Online Supplementary Appendix for: Phillips-Solo Asymptotics for Continuous Time ARMA Processes Sampled in Discrete Time

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This document contains the proofs of Theorem 2 and 4 as well as some supplementary results that are required in these proofs. Note: PS92 refers to Phillips and Solo (1992).

Proof of Theorem 2. (a) The result holds if the conditions of either Theorems 3.1 ($\mathscr{A}_2$ and $\mathscr{S}_1$), 3.2 ($\mathscr{A}_1$ and $\mathscr{S}_3$) or 3.10 ($\mathscr{A}_2$ and $\mathscr{S}_6$) of PS92 hold. We begin with Theorem 3.1. Under $\mathscr{A}_2^c$ it is clear that ϵ_{th} is a zero mean i.i.d. process. From the definition of ϵ_{th} the i'th element of ϵ_{th} is simply

$$\epsilon_{i,th} = \int_{th-h}^{th} \Theta_{ip}(th-r)\zeta(dr), \quad i = 1, \ldots, p. \tag{1}$$

We therefore have

$$\begin{aligned} E(\|\epsilon_{th}\|^2) = \sum_{i=1}^{p} E(\epsilon_{i,th}^2) &= \sum_{i=1}^{p} E\left(\int_{th-h}^{th} \Theta_{ip}(th-r)\zeta(dr)\right)^2 \\ &= \sigma^2 \sum_{i=1}^{p} \int_0^h \Theta_{ip}(r)^2 dr < \infty, \end{aligned}$$

hence $\mathscr{A}_2^c$ implies $\mathscr{A}_2$. Applying the Beveridge-Nelson (BN) decomposition to Y_{th} we obtain

$$Y_{th} = C_h(1)\epsilon_{th} + \tilde{\epsilon}_{th-h} - \tilde{\epsilon}_{th} \tag{2}$$

where

$$\tilde{\epsilon}_{th} = \tilde{C}_h(L^h)\epsilon_{th} = \sum_{j=0}^{\infty} \tilde{C}_{h,j}\epsilon_{th-jh}, \quad \tilde{C}_{h,j} = \sum_{k=j+1}^{\infty} C_{h,k}.$$

Let $\tilde{\Omega}_h = E(\tilde{\epsilon}_{th}\tilde{\epsilon}'_{th}) = \sum_{j=0}^{\infty} \tilde{C}_{h,j}\Omega_h\tilde{C}'_{h,j}$; it follows that $\|\tilde{\Omega}_h\| < \infty$ provided $\sum_{j=0}^{\infty}\|\tilde{C}_{h,j}\|^2 < \infty$ which holds under $\mathscr{S}_1^c$ (and hence Assumption 2) using the arguments in the proof of Lemma 2.1 of PS92. Summing (2) and pre-multiplying by b' yields

$$\frac{1}{n}\sum_{t=1}^{n} X_{th} = b'\frac{1}{n}\sum_{t=1}^{n} Y_{th} = b'C_h(1)\frac{1}{n}\sum_{t=1}^{n}\epsilon_{th} + b'\frac{1}{n}(\tilde{\epsilon}_0 - \tilde{\epsilon}_{nh}). \tag{3}$$

The first term on the right-hand side converges a.s. to zero and so we require the terms involving $\tilde{\epsilon}_0$ and $\tilde{\epsilon}_{nh}$ to be $o_{a.s.}(1)$ for the stated result to hold. But this will hold if both $E(b'\tilde{\epsilon}_0)^2 < \infty$ and $E(b'\tilde{\epsilon}_{nh})^2 < \infty$ (see PS92, p.976). We have

$$\begin{aligned} E(b'\tilde{\epsilon}_{th})^2 &= E(b'\tilde{\epsilon}_{th}\tilde{\epsilon}'_{th}b) \\ &= b'\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\tilde{C}_{h,j}E(\tilde{\epsilon}_{th-jh}\tilde{\epsilon}'_{th-kh})\tilde{C}'_{h,k}b \\ &= b'\sum_{j=0}^{\infty}\tilde{C}_{h,j}\tilde{\Omega}_h\tilde{C}'_{h,j}b \\ &\leq \|b\|^2\|\tilde{\Omega}_h\|\sum_{j=0}^{\infty}\|\tilde{C}_{h,j}\|^2 < \infty \end{aligned}$$

under $\mathscr{S}_1^c$; taking $t = 0$ and $t = n$ yields the required result.

For showing that Theorem 3.2 holds, it is clear that ϵ_{th} is i.i.d. with zero mean under $\mathscr{A}_1^c$ so we need to show that $E\|\epsilon_{th}\| < \infty$ in order for $\mathscr{A}_1$ to hold. First, note that, $th-h < r \leq th$ implies $th - r < h$ and so

$$\|e^{A(th-r)}\| = \left\|\sum_{j=0}^{\infty}\frac{A^j(th-r)^j}{j!}\right\| \leq \sum_{j=0}^{\infty}\frac{\|A\|^j(th-r)^j}{j!} = e^{\|A\|(th-r)} \leq e^{\|A\|h}.$$

We then have (noting that $\|e_p\| = 1$)

$$E\|\epsilon_{th}\| \leq E\int_{th-h}^{th}\|e^{A(th-r)}\||\zeta(dr)| \leq e^{\|A\|h}\int_{th-h}^{th}E|\zeta(dr)| \leq K_0he^{\|A\|h} < \infty$$

using $\mathscr{A}_1^c$; hence $\mathscr{A}_1$ is satisfied for ϵ_{th}. We next return to the BN decomposition in (3). The first and second terms on the right-hand side converge a.s. to zero while the third term is

$$\frac{b'}{n}\tilde{\epsilon}_{nh} = \frac{b'}{n}\sum_{j=0}^{\infty}\tilde{C}_{h,j}\epsilon_{nh-jh} = \frac{b'}{n}\sum_{s=-\infty}^{n}\tilde{C}_{h,n-s}\epsilon_{sh} = \frac{b'}{n}\sum_{s=0}^{n}\tilde{C}_{h,n-s}\epsilon_{sh} + \frac{b'}{n}\sum_{t=1}^{\infty}\tilde{C}_{h,n+t}\epsilon_{-th}.$$

Let $\bar{C}_{h,t} = \sum_{s=t+1}^{\infty} \|C_{h,s}\|$ and note that

$$\sum_{t=1}^{\infty} \|\tilde{C}_{h,t}\| \leq \sum_{t=1}^{\infty} \bar{C}_{h,t} = \sum_{t=1}^{\infty} \sum_{s=t+1}^{\infty} \|C_{h,s}\| \leq \sum_{s=1}^{\infty} s\|C_{h,s}\| < \infty$$

under $\mathscr{S}_3^c$. Then

$$\begin{aligned} \frac{1}{n}\|b'\tilde{\epsilon}_{nh}\| &\leq \frac{\|b\|}{n} \sum_{t=0}^{n} \|\tilde{C}_{h,t}\| \|\epsilon_{nh-th}\| + \frac{\|b\|}{n} \sum_{t=1}^{\infty} \bar{C}_{h,n+t} \|\epsilon_{-th}\| \\ &\leq \frac{\|b\|}{n} \left(\max_{0 \leq t \leq n} \|\epsilon_{th}\| \right) \left(\sum_{t=0}^{\infty} \|\tilde{C}_{h,t}\| \right) + \frac{\|b\|}{n} \sum_{t=1}^{\infty} \bar{C}_{h,t} \|\epsilon_{-th}\|. \end{aligned} \tag{4}$$

But $\max_{0 \leq t \leq n} \|\epsilon_{th}\| \overset{a.s.}{\to} 0$ for the reasons stated in PS92 (pp.988–989) and the last term of (4) is finite as its expectation is finite, so both terms in (4) converge a.s. to zero and the result is established.

