Correlated Equilibrium and the Pricing of Public Goods*

Joseph M. Ostroy†
Department of Economics
UCLA

Joon Song‡
Department of Economics
University of Essex

Autumn 2006

Abstract

Lindahl equilibrium is an application of price-taking behavior to achieve efficiency in the allocation of public goods. Such an equilibrium requires individuals to be strategically naive, i.e., Lindahl equilibrium is not incentive compatible. Correlated equilibrium is defined precisely to take account of strategic behavior and incentive compatibility. Using the duality theory of linear programming, we show that these two seemingly disparate notions can be combined to give a public goods, Lindahl pricing characterization of efficient correlated equilibria. We also show that monopoly theory can be used to characterize inefficient correlated equilibria.

1 Introduction

A public good is an outcome that is jointly consumed by everyone. Compared to private goods, duality theory for public goods exhibits the following well-known reversal: Instead of outcomes that are personalized (different allocations to different individuals) and prices that are impersonal (the same to all), with public goods the outcome is impersonal, while prices are personalized. The fact that the prices each individual faces are personalized — and depend on the individual's tastes — is evidence of their strategic manipulability, i.e., implicit in the formulation are incentives to deviate.

*Thanks to David Rahman for helpful discussions.
†Contact: ostroy@ucla.edu
‡Contact: joonsong@ucla.edu

For a game in normal form, each individual's payoff depends on the play of the game. Hence, a play of the game fits the definition of a public good. A correlated equilibrium (Aumann [1974]) is a (randomized) play from which no individual has an incentive to deviate. Hart and Schmeidler [1989] and Nau and McCardle [1990] have shown that the existence of correlated equilibrium can be demonstrated using the duality techniques of linear programming.

The goal of this paper is to integrate the public goods and correlated equilibrium dualities.

First we shall characterize Lindahl equilibrium for a game in normal form without incentive constraints as a linear programming problem. The model can be described as individual consumers (the players) purchasing the public good (a randomized play of the game) from a supplier with a zero cost, constant returns to scale technology.

Next, we add the incentive constraints that define correlated equilibrium to the above LP problem. Their burden falls on the supplier whose technology will, in effect, be reduced by the imposition of these incentive constraints. A dual consequence is that prices will change from those characterizing first-best outcomes to prices that take account of strategic behavior. This leads to a characterization of the second-best, or incentive efficient, correlated equilibrium — those achieving a maximum weighted sum of payoffs — as Lindahl equilibria for the incentive constrained model of public goods.

Incentive constrained efficient Lindahl equilibria have the property that the supplier of the public good is a price-taker earning zero profits. Other correlated equilibria do not share that property. We focus on worst cases — those correlated equilibria achieving a minimum weighted sum of payoffs. We characterize such outcomes as equilibria in which the supplier acts as a monopolist whose goal is to extract as much as possible from his customers by imposing quantity constraints on their purchases.

Myerson [1991] has described the implementation of correlated equilibria via a mediator. To summarize our results in these terms, distinguish between a mediator facing competition from a elastic supply of identical mediators and a mediator for whom there are no substitutes. The mediator-supplier facing competition will implement a Lindahl equilibrium while a mediator-supplier without competitors will implement a monopoly equilibrium.

2 Pricing and efficiency in normal form game

Let $A = A_1 \times A_2 \times \cdots \times A_n$ be the finite set of strategies and

$$Z(A) = \{z \in \mathbb{R}^{|A|} : z(a) \geq 0, \sum_a z(a) = 1\}$$

the set of randomized play of the game. The payoff function for i is $v_i : A \to \mathbb{R}$. The utility of z to i is

$$v_i \cdot z = \sum_a v_i(a) z(a)$$

Fixing A, a game is defined by $\mathbf{v} = (v_1, \ldots, v_n)$.

2.1 A model of public goods

We ignore incentive compatibility in this section and focus on the public good interpretation of game. Therefore, we are not analyzing a game *per se*.

The list $\mathbf{v}$ can also be regarded as the utility functions of an economic model. For this interpretation, A will denote an index set of $|A|$ objects, e.g. houses. A distinctive feature of the analysis in this section is that there is no particular connection between i and A_i. Without changing notation, extend the definition of v_i from A to $v_i : \mathbb{R}^{|A|} \to \mathbb{R} \cup \{-\infty\}$ as

$$v_i(z) = \begin{cases} v_i \cdot z & \text{if } z \in Z(A) \\ -\infty & \text{otherwise} \end{cases}$$

Denote by $e_a \in Z(A)$ the element such that $e_a(a') = 0$ for all $a' \neq a$. The utility function v_i states that i can live in one house or split his "time" among them. In the following, the aim is to establish an equilibrium in which *all* individuals choose to live in the same house, or more generally, the *same mixture* of houses.[1]

Let $p_i : A \to \mathbb{R}$ be the (rental) prices charged to i for each $a \in A$. The opportunity to purchase z yields the indirect utility function

$$v_i^*(p_i) = \max_{z \in Z(A)} \{v_i \cdot z - p_i \cdot z\}$$

The maximization underlying $v_i^*(p_i)$ presumes that the individual can choose any (mixture of) a; and, prices are inescapable in that the choice z cannot be separated from the money payment

[1] The function v_i is similar to a buyer's utility in a commodity representation of the assignment model, e.g. Shapley and Shubik [1971].

$p_i \cdot z$. The function v_i^* is the indirect utility or conjugate of v_i which describes the preferences of an individual with quasi-linear utility $u_i(z, m) = v_i \cdot z + m$ facing a budget constraint $p_i \cdot z + m = 0$.

Introduce a market by adding individual 0, the supplier of z. The supplier's cost function is

$$c_0(z) = \begin{cases} 0 & \text{if } z \in \mathbb{R}_+^{|A|} \\ \infty & \text{otherwise} \end{cases}$$

Hence, the supplier can costlessly provide any e_a or non-negative linear combination thereof.

The supplier's objective at prices $p_0 : A \to \mathbb{R}$ is to maximize profits,

$$\sup_z \{p_0 \cdot z - c_0(z)\} = \sup_{z \geq 0} \{p_0 \cdot z\} = \sup_{z(a) \geq 0} \{p_0(a) z(a)\}.$$

If we use the sign convention that the output price (input price) is negative (positive), we can write the supplier's objective in a similar form to the buyers' as $v_0^*(p_0) = \sup_z \{v_0(z) - p_0 \cdot z\}$ where $v_0(z) = -c_0(z)$.

