the new definition and those currently in the literature.

Let $G$ be a Kac-Moody group and $\mathscr{C} = \mathscr{C}_G$ its twin citadel. Suppose $H$ is a subgroup of $G$. Then $H$ acts on $\mathscr{C}$ and one can consider the fixed points $\mathscr{C}^H$ under this action. This can be computed on each $\mathscr{B}(\Psi) \subseteq \mathscr{C}$ independently. Then

$$\mathscr{B}(\Psi)^H = \{R \text{ residue of } \mathscr{B}_\Psi \subseteq \mathscr{B}(\Psi) \mid h \cdot R = R \text{ for all } h \in H\}.$$

Altogether, one has

$$\begin{aligned} \mathscr{C}^H &= \bigsqcup_{\Omega} \mathscr{B}(\Psi)^H, \\ &= \{R \text{ residue of } \mathscr{C} \mid h \cdot R = R \text{ for all } h \in H\}. \end{aligned}$$

Therefore $\mathscr{C}^H$ corresponds to the facial quasi-parabolic subgroups $P$ of $G$ such that $H \subseteq P$.

As one would hope, the fixed points $\mathscr{C}^H$ form a convex set in $\mathscr{C}$. Indeed, suppose $R_\varepsilon$ and $R_\Psi$ are spherical residues of $\mathscr{B}_\varepsilon$ and $\mathscr{B}(\Psi)$ respectively, contained in $\mathscr{C}^H$. One must show that the projection of $R_\Psi$ onto $R_\varepsilon$, and vice versa, is also contained in $\mathscr{C}^H$. Let $h \in H$. Then

$$h \cdot \mathrm{proj}_{R_\varepsilon} R_\Psi = \mathrm{proj}_{h \cdot R_\varepsilon} h \cdot R_\Psi$$

because the action of $G$ on $\mathscr{C}$ preserves $\delta$. But $h \cdot R_\varepsilon = R_\varepsilon$ and $h \cdot R_\Psi = R_\Psi$. Therefore

$$h \cdot \mathrm{proj}_{R_\varepsilon} R_\Psi = \mathrm{proj}_{R_\varepsilon} R_\Psi \implies \mathrm{proj}_{R_\varepsilon} R_\Psi \in \mathscr{C}^H.$$

After replacing $R_\varepsilon$ with $R_\Psi$, the same argument yields the fact that $\mathrm{proj}_{R_\Psi} R_\varepsilon \in \mathscr{C}^H$. The case of two residues in the same building, or some twin building, follows in the same way. In this instance, since $H$ fixes the residues it must fix the $W$-metric between them, even if $H$ does not fix the entire (twin) building. This is recorded in the following lemma.

**Lemma 5.12.** *Let $G$ be a Kac-Moody group and $\mathscr{C} = \mathscr{C}_G$ its twin citadel. Suppose $H$ is a subgroup of $G$. Then $\mathscr{C}^H$ is convex.*

In particular, there is a well-defined notion of complete reducibility for fixed point sets of twin citadels $\mathscr{C} = \mathscr{C}_G$ under the action of a subgroup $H$ of $G$. The next definition characterises this property using the subgroup structure of $G$.

**Definition 5.13.** Let $G$ be a Kac-Moody group and $H$ a subgroup of $G$. Then $H$ is *$G$-completely reducible* (*$G$-cr* for short) if whenever $H$ is contained in a facial quasi-parabolic subgroup $P$ of $G$, then $H$ is contained in a quasi-Levi subgroup $L$ of $P$.

It is straightforward to see that a subgroup $H$ is $G$-cr if and only if $\mathscr{C}^H$ is $\mathscr{C}$-cr. Similarly, if $H$ is $G$-cr then $H$ is basically $G$-cr. The converse is not true. This is illustrated in the following example.

**Example 5.14.** Let $G = \mathrm{SL}_n(k[t,t^{-1}]) \rtimes k^*$. Suppose that $U'$ is the subgroup from Example 5.8, so $U'$ is basically $G$-cr. But $U' = U_\lambda$ for some non-generic cocharacter $\lambda \in Y_G$. In particular, $P = P_\lambda$ is a quasi-Borel subgroup. Therefore the quasi-Levi subgroups of $P$ are, up to some discrete group, just maximal tori of $G$. It follows that $U'$ is not contained in any quasi-Levi subgroup of $P$. Hence $U'$ is non-$G$-cr.

A bounded subgroup $H$ of $G$ is boundedly $G$-cr if it is $G$-cr. The converse remains an open question. At least when $G$ is affine type, this question seems related to [12].