Turning to Theorem 3.10 of PS92, we have established that $\mathscr{A}_2$ holds under $\mathscr{A}_2^c$. Let

$$w_n = b' \sum_{t=1}^{n} \big(Y_{th} - C_h(1)\epsilon_{th} \big).$$

We need to show that

$$\frac{1}{n^2} \sum_{n=1}^{\infty} E(w_n^2) < \infty$$

so that $(1/n)w_n \overset{a.s.}{\to} 0$ and hence $(1/n)\sum_{t=1}^{n} X_{th} \overset{a.s.}{\to} 0$ as $(1/n)\sum_{t=1}^{n} \epsilon_{th} \overset{a.s.}{\to} 0$ under $\mathscr{A}_2$. The quantity of interest is

$$\begin{aligned} w_n^2 &= b' \sum_{t=1}^{n} \big(Y_{th} - C_h(1)\epsilon_{th} \big) \sum_{s=1}^{n} \big(Y_{sh} - C_h(1)\epsilon_{sh} \big)' b \\ &= b' \sum_{t=1}^{n} \sum_{s=1}^{n} Y_{th} Y_{sh}' b + b' C_h(1) \sum_{t=1}^{n} \sum_{s=1}^{n} \epsilon_{th} \epsilon_{sh}' C_h(1)' b - 2b' \sum_{t=1}^{n} Y_{th} \sum_{s=1}^{n} \epsilon_{sh}' C_h(1)' b. \end{aligned}$$

Now

$$\begin{aligned} E\left(\sum_{t=1}^{n} Y_{th} \sum_{s=1}^{n} \epsilon_{sh}' \right) = \sum_{t=1}^{n} E\left(Y_{th} \sum_{s=1}^{n} \epsilon_{sh}' \right) &= \sum_{t=1}^{n} \sum_{s=1}^{t} C_{h,t-s} \Omega_h \\ &= \sum_{t=1}^{n} \sum_{j=0}^{t-1} C_{h,j} \Omega_h \\ &= \sum_{t=1}^{n} \big(C_h(1) - \tilde{C}_{h,t-1} \big) \Omega_h \end{aligned}$$

so that

$$\begin{aligned}\frac{1}{n^2}E(w_n^2) &= \frac{1}{n^2}b'\sum_{t=1}^{n}\sum_{s=1}^{n}E\big(Y_{th}Y_{sh}'\big)b+\frac{1}{n}b'C_h(1)\Omega_hC_h(1)'b\\ &\quad -\frac{2}{n^2}b'\sum_{t=1}^{n}\big(C_h(1)-\tilde{C}_{h,t-1}\big)\Omega_hC_h(1)'b\\ &= \frac{1}{n^2}b'\sum_{t=1}^{n}\sum_{s=1}^{n}E\big(Y_{th}Y_{sh}'\big)b-\frac{1}{n}b'C_h(1)\Omega_hC_h(1)'b+\frac{2}{n^2}b'\sum_{t=1}^{n}\tilde{C}_{h,t-1}\Omega_hC_h(1)'b.\end{aligned}$$

Following the steps in PS92 (p.995) we obtain

$$\sum_{n=1}^{\infty}\left\|\frac{1}{n^2}\sum_{t=1}^{n}\tilde{C}_{h,t-1}\right\|\le\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{t=1}^{n}\|\tilde{C}_{h,t-1}\|\le\sum_{s=1}^{\infty}\|C_{h,s}\|(1+\log s)+\left(\sum_{s=1}^{\infty}\|C_{h,s}\|\right)\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)$$

which is finite under $\mathscr{S}_6^c$. Now let $\Gamma_{h,r}=E(Y_{th}Y_{th+rh})$ and define

$$\begin{aligned}a_n &= \frac{1}{n^2}b'\sum_{t=1}^{n}\sum_{s=1}^{n}E\big(Y_{th}Y_{sh}'\big)b-\frac{1}{n}b'C_h(1)\Omega_hC_h(1)'b\\ &= \frac{1}{n}b'\big(\Gamma_{h,0}+\sum_{r=1}^{n-1}(\Gamma_{h,r}+\Gamma_{h,r}')\big)b-\frac{1}{n^2}b'\sum_{r=1}^{n-1}r(\Gamma_{h,r}+\Gamma_{h,r}')b\\ &\quad -\frac{1}{n}b'\big(\Gamma_{h,0}+\sum_{r=1}^{\infty}(\Gamma_{h,r}+\Gamma_{h,r}')\big)b\\ &= -\frac{1}{n}b'\sum_{r=n}^{\infty}(\Gamma_{h,r}+\Gamma_{h,r}')b-\frac{1}{n^2}b'\sum_{r=1}^{n-1}r(\Gamma_{h,r}+\Gamma_{h,r}')b.\end{aligned}$$

From PS92 (p.995) again we have that

$$\sum_{n=1}^{\infty}|a_n|\le 2\|b\|^2\sum_{r=1}^{\infty}\|\Gamma_{h,r}\|(1+\log r)+2\|b\|^2\sum_{r=1}^{\infty}\|\Gamma_{h,r}\|$$

while from the argument on PS92 (p.995) it can be shown that

$$\sum_{r=1}^{\infty}\log r\|\Gamma_{h,r}\|\le\|\Omega_h\|\sum_{r=1}^{\infty}\log r\|C_{h,r}\|\sum_{s=0}^{\infty}\|C_{h,s}\|(1+\log s)<\infty$$

under $\mathscr{S}_6^c$ and, hence, $\sum_{n=1}^{\infty}(1/n^2)E(w_n^2)<\infty$ as required.

(b) The result holds by Theorem 3.4 of PS92 which requires $\mathscr{A}_2$ and $\mathscr{S}_1$; the former is

satisfied under $\mathscr{A}_2^c$ as shown previously. Again using the BN decomposition we obtain

$$\frac{1}{n^{1/2}}\sum_{t=1}^{[nr]} X_{th} = b'C_h(1)\frac{1}{n^{1/2}}\sum_{t=1}^{[nr]}\epsilon_{th} + \frac{1}{n^{1/2}}b'\tilde{\epsilon}_0 - \frac{1}{n^{1/2}}b'\tilde{\epsilon}_{[nr]h}.$$

Condition $\mathscr{A}_2$ ensures that the sum involving ϵ_{th} satisfies the CLT and IP so we need to deal with the remaining terms. For the CLT, setting $r = 1$ above, it suffices to show that $(1/n)(b'\tilde{\epsilon}_0)^2 \xrightarrow{p} 0$ and $(1/n)(b'\tilde{\epsilon}_{nh})^2 \xrightarrow{p} 0$, which hold if $E(b'\tilde{\epsilon}_0)^2 = E(b'\tilde{\epsilon}_{nh})^2 < \infty$, which in turn has already been established under $\mathscr{S}_1^c$ in the demonstration of the validity of Theorem 3.1 of PS92 in part (a). For the IP we require

$$\sup_r \left| \frac{1}{n^{1/2}}\sum_{t=1}^{[nr]} X_{th} - b'C_h(1)\frac{1}{n^{1/2}}\sum_{t=1}^{[nr]}\epsilon_{th} \right| \leq \frac{1}{n^{1/2}}|b'\tilde{\epsilon}_0| + \sup_r \left| \frac{1}{n^{1/2}} b'\tilde{\epsilon}_{[nr]h} \right| \xrightarrow{p} 0$$

which holds if $\max_{1\leq k\leq n}\{(1/n)(b'\tilde{\epsilon}_{kh})^2\} \xrightarrow{p} 0$. The argument on p.978 of PS92 ensures this holds under $\mathscr{S}_1$ which is implied by $\mathscr{S}_1^c$ and the required result is established.

(c) Here we will establish that the steps in the proof of Theorem 3.7 of PS92, which rely on $\mathscr{A}_2$ and $\mathscr{S}_5$, hold for $X_{th} = b'Y_{th}$. We have already shown that $\mathscr{A}_2$ holds under $\mathscr{A}_2^c$. We begin with $X_{th}^2 = (b'C_h(L^h)\epsilon_{th})^2 = X_{a,th} + 2X_{b,th}$ where

$$\begin{aligned}
X_{a,th} &= b'\sum_{j=0}^{\infty} C_{h,j}\epsilon_{th-jh}\epsilon'_{th-jh}C'_{h,j}b \\
&= \operatorname{tr}\left\{\sum_{j=0}^{\infty} C'_{h,j}bb'C_{h,j}\epsilon_{th-jh}\epsilon'_{th-jh}\right\} \\
&= \operatorname{tr}\left\{F_0(L^h)\epsilon_{th}\epsilon'_{th}\right\}, \\
X_{b,th} &= b'\sum_{r=1}^{\infty}\sum_{j=0}^{\infty} C_{h,j}\epsilon_{th-jh}\epsilon'_{th-jh-rh}C'_{h,j+r}b \\
&= \sum_{r=1}^{\infty}\operatorname{tr}\left\{\sum_{j=0}^{\infty} C'_{h,j+r}bb'C_{h,j}\epsilon_{th-jh}\epsilon'_{th-jh-rh}\right\} \\
&= \sum_{r=1}^{\infty}\operatorname{tr}\left\{F_r(L^h)\epsilon_{th}\epsilon'_{th-rh}\right\},
\end{aligned}$$

where

$$F_j(z) = \sum_{k=0}^{\infty} C'_{h,k+j}bb'C_{h,k}z^k = \sum_{k=0}^{\infty} F_{jk}z^k, \quad F_{jk} = C'_{h,k+j}bb'C_{h,k}.$$

Applying the BN decomposition to $F_j(z)$ yields

$$F_j(z) = F_j(1) - (1-z)\tilde{F}_j(z), \quad \tilde{F}_j(z) = \sum_{k=0}^{\infty} \tilde{F}_{jk} z^k, \quad \tilde{F}_{jk} = \sum_{s=k+1}^{\infty} F_{js} = \sum_{s=k+1}^{\infty} C'_{h,s+j} b b' C_{h,s}.$$