The hypothesis that the supplier can costlessly provide any non-negative quantities is adopted to guarantee that profits will be zero in equilibrium. Thus, if profits are maximized at z_0,

$$(1) \qquad v_0^*(p_0) = v_0(z_0) - p_0 \cdot z_0 = -p_0 \cdot z_0 = 0$$

$$(2) \qquad p_0(z) \geq 0$$

$$(3) \qquad z_0(a) > 0 \text{ implies } p_0(a) = 0.$$

Formally, $v_0 : \mathbb{R}^{|A|} \to \{0\} \cup \{-\infty\}$ is the (concave) indicator function of the cone $\mathbb{R}_+^{|A|}$ and $v_0^* : \mathbb{R}^{|A|} \to \{0\} \cup \{\infty\}$ is its support function. As the support function of a cone, v_0^* is the (convex) indicator function of the polar cone $\{p : pz \leq 0, \forall z \in \mathbb{R}_+^{|A|}\}$.

Definition 1 *A Lindahl equilibrium* WITH TRANSFERS *for* $\mathbf{v}$ *is a* $([z_i, p_i], z_0, p_0)$ *such that:*

$$(a) \qquad v_i(z_i) - p_i \cdot z_i = v_i^*(p_i) \text{ for all } i$$

$$(b) \qquad v_0(z_0) - p_0 \cdot z_0 = v_0^*(p_0)$$

$$(c) \qquad z_i = z_0 \text{ for all } i$$

$$(d) \qquad \sum_i p_i + p_0 = 0.$$

This price-taking equilibrium satisfies the usual conditions of (a) utility maximization for the buyers, (b) profit maximization for the supplier, and (c) "market clearance" for public goods in which the supplier simultaneously supplies z_0 to each i while (d) receiving the sum of the prices paid by the buyers. Conditions (c) and (d) are familiar conditions for Lindahl equilibrium.

We distinguish between equilibrium with and without money transfers. It was remarked above that a necessary condition for equilibrium is that the money the supplier receives from buyers, his profits, should be zero. But, the fact that $\sum_i p_i \cdot z_i = -p \cdot z_0 = 0$ in equilibrium does not imply that

$$(e) \qquad p_i \cdot z_i = 0, \forall i.$$

Definition 2 *A Lindahl equilibrium for* $\mathbf{v}$ *is a* $([z_i, p_i], z_0, p_0)$ *satisfying (a)–(e).*

The distinction between equilibrium and equilibrium with transfers is a well-known method of using the simplifying hypothesis of quasi-linearity, i.e., when transfers are possible, to obtain results that apply to non-transferable utility models that do not, in fact, require quasi-linearity. See Shapley (1969). Except when explicitly mentioned otherwise (see Corollary 1, in Section 3.2), Lindahl equilibrium is restricted to Definition 2.

The definition of efficiency, or Pareto-optimality, in $\mathbf{v}$ also treats A as an index set, ignoring the relation between i and A_i.

Definition 3 *A* $z \in Z(A)$ *is efficient for* $\mathbf{v}$ *if there is no* $z' \in Z(A)$ *such that* $v_i \cdot z' \geq v_i \cdot z$ *for all* i *and at least one inequality is strict. Regarding* $\mathbf{v}$ *as an economic model to which a supplier added, the statement of efficiency is: there exists* $[(z_i), z_0]$ *such that (i)* $z_i = z_0$ *for all* i *and (ii)* $z_i \in Z(A)$ *and there is no* $[(z_i'), z_0']$ *satisfying (i) and (ii) for which utility is at least as large for all individuals and strictly greater for at least one* i*.*

The utility outcomes in $\mathbf{v}$ are

$$\{(v_1 \cdot z, v_2 \cdot z, \ldots, v_n \cdot z) : z \in Z(A)\},$$

a set that is convex because $Z(A)$ is. Let $\Delta = \{\lambda = (\lambda_i) : \lambda_i \geq 0, \sum_i \lambda_i = 1\}$. If $\lambda \gg 0$, then any $z \in Z(A)$ such that

$$\left(\sum_i \lambda_i v_i\right) \cdot z = \max_{z' \in Z(A)} \left(\sum_i \lambda_i v_i\right) \cdot z' = \max_a \left(\sum_i \lambda_i v_i\right) \cdot (a)$$

is efficient.

If z achieves the above maximum when one or more $\lambda_i = 0$, we can only conclude that z is weakly efficient, i.e., there is no $z' \in Z(A)$ such that $v_i \cdot z' > v_i \cdot z$ for all i. Here we concentrate on efficient z associated with $\lambda \gg 0$.

Definition 4 *An efficient z associated with $\lambda \gg 0$ is one that solves*

$$\max_z \left(\sum_i \lambda_i v_i \right) \cdot z$$

when $\lambda_i > 0$ for each i.

Efficiency does not require randomization and is consistent with it only if two or more plays of the game yield the same highest weighted total utility. Because concepts used in section 2.3 will require randomization, efficiency is also described in these terms.

Lindahl equilibrium describes a decentralized way to establish efficiency with public goods.

Theorem 1 [First Welfare Theorem] *A Lindahl equilibrium is efficient.* [Second Welfare Theorem] *For any efficient z associated with $\lambda \gg 0$, there are prices p_i and p_0 such that $([p_i, z_i], p_0, z_0)$ is a Lindahl equilibrium with $z = z_i = z_0$.*

2.1.1 Proof of Theorem 1 and Characterization of Welfare Theorems

The existence of Lindahl equilibrium is an application of linear programming in which optimal solutions to the primal describe the set of efficient z's and optimal solutions to the dual solution define Lindahl prices. The primal linear programming (the planner's problem) is

$$\begin{aligned} \max_{z_i, z_0} \quad & \sum_i \lambda_i \left(v_i \cdot z_i \right) \\ \text{subject to} \quad & \sum_a z_i(a) = 1, \quad i = 1, \ldots, n \\ & z_i(a) - z_0(a) = 0, \quad \forall a \\ & z_i(a), z_0(a) \geq 0 \end{aligned}$$

The first constraint states that the mixture of outcomes for each player has to sum to one and the second constraint states that the choice of each player has to be the same since the outcome of a game is interpreted as a public good. The third constraint is the mixtures have to be non-negative. So, the first and the third imply that mixtures are probabilities.

It is known that the (maximization) linear programming has a dual (minimization) linear programming that achieves the same value as the primal linear programming. The dual linear programming (the decentralization) is

$$\begin{array}{ll} \min\limits_{y_i, p_i} & \sum\limits_i y_i \\ \text{subject to} & y_i \geq \lambda_i v_i(a) - p_i(a), \quad \forall i, a \\ & 0 \geq \sum\limits_i p_i(a) \end{array}$$

For the characterization of the welfare theorems, we prove a lemma.

Lemma 1 *For each $\lambda \gg 0$ and an efficient z associated with the λ, there exists a dual solution such that $p_i \cdot z = 0$ for all i.*

Proof. Suppose not, so that $p_i \cdot z_i \neq 0$ for some $i \in I$. Define new dual variables as in the following.

$$\begin{aligned} \hat{y}_i &:= y_i + p_i \cdot z_i \\ \hat{p}_i(a) &:= p_i(a) - p_i \cdot z_i \end{aligned}$$

where $z(a)$ is the optimal solution of the primal.