**Conjecture 5.15.** *A bounded subgroup $H$ of $G$ is $G$-cr if and only if it is boundedly $G$-cr.*

The next example details the flexibility of studying $G$-cr subgroups using quasi-parabolic subgroups rather than parabolic subgroups.

**Example 5.16.** Let $G = \mathrm{SL}_n(k[t,t^{-1}]) \rtimes k^*$. Suppose that $S$ and $T$ are the tori of $G$ from Example 5.7. Then $C_G(T) = T$ is a bounded subgroup of $G$, hence stabilises a pair of spherical residues in the twin keep $\mathscr{B} = \mathscr{B}_G$. In fact, it is well-known that $\mathscr{B}^T$ is a twin apartment of $\mathscr{B}$, hence completely reducible in $\mathscr{B}$. On the other hand, the centraliser $C_G(S) \cong (S \times \mathbb{Z}) \rtimes k^*$ is unbounded, therefore the only residue in $\mathscr{B}^{C_G(S)}$ is $\mathscr{B}$ itself. However, $C_G(S)$ is a quasi-Levi subgroup of a quasi-Borel subgroup. This implies that the fixed point set of $\mathscr{C}$ under $C_G(S)$ contains a building from each $\mathscr{B}(\Psi)$. That is to say, $\mathscr{C}^{C_G(S)}$ is non-empty and completely reducible. Note that this includes a trivial spherical building from $\mathscr{B}(\Psi)$ when $\Psi$ is the natural partition.

The final result of this section provides a group-theoretic analogue of Theorem 5.11.

**Theorem 5.17.** *Let $G$ be a Kac-Moody group and $H$ a subgroup of $G$. Suppose that whenever $H$ is contained in a maximal facial quasi-parabolic subgroup $P_\Psi$ of $G$ then $H$ is contained in finite type quasi-Levi subgroup $L_\Psi$ of some facial quasi-parabolic subgroup $P'_\Psi \subseteq P_\Psi$. Then $H$ is $G$-cr if and only if $H$ is $L_\Psi$-cr for all such $L_\Psi$.*

The upshot of Theorem 5.17 is that the framework of citadels and quasi-parabolic subgroups appears to be the natural setting in which one can study complete reducibility for Kac-Moody groups. In particular, the theorem suggests that many structural results from the theory for reductive groups admit analogues in Kac-Moody groups once parabolic and Levi subgroups are replaced by their quasi-parabolic and quasi-Levi counterparts. For instance, it is likely that one can obtain results about normalisers and centralisers of $G$-completely reducible subgroups similar to those first proved in [4, Theorem 3.1]. These extensions will be addressed in forthcoming work.

CHAPTER 6

# Future directions

The thesis concludes by suggesting three avenues for further research, each motivated by the findings in the preceding chapters.

First of all, it would be interesting to see how far the analogy between twin and twisted buildings extends. This thesis defines twisted buildings using the $W$-metric approach which provides a way to define building-like objects without reference to an apartment system. The first question one might like to ask is whether there is a suitable notion of an apartment in a twisted building. The author expects this to be true, and the globally defined twisting map ought to imply that the common apartment property holds. Extending this further, one might expect a notion of an apartment in citadels and twin citadels. However, the failure of Bruhat-type decompositions for quasi-Borel subgroups suggests that the common apartment property fails for arbitrary chambers outside of the keep.

It is a fundamental result in the theory of buildings that a group with a BN-pair admits a thick building with a strongly transitive action. This is a powerful axiomatic tool that can be used to prove simplicity results for the group. There are similar structures, such as twin BN-pairs, that capture the notion of a twin building associated to a group. By looking at the Billig-Dyer decompositions one could naively define the notion of a *twisted BN-pair*. It is plausible that a group admitting such a structure would admit a twisted building, and this could lead to new simplicity results. It would be interesting to see whether such a definition was independent of the various BN-pairs in the literature, or whether possessing a certain BN-pair implies the existence of a twisted BN-pair.

A related problem is whether it makes sense to extend the definition of a twisted building beyond those arising from sets of positive roots. In particular, Dyer's theory of twisted Bruhat orders and reflection subgroups is phrased in the more general setting of initial sections of reflection orders, not only those arising from sets of positive roots. Moreover, many of the results presented about minimal length elements in cosets hold in this generality. This suggests that one may be able to obtain a well-behaved theory of twisted buildings after replacing $A_\Psi$ with an arbitrary initial section of a reflection order. However, the collection of initial sections of reflection orders is wild. An interesting subclass to consider would be the initial sections that appear in Dyer's formulation of the Kazhdan-Lusztig conjecture for Kac-Moody algebras [15, Conjecture 6.15].

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