We can then express $X_{a,th}$ and $X_{b,th}$ in the form

$$\begin{aligned} X_{a,th} &= \operatorname{tr}\left\{F_0(1)\epsilon_{th}\epsilon'_{th}\right\} - (1-L^h)\tilde{X}_{a,th}, \quad \tilde{X}_{a,th} = \operatorname{tr}\left\{\tilde{F}_0(L^h)\epsilon_{th}\epsilon'_{th}\right\}, \\ X_{b,th} &= \sum_{r=1}^{\infty} \operatorname{tr}\left\{F_r(1)\epsilon_{th}\epsilon'_{th-rh}\right\} - (1-L^h)\tilde{X}_{b,th}, \quad \tilde{X}_{b,th} = \sum_{r=1}^{\infty} \operatorname{tr}\left\{\tilde{F}_r(L^h)\epsilon_{th}\epsilon'_{th-rh}\right\}. \end{aligned}$$

From the decomposition of X_{th}^2 we need to show that

$$\frac{1}{n}\sum_{t=1}^{n} X_{a,th} \xrightarrow{a.s.} \gamma_{h,0}, \quad \frac{1}{n}\sum_{t=1}^{n} X_{b,th} \xrightarrow{a.s.} 0.$$

Beginning with $X_{a,th}$ we need to consider

$$\frac{1}{n}\sum_{t=1}^{n} X_{a,th} = \operatorname{tr}\left\{F_0(1)\frac{1}{n}\sum_{t=1}^{n}\epsilon_{th}\epsilon'_{th}\right\} - \frac{1}{n}\tilde{X}_{a,nh}.$$

Under $\mathscr{A}_2$, which is implied by $\mathscr{A}_2^c$, the first term converges to

$$\operatorname{tr}\left\{F_0(1)\Omega_h\right\} = \operatorname{tr}\left\{\sum_{j=0}^{\infty} C'_{h,j} b b' C_{h,j}\Omega_h\right\} = b'\sum_{j=0}^{\infty} C_{h,j}\Omega_h C'_{h,j} b = \gamma_{h,0},$$

while for the second term we have

$$\begin{aligned} \frac{1}{n}\tilde{X}_{a,nh} = \operatorname{tr}\left\{\tilde{F}_0(L^h)\frac{1}{n}\epsilon_{nh}\epsilon'_{nh}\right\} &= \operatorname{tr}\left\{\frac{1}{n}\sum_{k=0}^{\infty}\tilde{F}_{0k}\epsilon_{nh-kh}\epsilon'_{nh-kh}\right\} \\ &\leq \frac{1}{n}\left(\max_{0\leq t\leq n}\|\epsilon_{th}\|^2\right)\left(\sum_{k=0}^{\infty}\|\tilde{F}_{0k}\|\right) + \frac{1}{n}\sum_{k=1}^{\infty}\|\tilde{F}_{0k}\|\|\epsilon_{-k}\|^2. \end{aligned}$$

Now $(1/n)\max_{0\leq t\leq n}\|\epsilon_{th}\|^2 \xrightarrow{a.s.} 0$ as $(1/n)\|\epsilon_{nh}\|^2 \xrightarrow{a.s.} 0$ while

$$\begin{aligned} \sum_{k=0}^{\infty}\|\tilde{F}_{0k}\| = \sum_{k=0}^{\infty}\left\|\sum_{s=k+1}^{\infty} C'_{h,j+s} b b' C_{h,s}\right\| &\leq \sum_{k=0}^{\infty}\sum_{s=k+1}^{\infty}\|C_{h,s}\|^2\|b\|^2 \\ &\leq \|b\|^2\sum_{s=0}^{\infty} s\|C_{h,s}\|^2 < \infty \end{aligned}$$

under $\mathscr{S}_5$ which is implied by $\mathscr{S}_5^c$. Hence $(1/n)\tilde{X}_{a,nh} \stackrel{a.s.}{\to} 0$ as required.

Turning to $X_{b,th}$ we have

$$\frac{1}{n}\sum_{t=1}^{n} X_{b,th} = \sum_{r=1}^{\infty} \mathrm{tr}\left\{ F_r(1)\frac{1}{n}\sum_{t=1}^{n} \epsilon_{th}\epsilon'_{th-rh} \right\} - \frac{1}{n}\tilde{X}_{b,nh}$$

and the objective is to show that both terms converge a.s. to zero. The first term is

$$\mathrm{tr}\left\{ \frac{1}{n}\sum_{t=1}^{n} \epsilon_{th}\epsilon^{\dagger\prime}_{th-h} \right\} \quad \text{where} \quad \epsilon^{\dagger\prime}_{th-h} = \sum_{r=1}^{\infty} \epsilon'_{th-rh}F_r(1).$$

By Kronecker's lemma this term converges a.s. to zero if

$$T_n = \mathrm{tr}\left\{ \sum_{t=1}^{n} \frac{1}{t}\epsilon_{th}\epsilon^{\dagger\prime}_{th-h} \right\} = \sum_{t=1}^{n} \frac{1}{t}\epsilon^{\dagger\prime}_{th-h}\epsilon_{th}$$

converges a.s. to zero. But T_n is a martingale so converges if $\sup_n E(T_n^2) < \infty$; we have

$$\begin{aligned} E(T_n^2) &= E\left(\sum_{t=1}^{n}\sum_{s=1}^{n} \frac{1}{t}\frac{1}{s}\epsilon^{\dagger\prime}_{th-h}\epsilon_{th}\epsilon'_{sh}\epsilon^{\dagger}_{th-h} \right) \\ &= \sum_{t=1}^{n}\sum_{s=1}^{n} \frac{1}{t}\frac{1}{s} E\left(\epsilon_{th}\epsilon'_{sh}\epsilon^{\dagger}_{th-h}\epsilon^{\dagger\prime}_{th-h} \right) \\ &= \mathrm{tr}\left\{ \Omega_h\Omega_h^{\dagger} \sum_{t=1}^{n} \frac{1}{t^2} \right\} < \infty, \end{aligned}$$

where $\Omega_h^{\dagger} = E\big(\epsilon^{\dagger}_{th-h}\epsilon^{\dagger\prime}_{th-h}\big) = \sum_{r=1}^{\infty} F_r(1)'\Omega_h F_r(1)$.

The term involving $\tilde{X}_{b,nh}$ converges a.s. to zero if $\sum_{n=1}^{\infty}(1/n^2)E(\tilde{X}^2_{b,nh}) < \infty$ which in turn holds if $E(\tilde{X}^2_{b,nh}) < \infty$. Now

$$\tilde{X}_{b,nh} = \sum_{r=1}^{\infty} \mathrm{tr}\left\{ \tilde{F}_r(L^h)\epsilon_{nh}\epsilon'_{nh-rh} \right\} = \sum_{r=1}^{\infty} \epsilon'_{nh-rh}\tilde{F}_r(L^h)\epsilon_{nh}$$

which implies that

$$\tilde{X}^2_{b,nh} = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \epsilon'_{nh-rh}\tilde{F}_r(L^h)\epsilon_{nh}\epsilon'_{nh}\tilde{F}_s(L^h)'\epsilon_{nh-sh}.$$

Writing each of the quadratic forms in terms of the individual elements yields

$$\tilde{X}^2_{b,nh} = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \epsilon_{i,nh-rh}\tilde{F}_{r,ij}(L^h)\epsilon_{j,nh}\epsilon_{k,nh}\tilde{F}_{s,kl}(L^h)\epsilon_{l,nh-sh}.$$

But $\tilde{F}_{r,ij}(z) = \sum_{u=0}^{\infty}\tilde{F}_{ru,ij}z^u$ and $\tilde{F}_{s,kl}(z) = \sum_{v=0}^{\infty}\tilde{F}_{sv,kl}z^v$ so that

$$\tilde{X}^2_{b,nh} = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\sum_{u=0}^{\infty} \tilde{F}_{ru,ij}\epsilon_{j,nh-uh}\epsilon_{i,nh-rh-uh}\sum_{v=0}^{\infty}\tilde{F}_{sv,kl}\epsilon_{k,nh-vh}\epsilon_{l,nh-sh-vh},$$

from which

$$E(\tilde{X}^2_{b,nh}) = \sum_{i,j,k,l=1}^{p}\sum_{r,s=1}^{\infty}\sum_{u,v=0}^{\infty} \tilde{F}_{ru,ij}\tilde{F}_{sv,kl}E\big(\epsilon_{j,nh-uh}\epsilon_{i,nh-rh-uh}\epsilon_{k,nh-vh}\epsilon_{l,nh-sh-vh}\big)$$

where indices on summations are combined for notational convenience. From Lemma B1(a)

$$\begin{aligned} E(\tilde{X}^2_{b,nh}) &= \sigma^4\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p}\phi_{h,il}\phi_{h,jk}\sum_{r=1}^{\infty}\sum_{u=0}^{\infty}\tilde{F}_{ru,ij}\tilde{F}_{ru,kl} \\ &+ \sigma^4\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p}\phi_{h,ik}\phi_{h,jl}\sum_{u,v=0}^{\infty}\tilde{F}_{v-u,u,ij}\tilde{F}_{u-v,v,kl}. \end{aligned}$$