Then, the first dual constraints are trivially satisfied. For the second dual constraints, observe that $\sum_i p_i(a) = 0$ at the optimal value of the dual by the fundamental theorem of linear programming. Therefore, $\sum_i p_i \cdot z_0 = \sum_a \sum_i p_i(a) z_0(a) = 0$; hence, we have

$$\sum_i \hat{p}_i(a) = \sum_i p_i(a) - \sum_i p_i \cdot z_i = \sum_i p_i(a),$$

which shows that the second dual constraints hold too.

Moreover, the value of the dual will be the same as before since

$$\sum_i \hat{y}_i = \sum_i y_i + \sum_i \hat{p}_i \cdot z = \sum_i y_i$$

from the constraint for the producer. ∎

Note that $p_0 \cdot z = 0$ is automatically derived from Lemma 1. After taking an optimal solution (p_i, p_0) such that $p_i \cdot z = 0$ and $p_0 \cdot z = 0$, we show how to interpret the dual constraints in terms of players' and the supplier's optimization problems.

Players' Optimization

y_i is the dual variable for individual defined by his probability constraint; hence, it measures the value of the presence of individual i.

To illustrate how the first constraints are interpreted as optimization problem of players, sum the first dual constraints weighted by $z_i(a)$ to get

$$y_i \geq \lambda_i v_i \cdot z_i - p_i \cdot z_i.$$

By the fundamental theorem of linear programming, if z_i were an optimal choice of the primal linear programming, we have the following.

$$y_i = \lambda_i v_i \cdot z_i - p_i \cdot z_i$$

Since we have chosen p_i so that $p_i \cdot z = 0$, we get $y_i = \lambda_i v_i \cdot z_i$. Therefore,

- if $p_i \cdot z_i < 0$, i.e. if player i saved some money by trading probability, his choice would not be optimal since

  $$y_i \geq \lambda_i v_i \cdot z_i - p_i \cdot z_i \geq \lambda_i v_i \cdot z_i.$$

  In other words, player i does not achieve utility level of y_i/λ_i.
- if $p_i \cdot z_i > 0$, z_i is not feasible to player i.
- if $p_i \cdot z_i = 0$, and player i chooses other probability than the optimal probability,

  $$y_i \geq \lambda_i v_i \cdot z_i$$

  In other words, player i cannot get more utility by choosing a probability other than the optimal one solved by the planner.

Therefore, we have shown that the first dual constraints are interpreted as players' optimization problems as long as $\lambda \gg 0$.

Supplier's Optimization

For the supplier, by adding the second constraints with a certain probability $z_0(a)$, we get

$$0 \geq \sum_i p_i \cdot z_0.$$

- if $z_0 \notin \mathbb{R}_+^{|A|}$, then the inequality does not hold since the direction of inequality would not be guaranteed.
- if $z_0 \in \mathbb{R}_+^{|A|}$, and if the supplier chooses other probability than z,

  $$0 \geq \sum_i p_i \cdot z_0.$$

In other words, the supplier cannot get more utility by choosing a probability other than the optimal one solved by the planner.

We have shown that the choice of the supplier is consistent with the planner's.

Proof of Welfare Theorems

Again by the fundamental theorem of linear programming, the value of the primal and the value of the dual are the same; hence we proved the second welfare theorem that the planner's problem is decentralized by the Lindahl price. The proof of the first welfare theorem also follows from the proof of the second welfare theorem.

Remark: Note that the decentralization possibility does not depend upon weights λ. In a private good economy, decentralization without transfer of money among agents is possible only when weights are carefully chosen. For example, in a two person trade economy with strictly concave utility functions, only one point on the contract curve can be decentralized without money transfers. However, in the public goods model, above, the entire frontier can be decentralized.

The second constraint of the primal

$$z_i(a) - z_0(a) = 0, \forall i, \forall a,$$

makes the individual probability constraints redundant. For example, the value of the primal would be the same even if we have individual probability constraints only for one individual. This redundancy implies a considerable degree of freedom in choosing the value of the dual variables; hence, the entire utility frontier can be decentralized.

[**Note to Authors themselves**] The crucial reason for decentralization of the entire utility frontier are: (i) public good situation and (ii) non-existing algebraic structure in the choice set.

2.2 Correlated Equilibrium: Examples

Definition 5 *Correlated equilibrium is a probability distribution z_0 on set A satisfying the following incentive compatibility conditions*

$$\sum_{a_{-i}} [v_i(a_{-i}, d_i) - v_i(a_{-i}, a_i)]\, z_0(a) \leq 0, \forall i, a_i, d_i.$$

Typical interpretation of correlated equilibrium is: (i) there is a randomization device with probability distribution z_0, (ii) when $a = (a_i)_{i \in \{1,\ldots,n\}}$ is realized by the randomization device, player i is recommended to play a_i without knowing which actions were recommended to other players. Therefore, the incentive compatibility constraints (divided by $\sum_{\tilde{a}:\tilde{a}_i=a_i} z_0(\tilde{a})$) describes that expected utility gain by deviating to d_i from recommended a_i is smaller or equal to zero; hence, there is no incentive to deviate.

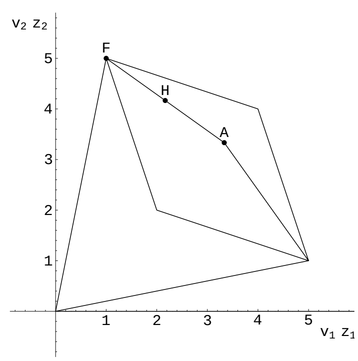
Before formulating a model with the incentive compatibility constraints of correlated equilibria, we present an example showing informally how correlated equilibrium can be decentralized.

Let the game be the following.

	b_1	b_2
a_1	$5, 1$	$0, 0$
a_2	$4, 4$	$1, 5$

Here, $A = \{a_1, a_2\} \times \{b_1, b_2\} = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)\}$.

All the possible correlated equilibria are illustrated in terms of payoff in the following graph.



The outside quadrangle represents all the possible outcomes by randomization. The inner quadrangle represents all the correlated equilibria. Without asymmetric information, efficient allocation when weights on the two players are the same is $z(a_2, b_1) = 1$ that can achieve payoff $(4, 4)$. However, the best correlated equilibrium with the same weights on the two players is A. We say that A is *constrained efficient*. Formal definition will follow in section 2.3.

The points A, H, and F can be decentralized by the following prices.

	A	H	F
Payoffs	(10/3, 10/3)	(13/6,25/6)	(1, 5)
Probability	$\begin{matrix} 1/3 & 0 \\ 1/3 & 1/3 \end{matrix}$	$\begin{matrix} 1/6 & 0 \\ 1/6 & 2/3 \end{matrix}$	$\begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}$
Price for 1	$\begin{matrix} 5/3 & -10/3 \\ 2/3 & -7/3 \end{matrix}$	$\begin{matrix} 17/6 & -13/6 \\ 11/6 & -7/6 \end{matrix}$	$\begin{matrix} 4 & -1 \\ 3 & 0 \end{matrix}$
Price for 2	$\begin{matrix} -49/15 & -14/3 \\ 14/15 & 7/3 \end{matrix}$	$\begin{matrix} -133/30 & -35/6 \\ -7/30 & 7/6 \end{matrix}$	$\begin{matrix} -28/5 & -7 \\ -7/5 & 0 \end{matrix}$
Price for the supplier	$\begin{matrix} -8/5 & -8 \\ 8/5 & 0 \end{matrix}$	$\begin{matrix} -8/5 & -8 \\ 8/5 & 0 \end{matrix}$	$\begin{matrix} -8/5 & -8 \\ 8/5 & 0 \end{matrix}$

Note that the prices are normalized so that the marginal utility of income is 1 for player 1, and 7/5 for player 2. In other words, if any of them were given ϵ amount of money, their utility will increase by ϵ or $7\epsilon/5$. Also note that the prices are such that the supplier's profit is zero, i.e. actuarially fair prices, which could be argued through the competition of the suppliers.

Player i's problem is

$$\max_{z_i} v_i \cdot z_i \quad \text{subject to} \quad \sum_a z_i(a) = 1, p_i \cdot z_i = 0$$

For point A, it can be easily verified that $(z_i(a_1, b_1), z_i(a_1, b_2), z_i(a_2, b_1), z_i(a_2, b_2)) = (1/3, 0, 1/3, 1/3)$ is an optimal solution of player 1. The similar can be shown for point F and H.

The supplier also would choose the same probability as players. For point A, the supplier wants to sell more (a_2, b_1) since the profit is higher. However, if she sells more (a_2, b_1), the incentive compatibility condition would break. Under the premise that the supplier gets very large negative utility when she sells incentive non-compatible probability, the supplier would not sell (a_2, b_1) more than 1/3 amount. Rigorously, the supplier's problem is

$$\max_{z_0} p_0 \cdot z_0 \quad \text{subject to} \quad z_0(a) \geq 0, \sum_{a_{-i}} [v_i(a_{-i}, d_i) - v_i(a_{-i}, a_i)]\, z_0(a) \leq 0, \forall i, a_i, d_i.$$

It is easily verified that $(z_0(a_1, b_1), z_0(a_1, b_2), z_0(a_2, b_1), z_0(a_2, b_2)) = (1/3, 0, 1/3, 1/3)$ is an optimal solution of the supplier. And the similar can be shown for point F and H.

2.3 Correlated equilibrium and public goods

The modeling is the same to that of section 2.1 with the important exception that the supplier can now sell only those z_0 that are incentive compatible. To incorporate that restriction, the supplier's cost function is redefined as,

$$c_0^I(z) = \begin{cases} 0 & \text{if } z \in \mathbb{R}_+^{|A|}, \sum_{a_{-i}} [v_i(a_{-i}, d_i) - v_i(a_{-i}, a_i)]\, z_0(a) \leq 0, \forall i, a_i, d_i \\ \infty & \text{otherwise} \end{cases}$$

So the supplier's maximization problem is

$$(b') \qquad \max_{z_0} p_0 \cdot z_0 - C_0^I(z).$$

Definition 6 *A Lindahl equilibrium of correlated equilibrium game for* $\mathbf{v}$ *is a* $([z_i, p_i], z_0, p_0)$ *satisfying (a), (b′), and (c)–(e).*

Define the set of incentive feasible probabilities for each i

$$C_i(A) := \{z : \forall d_i, \sum_{a_{-i}} [v_i(a_{-i}, d_i) - v_i(a_{-i}, a_i)]\, z(a) \leq 0\}.$$

The set of incentive feasible probabilities is therefore,

$$C(A) := \cap_{i \in I} C_i(A).$$

The definition of constrained efficiency under the asymmetric information is stated.

Definition 7 *An* incentive compatible $z \in Z(A) \cap C(A)$ *is constrained efficient for* $\mathbf{v}$ *if there is no* incentive compatible $z' \in Z(A) \cap C(A)$ *such that* $v_i \cdot z' \geq v_i \cdot z$ *for all* i *and at least one inequality is strict. Regarding* $\mathbf{v}$ *as an economic model to which a supplier added, the statement of constrained efficiency is: there exists* $[(z_i), z_0]$ *such that (i)* $z_i = z_0$ *for all* i *and (ii)* $z_i \in Z(A) \cap C(A)$*, and there is no* $[(z_i'), z_0']$ *satisfying (i) and (ii) for which utility is at least as large for all individuals and strictly greater for at least one* i*.*

Lindahl equilibrium of correlated equilibrium game describes a decentralized way to establish constrained efficiency with games.

Theorem 2 [First Welfare Theorem] *A Lindahl equilibrium of correlated equilibrium game is efficient.* [Second Welfare Theorem] *For any efficient* z *associated with* $\lambda \gg 0$*, there are prices* p_i *and* p_0 *such that* $([p_i, z_i], p_0, z_0)$ *is a Lindahl equilibrium of correlated equilibrium game with* $z = z_i = z_0$*.*

2.3.1 Proof of Theorem 2 and Characterization of Welfare Theorems

The primal linear programming (the planner's problem) is

$$\begin{aligned}
\max_{z_i, z_0} \quad & \sum_i \lambda_i \left(v_i \cdot z_i\right) \\
\text{subject to} \quad & \sum_a z_i(a) = 1, \quad i = 1, \ldots, n \\
& z_i(a) - z_0(a) = 0, \quad \forall a \\
& \lambda_i \sum_{a_{-i}} \left[v_i(a_{-i}, d_i) - v_i(a_{-i}, a_i)\right] z_0(a) \leq 0, \forall i, a_i, d_i \\
& z_i(a), z_0(a) \geq 0
\end{aligned}$$

The first constraint states that the probability allocation of outcomes for each player has to sum to one. The second constraint states that the probability choice of each player has to be the same since the outcome of a game is interpreted as a public good. The third is the incentive compatibility constraint. Note that $z_0(a)$ was used instead of $z_i(a)$ for the incentive compatibility constraints so that the supplier bears the cost of the incentive compatibility constraints.

The dual linear programming (the decentralization) is

$$\begin{aligned}
\min_{y_i, p_i} \quad & \sum_i y_i \\
\text{subject to} \quad & y_i \geq \lambda_i v_i(a) - p_i(a), \quad \forall i, a \\
& 0 \geq \sum_i p_i(a) - \sum_i \lambda_i \sum_{d_i} \left[v_i(a_{-i}, d_i) - v_i(a_{-i}, a_i)\right] \alpha_i(d_i|a_i), \forall a \\
& \alpha_i(d_i|a_i) \geq 0
\end{aligned}$$

The dual programming is identical to that of section 2.1 except for the last term of the second dual constraint.

For the characterization of the welfare theorems, we prove a lemma.