The first term is bounded as $\phi_{h,il}$ and $\phi_{h,jk}$ are finite and

$$\begin{aligned} \left|\sum_{r=1}^{\infty}\sum_{u=0}^{\infty}\tilde{F}_{ru,ij}\tilde{F}_{ru,kl}\right| &\le \sum_{r=1}^{\infty}\sum_{u=0}^{\infty}|\tilde{F}_{ru,ij}||\tilde{F}_{ru,kl}| \\ &\le \sum_{r=1}^{\infty}\sum_{u=0}^{\infty}\|\tilde{F}_{ru}\|^2 \\ &\le \sum_{r=1}^{\infty}\sum_{u=0}^{\infty}\left\|\sum_{s=u+1}^{\infty}C'_{h,r+s}bb'C_{h,s}\right\|^2 \\ &\le \|b\|^2\sum_{r=1}^{\infty}\sum_{u=0}^{\infty}\left(\sum_{s=u+1}^{\infty}\|C_{h,r+s}\|^2\right)\left(\sum_{s=u+1}^{\infty}\|C_{h,s}\|^2\right) \\ &\le \|b\|^2\left(\sum_{u=0}^{\infty}\sum_{s=u+1}^{\infty}\|C_{h,s}\|^2\right)\left(\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\|C_{h,r+s}\|^2\right) \\ &\le \|b\|^2\left(\sum_{s=1}^{\infty}\|C_{h,s}\|^2\sum_{u=1}^{s-1}1\right)^2 \\ &\le \|b\|^2\left(\sum_{s=1}^{\infty}s\|C_{h,s}\|^2\right)^2 < \infty \end{aligned}$$

under $\mathscr{S}_5^c$. The second term is also bounded in view of the finiteness of $\phi_{h,ik}$ and $\phi_{h,jl}$ while

$$\begin{aligned}\left(\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\tilde{F}_{v-u,u,ij}\tilde{F}_{u-v,v,kl}\right)^2 &\leq \left(\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\|\tilde{F}_{v-u,u}\|\|\tilde{F}_{u-v,v}\|\right)^2 \\ &\leq \left(\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\|\tilde{F}_{v-u,u}\|^2\right)^2 \\ &\leq \left(\sum_{v=0}^{\infty}\sum_{u=0}^{v}\|\tilde{F}_{v-u,u}\|^2\right)^2 \\ &\leq \left(\sum_{r=1}^{\infty}\sum_{u=0}^{\infty}\|\tilde{F}_{ru}\|^2+\|\tilde{F}_{00}\|^2\right)^2<\infty\end{aligned}$$

using arguments from the first term. Hence $E(\tilde{X}_{b,nh}^2)<\infty$ implying that

$$\sum_{n=1}^{\infty}E(\tilde{X}_{b,nh}^2)\frac{1}{n^2}\leq K\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty$$

and $(1/n)\tilde{X}_{b,nh}\overset{a.s.}{\to}0$ as required.

(d) Here, we follow the proof of Theorem 3.8 in PS92 under $\mathscr{A}_3^c$ and $\mathscr{S}_5^c$. From the decomposition $X_{th}^2=X_{a,th}+2X_{b,th}$ and the definitions of $X_{a,th}$ and $X_{b,th}$ we obtain

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left(X_{th}^2-\gamma_{h,0}\right) &= \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\mathrm{tr}\left\{F_0(1)\epsilon_{th}\epsilon_{th}'-\gamma_{h,0}+2\epsilon_{th}\epsilon_{th-h}^{\dagger\prime}\right\} \\ &\quad+\frac{1}{\sqrt{n}}\left(\tilde{X}_{a0}-\tilde{X}_{a,nh}\right)+\frac{2}{\sqrt{n}}\left(\tilde{X}_{b0}-\tilde{X}_{b,nh}\right).\end{aligned}$$

In part (c) we established that $E(\tilde{X}_{b,nh}^2)<\infty$ and hence the final term converges to zero in probability. For the second term we have

$$E(\tilde{X}_{a,th})=E\left(\mathrm{tr}\left\{\sum_{k=0}^{\infty}\tilde{F}_{0k}\epsilon_{th-kh}\epsilon_{th-kh}'\right\}\right)=\mathrm{tr}\left\{\sum_{k=0}^{\infty}\tilde{F}_{0k}\Omega\right\}.$$

But $\sum_{k=0}^{\infty}\tilde{F}_{0k}=\sum_{k=0}^{\infty}\sum_{s=k+1}^{\infty}C_{h,s}'bb'C_{h,s}$ and so

$$\left|\mathrm{tr}\left\{\sum_{k=0}^{\infty}\tilde{F}_{0k}\Omega\right\}\right|\leq\sum_{k=0}^{\infty}\left|\mathrm{tr}\left\{\tilde{F}_{0k}\Omega\right\}\right|\leq\|\Omega\|\sum_{k=0}^{\infty}\|\tilde{F}_{0k}\|<\infty$$

since $\|\Omega\|<\infty$ and $\sum_{k=0}^{\infty}\|\tilde{F}_{0k}\|\leq\|b\|^2\sum_{s=1}^{\infty}s\|C_{h,s}\|^2<\infty$ under $\mathscr{S}_5^c$. Hence $|E(\tilde{X}_{a0})|<\infty$ and $|E(\tilde{X}_{a,nh})|<\infty$ and the second term converges in probability to zero. We therefore need

to show that the first term converges weakly to $N(0, v_h(0))$ which we shall do by verifying (see PS92, p.992) that

$$\frac{1}{n}\sum_{t=1}^{n} Z_{th}^2 \xrightarrow{p} v_h(0),$$

$$\frac{1}{n}\sum_{t=1}^{n} E\big(Z_{th}^2 1(Z_{th}^2 > \delta n)\big) \to 0$$

where $Z_{th} = \mathrm{tr}\left\{F_0(1)\epsilon_{th}\epsilon_{th}' - \gamma_{h,0} + 2\epsilon_{th}\epsilon_{th-h}^{\dagger\prime}\right\}$. Following the arguments in PS92 the second condition holds if $E(Z_{th}^2) = v_h(0)$ and the first if $(1/n)\sum_{t=1}^{n} E(Z_{th}^2|\mathscr{F}_{th-h}) \xrightarrow{p} v_h(0)$. Writing $Z_{th} = \epsilon_{th}'F_0(1)\epsilon_{th} - \gamma_{h,0} + 2\epsilon_{th-h}^{\dagger\prime}\epsilon_{th}$ we therefore need to consider

$$Z_{th}^2 = \big(\epsilon_{th}'F_0(1)\epsilon_{th} - \gamma_{h0}\big)^2 + 4\big(\epsilon_{th-h}^{\dagger\prime}\epsilon_{th}\big)^2 + 4\big(\epsilon_{th}'F_0(1)\epsilon_{th} - \gamma_{h0}\big)\big(\epsilon_{th-h}^{\dagger\prime}\epsilon_{th}\big). \tag{5}$$

The first term can be written in the form

$$\big(\epsilon_{th}'F_0(1)\epsilon_{th} - \gamma_{h0}\big)^2 = \big(\epsilon_{th}'F_0(1)\epsilon_{th}\big)^2 + \gamma_{h0}^2 - 2\gamma_{h0}\epsilon_{th}'F_0(1)\epsilon_{th}.$$

It is straightforward to show that $E(\epsilon_{th}'F_0(1)\epsilon_{th}) = E\mathrm{tr}\{F_0(1)\epsilon_{th}\epsilon_{th}'\} = \mathrm{tr}\{F_0(1)\Omega_h\} = \gamma_{h,0}$ while the square of the quadratic form has the representation

$$\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} F_{0,ij}(1)F_{0,kl}(1)\epsilon_{i,th}\epsilon_{j,th}\epsilon_{k,th}\epsilon_{l,th}.$$

Using Lemma B1(b) for $E(\epsilon_{i,th}\epsilon_{j,th}\epsilon_{k,th}\epsilon_{l,th})$ results in

$$\begin{aligned} E\big(\epsilon_{th}'F_0(1)\epsilon_{th}\big)^2 &= K_4\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p}\phi_{h,ijkl}F_{0,ij}(1)F_{0,kl}(1) \\ &\quad +\sigma^4\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p}(\phi_{h,ij}\phi_{h,kl} + \phi_{h,ik}\phi_{h,jl} + \phi_{h,il}\phi_{h,jk})F_{0,ij}(1)F_{0,kl}(1). \end{aligned}$$

The second term in (5) has expectation

$$E\big(\epsilon_{th-h}^{\dagger\prime}\epsilon_{th}\big)^2 = E\mathrm{tr}\left\{\epsilon_{th-h}^{\dagger}\epsilon_{th-h}^{\dagger\prime}\epsilon_{th}\epsilon_{th}'\right\} = \mathrm{tr}\left\{E\big(\epsilon_{th-h}^{\dagger}\epsilon_{th-h}^{\dagger\prime}\big)E\big(\epsilon_{th}\epsilon_{th}'\big)\right\} = \mathrm{tr}\{\Omega_h^{\dagger}\Omega_h\}$$

where $\Omega_h^{\dagger}$ was defined in the proof of part (c). This establishes that $E(Z_{th}^2) = v_h(0)$ as the remaining terms in (5) have expectation zero in view of ϵ_{th} and $\epsilon_{th-h}^{\dagger}$ being independent under $\mathscr{A}_3^c$.