Lemma 2 *For each $\lambda \gg 0$ and an efficient z associated with the λ, there exists a dual solution such that $p_i \cdot z = 0$ for all i.*

Proof. Identical to that of Lemma 1 ∎

Note that $p_o \cdot z = 0$ is automatically derived from Lemma 2. After taking an optimal solution (p_i, p_0) such that $p_i \cdot z = 0$ and $p_0 \cdot z = 0$, we show how to interpret the dual constraints in terms of players' and the supplier's optimization problems.

Players' Optimization

Identical to that of Section 2.1.1.

Supplier's Optimization

We prove a lemma that will be useful to interpret the second constraints as the supplier's optimization problem.

Lemma 3 *If $z_0(a)$ is incentive compatible,*

$$\sum_a \sum_i \lambda_i \sum_{d_i} [v_i(a_{-i}, d_i) - v_i(a_{-i}, a_i)]\, \alpha_i(d_i|a_i) z_0(a) \leq 0$$

Moreover, if $z_0(a)$ is an optimal solution of the primal, the inequality is equality.

Proof. From the incentive compatibility constraints, we conclude

$$\begin{aligned} &\sum_i \lambda_i \sum_{d_i} \sum_{a_i} \sum_{a_{-i}} [v_i(a_{-i}, d_i) - v_i(a_{-i}, a_i)]\, \alpha_i(d_i|a_i) z_0(a) \\ = &\sum_i \lambda_i \sum_{a_i} \sum_{d_i} \alpha_i(d_i|a_i) \left[\sum_{a_{-i}} [v_i(a_{-i}, d_i) - v_i(a_{-i}, a_i)]\, z_0(a) \right] \leq 0 \end{aligned}$$

Moreover, by the fundamental theorem of linear programming, we have

$$\lambda_i \sum_{a_{-i}} [v_i(a_{-i}, d_i) - v_i(a_{-i}, a_i)]\, z_0(a) \alpha_i(d_i|a_i) = 0,$$

which implies equality. Therefore, the result follows. ∎

The supplier's optimization problem is illustrated by the following inequalities

$$0 \geq \sum_i p_i \cdot z_0 - \sum_a \sum_i \lambda_i \sum_{d_i} [v_i(a_{-i}, d_i) - v_i(a_{-i}, a_i)]\, \alpha_i(d_i|a_i) z_0(a), \tag{$**$}$$

which are derived by adding the second constraints with a certain probability $z_0(a)$. By the fundamental theorem of linear programming and Lemma 3, if z_0 were an optimal choice of the primal linear programming, we have

$$0 = \sum_i p_i \cdot z_0.$$

Therefore,

- if $z_0 \in \mathbb{R}_+^{|A|}$ but $z_0 \notin C$, then z_0 is infeasible to the supplier.

- if $z_0 \in \mathbb{R}_+^{|A|}$ and $z_0 \in C$, then z_0 does not increase the supplier's profit by the following inequalities derived from Lemma 3.

$$0 \geq \sum_i p_i \cdot z_0 - \sum_a \sum_i \lambda_i \sum_{d_i} \left[v_i(a_{-i}, d_i) - v_i(a_{-i}, a_i) \right] \alpha_i(d_i|a_i) z_0(a) \geq \sum_i p_i \cdot z_0$$

  In other words, the supplier cannot get more utility by choosing a probability other than the optimal one solved by the planner.

We have shown that the choice of the supplier is consistent with the planner's.

Again by the fundamental theorem of linear programming, the value of the primal and the value of the dual are the same; hence we proved the second welfare theorem that the planner's problem is decentralized by the Lindahl price. As in the last section, we also get the result that the decentralization is always possible for any weights $\lambda \gg 0$ as was in the case without incentive compatibility constraints. The proof of the first welfare theorem follows from the proof of the second welfare theorem.

3 Monopolistic supplier of CE

In Section 2, we characterized the best correlated equilibria by Lindahl pricing; hence, proved the welfare theorems. In the classical monopoly problem, it is known that the efficient outcome is not obtained since the monopolist does not behave as a price-taker.

The environment that the classical monopolist is situated in is characterized by the followings: (i) she is the sole producer, and (ii) her ability to price discriminate is constrained. Therefore, (iii) she ends up with producing inefficient quantity. Note that, if she was not constrained to put uniform price to consumers, she produces efficient quantity, and extracts all the gains from production by setting prices equal to consumers utilities.

For an analogy to the above observation, we let a monopolistic supplier of correlated equilibrium to be situated in where (i) she is the sole supplier, and (ii) her ability to set price is limited by a regulation that will be described in the below. Therefore, (iii) she ends up with suppling inefficient correlated equilibrium. If her ability to set price is not constrained by a regulation, she will implement efficient correlated equilibrium, and will extract all the surplus from the players.

Without assumption of quasi-linear utility, it is impossible for the monopolist to extract money from the consumers as in the classical monopolist problem. So, we will assume that players' utility functions are quasi-linear.

Formally, player i's utility function is

$$v_i \cdot z_i - p_i \cdot z_i.$$

where $p_i \cdot z_i$ is money expenditure. Also, money cannot be used for incentive compatibility constraints, i.e. incentive compatibility constraints remain the same, i.e.

$$\sum_{a_{-i}} [v_i(a_{-i}, d_i) - v_i(a_{-i}, a_i)]\, z_0(a) \leq 0, \forall i, a_i, d_i.$$

We also assume individual rationality of players in the sense that monopolist cannot extract too much from the consumer.

Assumption 1 (Individual Rationality) *Monopolist cannot extract too much so that players' payoff becomes negative, i.e.*

$$v_i \cdot z_i - p_i \cdot z_i \geq 0.$$

So, monopolist's problem without constraint on price is

$$\begin{array}{ll} \max\limits_{p_i, z_0} & \sum\limits_i p_i \cdot z_0 - c_0^I(z_0) \\ s.t. & z_0 \in \operatorname*{argmax}\limits_{z_i : \sum_a z_i(a) = 1} \; v_i \cdot z_i - p_i \cdot z_i \\ & v_i \cdot z_i - p_i \cdot z_i \geq 0 \end{array}$$

We illustrate by an example how a monopolist without any constraint on price implements efficient correlated equilibrium, and extracts all the surplus from the players.

Example 1 *Let the game be the following.*

	b_1	b_2
a_1	$5, 1$	$0, 0$
a_2	$4, 4$	$1, 5$

The following is zero profit price for the monopolist to implement the best correlated equilibrium.

Payoffs	*Probability*	*Price for 1*	*Price for 2*	*Price for the monopolist*
(10/3, 10/3)	$\begin{array}{c\|c} 1/3 & 0 \\ \hline 1/3 & 1/3 \end{array}$	$\begin{array}{c\|c} 5/3 & -5/3 \\ \hline 2/3 & -7/3 \end{array}$	$\begin{array}{c\|c} -7/3 & -5/3 \\ \hline 2/3 & 5/3 \end{array}$	$\begin{array}{c\|c} -2/3 & -10/3 \\ \hline 4/3 & -2/3 \end{array}$

It is trivial that the price implements the correlated equilibrium.