For the IP we can write

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]h}\left(X_{th}^2-\gamma_{h,0}\right)=\frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]h}Z_{th}+\frac{1}{\sqrt{n}}\left(\tilde{X}_{a0}-\tilde{X}_{a,[nr]h}\right)+\frac{2}{\sqrt{n}}\left(\tilde{X}_{b0}-\tilde{X}_{b,[nr]h}\right).$$

In view of earlier results we need to show that

$$\max_{1\leq k\leq n}\frac{1}{n}\tilde{X}_{a,kh}\xrightarrow{p}0,\quad \max_{1\leq k\leq n}\frac{1}{n}\tilde{X}_{b,kh}\xrightarrow{p}0.$$

These conditions hold using the arguments in PS92 (p.993). □

Proof of Theorem 4. (a) We begin by verifying that the conditions for Theorem 3.1 of PS92 hold. As $\xi_{th}=A^{-1}\epsilon_{th}$ it follows immediately that $\mathscr{A}_2^c$ implies $\mathscr{A}_2$ using the result from the proof of Theorem 2(a). The BN decomposition is

$$Y_{th}^f=\tilde{\xi}_{th-h}-\tilde{\xi}_{th}+\eta_{th} \tag{6}$$

and it is clear that $\|\tilde{\Omega}_h^f\|<\infty$ and $\|\Omega_h^\eta\|<\infty$, where $\Omega_h^f=E(\tilde{\xi}_{th}\tilde{\xi}_{th}')$ and $\Omega_h^\eta=E(\eta_{th}\eta_{th}')$. Summing (6) and pre-multiplying by b' yields

$$\frac{1}{n}\sum_{t=1}^{n}X_{th}^f=b'\frac{1}{n}\sum_{t=1}^{n}Y_{th}^f=b'\frac{1}{n}\left(\tilde{\xi}_0-\tilde{\xi}_{nh}\right)+b'\frac{1}{n}\sum_{t=1}^{n}\eta_{th}.$$

Following the arguments in the proof of Theorem 2(a) it is possible to show that the term involving $\tilde{\xi}_0-\tilde{\xi}_{nh}$ is $o_{a.s.}(1)$ while $b'\eta_{th}=-\zeta_{th}/a_0$ and hence the sum involving η_{th} converges to zero, yielding the required result.

Turning to Theorem 3.2 of PS92 we again rely on the results in the proof of Theorem 2(a) to show that $\mathscr{A}_1$ is satisfied for ξ_{th} under $\mathscr{A}_1^c$ in view of $E\|\xi_{th}\|\leq\|A^{-1}\|E\|\epsilon_{th}\|<\infty$. The summability results apply again, replacing $\tilde{C}_{h,j}$ with $\tilde{B}_{h,j}$, and the result follows noting again that the normalised sum involving η_{th} converges to zero.

The proof of the validity of Theorem 3.10 in PS92 follows in the same way as in the proof of Theorem 2(a) where now

$$w_n=b'\sum_{t=1}^{n}(Y_{th}^f-\eta_{th})$$

so the proof simplifies somewhat.

(b) To validate Theorem 3.4 of PS92 we need

$$\sup_r \left| \frac{1}{n^{1/2}} \sum_{t=1}^{[nr]} X_{th}^f - b' \frac{1}{n^{1/2}} \sum_{t=1}^{[nr]} \eta_{th} \right| \leq \frac{1}{n^{1/2}} |b' \tilde{\xi}_0| + \sup_r \left| \frac{1}{n^{1/2}} b' \tilde{\xi}_{[nr]h} \right| \xrightarrow{p} 0.$$

The right-hand side converges to zero while

$$b' \frac{1}{n^{1/2}} \sum_{t=1}^{[nr]} \eta_{th} = -\frac{1}{a_0 n^{1/2}} \int_0^{[nr]h} \zeta(dr) \xrightarrow{d} \sigma_f W(r)$$

under $\mathscr{A}_2^c$ as required. When $r = 1$ the limit distribution is $N(0, \sigma_f^2)$.

(c) To demonstrate the validity of Theorem 3.7 in PS92 we now have

$$\left(X_{th}^f\right)^2 = \left(b' B_h(L^h) \xi_{th} + b' \eta_{th}\right)^2 = X_{a,th}^f + 2X_{b,th}^f + X_{c,th}^f$$

where $X_{a,th}^f$ and $X_{b,th}^f$ are defined analogously to $X_{a,th}$ and $X_{b,th}$ in the proof of Theorem 2(c) and

$$X_{c,th}^f = (b' \eta_{th})^2 + 2b' B_h(L^h) \xi_{th} \eta_{th}' b.$$

The same arguments as before apply to $X_{a,th}^f$ and $X_{b,th}^f$ so that

$$\frac{1}{n} \sum_{t=1}^n X_{a,th}^f \xrightarrow{a.s.} b' \sum_{j=0}^\infty B_{h,j} \Omega_h^f B_{h,j}' b \quad \text{and} \quad \frac{1}{n} \sum_{t=1}^n X_{b,th}^f \xrightarrow{a.s.} 0.$$

For the first component of $X_{c,th}^f$ we have, under $\mathscr{A}_2^c$,

$$\frac{1}{n} \sum_{t=1}^n (b' \eta_{th})^2 = \frac{1}{a_0^2 n} \sum_{t=1}^n \zeta_{th}^2 \xrightarrow{a.s.} \frac{\sigma^2 h}{a_0^2}$$

while for the second component we have

$$\frac{1}{n} b' \sum_{t=1}^n \sum_{j=0}^\infty B_{h,j} \xi_{th-jh} \eta_{th}' b \xrightarrow{a.s.} b' E(\xi_{th} \eta_{th}') b = -\frac{\sigma^2}{a_0} b' A^{-1} \phi_{h,\bullet p} b$$

using Lemma B for the expectation. Hence

$$\frac{1}{n} \sum_{t=1}^n \left(X_{a,th}^f + X_{c,th}^f\right) \xrightarrow{a.s.} b' \sum_{j=0}^\infty B_{h,j} \Omega_h^f B_{h,j}' b + \frac{\sigma^2 h}{a_0^2} - 2\frac{\sigma^2}{a_0} b' A^{-1} \phi_{h,\bullet p} b = \gamma_{h,0}^f$$

as required.

(d) We need to verify that Theorem 3.8 of PS92 holds. The quantity of interest can be

written

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left(\left(x_{th}^{f}\right)^{2}-\gamma_{h,0}^{f}\right) = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\Big(\mathrm{tr}\left\{F_0(1)\xi_{th}\xi_{th}'-\gamma_{h,0}^{f}+2\xi_{th}\xi_{th-h}^{\dagger\prime}\right\}+(b'\eta_{th})^{2} +2b'B_h(L^h)\xi_{th}\eta_{th}'b\Big)+\frac{1}{\sqrt{n}}(\tilde{X}_{a0}^{f}-\tilde{X}_{a,nh}^{f})+\frac{1}{\sqrt{n}}(\tilde{X}_{b0}^{f}-\tilde{X}_{b,nh}^{f})$$

where $\tilde{X}_{a,th}^{f}$ and $\tilde{X}_{b,th}^{f}$ are defined analogously to $\tilde{X}_{a,th}$ and $\tilde{X}_{b,th}$ in the proof of Theorem 2(c) and $\xi_{th-h}^{\dagger}=\sum_{s=1}^{\infty}F_s(1)'\xi_{th-sh}$. In fact, the last two terms are asymptotically negligible as before, and so we need to consider

$$Z_{th}^{f}=\xi_{th}'F_0(1)\xi_{th}-\gamma_{h,0}^{f}+2\xi_{th-h}^{\dagger\prime}\xi_{th}+\frac{1}{a_0^2}\zeta_{th}^{2}-\frac{2}{a_0}b'B_h(L^h)\xi_{th}\zeta_{th},$$

recalling that $b'\eta_{th}=-\zeta_{th}/a_0$. The arguments in the proof of Theorem 2(d) are equally valid here and our goal is to demonstrate that $E(Z_{th}^{f})^2=v_h^{f}(0)$. We can write

$$\left(Z_{th}^{f}\right)^{2}=\sum_{i=1}^{10}Z_{i,th} \quad\longrightarrow\quad v_h^{f}(0)=\sum_{i=1}^{10}E(Z_{i,th})$$

and we will define each term and derive their expectations as follows:

$\underline{Z_{1,th}}$: The first component is

$$Z_{1,th}=(\xi_{th}'F_0(1)\xi_{th}-\gamma_{h,0}^{f})^2=(\xi_{th}'F_0(1)\xi_{th})^2+(\gamma_{h,0}^{f})^2-2\gamma_{h,0}^{f}\xi_{th}'F_0(1)\xi_{th}.$$

We have, straightforwardly,

$$E(\xi_{th}'F_0(1)\xi_{th})=E\mathrm{tr}\{F_0(1)\xi_{th}\xi_{th}'\}=\mathrm{tr}\{F_0(1)\Omega_h^{f}\}=b'\sum_{j=0}^{\infty}B_{h,j}\Omega_h^{f}B_{h,j}'b$$

in view of $F_0(1)=\sum_{j=0}^{\infty}B_{h,j}'bb'B_{h,j}$. For the first term, noting that $\xi_{th}=A^{-1}\epsilon_{th}$, we can write

$$\xi_{th}'F_0(1)\xi_{th}=\epsilon_{th}'(A^{-1})'F_0(1)A^{-1}\epsilon_{th}=\epsilon_{th}'F_A\epsilon_{th}$$

where $F_A=(A^{-1})'F_0(1)A^{-1}$. It follows that

$$(\xi_{th}'F_0(1)\xi_{th})^2=\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p}F_{A,ij}F_{A,kl}\epsilon_{i,th}\epsilon_{j,th}\epsilon_{k,th}\epsilon_{l,th}$$

and, using Lemma B1, $E(\xi'_{th}F_0(1)\xi_{th})^2 = \mu_1$ where

$$\mu_1 = \sum_{i,j,k,l=1}^{p} \left\{K_4\phi_{h,ijkl} + \sigma^4(\phi_{h,ij}\phi_{h,kl} + \phi_{h,ik}\phi_{h,jl} + \phi_{h,il}\phi_{h,jk})\right\}F_{A,ij}F_{A,kl}.$$

Hence

$$E(Z_{1,th}) = \mu_1 + (\gamma^f_{h,0})^2 - 2\gamma^f_{h,0}b'\sum_{j=0}^{\infty} B_{h,j}\Omega^f_h B'_{h,j}b.$$

$\underline{Z_{2,th}}$: Using the approach in the proof of Theorem 2(d) we obtain $E(Z_{2,th}) = \mu_2$ where

$$\mu_2 = 4E(\xi^{\dagger\prime}_{th-h}\xi_{th})^2 = 4\mathrm{tr}\left\{E(\xi^{\dagger}_{th-h}\xi^{\dagger\prime}_{th-h})E(\xi_{th}\xi'_{th})\right\} = 4\mathrm{tr}\{\Omega^{f\dagger}_h\Omega^f_h\}$$

and where $\Omega^{f\dagger}_h = E(\xi^{\dagger}_{th-h}\xi^{\dagger\prime}_{th-h}) = \sum_{s=1}^{\infty} F_s(1)'\Omega^f_h F_s(1)$.

$\underline{Z_{3,th}}$: Using $\mathscr{A}_3^c$ we obtain $E(Z_{3,th}) = \mu_3$ where

$$\mu_3 = E\left(\frac{\zeta^4_{th}}{a_0^4}\right) = \frac{K_4 h}{a_0^4}.$$

$\underline{Z_{4,th}}$: Here we have

$$\begin{aligned} E(Z_{4,th}) &= \frac{4}{a_0^2}E\left(b'B_h(L^h)\xi_{th}\zeta^2_{th}\xi'_{th}B_h(L^h)'b\right) \\ &= \frac{4}{a_0^2}E\left(b'\sum_{r=0}^{\infty}B_{h,r}\xi_{th-rh}\zeta^2_{th}\sum_{s=0}^{\infty}\xi'_{th-sh}B'_{h,s}b\right). \end{aligned}$$

The only non-zero expectations occur when $r = s$ and so

$$E(Z_{4,th}) = \frac{4}{a_0^2}b'\sum_{r=0}^{\infty}B_{h,r}E\left(\zeta^2_{th}\xi_{th-rh}\xi'_{th-rh}\right)B'_{h,r}b.$$

The required expectations for $r = 0$ and $r \geq 1$ are provided in Lemma B2 and we obtain $E(Z_{4,th}) = \mu_4$ where

$$\begin{aligned} \mu_4 &= \frac{4}{a_0^2}b'A^{-1}\left(\left(K_4 + \sigma^4 h\right)\Phi_h + 2\sigma^4\phi_{h,\bullet p}\phi'_{h,\bullet p}\right)(A^{-1})'b \\ &\quad + \frac{4\sigma^4 h}{a_0^2}b'\sum_{r=1}^{\infty}B_{h,r}A^{-1}\Phi_h(A^{-1})'B'_{h,r}b. \end{aligned}$$

$\underline{Z_{5,th}}$: It is straightforward to show that

$$E(Z_{5,th}) = 4E\big((\xi'_{th}F_0(1)\xi_{th} - \gamma^f_{h,0})\xi^{\dagger\prime}_{th-h}\xi_{th}\big) = 0$$

owing to the independence of $\xi^{\dagger}_{th-h}$ and ξ_{th}.

$\underline{Z_{6,th}}$: The expectation of interest is

$$E(Z_{6,th}) = \frac{2}{a_0^2}E\left((\xi'_{th}F_0(1)\xi_{th} - \gamma^f_{h,0})\zeta^2_{th}\right).$$

Using Lemma B2(b) we have

$$\begin{aligned}
\mu_6 &= \frac{2}{a_0^2}E(\xi'_{th}F_0(1)\xi_{th}\zeta^2_{th}) \\
&= \frac{2}{a_0^2}\mathrm{tr}\left\{F_0(1)E(\zeta^2_{th}\xi_{th}\xi'_{th})\right\} \\
&= \frac{2}{a_0^2}\mathrm{tr}\left\{F_0(1)A^{-1}\left(\big(K_4+\sigma^4 h\big)\Phi_h + 2\sigma^4\phi_{h,\bullet p}\phi'_{h,\bullet p}\right)(A^{-1})'\right\}
\end{aligned}$$

and hence

$$E(Z_{6,th}) = \mu_6 - \frac{2\sigma^2 h}{a_0^2}\gamma^f_{h,0}.$$

$\underline{Z_{7,th}}$: For this term we have

$$\begin{aligned}
E(Z_{7,th}) &= -\frac{4}{a_0}E\left(\big(\xi'_{th}F_0(1)\xi_{th} - \gamma^f_{h,0}\big)b'B_h(L^h)\xi_{th}\zeta_{th}\right) \\
&= -\frac{4}{a_0}E\left(\big(\xi'_{th}F_0(1)\xi_{th} - \gamma^f_{h,0}\big)b'\sum_{r=0}^{\infty}B_{h,r}\xi_{th-rh}\zeta_{th}\right).
\end{aligned}$$

For the first component he only non-zero expectation occurs when $r = 0$ and so we need to consider

$$E\left(\xi'_{th}F_0(1)\xi_{th}b'\sum_{r=0}^{\infty}B_{h,r}\xi_{th-rh}\zeta_{th}\right) = \sum_{i=1}^{p}\sum_{j=1}^{p}F_{A,ij}\sum_{k=1}^{p}b_k E(\zeta_{th}\epsilon_{i,th}\epsilon_{j,th}\epsilon_{k,th})$$

where F_A was defined in the treatment of $Z_{1,th}$. The expectation of interest is given in Lemma B1(b) and the overall expectation is

$$\mu_7 = -\frac{4}{a_0}\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}F_{A,ij}b_k\left(K_4\phi_{h,ijk} + \sigma^4\big(\phi_{h,ij}\phi_{h,k} + \phi_{h,ik}\phi_{h,j} + \phi_{h,i}\phi_{h,jk}\big)\right).$$

The second component involves

$$E\left(\sum_{r=0}^{\infty} B_{h,r}\xi_{th-rh}\zeta_{th}\right) = E\big(\xi_{th}\zeta_{th}\big) = \sigma^2 A^{-1}\phi_{h,\bullet p}$$

using Lemma B2(a). It follows that

$$E(Z_{7,th}) = \mu_7 + 4\sigma^2\frac{\gamma^f_{h,0}}{a_0}b'A^{-1}\phi_{h,\bullet p}.$$

$\underline{Z_{8,th}}$: We have, straightforwardly, that

$$E(Z_{8,th}) = \frac{4}{a_0^2}E\big(\xi^{\dagger\prime}_{th-h}\xi_{th}\zeta^2_{th}\big) = 0$$

due to the independence of $\xi^{\dagger}_{th-h}$ from ξ_{th} and ζ^2_{th}.