However, the monopolist will set price as in the following, and extracts $20/3 = 10/3 + 10/3$.

Payoffs	*Probability*	*Price for 1*	*Price for 2*	*Price for the monopolist*
(0, 0)	$\begin{array}{c\|c} 1/3 & 0 \\ \hline 1/3 & 1/3 \end{array}$	$\begin{array}{c\|c} 15/3 & 5/3 \\ \hline 12/3 & 3/3 \end{array}$	$\begin{array}{c\|c} 3/3 & 5/3 \\ \hline 12/3 & 15/3 \end{array}$	$\begin{array}{c\|c} 18/3 & 10/3 \\ \hline 24/3 & 18/3 \end{array}$

In other words, monopolist will simply increase the price implementing the efficient correlated equilibrium by 10/3 for each of the players. Note that, if the monopolist implements less efficient correlated equilibrium, the amount that she can extract diminish. Therefore, she always implements the best correlated equilibrium.

The intuition comes from the classical monopolist's problem: the classical monopolist with price discrimination implements efficient quantity in terms of social welfare, then extract all the surplus from the consumer by charging higher prices.

3.1 The monopolist's problem

If the classical monopolist loses freedom to charge different prices, she implements less efficient outcome, i.e. monopoly outcome. In the current context of correlated equilibrium, if the monopolistic correlated equilibrium supplier faces a constraint restricting certain prices, she would implement less efficient correlated outcome. The following assumption is one that restricts the freedom of price setting.

Assumption 2 *Let $\left(p_i^{AF}(a;z)\right)_{a\in A}$ be actuarially fair prices for correlated equilibrium z, i.e. prices that gives zero profit for the monopolist. Define $\overline{p}_i(z) = \max_a p_i^{AF}(a;z)$. The monopolist cannot price any outcome a at more than $(1+\beta)\overline{p}_i(z)$, i.e. $p_i(a) \leq (1+\beta)\overline{p}_i(z)$ for given $\beta > 0$.*

The assumption can be rationalized as 'minimal consumer protection', where a regulator limits "mark-up" not to be more than β. Therefore, parameter β measures the strictness of the regulation. If β is too high, the regulation would not be meaningful, so implemented correlated equilibrium will be the best one, and the monopolist will extract all the surplus.

Formally, monopolist's problem with constraint on price is

$$\begin{aligned} \max_{p_i, z_0} \quad & \sum_i p_i \cdot z_0 - c_0^I(z_0) \\ s.t. \quad & z_0 \in \underset{z_i : \sum_a z_i(a) = 1}{\operatorname{argmax}} \; v_i \cdot z_i - p_i \cdot z_i \\ & v_i \cdot z_i - p_i \cdot z_i \geq 0 \\ & p_i(a) \leq (1+\beta)\overline{p}_i(z_0) \end{aligned}$$

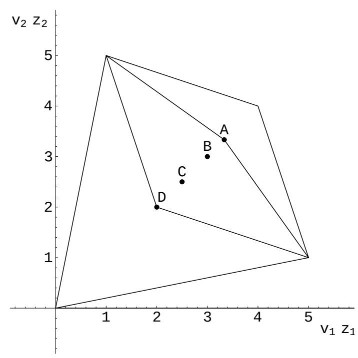
Unlike a price-taker who chooses only quantity, a monopolist chooses price and quantity. Consequently, the monopolist's problem is non-linear.

An example illustrates how the monopolist implements less efficient correlated equilibrium.

Example 2 *Again, let the game be the following.*

	b_1	b_2
a_1	$5, 1$	$0, 0$
a_2	$4, 4$	$1, 5$

All the possible correlated equilibria are illustrated in terms of payoff in the following graph.



Actuarially fair prices, payoffs, and probabilities for each correlated equilibrium are the followings.

	A	*B*	*C*	*D*
Payoffs	*(10/3, 10/3)*	*(3,3)*	*(2.5, 2.5)*	*(2,2)*
Probability	$\begin{array}{c\|c} 1/3 & 0 \\ \hline 1/3 & 1/3 \end{array}$	$\begin{array}{c\|c} 1/2 & 0 \\ \hline 0 & 1/2 \end{array}$	$\begin{array}{c\|c} 1/4 & 1/4 \\ \hline 1/4 & 1/4 \end{array}$	$\begin{array}{c\|c} 1/3 & 1/3 \\ \hline 0 & 1/3 \end{array}$
Price for 1	$\begin{array}{c\|c} 5/3 & -5/3 \\ \hline 2/3 & -7/3 \end{array}$	$\begin{array}{c\|c} 2 & -3 \\ \hline 1 & -2 \end{array}$	$\begin{array}{c\|c} 5/2 & -5/2 \\ \hline 3/2 & -3/2 \end{array}$	$\begin{array}{c\|c} 3 & -2 \\ \hline 2 & -1 \end{array}$
Price for 2	$\begin{array}{c\|c} -7/3 & -5/3 \\ \hline 2/3 & 5/3 \end{array}$	$\begin{array}{c\|c} -2 & -3 \\ \hline 1 & 2 \end{array}$	$\begin{array}{c\|c} -3/2 & -5/2 \\ \hline 3/2 & 5/2 \end{array}$	$\begin{array}{c\|c} -1 & -2 \\ \hline 2 & 3 \end{array}$
Price for the supplier	$\begin{array}{c\|c} -2/3 & -10/3 \\ \hline 4/3 & -2/3 \end{array}$	$\begin{array}{c\|c} 0 & -6 \\ \hline 2 & 0 \end{array}$	$\begin{array}{c\|c} 1 & -5 \\ \hline 3 & 1 \end{array}$	$\begin{array}{c\|c} 2 & -4 \\ \hline 4 & 2 \end{array}$

We make three observations.

1: *Except at point A that is on the utility frontier, the supplier is not selling the optimal probability under the given prices. For example, at point B, the supplier coulud sell more of* (a_2, b_1) *to increase profit. The monopolist in the classical monopoly model limits the quantity of production to increase the price, we argue that the monopolist in our model limits the probability in a certain way to raise the prices. Detailed illustration will follow after the observations.*

2: *By shifting up the prices for the player by* ϵ*, the monopolist can make player* i *to choose the same* z_0*; hence, incentive compatibility would be preserved.*

3: *The prices for (a_1, b_1) are increasing as players' utilities from the game decrease (from A to D). This is not coincidence. From the dual constraint, we had*

$$v_i \cdot z_0 = y_i = v_i(a) - p_i(a), \text{ if } z_0(a) > 0.$$

Therefore, as y_i (expected utility) increases $p_i(a)$ decreases. As a matter of fact, the prices were chosen to satisfy the above equality even when a is not in the support of z_0.

Now let us illustrate how a monopolist regulated by Assumption 2 extracts money from players. Let $\beta = 0.5$. At point A, the $p_i^M(z) = 5/3$ for both players. The highest price that the monopolist can charge is $(1+\beta)\frac{5}{3} = \frac{5}{2}$. Therefore, the amount of money that the monopolist can extract is at most $5/2 - 5/3 = 5/6$ by increasing all the prices $p_i(a)$ by $5/6$. At point B, the monopolist can charge up to $2(1+\beta) = 3$; hence, she can extract up to $3 - 2 = 1$. Eventually, the monopolist will implement correlated equilibrium D with the following prices and probability.

Prob.		*AF Price for 1*		*AF Price for 2*		*Price for 1*		*Price for 2*		*Revenue*	
1/3	1/3	3	−2	−1	−2	4.5	−0.5	0.5	−0.5	5	−1
0	1/3	2	−1	1	3	3.5	0.5	3.5	4.5	7	5

Hence, the profit is $1/3 \cdot 5 + 1/3 \cdot 5 + 1/3 \cdot (-1) = 3$. The utility each player gets is $(1/3 \cdot 5 + 1/3 \cdot 1 + 1/3 \cdot 0) - (1/3 \cdot 4.5 + 1/3 \cdot 0.5 + 1/3 \cdot (-0.5)) = 0.5$. Note that individual rationality condition is not binding in this example.

For simplicity of exposition, we ignore individual rationality condition for the time being, and state the main theorem. In the Section 3.3, we illustrate how individual rationality interacts with Assumption 2.

Theorem 3 *The monopolist chooses the worst correlated equilibrium.*

Proof. See Section 3.2 ∎

3.2 Proof of Theorem 3

Player i's optimization can be represented by the following.

$$v_i(a) - p_i(a) - \rho_i \leq 0, \quad \text{equality if} \quad z_i(a) > 0$$
$$\sum z_0(a) = 1$$

where ρ_i is the multiplier of the probability constraint.

It is without loss of generality to have

$$v_i(a) - p_i(a) - \rho_i = 0, \forall a$$

since the monopolist can set $p_i(a)$ lower to make the equality hold even when a is not on the support of the correlated equilibrium that the monopolist wants to set. Because the utility function is linear, player i would choose the correlated equilibrium that the monopolist wants to implement.

Adding the FOC with weight $z_0(a)$ that the monopolist wants to implement, we get

$$v_i \cdot z_0 - p_i \cdot z_0 - \rho_i = 0$$

Therefore,

$$\rho_i = v_i \cdot z_0 - p_i \cdot z_0 = v_i(a) - p_i(a), \forall a \in A$$

Actuarially fair price is defined as the zero profit prices for the monopolist. Therefore, *actuarially fair price* $\tilde{p}_i(a)$ is defined by

$$v_i(a) - \tilde{p}_i(a) = \rho_i = v_i \cdot z_0 - \tilde{p}_i \cdot z_0 = v_i \cdot z_0$$

For the regulation of Assumption 2, we derive

$$\begin{aligned} p_i(a) &\leq (1+\beta)\overline{p}_i(z) = (1+\beta) \max_a \tilde{p}_i(a) \\ &= (1+\beta) \max_a \left[v_i(a) - v_i \cdot z_0\right] = (1+\beta) \left[\max_a [v_i(a)] - v_i \cdot z_0\right] \end{aligned} \tag{$***$}$$

Therefore, the program is equivalent to

$$\begin{aligned} \max_{p_i, z_0} \quad & \sum_i p_i \cdot z_0 \\ s.t. \quad & v_i \cdot z_0 - p_i \cdot z_0 = v_i(a) - p_i(a) \\ & \sum z_0(a) = 1 \\ & p_i(a) \leq \left[\max_a [v_i(a)] - v_i \cdot z_0\right] (1+\beta) \\ & \sum_{a_{-i}} \left[v_i(a_{-i}, d_i) - v_i(a_{-i}, a_i)\right] z_0(a) \leq 0, \forall i, \forall a_i, \forall d_i \end{aligned}$$

By substituting $I_i = p_i \cdot z_0$ into the objective function and the first constraint, we get

$$\begin{aligned}
\max_{p_i, z_0, I_i} \quad & \sum_i I_i \\
s.t. \quad & v_i \cdot z_0 - I_i = v_i(a) - p_i(a) \\
& I_i = p_i \cdot z_0 \\
& \sum z_0(a) = 1 \\
& p_i(a) \leq \left[\max_a [v_i(a)] - v_i \cdot z_0\right](1+\beta) \\
& \sum_{a_{-i}} [v_i(a_{-i}, d_i) - v_i(a_{-i}, a_i)]\, z_0(a) \leq 0, \forall i, \forall a_i, \forall d_i
\end{aligned}$$

However, the second constraint is redundant since it can be derived from the first and the third constraints by adding the first constraints with weight $z_0(a)$. Therefore, we get a linear programming formulation.

$$\begin{aligned}
\max_{p_i, z_0, I_i} \quad & \sum_i I_i \\
s.t. \quad & v_i \cdot z_0 - I_i = v_i(a) - p_i(a) \\
& \sum z_0(a) = 1 \\
& p_i(a) \leq \left[\max_a [v_i(a)] - v_i \cdot z_0\right](1+\beta) \\
& \sum_{a_{-i}} [v_i(a_{-i}, d_i) - v_i(a_{-i}, a_i)]\, z_0(a) \leq 0, \forall i, \forall a_i, \forall d_i
\end{aligned}$$

- **(Step 1) For given correlated equilibrium z_0, $I_i = \beta\left[\max_a [v_i(a)] - v_i \cdot z_0\right]$ and $p_i(a) := v_i(a) - v_i \cdot z_0 + I_i$ are optimal.**

  The first constraint is satisfied by the choice of $p_i(a)$. The second constraint is also satisfied since z_0 is a probability. The fourth is also satisfied since z_0 is a correlated equilibrium. The third constraint is also satisfied by the following.

  $$\begin{aligned}
  p_i(a) = v_i(a) - v_i \cdot z_0 + I_i \quad &= \quad v_i(a) - v_i \cdot z_0 + \beta\left[\max_{\tilde{a}} v_i(\tilde{a}) - v_i \cdot z_0\right] \\
  &\leq \quad (1+\beta)\left[\max_{\tilde{a}} v_i(\tilde{a}) - v_i \cdot z_0\right].
  \end{aligned}$$

  Also, the third constraint is binding for $a \in \operatorname{argmax}_{\tilde{a}} v_i(\tilde{a})$. It is impossible to increase I_i more without breaking any constraints; hence, the optimum is achieved.