$\underline{Z_{9,th}}$: The expectation of interest is

$$E(Z_{9,th}) = -\frac{8}{a_0}E\big(\xi^{\dagger\prime}_{th-h}\xi_{th}b'B_h(L^h)\xi_{th}\zeta_{th}\big).$$

Using the definitions of $\xi^{\dagger}_{th-h}$ and $B_h(L^h)$ this involves the expectation

$$\begin{aligned}
E\left(\xi^{\dagger\prime}_{th-h}\xi_{th}b'B_h(L^h)\xi_{th}\zeta_{th}\right) &= E\left(\sum_{s=1}^{\infty}\xi'_{th-sh}F_s(1)\xi_{th}b'\sum_{r=0}^{\infty}\xi_{th-rh}\zeta_{th}\right)\\
&= E\left(\mathrm{tr}\left\{\zeta_{th}\xi_{th}b'\sum_{r=0}^{\infty}\xi_{th-rh}\sum_{s=1}^{\infty}\xi'_{th-sh}F_s(1)\right\}\right)\\
&= \mathrm{tr}\left\{\sigma^2A^{-1}\phi_{h,\bullet p}b'\sum_{r=1}^{\infty}B_{h,r}\Omega^f_hF_s(1)\right\}\\
&= \sigma^2b'\sum_{r=1}^{\infty}B_{h,r}\Omega^f_hF_s(1)A^{-1}\phi_{h,\bullet p}
\end{aligned}$$

where we have used Lemma B2(a). Hence $E(Z_{9,th}) = \mu_9$ where

$$\mu_9 = -\frac{8\sigma^2}{a_0}b'\sum_{r=1}^{\infty}B_{h,r}\Omega^f_hF_s(1)A^{-1}\phi_{h,\bullet p}.$$

$\underline{Z_{10,th}}$: The final component is

$$E(Z_{10,th}) = -\frac{4}{a_0^3}E\left(b'B_h(L^h)\xi_{th}\zeta_{th}^3\right) = -\frac{4}{a_0^3}E\left(b'\sum_{r=0}^{\infty}B_{h,r}\xi_{th-rh}\zeta_{th}^3\right).$$

The only non-zero term is that for which $r = 0$ and so $E(Z_{10,th}) = \mu_{10}$ where

$$\begin{aligned}\mu_{10} = -\frac{4}{a_0^3}b'E(\xi_{th}\zeta_{th}^3) &= -\frac{4}{a_0^3}b'A^{-1}E(\epsilon_{th}\zeta_{th}^3)\\ &= -\frac{4}{a_0^3}(K_4+3h\sigma^4)b'A^{-1}\phi_{h,\bullet p}\end{aligned}$$

using Lemma B1(b).

We have therefore established that

$$\begin{aligned}v_h^f(0) = \sum_{i=1}^{10}E(Z_{i,th}) &= \mu_1 + (\gamma_{h,0}^f)^2 - 2\gamma_{h,0}^f b'\sum_{j=0}^{\infty}B_{h,j}\Omega_h^f B_{h,j}'b + \mu_2 + \mu_3\\ &\quad +\mu_4 + \mu_6 - \frac{2\sigma^2 h}{a_0^2}\gamma_{h,0}^f + \mu_7 + 4\sigma^2\frac{\gamma_{h,0}^f}{a_0}b'A^{-1}\phi_{h,\bullet p} + \mu_9 + \mu_{10}.\end{aligned}$$

It can be shown that

$$\begin{aligned}&-2\gamma_{h,0}^f b'\sum_{j=0}^{\infty}B_{h,j}\Omega_h^f B_{h,j}'b - \frac{2\sigma^2 h}{a_0^2}\gamma_{h,0}^f + \frac{4\sigma^2\gamma_{h,0}^f}{a_0}b'A^{-1}\phi_{h,\bullet p}\\ &\quad= -2\gamma_{h,0}^f\underbrace{\left(b'\sum_{j=0}^{\infty}B_{h,j}\Omega_h^f B_{h,j}'b + \frac{\sigma^2 h}{a_0^2} - \frac{2\sigma^2}{a_0}b'A^{-1}\phi_{h,\bullet p}\right)}_{\gamma_{h,0}^f}\\ &\quad= -2\left(\gamma_{h,0}^f\right)^2\end{aligned}$$

and hence

$$v_h^f(0) = \mu_1+\mu_2+\mu_3+\mu_4+\mu_6+\mu_7+\mu_9+\mu_{10}-(\gamma_{h,0}^f)^2$$

as required. □

Supplementary results

Lemma B1. *Let ϵ_{th} be defined as in Theorem 1(c). Then:*

(a) Under Assumption 2 or $\mathscr{A}_2^c$,

$$E\big(\epsilon_{j,nh-uh}\epsilon_{i,nh-rh-uh}\epsilon_{k,nh-vh}\epsilon_{l,nh-sh-vh}\big) = \sigma^4\big(\phi_{h,il}\phi_{h,jk} + \phi_{h,ik}\phi_{h,jl}\big).$$

(b) Under $\mathscr{A}_3^c$,

$$\begin{aligned}
E(\epsilon_{i,th}\epsilon_{j,th}\epsilon_{k,th}\epsilon_{l,th}) &= K_4\phi_{h,ijkl} + \sigma^4\big(\phi_{h,ij}\phi_{h,kl} + \phi_{h,ik}\phi_{h,jl} + \phi_{h,il}\phi_{h,jk}\big),\\
E\big(\zeta_{th}\epsilon_{i,th}\epsilon_{j,th}\epsilon_{k,th}\big) &= K_4\phi_{h,ijk} + \sigma^4\big(\phi_{h,ij}\phi_{h,k} + \phi_{h,ik}\phi_{h,j} + \phi_{h,i}\phi_{h,jk}\big),\\
E\big(\zeta_{th}^3\epsilon_{th}\big) &= (K_4 + 3h\sigma^4)\phi_{h,\bullet p},
\end{aligned}$$

where $\phi_{h,ij}$ and $\phi_{h,ijkl}$ are defined in Theorem 2(d),

$$\begin{aligned}
\phi_{h,i} &= \int_0^h \Theta_{ip}(r)dr,\\
\phi_{h,ijk} &= \int_0^h \Theta_{ip}(r)\Theta_{jp}(r)\Theta_{kp}(r)dr,\\
\phi_{h,\bullet p} &= \int_0^h \Theta_{\bullet p}(r)dr,
\end{aligned}$$

and K_4 is defined in Table 1.

Proof of Lemma B1. (a) The expectation of interest is,using (1),

$$\begin{aligned}
&E\big(\epsilon_{j,nh-uh}\epsilon_{i,nh-rh-uh}\epsilon_{k,nh-vh}\epsilon_{l,nh-sh-vh}\big)\\
&\quad= \int_{nh-uh-h}^{nh-uh}\Theta_{jp}(nh-uh-r_j)\int_{nh-rh-uh-h}^{nh-rh-uh}\Theta_{ip}(nh-rh-uh-r_i)\\
&\quad\times\int_{nh-vh-h}^{nh-vh}\Theta_{kp}(nh-vh-r_k)\int_{nh-sh-vh-h}^{nh-sh-vh}\Theta_{lp}(nh-sh-vh-r_l)\\
&\quad\times E\big(\zeta(dr_j)\zeta(dr_i)\zeta(dr_k)\zeta(dr_l)\big).
\end{aligned}$$

The non-zero expectations occur when: (i) $u = v$, $r = s$, $r_i = r_l$ and $r_j = r_k$, and (ii)

$u = s + v$ and $v = r + u$ with $r_i = r_k$ and $r_j = r_l$. In case (i) the resulting expectation is

$$\begin{aligned}
&\int_{nh-rh-uh-h}^{nh-rh-uh} \Theta_{ip}(nh-rh-uh-r_i) \int_{nh-rh-uh-h}^{nh-rh-uh} \Theta_{lp}(nh-rh-uh-r_i) E\big(\zeta(dr_i)^2\big) \\
&\quad \times \int_{nh-uh-h}^{nh-uh} \Theta_{jp}(nh-uh-r_j) \int_{nh-uh-h}^{nh-uh} \Theta_{kp}(nh-uh-r_j) E\big(\zeta(dr_j)^2\big) \\
&\quad = \sigma^2 \int_{nh-rh-uh-h}^{nh-rh-uh} \Theta_{ip}(nh-rh-uh-r_i)\Theta_{lp}(nh-rh-uh-r_i) dr_i \\
&\quad \times \sigma^2 \int_{nh-uh-h}^{nh-uh} \Theta_{jp}(nh-uh-r_j)\Theta_{kp}(nh-uh-r_j) dr_j \\
&\quad = \sigma^4 \int_0^h \Theta_{ip}(s)\Theta_{lp}(s) ds \int_0^h \Theta_{jp}(s)\Theta_{kp}(s) ds = \sigma^4 \phi_{h,il}\phi_{h,jk}
\end{aligned}$$

using the notation in Theorem 2(d). In case (ii) we obtain, using similar arguments,

$$\begin{aligned}
&\int_{nh-rh-sh-vh-h}^{nh-rh-sh-vh} \Theta_{ip}(nh-rh-sh-vh-r_i) \\
&\quad \times \int_{nh-rh-sh-vh-h}^{nh-rh-sh-vh} \Theta_{kp}(nh-rh-sh-vh-r_i) E\big(\zeta(dr_i)^2\big) \\
&\quad \times \int_{nh-sh-vh-h}^{nh-sh-vh} \Theta_{jp}(nh-sh-vh-r_j) \int_{nh-sh-vh-h}^{nh-sh-vh} \Theta_{lp}(nh-sh-vh-r_j) E\big(\zeta(dr_j)^2\big) \\
&\quad = \sigma^4 \int_0^h \Theta_{ip}(s)\Theta_{kp}(s) ds \int_0^h \Theta_{jp}(s)\Theta_{lp}(s) ds = \sigma^4 \phi_{h,ik}\phi_{h,jl}.
\end{aligned}$$

Combining these two expressions yields the desired result.