- **(Step 2) The worst equilibrium z_0 (in the sense that $\sum_i v_i \cdot z_0$ is minimized with incentive compatibility constraint) is an optimal solution to the monopolist's program with $I_i = \beta\left[\max_a [v_i(a)] - v_i \cdot z_0\right]$ and $p_i(a) := v_i(a) - v_i \cdot z_0 + I_i$.**

The objective function of the monopolist is

$$\sum_i I_i = \beta \left[\sum_i \max_a [v_i(a)] - \sum_i v_i \cdot z_0 \right] = \beta \sum_i \max_a [v_i(a)] - \beta \sum_i v_i \cdot z_0$$

Therefore, the optimality is shown. ∎

3.3 Individual Rationality Condition – Revisited

For the monopolist without constraint on price setting, individual rationality condition was introduced as a mean to deter the monopolist to extract infinite amount of welfare from the players. Under Assumption 2, infinite amount of welfare extraction is already impossible because of condition $(***)$, i.e. there is upper bound for price.

For the game that payoff of each outcome is high enough as in Example 2, individual rationality is not binding. Individual rationality matters only when payoff for some outcome is close to zero or negative as in the following example.

Example 3 *Let the game be the following.*

	b_1	b_2
a_1	$3, -1$	$-2, -2$
a_2	$2, 2$	$-1, 3$

Each player's payoff shifted down by 2 *from Example 2. Let* $\beta = 0.5$.

The following price is actuarially fair.

Payoffs	*Probability*	*Price for 1*	*Price for 2*	*Price for the monopolist*
(1, 1)	$\begin{array}{c\|c} 1/2 & 0 \\ \hline 0 & 1/2 \end{array}$	$\begin{array}{c\|c} 2 & -3 \\ \hline 1 & -2 \end{array}$	$\begin{array}{c\|c} -2 & -3 \\ \hline 1 & 2 \end{array}$	$\begin{array}{c\|c} 0 & -6 \\ \hline 2 & 0 \end{array}$

We show that the following correlated equilibrium is a solution of the monopolist.

Payoffs	*Probability*	*Price for 1*	*Price for 2*	*Price for the monopolist*
(0, 0)	$\begin{array}{c\|c} 1/2 & 0 \\ \hline 0 & 1/2 \end{array}$	$\begin{array}{c\|c} 3 & -2 \\ \hline 2 & -1 \end{array}$	$\begin{array}{c\|c} -1 & -2 \\ \hline 2 & 3 \end{array}$	$\begin{array}{c\|c} 2 & -4 \\ \hline 4 & 2 \end{array}$

Since payoff is 0*, individual rationality constraints are binding. Also note that condition of the regulation is binding too, i.e.*

$$3 = p_1(a_1, b_1) = (1+\beta) \max_{(\tilde{a}, \tilde{b})} \overline{p}_1(\tilde{a}, \tilde{b}) = (1+\beta)\overline{p}_1(a_1, b_1) = 3,$$

$$3 = p_2(a_2, b_2) = (1+\beta) \max_{(\tilde{a}, \tilde{b})} \overline{p}_2(\tilde{a}, \tilde{b}) = (1+\beta)\overline{p}_2(a_2, b_2) = 3.$$

Therefore, if there is a way to increase profit more, it must be by increasing $v_i \cdot z_i$. However, if it is done, $p_i(a)$ has to decrease further because of

$$p_i(a) \leq \left[\max_a[v_i(a)] - v_i \cdot z_0\right](1+\beta)$$

Considering condition

$$v_i \cdot z_0 - I_i = v_i(a) - p_i(a) \Leftrightarrow I_i = v_i \cdot z_0 - v_i(a) + p_i(a)$$

the profit cannot go up since increase in $v_i \cdot z$ is dwarfed by decrease in $p_i(a)$ that I_i decreases. ∎

When individual rationality is binding, worst correlated equilibrium is (typically) not implemented. Formally, let us add individual rationality condition back to the monopolist's problem in Section 3.2. If individual rationality is not binding, we are done. If individual rationality does not hold, we have to decrease $p_i(a)$ since

$$v_i \cdot z_0 - I_i = v_i(a) - p_i(a).$$

Now condition $p_i(a) \leq [\max_a[v_i(a)] - v_i \cdot z_0](1+\beta)$ is relaxed. By increasing $v_i \cdot z_0$, we make it tighter again. This procedure of binding the constraint is beneficial to the monopolist since she can extract larger I_i considering $v_i \cdot z_0 - I_i = v_i(a) - p_i(a)$. Therefore, we have shown that a correlated equilibrium better than the worst one is (typically[2]) implemented when individual rationality is binding.

Proposition 1 *When individual rationality condition is binding, correlated equilibrium (typically) better than the worst one is implemented by the monopolist.*

We also note a relationship between β and $v_i \cdot z_0$.

Proposition 2 *When individual rationality condition is binding,*

$$\beta = \frac{\max_a[v_i(a)]}{\max_a[v_i(a)] - v_i \cdot z_0} - 1$$

Proof. For at least one a, $p_i(a) = (1+\beta)[\max_{\tilde{a}} v_i(\tilde{a}) - v_i \cdot z_0]$. Because of $v_i \cdot z_0 - I_i = v_i(a) - p_i(a)$, we have

$$\begin{aligned} & v_i(a) - v_i \cdot z_0 + I_i = (1+\beta)[v_i(a) - v_i \cdot z_0] \\ \Rightarrow \quad & 1+\beta = \frac{v_i(a) - v_i \cdot z_0 + I_i}{v_i(a) - v_i \cdot z_0} = 1 + \frac{I_i}{v_i(a) - v_i \cdot z_0} \\ \Rightarrow \quad & \beta = \frac{I_i}{v_i(a) - v_i \cdot z_0} = \frac{v_i(a)}{v_i(a) - v_i \cdot z_0} - 1 \qquad \text{because of } v_i \cdot z_0 - p_i \cdot z_0 = 0 \end{aligned}$$

[2]There are rare cases that individual rationality is binding, and the worst correlated equilibrium is implemented.

for $a \in \underset{\tilde{a}}{\operatorname{argmax}} v_i(\tilde{a})$. ∎

Therefore, stricter regulation (smaller β) means worse correlated equilibrium from Proposition 2. However, note that when $\beta = 0$, the monopolist is indifferent between any correlated equilibrium since she cannot extract any surplus because of the regulation.

References

[1] Aumann, R. J., "Subjectivity and Correlation in Randomized Strategies," *Journal of Mathematical Economics* 1: 67-96 (1974)

[2] Hart, S., and D. Schmeidler, "Existence of Correlated Equilibria," *Mathematics of Operations Research* 14: 18-25 (1989)

[3] Myerson, R. B., "Game Theory: Analysis of Conflict," Harvard University Press (1991)

[4] Nau, R. F., and K. F. McCardle, "Coherent Hehavior in Noncooperative Games," *Journal of Economic Theory* 50: 424-444 (1990)

[5] Shapley, L. S., "Utility Comparison and the Theory of Games", *La Decision*: 251-263, Paris: Editions du Centre National de la Recherche Scientifique. (1969) (Reprinted on pp.307-319 of The Shapley Value (Alvin E. Roth, ed.), Cambridge: Cambridge University Press, 1989)

[6] Shapley, L. S. and M. Shubik, "The assignment game I: the core," *International Journal of Game Theory* 1: 111-130 (1971)