(b) The first expectation of interest is

$$\begin{aligned}
E(\epsilon_{i,th}\epsilon_{j,th}\epsilon_{k,th}\epsilon_{l,th}) \;=\; & E\left(\int_{th-h}^{th} \Theta_{ip}(th-r_i)\zeta(dr_i) \int_{th-h}^{th} \Theta_{jp}(th-r_j)\zeta(dr_j) \right. \\
& \left. \times \int_{th-h}^{th} \Theta_{kp}(th-r_k)\zeta(dr_k) \int_{th-h}^{th} \Theta_{lp}(th-r_l)\zeta(dr_l) \right).
\end{aligned}$$

The non-zero expectations occur if: (i) $r_i = r_j = r_k = r_l$; (ii) $r_i = r_j \neq r_k = r_l$; (iii) $r_i = r_k \neq r_j = r_l$; and (iv) $r_i = r_l \neq r_j = r_k$. In case (i) the expectation is

$$\int_{th-h}^{th} \Theta_{ip}(th-r)\Theta_{jp}(th-r)\Theta_{kp}(th-r)\Theta_{lp}(th-r) E\big(\zeta(dr)^4\big) = K_4 \phi_{h,ijkl}$$

while in case (ii) we obtain

$$\int_{th-h}^{th} \Theta_{ip}(th-r_i)\Theta_{jp}(th-r_i) E\big(\zeta(dr_i)^2\big) \int_{th-h}^{th} \Theta_{kp}(th-r_k)\Theta_{lp}(th-r_k) E\big(\zeta(dr_k)^2\big) = \sigma^4 \phi_{h,ij}\phi_{h,kl}.$$

Cases (iii) and (iv) follow in a similar fashion, yielding the stated result.

For the second expectation we have

$$\begin{aligned} E(\zeta_{th}\epsilon_{i,th}\epsilon_{j,th}\epsilon_{k,th}) &= E\left(\int_{th-h}^{th}\zeta(dr_l)\int_{th-h}^{th}\Theta_{ip}(th-r_i)\zeta(dr_i)\right. \\ &\quad \left.\times\int_{th-h}^{th}\Theta_{jp}(th-r_j)\zeta(dr_j)\int_{th-h}^{th}\Theta_{kp}(th-r_k)\zeta(dr_k)\right). \end{aligned}$$

This is just a special case of the first expectation in which $\Theta_{lp}(r) = 1$ and so the result is immediate.

For the final expectation we need to consider

$$E(\zeta_{th}^3\epsilon_{th}) = E\left(\int_{th-h}^{th}\zeta(dr_i)\int_{th-h}^{th}\zeta(dr_j)\int_{th-h}^{th}\zeta(dr_k)\int_{th-h}^{th}\Theta(th-r_l)e_p\zeta(dr_l)\right).$$

The non-zero expectations occur for the same four combinations of $r_i - -r_l$ as for the first expectation in part (b). Evaluating these in turn and recalling that $\Theta(r)e_p = \Theta_{\bullet p}(r)$ yields the desired result. □

Lemma B2. *Let ξ_{th} be defined as in Theorem 2(c). Then:*
(a) Under Assumption 2 or $\mathscr{A}_2^c$,

$$\begin{aligned} E\big(\zeta_{th}\xi_{th}\big) &= \sigma^2 A^{-1}\phi_{h,\bullet p}, \\ E\big(\zeta_{th}^2\xi_{th-rh}\xi_{th-rh}'\big) &= \sigma^4 h A^{-1}\Phi_h(A^{-1})', \quad r \geq 1. \end{aligned}$$

(b) Under $\mathscr{A}_3^c$,

$$E\big(\zeta_{th}^2\xi_{th}\xi_{th}'\big) = A^{-1}\Big(\big(K_4 + \sigma^4 h\big)\Phi_h + 2\sigma^4\phi_{h,\bullet p}\phi_{h,\bullet p}'\Big)(A^{-1})',$$

where $\phi_{h,\bullet p}$ is defined in Lemma B1,

$$\Phi_h = \int_0^h \Theta_{\bullet p}(s)\Theta_{\bullet p}(s)'ds,$$

and K_4 is defined in Table 1.

Proof of Lemma B2. (a) We have

$$\begin{aligned} E\big(\zeta_{th}\xi_{th}\big) &= A^{-1}E(\zeta_{th}\epsilon_{th}) \\ &= A^{-1}E\left(\int_{th-h}^{th}\zeta(dr)\int_{th-h}^{th}\Theta_{\bullet p}(th-s)\zeta(ds)\right) \\ &= \sigma^2 A^{-1}\int_0^h \Theta_{\bullet p}(r)dr. \end{aligned}$$

For the second expectation, as $r \geq 1$,

$$\begin{aligned} E\big(\zeta_{th}^2\xi_{th-rh}\xi'_{th-rh}\big) &= A^{-1}E\left(\int_{th-h}^{th}\zeta(dr_i)\int_{th-h}^{th}\zeta(dr_j)\int_{th-rh-h}^{th-rh}\Theta_{\bullet p}(th-rh-r_k)\zeta(dr_k)\right. \\ &\quad \left.\times\int_{th-rh-h}^{th-rh}\Theta_{\bullet p}(th-rh-r_l)'\zeta(dr_l)\right)(A^{-1})' \\ &= E\left(\int_{th-h}^{th}\zeta(dr_i)\int_{th-h}^{th}\zeta(dr_j)\right)A^{-1}E\left(\int_{th-rh-h}^{th-rh}\Theta_{\bullet p}(th-rh-r_k)\right. \\ &\quad \left.\times\zeta(dr_k)\int_{th-rh-h}^{th-rh}\Theta_{\bullet p}(th-rh-r_l)'\zeta(dr_l)\right)(A^{-1})' \\ &= \sigma^4 hA^{-1}\int_0^h \Theta_{\bullet p}(s)\Theta_{\bullet p}(s)'ds(A^{-1})' \end{aligned}$$

as required.

(b) The expectation under consideration is

$$\begin{aligned} E\big(\zeta_{th}^2\xi_{th-rh}\xi'_{th-rh}\big) &= A^{-1}E\left(\int_{th-h}^{th}\zeta(dr_i)\int_{th-h}^{th}\zeta(dr_j)\int_{th-rh-h}^{th-rh}\Theta_{\bullet p}(th-rh-r_k)\zeta(dr_k)\right. \\ &\quad \left.\times\int_{th-rh-h}^{th-rh}\Theta_{\bullet p}(th-rh-r_l)'\zeta(dr_l)\right)(A^{-1})'. \end{aligned}$$

The non-zero expectations occur if: (i) $r_i = r_j = r_k = r_l$; (ii) $r_i = r_j \neq r_k = r_l$; (iii) $r_i = r_k \neq r_j = r_l$; and (iv) $r_i = r_l \neq r_j = r_k$. In case (i) the expectation is

$$A^{-1}\int_{th-h}^{th}\Theta_{\bullet p}(th-r)\Theta_{\bullet p}(th-r)'E\big(\zeta(dr)^4\big)(A^{-1})' = K_4A^{-1}\int_0^h\Theta_{\bullet p}(s)\Theta_{\bullet p}(s)'ds(A^{-1})'.$$

In case (ii) we find the expectation to be

$$\begin{aligned} &A^{-1}\int_{th-h}^{th}E\big(\zeta(ds)^2\big)\int_{th-h}^{th}\Theta_{\bullet p}(th-r)\Theta_{\bullet p}(th-r)'E\big(\zeta(dr)^2\big)(A^{-1})' \\ &\qquad = \sigma^4 hA^{-1}\int_0^h\Theta_{\bullet p}(s)\Theta_{\bullet p}(s)'ds(A^{-1})'. \end{aligned}$$

In case (iii) the expectation is

$$A^{-1}\int_{th-h}^{th}\Theta_{\bullet p}(th-s)E\big(\zeta(ds)^2\big)\int_{th-h}^{th}\Theta_{\bullet p}(th-r)'E\big(\zeta(dr)^2\big)(A^{-1})'$$
$$=\sigma^4A^{-1}\int_0^h\Theta_{\bullet p}(s)ds\int_0^h\Theta_{\bullet p}(s)'ds(A^{-1})',$$

which is also the expectation in case (iv). Combining these four expressions yields the desired result. □

Reference

Phillips, P.C.B., Solo, V., 1992. Asymptotics for linear processes, *Annals of Statistics* 20, 971–1001.